Master thesis report

# Spectrum of unbounded operators and applications to PDEs.

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Master thesis supervised by Romain Joly and co-supervised by Eloi Martinet.

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**Abstract**

This manuscript is the outcome of my long research project, conducted as part of the conclusion of my Ensimag engineering training and the preparation for my MSIAM double university degree.

All the studies conducted during this master thesis are based on the fundamental result of the diagonalization of the Laplacian with homogeneous Dirichlet boundary conditions. The second section first explores the asymptotic behavior of eigenvalues through the proof of Weyl's law. We then address an optimization problem by proving that the domain minimizing the fundamental eigenvalue of the Laplacian at constant volume is the ball, according to Faber-Krahn inequality theorem.

The final section presents the numerical results obtained. The first objective was to develop an algorithm that could take any domain in $\mathbb{R}^2$ as input and calculate the eigenvalues and eigenfunctions of the Laplacian. A step-by-step study is carried out; first, we solve the one-dimensional case, then extend the reasoning to the two-dimensional rectangle, and finally, we generalize it to any domain in $\mathbb{R}^2$. From this algorithm, we design a numerical optimization procedure to establish the Faber-Krahn inequality and extend it to the second eigenvalues. For this shape optimization problem, we represent the domain using the phase field method. My code is available on the git repository associated with the thesis.

In the appendix, we present a brief proof of the diagonalization result of the Laplacian. This allows us to highlight the essential results of functional analysis necessary for our study and to include, for the sake of completeness, a sketch of the proof of the central result of our studies.

The final appendix compiles various results from measure theory, thus clarifying and formalizing the statements of the results used throughout our study.

***Keywords:*** *Laplacian, Diagonalization, Functional analysis, Weyl's law, Riesz rearrangement inequality, Faber-Krahn inequality, Finite differences, Phase-field.*

**Résumé**

Ce manuscrit constitue l'aboutissement de mon projet long de recherche, mené dans le cadre de la conclusion de ma formation d'ingénieur Ensimag et de la préparation de mon double diplôme universitaire MSIAM.

L'intégralité des études réalisées durant ce stage s'appuie sur le résultat fondamental de diagonalisation du Laplacien avec conditions au bord de Dirichlet homogènes. La seconde section explore d'abord le comportement asymptotique des valeurs propres par la démonstration de la loi de Weyl. Nous abordons ensuite un problème d'optimisation en démontrant que le domaine qui minimise la valeur propre fondamentale du Laplacien à volume constant est la boule, selon le célèbre théorème de l'inégalité de Faber-Krahn.

La dernière section expose les résultats numériques obtenus. Le premier objectif était de développer un algorithme capable de prendre en entrée un domaine quelconque de $\mathbb{R}^2$ et de calculer les valeurs et fonctions propres associées. Une étude pas à pas est réalisée; d'abord, nous résolvons le cas unidimensionnel, nous étendons le raisonnement au rectangle bidimensionnel, et enfin, nous généralisons au cas de tout domaine de $\mathbb{R}^2$. À partir de cet algorithme, nous mettons en place une procédure d'optimisation numérique afin d'établir l'inégalité de Faber-Krahn et l'étendons aux valeurs propres suivantes. Pour ce problème d'optimisation de forme, nous représentons le domaine à l'aide de la méthode du champ de phase (phase-field). Mon code est disponible sur le dépot git associé au stage.

En annexe, nous présentons premièrement une démonstration succincte du résultat de diagonalisation du Laplacien. Cela nous permet de mettre en lumière les résultats essentiels de l'analyse fonctionnelle nécessaires à notre étude, et d'inclure, par souci de complétude, une esquisse de preuve du résultat central de notre recherche.

La dernière annexe compile divers résultats de la théorie de la mesure, clarifiant et formalisant ainsi les énoncés des résultats utilisés tout au long de notre étude.

***Mots clés:*** *Laplacien, Diagonalisation, Analyse fonctionnelle, Loi de Weyl, Inégalité de réarrangement de Riesz, Inégalité de Faber-Krahn, Différences finies, Phase-field.*

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Lastly, I cannot forget to thank sincerely Éric Dumas, professor and researcher at the Institut Fourier, for kindly agreeing to supervise my first research project at the end of my Licence 3. This experience and his support over the last few years have been decisive in confirming my orientation toward theoretical studies.

# Contents

# I Study of some evolution properties of eigenvalues

## I.1 Introduction

The core of this manuscript revolves around the study of solutions to the eigenvalue problem of the Laplacian with homogeneous Dirichlet boundary conditions. Throughout this document, unless otherwise specified, we will consider $\Omega$ as an open subset of $\mathbb{R}^N$, where $N \in \mathbb{N}$. The problem can be formulated as follows

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma := \partial\Omega. \end{cases} \tag{1}$$

In Appendix A a brief overview of the fundamental concepts and results of functional analysis is presented, along with a sketch of the proof of Theorem A.1 which is the central result of our study. This theorem establishes the spectral decomposition on $L^2(\Omega)$ of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions. More precisely, it ensures the existence of a Hilbert basis of $L^2(\Omega)$ of eigenfunctions of the Dirichlet Laplacian. In other words, there exists a sequence of pairwise orthonormal eigenfunctions that cover a dense space in $L^2(\Omega)$ such that these functions are solutions of (1). We denote the eigenfunctions as $(e_n)$ and the associated eigenvalues as $(\lambda_n)$. Furthermore, it also specifies that we can find representatives of these eigenfunctions in $H_0^1(\Omega) \cap C^\infty(\Omega)$ and that the associated eigenvalues are positive and goes to infinity. The studies carried out here aim to better understand the behavior of eigenvalues and the link they have with the domain $\Omega$. More precisely, we will first focus on what their asymptotic evolution reveals on $\Omega$ and then on the problem of minimizing the first eigenvalue at constant volume. But first, let us introduce the problem so that we can better understand what is at stake.

### I.1.1 Genesis of this problem

Let us introduce the eigenvalue problem through a two-dimensional interpretation. A well-established fact is that the perpendicular motion, denoted by $v(x,t)$, of a membrane constrained within $\Omega$ along its boundary $\Gamma$, satisfies the wave equation (2), where $x$ denotes the spatial variable and $t$ denotes time.

$$\frac{\partial^2 v}{\partial t^2} = c^2 \Delta v \qquad \text{in } \Omega \times \mathbb{R}^+. \tag{2}$$

The constant $c$ represents a value determined by the physical attributes of the membrane and the tension applied to it. Without loss of generality, from now on we will choose $c = 1$. Indeed, subject to time dilation, we can adjust the value of $c$. The solutions we will focus on are the stationary solutions. This choice restricts our study to a subset of all possible solutions. For instance, in the 1D case, while the solution on $\mathbb{R}$ follows the d'Alembert formula, the separation of variables leads to a more specific solution. However, since we don't know the general solution of the wave equation in arbitrary domains of $\mathbb{R}^d$, this case remains fascinating to explore. This last choice leads us to solutions of the following form

$$v(x,t) = u(x)w(t) \qquad x \in \Omega,\ t \in \mathbb{R}^+.$$

To derive the expression for such a solution, we substitute this expression into the wave equation, resulting in

$$v(x,t) = u(x)e^{i\omega t} \qquad x \in \Omega,\ t \in \mathbb{R}^+. \tag{3}$$

These solutions, named normal modes, being harmonic in time embody the pure tones that the membrane can generate. Moreover, $u$ must satisfy

$$-\Delta u = \omega^2 u \qquad \text{in } \Omega, \tag{4}$$

with the boundary condition $u = 0$ on the boundary $\Gamma$, corresponding to the membrane being fixed along boundary. This equation is also known as the Helmholtz equation. To summarize, the problem is indeed based on the eigenvalues of the Laplacian operator under the Dirichlet boundary condition.

### I.1.2 Importance of the inverse problem

We will now turn our attention to the inverse problem, i.e. what the sequence $(\lambda_n)_{n\geq 1}$ reveals about $\Omega$. The mathematical beauty of this problem lies in its ability to highlight the profound relationships between the geometry of a space and its spectral characteristics. In particular, it closely links spectral analysis, the theory of partial differential equations and geometry. A non-mathematical interpretation of this question could be the following. The sounds a drum makes when it is struck are determined by its physical characteristics, such as the material used, its tautness, and its size and shape. Drums vibrate at certain distinct frequencies called normal modes as seen previously. Suppose a drum is being played in one room, and a person with perfect pitch hears it but cannot see the drum. Is it possible for her to deduce the precise shape of the drum just from hearing the fundamental tone and all the overtones? The solution to such a problem has many direct applications. For example, reconstructing the geometry of a shape or domain from its eigenfrequencies offers applications in medical imaging and pattern recognition.

### I.1.3 State-of-the-art and proposed study

This problem of reconstructing a domain from its vibration spectrum was popularized by Mark Kac in his 1966 paper "Can one hear the shape of a drum?" [Kac66]. In this landmark publication, he specifies the following question: does a sequence of eigenvalues (the harmonics of the drum) characterize, up to an isometry, the manifolds (the geometry of the drum)? He proved that certain geometric properties of domains in $\mathbb{R}^2$, such as volume, can be deduced from an examination of the asymptotic behavior of the Laplacian spectrum. Furthermore, he conjectured that the domain might be completely determined by the spectral characteristics of the Laplacian. This conjecture was formulated specifically for dimension 2. Indeed, shortly before the publication of his paper, a counterexample was proposed by John Milnor in [Mil64]. He found a pair of tori in $\mathbb{R}^{16}$ with identical but non-isometric spectra. It wasn't until 1992 that the problem in two dimensions was finally solved by Carolyn Gordon, David Webb, and Scott Wolpert in [GWW92]. They managed to create two different flat shapes shown in Figure 1 with the same spectrum.

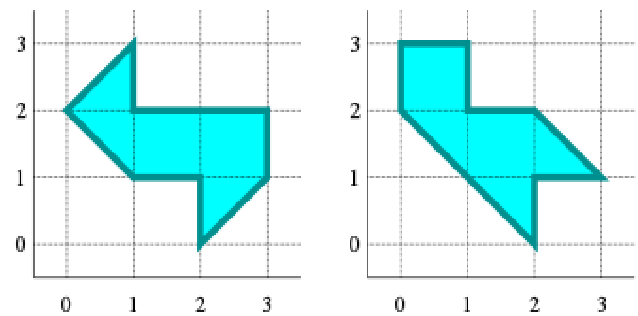


Figure 1: The first counterexample to Kac's conjecture, presented by Gordon, Webb, and Wolpert in 1992. Illustration taken from the Wikipedia page.

However, Steven Zelditch proved that Kac's question had a positive answer when limited to certain convex planar regions with analytic boundaries. The general conclusion regarding Kac's question is negative, one cannot ensure the shape just by listening to the tones. Meanwhile, the spectrum offers broader insights, enabling the identification of specific features such as area, dimension, and the number of holes for example. This is where our study begins. First, we will begin by proving Weyl's law. This law results from a conjecture made by David Hilbert, asserting that the asymptotic behavior of eigenvalues would be of paramount importance in such a problem. A result of this kind was formally proven for the first time by Hermann Weyl in 1911 in his publication [Wey11]. For $\Omega$ be a bounded open set in $\mathbb{R}^N$ such that $\partial\Omega$ is negligible, we have

$$\lambda_n \underset{n\to\infty}{\sim} 4\pi^2 \left(\frac{n}{B_N V}\right)^{2/N},$$

where $V$ is the measure of $\Omega$ and $B_N$ is the measure of the unit ball in $\mathbb{R}^N$. This formulation of the law is taken from the publication [Pro87], which gives an overview of known and important results concerning the inverse spectral problem of the Laplacian. However, there are other equivalent formulations, such as

$$N(\lambda) \underset{\lambda\ +\infty}{\sim} \frac{B_N V}{(2\pi)^N} \lambda^{N/2},$$

where $N(\lambda) = card\{\lambda_n \leq \lambda\}$. This result not only reveals the asymptotic growth pattern of eigenvalues as $n^{2/N}$ but also guarantees that two sets with different volumes cannot share the same spectrum! Thus, the asymptotic study of eigenvalues enables us to deduce both the dimension and the volume of the domain. This result is proven in Section I.2. Now that we have a better idea of the asymptotic behavior, the question naturally arises regarding the first eigenvalues. More specifically, we are going to study the influence of the shape of the domain on the fundamental eigenvalue, i.e. the first one. Rayleigh conjectured in 1877 that, among all fixed membranes with a specified area, the sphere would minimize the first eigenvalue, as documented in [Ray77]. This conjecture was subsequently proven by Faber and Krahn in the 1920s through a rearrangement technique. Since then, various proofs of this assertion have been presented in the academic literature. This result is known today as the Faber-Krahn inequality and is expressed as

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue associated with the bounded domains $\Omega$ of $\mathbb{R}^N$ and $\Omega^*$ is the centered Euclidian ball of the same volume. This result is proven in Section I.3.

## I.2 Estimation of the asymptotic evolution of eigenvalues

The proof we are about to present here is greatly inspired by the one presented in the book "PDE: An Introduction" by Walter A. Strauss [Str92] and also in "Methods of Mathematical Physics" by David Hilbert and Richard Courant [CH89]. We will begin by establishing the law for a rectangle $\Omega_R$ in the plane $\mathbb{R}^2$. The proof conducted for a two-dimensional rectangle can be generalized to blocks (rectangular cuboids) of any dimension $N \in \mathbb{N}$. Building on this result, we will extend the result to any bounded domain in $\mathbb{R}^N$ using a variational characterization of the eigenvalues of the Laplacian. To do this proof, we need to tackle the eigenvalue problem of the Laplacian operator with homogeneous Neumann boundary conditions. More precisely, we will simultaneously show Weyl's law for the Dirichlet and Neumann boundary conditions.

### I.2.1 Proof of Weyl's law in a rectangle

In this section, we will solve the eigenvalue problem for homogeneous Dirichlet conditions in a plane rectangle. Subsequently, we set

$$\Omega_R = \big\{(x, y) \in \mathbb{R}^2 \mid 0 < x < a,\ 0 < y < b\big\}$$

with $a > 0$ and $b > 0$. The natural choice is to use a Cartesian coordinate system. The eigenvalue problem becomes

$$\begin{cases} -\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{5}$$

Once again, we're going to use variable separation. However, as we will see later on, here this choice will allow us to consider all the solutions to our problem. We search a solution of the form

$$u(x, y) = X(x)Y(y). \tag{6}$$

This leads us to

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda.$$

In this equation, the spatial variables along each axis are separated, implying that each term is indeed constant. To see this, simply consider translations along each axis. Thus,

$$X'' + \lambda_x X = 0, \qquad Y'' + \lambda_y Y = 0 \qquad \text{and} \qquad \lambda = \lambda_x + \lambda_y.$$

Let us first focus on the equation for $X$. The boundary conditions on the left and right sides of the rectangle yield

$$X(0)Y(y) = 0 \qquad \text{and} \qquad X(a)Y(y) = 0 \qquad \forall y.$$

We require $Y$ not to be identically zero, otherwise our eigenfunction will be zero. So,

$$X'' + \lambda_x X = 0, \qquad X(0) = 0, \quad X(a) = 0.$$

Note that these equations ensure that $\lambda_x$ is an eigenvalue of the 1D Laplacian with Dirichlet boundary condition on certain domains, so $\lambda_x > 0$. The solution to this ordinary differential equation is expressed as a linear combination of cosine and sine functions,

$$X(x) = A\cos(\mu_x x) + B\sin(\mu_x x),$$

with $\mu_x^2 = \lambda_x$. Subsequently, the boundary condition provides us

$$X(0) = 0 = A \qquad \text{and} \qquad X(a) = 0 = B\sin(\mu_x a).$$

This leads us to the conclusion that $\sin(\mu_x a) = 0$, implying

$$\mu_x = \frac{l\pi}{a}, \qquad\qquad l \in \mathbb{N}.$$

Then,

$$\lambda_{x,l} = \frac{l^2\pi^2}{a^2}, \qquad X_l(x) = \sin\left(\frac{l\pi}{a}x\right), \qquad \text{for } l \in \mathbb{N}.$$

For $Y$, the same procedure yields

$$\lambda_{x,m} = \frac{m^2\pi^2}{b^2}, \qquad Y_m(y) = \sin\left(\frac{m\pi}{b}y\right), \qquad \text{for } m \in \mathbb{N}.$$

Thus, for $(l, m) \in \mathbb{N}^* \times \mathbb{N}^*$, the eigenvalues of the initial problem is

$$\lambda_{l,m} = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \tag{7}$$

with corresponding eigenfunctions

$$e_{l,m}(x, y) = \sin\left(\frac{l\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right).$$

It is important to note that if $l = 0$ or $m = 0$ we obtain a zero eigenfunction, which is not admissible. Moreover, the family of functions $(\sin(xl\pi/a)\sin(ym\pi/b))_{l,m\geq 1}$ forms a Hilbert basis of $L^2(\Omega_R)$. This result is a direct consequence of $(\sin(nx\pi/a))_{n\geq 1}$ forming a Hilbert basis of $L^2(]0, a[)$. Indeed, let $f \in L^2(\Omega_R)$, we set $f(\cdot, y)$ the function $x \mapsto f(x, y)$ and $f(x, \cdot)$ the function $y \mapsto f(x, y)$. The Fubini's theorem ensures us that $f(\cdot, y)$ is in $L^2(]0, a[)$ for almost every $y \in ]0, b[$ and $f(x, \cdot)$ is in $L^2(]0, b[)$ for almost every $x \in ]0, a[$. Therefore, we can write that for almost every $y \in ]0, b[$,

$$x \mapsto f(x, y) = \sum_l \mu_{x,l}(y)\sin(l\pi x/a), \qquad\qquad x \in ]0, a[, \tag{8}$$

and also for almost every $x \in ]0, a[$,

$$y \mapsto f(x, y) = \sum_m \mu_{y,m}(x)\sin(m\pi y/b), \qquad\qquad y \in ]0, b[.$$

Moreover,

$$\mu_{y,m}(x) = \langle y \mapsto \sin(m\pi y/b), f(x, \cdot)\rangle = \int_0^b f(x, y)\sin(m\pi y/b)dy.$$

Using equation Eq. (8) we have

$$\mu_{y,m}(x) = \int_0^b \sum_l \mu_{x,l}(y)\sin(l\pi x/a)\sin(m\pi y/b)dy.$$

We want to exchange the position of the integral and the sum. We know that $C_c^\infty(\Omega_R)$ is included in $L^2(\Omega_R)$, so we can assume for the moment that $f$ is in $C_c^\infty(\Omega_R)$. To invert sum and integral, we require the uniform convergence of the series $\sum_l \mu_{x,l}(y)\sin(l\pi x/a)\sin(m\pi y/b)$, which is equivalent to the uniform convergence of the series $\sum_l \mu_{x,l}(y)\sin(l\pi x/a)$ and the continuity of $]0, b[ \ni y \mapsto \mu_{x,l}(y)\sin(l\pi x/a)\sin(m\pi y/b)$.

The continuity of this function is equivalent to the continuity of $y \mapsto \mu_{x,l}(y) = \int_0^a f(x,y)\sin(l\pi y/a)dx$. We have that $y \mapsto f(x,y)\sin(l\pi y/a)$ is continuous on $]0,b[$ for all $x$ in $]0,a[$ and $|f(x,y)\sin(l\pi y/a)| \leq ||f||_\infty$. Then, Theorem B.8 gives us the continuity of $y \mapsto \mu_{x,l}(y)$. Regarding uniform convergence, we have

$$|\mu_{x,l}(y)\sin(l\pi x/a)| \leq |\mu_{x,l}(y)| = \left| \int_0^a f(x,y)\sin(l\pi y/a)dx \right|$$

and by a double integration by part we get

$$|\mu_{x,l}(y)\sin(l\pi x/a)| \leq \left| \frac{a^2}{l^2\pi^2} \int_0^a f''(x,y)\sin(l\pi y/a)dx \right| \leq \frac{a^2}{l^2\pi^2} \int_0^a |f''(x,y)|\, dx = \frac{\alpha^2}{l^2}.$$

Thus, the series $\sum_l \mu_{x,l}(y)\sin(l\pi x/a)$ converges normally and then uniformly. So, after interchanging the sum and the integral, we obtain

$$\mu_{y,m}(x) = \sum_l \sin(l\pi x/a) \int_0^b \mu_{x,l}(y)\sin(m\pi y/b)dy.$$

Since $\mu_{x,l}(y) = \langle x \mapsto \sin(l\pi x/a), f(\cdot,y)\rangle$ for every $x \in ]0,a[$, we have

$$\mu_{y,m}(x) = \sum_l \sin(l\pi x/a) \int_0^b \int_0^a f(x',y)\sin(\frac{l\pi x'}{a})\sin(\frac{m\pi y}{b})dx'dy = \sum_l \mu_{l,m}\sin(l\pi x/a).$$

This implies that for any $f \in C_c^\infty(\Omega_R)$, we have for all $(x,y) \in \Omega_R$,

$$f(x,y) = \sum_{l,m} \mu_{l,m}\sin(l\pi x/a)\sin(m\pi y/b).$$

Thanks to Theorem A.8, we know that $C_c^\infty(\Omega_R)$ is dense in $L^2(\Omega)$. Therefore, the vector space spanned by $(\sin(xl\pi/a)\sin(ym\pi/b))_{l,m\geq 1}$ is dense in $L^2(\Omega)$. It is easy to show that the elements of this family are pairwise orthogonal. We conclude that this family forms a Hilbert basis of $L^2(\Omega)$. This allows us to justify the separation of variables (6) and to assert that we have all the eigenvalues. Note that we have arbitrarily set the amplitude of all our functions to 1 for the sake of clarity. Meanwhile, it would be interesting to consider the question of normalizing these functions. Yet, here, only the eigenvalues are of interest to us. Let us take the liberty of designating $\lambda_{l,m}$ as $\lambda_n$ such that $(\lambda_n)_{n\geq 1}$ is an increasing sequence. We recall that $N(\lambda) := card\{\lambda_n \leq \lambda\}$, then $N(\lambda_n) = n$. We can also express $N(\lambda)$ thanks to Eq. (7) as the number of couple $(l,m) \in \mathbb{N}^2$ satisfying

$$\frac{l^2\pi^2}{\lambda a^2} + \frac{m^2\pi^2}{\lambda b^2} \leq 1.$$

In other words, it represents the number of points $(l,m) \in \mathbb{N}^2$ that lie in the upper right quarter of the ellipse with semi-axes $\frac{\sqrt{\lambda}a}{\pi}$ and $\frac{\sqrt{\lambda}b}{\pi}$. Hence, $N(\lambda)$ is bounded by the area of this domain. Indeed, each $(l,m)$ represents the upper-right corner of a unit square contained in the upper-right quarter ellipse, these points are represented in Figure 2. Then we obtain

$$N(\lambda) \leq \lambda\frac{ab}{4\pi}.$$

We now seek to obtain a lower bound that is precise enough to guarantee asymptotic growth. To achieve this, let us consider the same ellipse but with both semi-axes reduced by 1. This is depicted in blue in Figure 2. Such a reduction guarantees that its area is less than or equal to $N(\lambda)$. We therefore have

$$N(\lambda) \geq \frac{\pi}{4}(\frac{\sqrt{\lambda}a}{\pi} - 1)(\frac{\sqrt{\lambda}b}{\pi} - 1) \geq \lambda\frac{ab}{4\pi} - \frac{\sqrt{\lambda}}{4}(a+b).$$

That is, there is a constant C independent of $\lambda$ such that

$$\lambda\frac{ab}{4\pi} - C\sqrt{\lambda} \leq N(\lambda) \leq \lambda\frac{ab}{4\pi}.$$

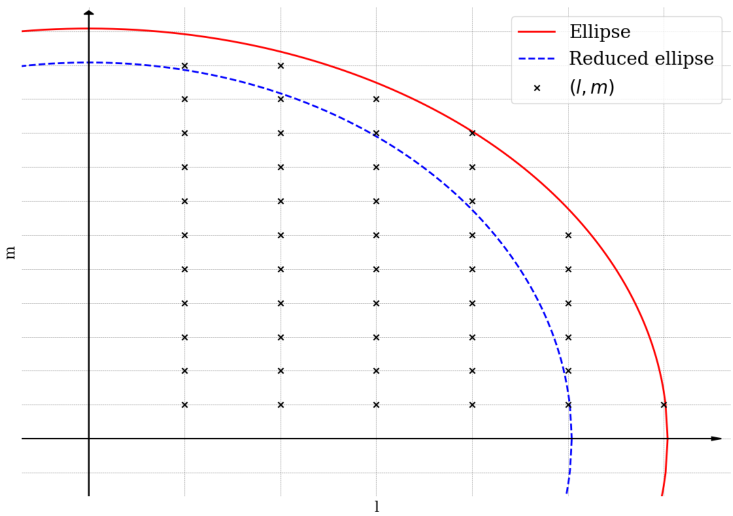



Figure 2: Counting the eigenvalues of the Laplacian for the rectangle. Illustration of $(l, m)$ pairs of $\lambda_{l,m}$ being counted in $N(\lambda)$.

We can interpret that the volume counted in excess is less than the length of the perimeter of the ellipse proportional to $\sqrt{\lambda}$, in order of magnitude. Finally, if we take $\lambda = \lambda_n$,

$$\lambda_n \frac{ab}{4\pi} - C\sqrt{\lambda_n} \leq n \leq \lambda_n \frac{ab}{4\pi}.$$

Recalling that $\sqrt{\lambda_n} \underset{n\to\infty}{=} o(\lambda_n)$ since $\lim\limits_{n\to\infty} \lambda_n = +\infty$, by dividing the previous equation by $n$, we finally have

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{ab}.$$

This proof can be extended to any blocks of dimension $N \in \mathbb{N}$ and also for the homogeneous Neumann boundary condition problem. For this boundary condition, we obtain the same eigenvalues, except that the associated eigenfunctions are $(\cos(xl\pi/a)\cos(ym\pi/b))_{l,m\geq 0}$. Except for this detail, the proof would be the same.

### I.2.2 Variational characterization of Laplacian eigenvalues

First, let us establish the notations we will use. We denote by the increasing sequence $(\lambda_n)_{n\geq 1}$ (resp. $(\mu_n)_{n\geq 1}$) the eigenvalues with pairwise orthogonal associated unitary eigenfunctions $(e_n)_{n\geq 1}$ (resp. $(a_n)_{n\geq 1}$) of $-\Delta$ with homogeneous Dirichlet boundary condition (resp. Neumann).

**Theorem I.1:** Minimax principle
*Let $G_0^{n-1} := H_0^1(\Omega) \cap span\{e_1, e_2, \ldots, e_{n-1}\}^\perp$. We set $\rho(e) := \frac{||\nabla e||_{L^2}^2}{||e||_{L^2}^2}$, called the Rayleigh quotient. Then,*

$$\lambda_n = \inf_{e \in G_0^{n-1}} \rho(e). \tag{9}$$

*Similarly, we denote $G^{n-1} := H^1(\Omega) \cap span\{a_1, a_2, \ldots, a_{n-1}\}^\perp$, and we have*

$$\mu_n = \inf_{a \in G^{n-1}} \rho(a). \tag{10}$$

*Proof.* First, let us consider $e \in H_0^1(\Omega)$. In one hand we have thanks to Theorem A.1 and Eq. (65),

$$||e||_{L^2}^2 = \left\langle \sum_{k=1}^{\infty} \langle e, e_k\rangle e_k, e \right\rangle = \sum_{k=1}^{\infty} \langle e, e_k\rangle^2.$$

On the other hand, thanks to Eq. (73),

$$||\nabla e||_{L^2}^2 = \int_\Omega \nabla e \nabla e = -\int_\Omega (\Delta e)e = \left\langle -\Delta\left(\sum_{k=1}^{\infty}\langle e, e_k\rangle e_k\right), e\right\rangle, \tag{11}$$

and thus,

$$||\nabla e||_{L^2}^2 = -\left\langle \sum_{k=1}^{\infty}\langle e, e_k\rangle \Delta e_k, e\right\rangle = \left\langle \sum_{k=1}^{\infty}\lambda_k\langle e, e_k\rangle e_k, e\right\rangle = \sum_{k=1}^{\infty}\lambda_k\langle e, e_k\rangle^2. \tag{12}$$

If we choose $e \in G_0^{n-1}$, we get

$$||\nabla e||_{L^2}^2 = \sum_{k=n}^{\infty}\lambda_n\langle e, e_k\rangle^2 \geq \lambda_n\sum_{k=n}^{\infty}\langle e, e_k\rangle^2 = \lambda_n\sum_{k=1}^{\infty}\langle e, e_k\rangle^2 = \lambda_n||e||_{L^2}^2.$$

Then, for all $e \in G_0^{n-1}$ we obtain $\rho(e) \geq \lambda_n$ and therefore

$$\inf_{e\in G_0^{n-1}} \rho(e) \geq \lambda_n.$$

Furthermore, we observe that $e_n \in G_0^{n-1}$ and thanks to Eq. (12) that $||\nabla e_n||_{L^2}^2 = \lambda_n$, so $\rho(e_n) = \lambda_n$. Since the infimum is reached, we have equality, thus proving Eq. (9). To prove Eq. (10) we can follow the same procedure, but with the distinction of considering $H^1(\Omega)$ and $G^{n-1}$. We invoke Theorem A.31 for the decomposition and Eq. (77) to validate the second equality in (11). □

We have also nearly proven the following result.

**Corollary I.2:** Rayleigh quotient of eigenfunctions
*Let $e \in G_0^{n-1}$, we have*

$$\rho(e) = \lambda_n \qquad \Longleftrightarrow \qquad e \in ES(\lambda_n).$$

*Proof.* Let $e \in G_0^{n-1}$, from Eq. (12) we derive

$$\rho(e) = \sum_{k=n}^{\infty}\lambda_k\left\langle \frac{e}{||e||_{L^2}^2}, e_k\right\rangle^2.$$

We suppose $\rho(e) = \lambda_n$. In other words, since $\sum_{k=n}^{m}\langle e, e_k\rangle^2 = ||e||_{L^2}^2$ that

$$\sum_{k=n}^{\infty}\lambda_k\left\langle \frac{e}{||e||_{L^2}^2}, e_k\right\rangle^2 = \sum_{k=n}^{\infty}\lambda_n\left\langle \frac{e}{||e||_{L^2}^2}, e_k\right\rangle^2.$$

This is equivalent to

$$\sum_{k=n}^{\infty}(\lambda_k - \lambda_n)\left\langle \frac{e}{||e||_{L^2}^2}, e_k\right\rangle^2 = 0.$$

Meanwhile, for all $k \geq n$, we have $(\lambda_k - \lambda_n) \geq 0$. Therefore, if $(\lambda_k - \lambda_n) \neq 0$ then $\langle e, e_k\rangle = 0$. This is equivalent to $e \in ES(\lambda_n)$. □

As a consequence of this principle, we derive the following corollary, which is a formulation that is similar but more general.

**Corollary I.3:** Minimax corollary
*Let $Q_0^n = \{V : n\text{-dimensional linear subspaces of } H_0^1(\Omega)\}$, we have*

$$\lambda_n = \inf_{V\in Q_0^n}\ \sup_{e\in V}\ \rho(e).$$

*Also, for $Q^n = \{V : n\text{-dimensional linear subspaces of } H^1(\Omega)\}$,*

$$\mu_n = \inf_{V\in Q^n}\ \sup_{a\in V}\ \rho(a).$$

*Proof.* Let us begin with
$$\lambda_n = \inf_{e\in G_0^{n-1}} \rho(e) = \sup_{e\in span\{e_1,e_2,\dots,e_n\}} \rho(e).$$
Indeed, thanks to the first results of the previous proof and by taking $e$ in $span\{e_1, e_2, \dots, e_n\} \subset H_0^1(\Omega)$, it follows that
$$||\nabla e||_{L^2}^2 = \sum_{k=1}^{n} \lambda_k \langle e, e_k\rangle^2 \leq \lambda_n \sum_{k=1}^{n} \langle e, e_k\rangle^2 = \lambda_n ||e||_{L^2}^2.$$
Then,
$$\sup_{e\in span\{e_1,e_2,\dots,e_n\}} \rho(e) \leq \lambda_n.$$
Since $e_n \in span\{e_1, e_2, \dots, e_n\}$ and $\rho(e_n) = \lambda_n$, we have
$$\sup_{e\in span\{e_1,e_2,\dots,e_n\}} \rho(e) = \lambda_n.$$
Similarly, we can show
$$\mu_n = \inf_{a\in G^{n-1}} \rho(a) = \sup_{a\in span\{a_1,a_2,\dots,a_n\}} \rho(a).$$
Now that we have these results, we can assert that
$$\lambda_n = \sup_{e\in span\{e_1,e_2,\dots,e_n\}} \rho(e) \geq \inf_{V\subset Q_0^n} \sup_{e\in V} \rho(e).$$
That is right, let $V \in Q_0^n$. We can represent $V$ as $span\{(e_k)_{k\in I}\}$ for a certain $I \in \mathbb{N}^n$ where all elements are distinct. Let $I_n$ denote its largest element. We have
$$||\nabla e||_{L^2}^2 = \sum_{k\in I} \lambda_k \langle e, e_k\rangle^2 \leq \lambda_{I_n} \sum_{k=1}^{n} \langle e, e_k\rangle^2 = \lambda_{I_n} ||e||_{L^2}^2,$$
and then
$$\sup_{e\in V} \rho(e) \leq \lambda_{I_n}.$$
Regardless of V, we have that $I_n \geq n$, then the minimal upper bound we can hope for this inequality is $\lambda_n$. This bound is achieved for $V = span\{e_1, e_2, \dots, e_n\}$, then
$$\inf_{V\in Q_0^n} \sup_{e\in V} \rho(e) = \lambda_n.$$
We can, of course, do the same for the data of the Neumann problem to obtain the second result of this corollary. □

This result will enable us to illustrate the general principle that the more constrained the system is, the larger the eigenvalues will be. For example, it enables us to contrast the data of the Neumann problem, which imposes fewer restrictions, with that of the Dirichlet problem, which imposes more.

**Lemma I.4:**
*For a same domain $\Omega$, for all $n \geq 1$,*
$$\mu_n \leq \lambda_n.$$

*Proof.* Since $H_0^1(\Omega) \subset H^1(\Omega)$, we have $Q_0^n \subset Q^n$ and therefore
$$\inf_{V\in Q^n} \sup_{a\in V} \rho(a) \leq \inf_{V\in Q_0^n} \sup_{e\in V} \rho(e).$$
Therefore, thanks to minimax corollary I.3 we obtain the expected result. In other words, minimax principle says that $\mu_n$ is given by minimizing the same expression as for $\lambda_n$ over a larger space, and hence $\mu_n \leq \lambda_n$. □

The minimax corollary also allows us to prove the monotonicity of the eigenvalues with respect to the inclusion of the considered space $\Omega$.

**Lemma I.5:**
*Given $\Omega \subset \Omega'$, we denote by $\lambda_n$ the eigenvalues associated with $\Omega$, and by $\lambda'_n$ those associated with $\Omega'$. We have the order relation, for all $n \geq 1$,*

$$\lambda'_n \leq \lambda_n.$$

*Similarly, if $H^1(\Omega) \subset H^1(\Omega')$, then*

$$\mu'_n \leq \mu_n.$$

*Proof.* We have easily $H_0^1(\Omega) \subset H_0^1(\Omega')$, thus $Q_0^n(H_0^1(\Omega)) \subset Q_0^n(H_0^1(\Omega'))$, which consequently leads us to

$$\inf_{V \in Q_0^n(H_0^1(\Omega'))} \sup_{e \in V} \rho(e) \leq \inf_{V \in Q_0^n(H_0^1(\Omega))} \sup_{e \in V} \rho(e).$$

If $H^1(\Omega) \subset H^1(\Omega')$, we obtain a similar result. This concludes our proof thanks to Corollary I.3. □

### I.2.3 Proof of the law on any domain

We will start by showing Weyl's law on any elementary set.

**Definition I.6:** Elementary set
*A part $\Omega$ of $\mathbb{R}^N$ is said to be an elementary set if it can be written as a finite union of almost disjoint blocks $\Omega_i \subset \mathbb{R}^N$ (i.e. their two-by-two intersections are negligible).*

So, let $\Omega$ be an elementary set. We denote by $(\lambda_n^i)_{n\geq 1}$ (resp. $(\mu_n^i)_{n\geq 1}$) the eigenvalues of the Laplacian with homogeneous Dirichlet (reps. Neumann) boundary condition on the block $\Omega_i$. We set the ordered collection of these sets of eigenvalues by $(\widetilde{\lambda_n})_{n\geq 1}$ (resp. $(\widetilde{\mu_n})_{n\geq 1}$) so that $\widetilde{\lambda_1} \leq \widetilde{\lambda_2} \leq ...$ (resp. $\widetilde{\mu_1} \leq \widetilde{\mu_2} \leq ...$). In Section I.2.1, we proved that on each block $\Omega_i$, we have when $\lambda$ converges to $+\infty$

$$N_{\lambda^i}(\lambda) = \frac{B_N|\Omega_i|}{(2\pi)^N}\lambda^{N/2} + o(\lambda^{N/2}),$$

and also

$$N_{\mu^i}(\lambda) = \frac{B_N|\Omega_i|}{(2\pi)^N}\lambda^{N/2} + o(\lambda^{N/2}),$$

where $|\Omega_i|$ is the measure of $\Omega_i$. Then,

$$N_{\widetilde{\lambda}}(\lambda) = \sum_i N_{\lambda^i}(\lambda) = \frac{B_N}{(2\pi)^N}\lambda^{N/2}\sum_i |\Omega_i| + o(\lambda^{N/2}) = \frac{B_N}{(2\pi)^N}\lambda^{N/2}|\Omega| + o(\lambda^{N/2}).$$

Similarly,

$$N_{\widetilde{\mu}}(\lambda) = \frac{B_N}{(2\pi)^N}\lambda^{N/2}|\Omega| + o(\lambda^{N/2}),$$

We define the sets $\widetilde{H_0^1}(\Omega) = \{u \in L^2(\Omega) \mid u_{|\Omega_i} \in H_0^1(\Omega_i),\ \forall i\}$ and $\widetilde{H^1}(\Omega) = \{u \in L^2(\Omega) \mid u_{|\Omega_i} \in H^1(\Omega_i),\ \forall i\}$. Thanks to Corollary I.3, we can give a variational characterization to these eigenvalues. Thus,

$$\widetilde{\lambda_n} = \inf_{V \in Q_0^n(\widetilde{H_0^1}(\Omega))} \sup_{e \in V} \widetilde{\rho}(e),$$

and

$$\widetilde{\mu_n} = \inf_{V \in Q^n(\widetilde{H^1}(\Omega))} \sup_{a \in V} \widetilde{\rho}(a),$$

with $\widetilde{\rho}(v) := \left(\sum_i ||\nabla v||^2_{L^2(\Omega_i)}\right)/||v||^2_{L^2}$.

**Property I.7:**
*We have the inclusions,*

$$\widetilde{H_0^1}(\Omega) \subset H_0^1(\Omega) \subset H^1(\Omega) \subset \widetilde{H^1}(\Omega). \tag{13}$$

*Proof.* We start with the first inclusion. Since $\Omega = \cup_{n=1}^{l}\Omega_i$ with Lebesgue measure $\lambda(\Omega_i \cap \Omega_j) = 0$ for $i \neq j$ it is sufficient to show the inclusion for $l = 2$. So we are going to show that if $u$ is in $\widetilde{H_0^1}(\Omega_1 \cup \Omega_2)$ with $\lambda(\Omega_1 \cap \Omega_2) = 0$, then $u$ is in $H_0^1(\Omega_1 \cup \Omega_2)$. First, $u$ is in $L^2(\Omega_1 \cup \Omega_2)$. For all $\phi \in C_C^{\infty}(\Omega)$, and all $i$ in $[\![1, N]\!]$ we have

$$\int_{\Omega_1\cup\Omega_2} u\frac{\partial \phi}{\partial x_i}dx = \int_{\Omega_1} u\frac{\partial \phi}{\partial x_i}dx + \int_{\Omega_2} u\frac{\partial \phi}{\partial x_i}dx - \int_{\Omega_1\cap\Omega_2} u\frac{\partial \phi}{\partial x_i}dx.$$

Meanwhile, since $\lambda(\Omega_1 \cap \Omega_2) = 0$, the last term is zero, then

$$\int_{\Omega_1\cup\Omega_2} u\frac{\partial \phi}{\partial x_i}dx = \int_{\Omega_1} u_{|\Omega_1}\frac{\partial \phi}{\partial x_i}dx + \int_{\Omega_2} u_{|\Omega_2}\frac{\partial \phi}{\partial x_i}dx.$$

Since $u_{|\Omega_1}$ and $u_{|\Omega_2}$ are respectively in $H_0^1(\Omega_1)$ and $H_0^1(\Omega_2)$, they have weak derivatives that we will note $g_i$ and $h_i$ respectively. So,

$$\int_{\Omega_1\cup\Omega_2} u\frac{\partial \phi}{\partial x_i}dx = \int_{\Omega_1} g_i\phi dx + \int_{\Omega_2} h_i\phi dx.$$

So $g_i\chi_{\Omega_1} + h_i\chi_{\Omega_2}$ is a weak derivative of $u$. Moreover, this function is clearly in $L^2(\Omega_1 \cup \Omega_2)$. It remains to prove that $u$ can be written as a limit of the element of $C_c^1(\Omega_1 \cup \Omega_2)$ in $H^1(\Omega_1 \cup \Omega_2)$. Let $a_n$ (resp. $b_n$) be a sequence in $C_c^1(\Omega_1)$ (resp. $C_c^1(\Omega_2)$) that converges to $u_{|\Omega_1}$ in $H^1(\Omega_1)$ (resp. to $u_{|\Omega_2}$ in $H^1(\Omega_2)$). The sequence $(a_n + b_n)$ is in $C_c^1(\Omega_1 \cup \Omega_2)$. Moreover, by a similar computation to the one performed previously, we have

$$||u - (a_n + b_n)||_{H^1(\Omega_1\cup\Omega_2)} = ||u - a_n||_{H^1(\Omega_1)} + ||u - b_n||_{H^1(\Omega_2)} = ||u_{|\Omega_1} - a_n||_{H^1(\Omega_1)} + ||u_{|\Omega_2} - b_n||_{H^1(\Omega_2)}.$$

Thus, $||u-(a_n+b_n)||_{H^1(\Omega_1\cup\Omega_2)}$ goes to 0 when $n \to \infty$, so there exists a sequence of elements in $C_c^1(\Omega_1\cup\Omega_2)$ that converges to $u$. Therefore, $u$ is in $H_0^1(\Omega_1 \cup \Omega_2)$. The second inclusion is true by definition of $H_0^1(\Omega)$. For the last inclusion, if $u$ is in $H^1(\Omega)$, then $u$ is in $L^2(\Omega)$. It follows that for any $i$, its restriction $u_{|\Omega_i}$ is in $L^2(\Omega_i)$. Furthermore, we have the existence of sets of functions $(g_j)_{j\in[\![1,N]\!]} \subset L^p(\Omega)$ such that

$$\int_\Omega u\frac{\partial \phi}{\partial x_j} = -\int_\Omega g_j\phi, \qquad \forall\phi \in C_C^{\infty}(\Omega),\ \forall j \in [\![1, N]\!].$$

So, by restricting our test functions $\phi$ to $C_c^{\infty}(\Omega_i)$ which is included in $C_c^{\infty}(\Omega)$, we obtain for all $i \in [\![1, N]\!]$,

$$\int_{\Omega_i} u\frac{\partial \phi}{\partial x_j} = \int_{\Omega_i} u_{|\Omega_i}\frac{\partial \phi}{\partial x_j} = -\int_{\Omega_i} g_j\phi.$$

Then, for all $i$, $u_{|\Omega_i}$ is in $H^1(\Omega_i)$ and thus $u$ is in $\widetilde{H^1}(\Omega)$. □

Thanks to Corollary I.3 and Property I.7 which indicate that we minimize over increasingly larger sets, we have

$$\widetilde{\mu_n} \leq \mu_n \leq \lambda_n \leq \widetilde{\lambda_n}.$$

Now we can see why we introduced Neumann eigenvalues. Therefore, we have by definition that

$$N_{\widetilde{\mu}}(\lambda) \geq N_\mu(\lambda) \geq N_\lambda(\lambda) \geq N_{\widetilde{\lambda}}(\lambda).$$

It follows,

$$\frac{B_N}{(2\pi)^N}\lambda^{N/2}|\Omega| + o(\lambda^{N/2}) \geq N_\mu(\lambda) \geq N_\lambda(\lambda) \geq \frac{B_N}{(2\pi)^N}\lambda^{N/2}|\Omega| + o(\lambda^{N/2}).$$

This allows us to conclude the proof of Weyl's law in dimension $N \in \mathbb{N}$ for elementary set. Now we consider an arbitrary domain $\Omega$ of $\mathbb{R}^N$. Our goal now is to show, for a given $\epsilon > 0$, the existence of $\Omega_i$ and $\Omega_o$ elementary sets of $\mathbb{R}^N$ such that $\Omega_i \subset \Omega \subset \Omega_o$ and $|\Omega_o \setminus \Omega_i| < \epsilon$.

**Theorem I.8:** Jordan's measure.
*Let $\Omega$ be a bounded part of $\mathbb{R}^N$. The inner and outer Jordan measures of $\Omega$ are defined by*

$$m_{i,j}(\Omega) = \sup_{V\subset\Omega\ \textit{elementary}} m(V),$$

*and*

$$m_{o,j}(\Omega) = \inf_{V\supset\Omega\ \textit{elementary}} m(V),$$

*respectively, where $m(V) := \sum_i |V_i|$ with $V_i$ are the blocks composing $V$ and $|V_i|$ the volume of the block. We then say that $\Omega$ is Jordan measurable if $m_{i,j}(\Omega) = m_{o,j}(\Omega)$.*

This is naturally equivalent to saying that $\Omega$ has a Jordan measure. Let us recall the result from measure theory. This result is derived from the article [Fri33].

**Theorem I.9:**
*A bounded open set $\Omega$ is Jordan measurable if and only if its topological boundary $\partial\Omega$ has Lebesgue measure zero.*

*Sketch of Proof.* Let $\Omega$ be a bounded set, so we can already state that $\Omega$ admits a Lebesgue measure. The inner Jordan measure is the Lebesgue measure of the topological interior of $\Omega$, i.e. $\mathring{\Omega}$. The outer Jordan measure is the Lebesgue measure of the topological closure of $\Omega$, i.e. $\overline{\Omega}$. Since the sets $\overline{\Omega}$ and $\mathring{\Omega}$ are bounded closed and open sets respectively, they are measurable. It immediately follows that a bounded set $\Omega$ is Jordan measurable if and only if its boundary $\partial\Omega$ has Lebesgue measure zero. □

Henceforth, we shall consider a set $\Omega$ such that $\partial\Omega$, is negligible in the sense of Lebesgue measure. This is a perfectly reasonable constraint in our case. We can, under this condition, ensure the existence of the elementary sets $\Omega_i$ and $\Omega_o$ as described above. Thanks to Lemma I.5, we have for any $n \in \mathbb{N}$,

$$\lambda_n^o \leq \lambda_n \leq \lambda_n^i,$$

where $(\lambda_n^i)_{n\geq 1}$ and $(\lambda_n^o)_{n\geq 1}$ are the ordered eigenvalues associated with $\Omega_i$ and $\Omega_o$ respectively. Therefore,

$$N_{\lambda^o}(\lambda) \geq N_\lambda(\lambda) \geq N_{\lambda^i}(\lambda).$$

This results in

$$\frac{B_N}{(2\pi)^N}\lambda^{N/2}|\Omega_o| + o(\lambda^{N/2}) \geq N_\lambda(\lambda) \geq \frac{B_N}{(2\pi)^N}\lambda^{N/2}|\Omega_i| + o(\lambda^{N/2}).$$

Meanwhile, we have

$$|\Omega| + \epsilon \geq |\Omega_o| \qquad \text{and} \qquad |\Omega_i| \geq |\Omega| - \epsilon,$$

which implies,

$$\frac{B_N}{(2\pi)^N}\lambda^{N/2}(|\Omega| + \epsilon) + o(\lambda^{N/2}) \geq N_\lambda(\lambda) \geq \frac{B_N}{(2\pi)^N}\lambda^{N/2}(|\Omega| - \epsilon) + o(\lambda^{N/2}).$$

As the choice of $\epsilon > 0$ is arbitrary, we conclude that

$$N_\lambda(\lambda) \underset{\lambda\ +\infty}{\sim} \frac{B_N|\Omega|}{(2\pi)^N}\lambda^{N/2}.$$

By choosing $\lambda = \lambda_n$, we conclude the proof of the Weyl's law,

$$\lambda_n \underset{n\to\infty}{\sim} 4\pi^2\left(\frac{n}{B_N V}\right)^{2/N}.$$

**Theorem I.10:** Weyl's law
*Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ such that $\partial\Omega$ is negligible. We consider $(\lambda_n)_{n\geq 1}$ as the increasing sequence of eigenvalues of the Laplacian operator with homogeneous Dirichlet boundary condition. The law states*

$$\lambda_n \underset{n\to\infty}{\sim} 4\pi^2\left(\frac{n}{B_N V}\right)^{2/N},$$

*where $V$ is the measure of $\Omega$ and $B_N$ is the measure of the unit ball in $\mathbb{R}^N$.*

We can apply precisely the same reasoning to the eigenvalues associated with the Neumann boundary condition to obtain

$$\mu_n \underset{n\to\infty}{\sim} 4\pi^2\left(\frac{n}{B_N V}\right)^{2/N}.$$

## I.3 Minimizing the first Dirichlet eigenvalue

The goal of this section is to prove the Faber-Krahn inequality. It suggests that, overall sets of equal volume, the ball has the lowest fundamental frequency. As previously, $\Omega$ will designate an open bounded set of $\mathbb{R}^N$ and $\lambda_n$ the $n^{\text{th}}$ Dirichlet eigenvalue.

### I.3.1 Schwarz's symmetrization

Let us start by introducing the notion of set symmetrization.

**Definition I.11:** Spherical rearrangement
*Let $A$ be a bounded open set of $\mathbb{R}^N$. We define its symmetrical rearrangement, denoted $A^*$, as the open ball of center 0 which has the same volume as $A$.*

The idea is to create a symmetrical, radial-dependent version of the functions. First, we note that for $f$ a non-negative measurable function that

$$f(x) = \int_0^{f(x)} dt = \int_0^{+\infty} \chi_{[0,f(x)]}(t)dt = \int_0^{+\infty} \chi_{\{x',\ t<f(x')\}}(x)dt. \tag{14}$$

This leads us to define the following symmetrization.

**Definition I.12:** Schwarz symmetrization
*Let $\Omega$ a bounded set of $\mathbb{R}^N$ and $f : \Omega \mapsto \mathbb{R}^+$ be measurable, such that its level sets $\Omega_t = \{x \in \Omega, f(x) > t\}$ have finite measures for all $t > 0$. We define its Schwarz symmetrization as $f^* : \Omega^* \mapsto \mathbb{R}^+$ by*

$$f^*(x) = \int_0^{+\infty} \chi_{\Omega_t^*}(x)dt.$$

An illustration is provided in Figure 3. Indeed, we have that $f^*$ depends only on the radius of its variable

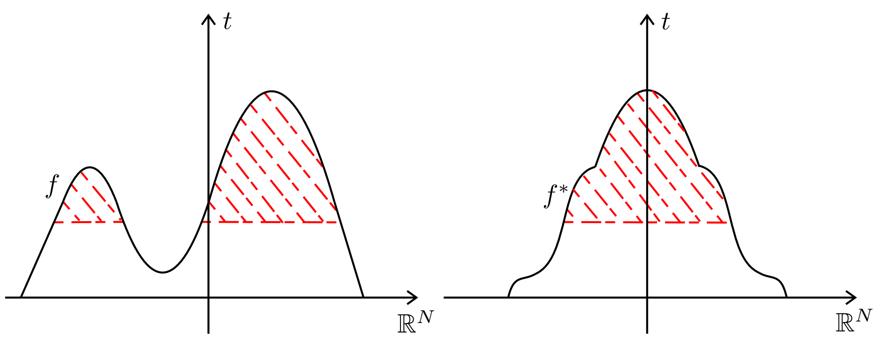



Figure 3: Illustration of Schwarz symmetrization. On the left figure, the red area represents $\Omega_t$, while in the figure on the right, it represents $\Omega_t^*$. By definition, they have the same volume.

and decreases as the radius increases. One of the main properties of this transformation is the link between its level sets and the rearranged level sets of the initial function.

**Property I.13:**
*Under same assumptions as in definition I.12, $f$ verifies*

$$\{x \in \Omega, f(x) > t\}^* = \{x \in \Omega^*, f^*(x) > t\}.$$

*Proof.* We will denote the set $\{x \in \Omega, f(x) > s\}$ by $\{f > s\}$. We have, by definition for $x \in \Omega^*$,

$$f^*(x) = \int_0^{+\infty} \chi_{\{f>s\}^*}(x)ds = \mathrm{Vol}(\{s,\ x \in \{f > s\}^*\}).$$

The set $\{s,\ x \in \{f > s\}^*\}$ is an interval with 0 as lower bound and

$$f^*(x) = \sup(\{s,\ x \in \{f > s\}^*\}).$$

Indeed, we have that $x$ is in $\{f > s\}^*$ if and only $B(0, |x|)$ is a subset of $\{f > s\}^*$ and this is true if and only if $s < f^*(x)$ by the definition of $f^*$. So, for all $t$ in $\mathbb{R}^+$, we have

$$x \in \{f^* > t\} \quad \Longleftrightarrow \quad t < f^*(x) = \sup(\{s,\ x \in \{f > s\}^*\}).$$

Meanwhile, we notice

$$t < \sup(\{s,\ x \in \{f > s\}^*\}) \quad \iff \quad t \in \{s,\ x \in \{f > s\}^*\}.$$

In conclusion, we get

$$x \in \{f^* > t\} \quad \iff \quad x \in \{f > t\}^*.$$

□

From this, we can deduce an elementary property of this transformation. It preserves the $L^2$ norm.

**Property I.14:** Conservation of the $L^2$ norm by symmetrization
*Let $f \in L^2(\Omega)$ and satisfy the assumptions of definition I.12. Then,*

$$||f||_2^2 = ||f^*||_2^2$$

*Proof.* We have, thanks to Eq. (14),

$$||f||_2^2 = \int_\Omega f(x)^2 dx = \int_\Omega \int_0^{+\infty} \chi_{\{f^2>t\}}(x) dt\ dx.$$

Fubini's theorem allows us to write

$$||f||_2^2 = \int_0^{+\infty} \int_\Omega \chi_{\{f^2>t\}}(x) dx\ dt = \int_0^{+\infty} \mathrm{Vol}(\{f^2 > t\}) dt.$$

Then, by using Property I.13, we have

$$||f||_2^2 = \int_0^{+\infty} \mathrm{Vol}(\{f^2 > t\}^*) dt = ||f^*||_2^2.$$

□

It is also interesting to note the following property.

**Property I.15:**
*Let $f$ be a non-negative function with measurable level sets. We also assume that $f$ is a radial function that is non-increasing with respect to the radius of its variable. Then, $f^* = f$.*

*Proof.* Let $x$ be in level set $\Omega_t = \{x \in \Omega,\ t < f(x)\}$. Then for all $x'$ in $\Omega$ such that $|x'| \leq |x|$ also satisfy $t < f(x')$, so $x'$ is in $\Omega_t$ too. Therefore, we have that the level sets depend only on the radius of $x$, and are therefore balls. This implies that for all $t > 0$, $\Omega_t = \Omega_t^*$ and then $f = f^*$. □

### I.3.2 Characterization of the gradient norm by the heat kernel

We recall that the Fourier transform on $L^2(\mathbb{R}^N)$ is constructed using the density of $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ as follows.

**Definition I.16:**
*The Fourier transform of a function $u \in L^2(\mathbb{R}^N)$ is defined as the limit in $L^2(\mathbb{R}^N)$ of the Fourier transform of any sequence of elements of $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ converging to $u$ in $L^2(\mathbb{R}^N)$.*

We recall the following fundamental results.

**Theorem I.17:** Plancherel's theorem
*Let $u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then $\widehat{u}$, his $L^1$ Fourier transform, is in $L^2(\mathbb{R}^N)$ and*

$$||\widehat{u}||_2 = ||u||_2.$$

*Additionally, for the Fourier transform of $L^2$ as defined above, this equality holds, indicating it is an isometry.*

**Corollary I.18:** Parseval's equality in $L^2$
*If $u$ and $v$ are in $L^2(\mathbb{R}^N)$, then Parseval's equality holds, i.e.*

$$(u, v)_{L^2} = (\widehat{u}, \widehat{v})_{L^2}.$$

**Theorem I.19:**
*The $L^2$ Fourier transform $\mathscr{F}$ is a bijective isometry from $L^2(\mathbb{R}^N)$ into itself. Let us denote its inverse $\mathscr{F}^{-1}$.*

Proofs of these results can be found in [LL97]. The following result will be useful to us and is closely related to the Theorem A.7.

**Theorem I.20:**
*Let $1 \leq p < \infty$ and $(f_n)$ a sequence of $L^p(\Omega)$ which converges to $f \in L^p(\Omega)$ in $L^p$. We can extract a subsequence $(f_m)$ such that, almost everywhere on $\Omega$,*

$$f_m(x) \underset{m\to\infty}{\longrightarrow} f(x).$$

*Proof.* We have that $(f_n)$ is a Cauchy sequence in $L^p(\Omega)$. Then, for all $j \in \mathbb{N}$, there exists $N_j$ such that for all $k, l \geq N_j$ we have

$$||f_l - f_f||_p \leq \frac{1}{2^j}.$$

We can therefore extract a subsequence $(f_{n_m})$ from $(f_n)$, which for simplicity will also be denoted $(f_m)$, satisfying for all $m \geq 1$

$$||f_{m+1} - f_m||_p \leq \frac{1}{2^m}.$$

Let define, for $n \in \mathbb{N}$ and for $x$ almost everywhere on $\Omega$,

$$g_n(x) = \sum_{k=0}^{n} |f_{k+1}(x) - f_k(x)|$$

We have thanks to Proposition A.10,

$$||g_n||_p \leq \sum_{k=0}^{n} ||f_{k+1} - f_k||_p \leq \sum_{k=0}^{n} \frac{1}{2^k} = 2 - 2^{-n} \leq 2,$$

and then

$$\sup_{n\in\mathbb{N}} \int_\Omega g_n^p \leq 2^p < \infty.$$

Therefore, the sequence $(g_n^p)$ is an increasing sequence of non-negative integrable functions on $\Omega$. Then, Theorem B.4 gives

$$\int_\Omega \lim_{n\to\infty} g_n^p(x)dx = \lim_{n\to\infty} \int_\Omega g_n^p(x)dx < \infty.$$

Thanks to Property B.3, we know that $\lim_{n\to\infty} g_n^p(x) = \left(\sum_{k=0}^{\infty} |f_{k+1}(x) - f_k(x)|\right)^p < \infty$ almost everywhere. Then, we have the pointwise convergence $g_n(x) \underset{n\to\infty}{\longrightarrow} g(x) := \sum_{k=0}^{\infty} |f_{k+1}(x) - f_k(x)|$ almost everywhere on $\Omega$. Let us now show that for $x$ almost everywhere in $\Omega$, the sequence $(f_m(x))$ is a Cauchy sequence. For $m \geq n$, we have

$$|f_m(x) - f_n(x)| = |\sum_{k=n}^{m-1} f_{k+1}(x) - f_k(x)| \leq \sum_{k=n}^{m-1} |f_{k+1}(x) - f_k(x)| = g_{m-1}(x) - g_{n-1}(x),$$

Since $(g_n)$ is an increasing sequence,

$$|f_m(x) - f_n(x)| \leq g(x) - g_{n-1}(x). \tag{15}$$

We know that for all $\epsilon > 0$, there exists $N_\epsilon$ such that for all $n \geq N_\epsilon$, $g(x) - g_{n-1}(x) \leq \epsilon$, then for all $m, n \geq N_\epsilon$,

$$|f_m(x) - f_n(x)| \leq \epsilon.$$

We can therefore deduce that, $x$ almost everywhere in $\Omega$, the sequence $(f_m(x))$ converges pointwise to $f_{\text{pw}}(x)$. Let us show that this pointwise limit is consistent with convergence to $f$ in $L^p$. Eq. (15) gives for all $n \in \mathbb{N}$

$$|f_{\text{pw}}(x) - f_n(x)|^p \leq g(x)^p,$$

where $g^p$ is integrable. By using Theorem B.5, we obtain

$$\lim_{n\to\infty} \int |f_{\text{pw}}(x) - f_n(x) - 0|^p dx = \lim_{n\to\infty} ||f_{\text{pw}} - f_n||_p^p = 0.$$

We deduce that $(f_n)$ also converge to $f_{\text{pw}}$ in $L^p$, then that

$$||f_{\text{pw}} - f||_p = ||f_{\text{pw}} - f_n + f_n - f||_p \leq ||f_{\text{pw}} - f_n||_p + ||f_n - f||_p \underset{n\to\infty}{\longrightarrow} 0,$$

i.e. $||f_{\text{pw}} - f||_p = 0$ and therefore $f_{\text{pw}}(x) = f(x)$ almost everywhere on $\Omega$. In conclusion, we have that $(f_n(x))$ converge almost everywhere to $f(x)$. □

Note that this result is also true for $\mathbb{R}^N$. Indeed, we can perform the same proof for $\mathbb{R}^N$ instead of $\Omega$. Now that we have recalled these results, we can introduce the following Fourier characterization of $H^1(\Omega)$.

**Theorem I.21:** Fourier characterization of $H^1(\mathbb{R}^N)$
*Let $u$ be in $L^2(\mathbb{R}^N)$. Then, $u$ is in $H^1(\mathbb{R}^N)$ if and only if $u \in L^2(\mathbb{R}^N)$ and the application $\xi \mapsto |\xi|\widehat{u}(\xi)$ is also in $L^2(\mathbb{R}^N)$. Moreover, in that case*

$$\widehat{\nabla u}(\xi) = 2i\pi\xi\widehat{u}(\xi).$$

*Proof.* Let us start with $u$ in $H^1(\mathbb{R}^N)$. By Proposition A.27, there exists a sequence $(u_n)$ of $C_C^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$. Thanks to Theorem I.17, we have

$$||\widehat{u_n} - \widehat{u}||_2 = ||u_n - u||_2 \to 0,$$

i.e. that $(\widehat{u_n})$ converge to $\widehat{u}$ in $L^2$. Similarly,

$$||\widehat{\nabla u_n} - \widehat{\nabla u}||_2 = ||\nabla u_n - \nabla u||_2 \to 0,$$

which means that $(\widehat{\nabla u_n})$ converge to $\widehat{\nabla u}$ in $L^2$ too. Thanks to Theorem I.20 we have almost everywhere on $\mathbb{R}^N$ that $\widehat{u_n}(\xi) \to \widehat{u}(\xi)$ and $\widehat{\nabla u_n}(\xi) \to \widehat{\nabla u}(\xi)$. Since $u_n \in C_c^\infty(\mathbb{R}^N)$, we can write

$$\widehat{\nabla u_n}(\xi) = \int_{\mathbb{R}^N} \nabla u_n(x) e^{-2i\pi\xi x} dx.$$

Green's formula A.26 gives

$$\widehat{\nabla u_n}(\xi) = \int_{\overline{\mathbb{R}^N}\setminus\mathbb{R}^N} (u_n, n) v d\sigma - \int_{\mathbb{R}^N} u_n(x) \nabla e^{-2i\pi\xi x} dx.$$

However, $u_n$ is compactly supported, so for $x \in \overline{\mathbb{R}^N} \setminus \mathbb{R}^N$, $u_n(x) = 0$. Then,

$$\widehat{\nabla u_n}(\xi) = 2i\pi\xi \int_{\mathbb{R}^N} u_n(x) e^{-2i\pi\xi x} dx = 2i\pi\xi\widehat{u_n}(\xi). \tag{16}$$

We deduce that $2i\pi\xi\widehat{u_n}(\xi) \to \widehat{\nabla u}(\xi)$ and we conclude that

$$\widehat{\nabla u}(\xi) = 2i\pi\xi\widehat{u}(\xi) \in L^2(\mathbb{R}^N). \tag{17}$$

So we've proved that $\xi \mapsto |\xi|\widehat{u}(\xi)$ is in $L^2(\mathbb{R}^N)$ since $\widehat{\nabla u}$ is in $L^2(\mathbb{R}^N)$. Now, let $u$ and $\xi \mapsto |\xi|\widehat{u}(\xi)$ be in $L^2(\mathbb{R}^N)$. Thanks to Parseval equality, we have for all $\phi \in C_c^\infty(\mathbb{R}^N)$ that

$$\int_{\mathbb{R}^N} \nabla\phi u = \int_{\mathbb{R}^N} \widehat{\nabla\phi}\widehat{u}.$$

Equation (16) allows us then to write

$$\int_{\mathbb{R}^N} \nabla \phi u = \int_{\mathbb{R}^N} 2i\pi\xi\widehat{\phi}\widehat{u}.$$

Finally, by using again Parseval equality and Theorem I.19, we get

$$\int_{\mathbb{R}^N} \nabla \phi u = \int_{\mathbb{R}^N} \phi \mathscr{F}^{-1}(2i\pi\xi\widehat{u}).$$

Then, the gradient of $u$, in the weak sense, is $\mathscr{F}^{-1}(2i\pi\xi\widehat{u})$ which is in $L^2(\mathbb{R}^N)$. □

Let us recall that the heat kernel is for $t \in ]0, +\infty[$, the function $K_t : \mathbb{R}^N \mapsto \mathbb{R}$ defined for every $x \in \mathbb{R}^N$ by

$$K_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}.$$

**Theorem I.22:** Heat kernel integral
*For all $t > 0$, the heat kernel $K_t$ is in $L^1$. More precisely,*

$$\int_{\mathbb{R}^N} K_t = 1.$$

*Proof.* Let us start with the case of dimension 1. Let $a > 0$, we begin by noting that

$$\int_{\mathbb{R}} e^{-ax^2} = \frac{1}{\sqrt{a}} e^{-z^2} dz.$$

We will therefore examine

$$\int_{\mathbb{R}} e^{-x^2}.$$

To estimate this quantity, the idea is to use Theorem B.6 on the application $(x, y) \mapsto e^{-x(1+y^2)}$ for $x, y \geq 0$. For $x \geq 0$,

$$\int_0^\infty e^{-x(1+y^2)} dx = \frac{1}{1+y^2},$$

then

$$\int_0^\infty \left( \int_0^\infty e^{-x(1+y^2)} dx \right) dy = \int_0^\infty \frac{1}{1+y^2} dy = [\arctan(x)]_0^\infty = \frac{\pi}{2}.$$

For $y > 0$,

$$\int_0^\infty e^{-x(1+y^2)} dy = e^{-x} \int_0^\infty e^{-xy^2} dy = \frac{e^{-x}}{\sqrt{x}} \int_0^\infty e^{-z^2} dz,$$

then,

$$\int_0^\infty \left( \int_0^\infty e^{-x(1+y^2)} dy \right) dx = \left( \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx \right) \left( \int_0^\infty e^{-y^2} dy \right) = 2 \left( \int_0^\infty e^{-z^2} dz \right)^2.$$

We deduce therefore

$$\int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2}.$$

In conclusion, we have indeed that $K_t$ is in $L^1(\mathbb{R}^N)$ since

$$\int_{\mathbb{R}^N} K_t(x) dx = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{x^2}{4t}} dx = \frac{1}{(4\pi t)^{N/2}} \left( \int_{\mathbb{R}} e^{-\frac{x_j^2}{4t}} dx_j \right)^N = \frac{1}{(4\pi t)^{N/2}} \sqrt{4t\pi}^N = 1.$$

□

So, the following result allows us to consider its convolution with any function in $L^2(\Omega)$.

**Theorem I.23:**
*If $u \in L^2(\mathbb{R}^N)$ and $v \in L^1(\mathbb{R}^N)$, then $u * v \in L^2(\mathbb{R}^N)$*

*Proof.* Subject to existence, we have that

$$||u*v||_2^2 = \int_{\mathbb{R}^N} (u*v)^2(x)dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x-y)||u(x-z)||v(y)||v(z)|dxdydz.$$

We note, for almost all $y,z \in \mathbb{R}^N$, thanks to Hölder's inequality that

$$\int_{\mathbb{R}^N} |u(x-y)||u(x-z)|dx \leq \left(\int_{\mathbb{R}^N} |u(x-y)|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^N} |u(x-z)|^2dx\right)^{1/2} = ||u||_2^2.$$

So,

$$||u*v||_2^2 = \int_{\mathbb{R}^N} (u*v)^2(x)dx \leq ||u||_2^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)||v(z)|dydz = ||u||_2^2||v||_1^2 < \infty. \tag{18}$$

□

We deduce that for $u \in L^2(\Omega)$, the function $K_t * u$ exists and is in $L^2(\mathbb{R}^N)$. One of the most important properties of convolution products is the following.

**Theorem I.24:**
*Let $u \in L^2(\mathbb{R}^N)$ and $v \in L^1(\mathbb{R}^N)$. Then $\widehat{u*v} = \widehat{u}\widehat{v}$.*

*Proof.* Let $(u_n)$ be a sequence of elements of $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ which converge to u. It is known, for $f,g \in L^1(\mathbb{R}^N)$, that $\widehat{f*g} = \widehat{f}\widehat{g}$. So,

$$\widehat{u_n * v} = \widehat{u_n}\widehat{v} \underset{n\to\infty}{\to} \widehat{u}\widehat{v},$$

by definition of $L^2$ Fourier transform. Moreover, thanks to Eq. (18) we have

$$||v*u_n - v*u||_2^2 = ||v*(u_n-u)||_2^2 \leq ||v||_1^2||u_n-u||_2^2 \to 0.$$

Then, $v*u_n$ goes to $v*u$ in $L^2$ and consequently $\widehat{u_n*v}$ to $\widehat{u*v}$ by definition of $L^2$ Fourier transform. □

The last result we will need to achieve our proof is the following.

**Theorem I.25:**
*We denote $\mathscr{G}_a : \mathbb{R}^N \ni x \mapsto e^{-ax^2}$ the Gaussian function with parameter $a \in \mathbb{R}$ with $a > 0$. We have, for almost all $\xi \in \mathbb{R}^N$,*

$$\widehat{\mathscr{G}_a}(\xi) = (\pi/a)^{N/2}e^{-\pi^2a^{-1}\xi^2}.$$

*Proof.* Let us start by noting that for $a > 0$, $\mathscr{G}_a$ is in $L^1(\mathbb{R}^N)$, so we can use the explicit Fourier transform formula. We can return to the 1D case by noting that for any $\xi \in \mathbb{R}^d$,

$$\widehat{\mathscr{G}_a}(\xi) = \int_{\mathbb{R}^N} e^{-ax^2}e^{-2i\pi\xi x}dx = \prod_{j=1}^N \int_{\mathbb{R}} \underbrace{e^{-ax_j^2}e^{-2i\pi\xi_jx_j}}_{\in L^1(\mathbb{R})}dx_j.$$

It is sufficient to prove the result for $N=1$, then we consider $\mathscr{G}_a : \mathbb{R} \ni x \mapsto e^{-ax^2}$. We first notice that for all $x \in \mathbb{R}$, the application $\xi \mapsto e^{-a\xi^2}e^{-2i\pi\xi x}$ is derivable and for all $\xi \in \mathbb{R}$ that

$$|-2i\pi xe^{-a\xi^2}e^{-2i\pi\xi x}| \leq Cxe^{-ax^2} \in L^1(\mathbb{R}).$$

Therefore, the dominated convergence theorem implies

$$\frac{\partial}{\partial\xi}\widehat{\mathscr{G}_a}(\xi) = -2i\pi\int_{\mathbb{R}} xe^{-ax^2}e^{-2i\pi x\xi}dx.$$

We notice, for all $y \in \mathbb{R}$,

$$\int_0^y xe^{-ax^2}dx = -\frac{1}{2a}\left[e^{-ax^2}\right]_0^y = -\frac{1}{2a}e^{-ay^2} + C,$$

with $C \in \mathbb{R}$. So, for all $\xi \in \mathbb{R}$, an integration by parts gives us

$$\frac{\partial}{\partial \xi}\widehat{\mathscr{G}_a}(\xi) = -2i\pi \left( \left[ -\frac{1}{2a} e^{-ax^2} e^{-2i\pi x\xi} \right]_{-\infty}^{+\infty} - \frac{i\pi\xi}{a} \int_R e^{-ax^2} e^{-2i\pi x\xi} dx \right),$$

and since $a > 0$, the [ ] part is null, then

$$\frac{\partial}{\partial \xi}\widehat{\mathscr{G}_a}(\xi) = -\frac{2\pi^2\xi}{a} \int_R e^{-ax^2} e^{-2i\pi x\xi} dx = -\frac{2\pi^2\xi}{a}\widehat{\mathscr{G}_a}(\xi).$$

Solving this ODE allows us to write, for all $\xi \in \mathbb{R}$,

$$\widehat{\mathscr{G}_a}(\xi) = \widehat{\mathscr{G}_a}(0) e^{-\frac{\pi^2}{a}\xi^2}.$$

To conclude, we need the value of

$$\widehat{\mathscr{G}_a}(0) = \int_{\mathbb{R}} e^{-ax^2} dx.$$

We define, for $s \in [0, 1]$, the function $f(s, \cdot) : \mathbb{R} \ni x \mapsto \frac{\exp(-a(s^2+1)x^2)}{s^2+1}$ and $I(x) = \int_0^1 f(s, x) ds$. We have for all $x$ in $\mathbb{R}$,

$$f_s'(x) = -2axe^{-a(s^2+1)x^2}$$

A simple function study gives us, for $b \in ]0, \infty[$, that $|xe^{-bx^2}| \leq \frac{\exp(-1/2)}{\sqrt{2b}}$. Then, there exists $C \in \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$|f_s'(x)| = \frac{C}{\sqrt{1+s^2}} \in L^1([0, 1]).$$

So,

$$I'(x) = -2axe^{-ax^2} \int_0^1 e^{-as^2x^2} ds = -2ae^{-ax^2} \int_0^x e^{-au} du = -2aJ'(x)J(x),$$

with $J(x) = \int_0^x e^{-au^2} du$. By integrating this last equality from 0 to $x \in \mathbb{R}$, we obtain

$$I(x) - I(0) = a(J(0)^2 - J(x)^2).$$

It is clear that $J(0) = 0$, $I(0) = \int_0^1 \frac{1}{1+s^2} ds = \frac{\pi}{4}$ and for all $x \in \mathbb{R}$ that

$$0 \leq I(x) \leq e^{-ax^2} \int_0^1 \frac{1}{1+s^2} ds \underset{|x|\to\infty}{\longrightarrow} 0.$$

We get

$$\lim_{|x|\to\infty} aJ(x)^2 = a \left( \int_0^{+\infty} e^{-au^2} du \right)^2 = a \left( \frac{1}{2} \int_{-\infty}^{+\infty} e^{-au^2} du \right)^2 = \frac{\pi}{4},$$

then,

$$a \left( \int_0^{+\infty} e^{-au^2} du \right)^2 = a \left( \frac{1}{2} \int_{-\infty}^{+\infty} e^{-au^2} du \right)^2 = \frac{\pi}{4}.$$

So,

$$\int_{\mathbb{R}} e^{-ax^2} dx = \left( \frac{\pi}{a} \right)^{\frac{1}{2}}.$$

If we go back to dimension $N$, we obtain

$$\widehat{\mathscr{G}_a}(0) = \int_{\mathbb{R}^N} e^{-ax^2} dx = \left( \frac{\pi}{a} \right)^{\frac{N}{2}}.$$

□

By choosing $a = 1/4t$, we obtain $\widehat{K_t}(\xi) = e^{-4\pi^2\xi^2 t}$. Then Theorem I.24 gives us that, for all $u$ in $L^2(\mathbb{R}^N)$

$$\widehat{K_t * u} = e^{-4\pi^2\xi^2 t}\widehat{u}(\xi) \tag{19}$$

This result enables us to prove the following characterization of the $L^2$ norm on $H^1$.

**Theorem I.26:** Characterization of the gradient norm by the heat kernel
*A function $u$ is in $H^1(\mathbb{R}^N)$ if and only if it is in $L^2(\mathbb{R}^N)$ and*

$$D_u(t) := \frac{1}{t}\left(\int_{\mathbb{R}^N} |u|^2 - \int_{\mathbb{R}^N} u(x)(K_t * u)(x)dx\right)$$

*is bounded in time, i.e. for all $t > 0$, $|D_u(t)| \leq M(u) < \infty$. In that case, we have the key characterization*

$$||\nabla u||_2^2 = \lim_{t\to 0} D_u(t).$$

*Proof.* We notice, for $u$ in $L^2(\mathbb{R}^N)$, that

$$D_u(t) = \frac{1}{t}\big(||u||_{L^2} - (u, K_t * u)_{L^2}\big).$$

Thanks to Parseval's and Plancherel's equality,

$$D_u(t) = \frac{1}{t}\big(||\widehat{u}||_{L^2} - (\widehat{u}, \widehat{K_t * u})_{L^2}\big).$$

We deduce from Eq. (19),

$$D_u(t) = \frac{1}{t}\int_{\mathbb{R}^N} (1 - e^{-4\pi^2\xi^2 t})|\widehat{u}(\xi)|^2 d\xi.$$

Note that for $a > 0$, the application $x \mapsto (1 - e^{-ax})/x$ is a decreasing function for $x > 0$, with limit $a$ when $x$ goes to 0. We deduce that

$$\sup_{t>0} D_u(t) = \lim_{t\to 0} D_u(t). \tag{20}$$

By posing $t = \frac{1}{n}$ we obtain

$$\lim_{t\to 0} D_u(t) = \lim_{n\to\infty} \int_{\mathbb{R}^N} \underbrace{n(1 - e^{-4\pi^2\xi^2/n})|\widehat{u}(\xi)|^2}_{:=f_n(\xi)} d\xi$$

We set the increasing sequence $(f_n)$ of non-negative and measurable functions. Indeed, $f_n$ is measurable since $\xi \mapsto n(1 - e^{-4\pi^2\xi^2/n})$ is continuous and then measurable and $\widehat{u}$ is in $L^2$ so measurable by definition. Since the product of measurable functions is measurable, we get our point. Moreover, for all $\xi$ in $\mathbb{R}^N$ we have the pointwise limit

$$\lim_{n\to\infty} f_n(\xi) = 4\pi^2\xi^2|\widehat{u}(\xi)|^2 := f(\xi).$$

From the monotone convergence Theorem B.4, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(\xi)d\xi = \int_{\mathbb{R}^N} f(\xi)d\xi = \int_{\mathbb{R}^N} 4\pi^2\xi^2|\widehat{u}(\xi)|^2 d\xi.$$

Therefore, we obtain

$$\sup_{t>0} D_u(t) = \lim_{t\to 0} D_u(t) = \int_{\mathbb{R}^N} 4\pi^2\xi^2|\widehat{u}(\xi)|^2 d\xi \tag{21}$$

We know from Theorem I.21 that $u$ is in $H^1(\mathbb{R}^N)$ if and only if $\int_{\mathbb{R}^N} |\xi|^2|\widehat{u}(\xi)|^2 d\xi < \infty$. Thanks to Eq. (21), this is also equivalent to $\lim_{t\to 0} D_u(t) < \infty$ and also equivalent to $D_u(t)$ being bounded in time. Finally, if we are in this case, thanks to Eq. (17) and Plancherel's equality, we get

$$\lim_{t\to 0} D_u(t) = \int_{\mathbb{R}^N} |2i\pi\xi\widehat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^N} |\widehat{\nabla u}(\xi)|^2 d\xi = ||\widehat{\nabla u}||_2^2 = ||\nabla u||_2^2.$$

□

### I.3.3 Riesz rearrangement inequality

We begin by proving the following useful property.

**Property I.27:**
*Let $f, g : \Omega \to \mathbb{R}^+$ be measurable functions. If $f \geq g$ almost everywhere, then $f^* \geq g^*$ almost everywhere.*

*Proof.* Let $t > 0$, we denote $\Omega_{f,t} = \{x, f(x) > t\}$ and $\Omega_{g,t} = \{x, g(x) > t\}$ Since $f(x) \geq g(x)$, if $g(x) > t$ then $f(x) > t$ and that for $x$ almost everywhere. So $\Omega_{g,t} \subset \Omega_{f,t}$ up to a set of measure zero, so $\lambda(\Omega_{g,t}) \leq \lambda(\Omega_{f,t})$. Therefore, $\Omega^*_{g,t} \subset \Omega^*_{f,t}$ which implies that for $x$ in $\Omega^*$ that $\chi_{\Omega^*_{f,t}}(x) \geq \chi_{\Omega^*_{g,t}}(x)$. By integrating over $t > 0$, we obtain $\int_{\mathbb{R}^+} \chi_{\Omega^*_{f,t}}(x)dt \geq \int_{\mathbb{R}^+} \chi_{\Omega^*_{g,t}}(x)dt$, i.e. that $f^*(x) \geq g^*(x)$. □

The Riesz rearrangement inequality will be useful in proving the Pólya-Szegő inequality presented in the next section. The proof presented below comes from [LL97]

**Theorem I.28:** Riesz rearrangement inequality
*Let $f, g$ and $h$ be non-negative measurable functions on $\mathbb{R}^N$. We define*

$$I(f, g, h) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)g(x-y)h(y)dxdy.$$

*We have*

$$I(f, g, h) \leq I(f^*, g^*, h^*),$$

*with $I(f^*, g^*, h^*) = \infty$ if $I(f, g, h) = \infty$.*

*Proof.* We start by reducing the problem to indicator functions of open measurable sets. Then, we will start by proving it in dimension 1 and after generalize it to any dimension.

**Problem reduction:** Given that $f, g$ and $h$ are non-negative measurable functions, thanks to Theorem B.1 we can set respectively $(f_n), (g_n)$ and $(h_n)$ increasing sequences of simple functions that converge pointwise to $f, g$ and $h$ respectively. Therefore, $(f^*_n), (g^*_n)$ and $(h^*_n)$ are also increasing sequences of simple functions that converge pointwise to $f^*, g^*$ and $h^*$ respectively. Let us show that. Given that $f_{n+1} \geq f_n$, thanks to Property I.27 we have $f^*_{n+1} \geq f^*_n$ almost everywhere. So $(f^*_n)$ is an increasing sequence. To prove the almost everywhere pointwise convergence of $(f^*_n)$ to $f^*$, we will start by giving a characterization of the Schwarz symmetrization of a simple function. For example, let $n$ an integer, by definition of simple function there exists $m$ in $\mathbb{N}$ such that

$$f_n = \sum_{i=1}^{m} \alpha_i \chi_{A_i},$$

where $A_i = f^{-1}(\alpha_i)$ which is measurable since $f$ is. It can be rewritten as follows

$$f_n = \sum_{i=1}^{m} \beta_i \chi_{B_i}, \tag{22}$$

with $B_{i+1} \subset B_i$. For example if the coefficients $\alpha_i$ are sorted to be increasing, we can consider $\beta_{i+1} = \alpha_{i+1} - \beta_i$ with $\beta_1 = \alpha_1$ and $B_i = \cap_{k \geq i} A_k$. It comes, for all $i$, that the level set associated to $\sum_{j<i} \beta_j$ with respect to $f_n$, i.e. $\Omega_{\sum_{j<i} \beta_j}$, is $B_i$. Then it comes

$$f^*_n = \sum_{i=1}^{m} \beta_i \chi_{B^*_i}, \tag{23}$$

with $B^*_{i+1} \subset B^*_i$. An illustration is provided in Figure 4. We know that for all $\epsilon > 0$ there exist $M$ such that for all $n \geq M$ we have $f(x) - f_n(x) \leq \epsilon$ almost everywhere on $\mathbb{R}$. Therefore, by using decomposition (22) we have almost everywhere,

$$f(x) \leq \sum_{i=1}^{m} \beta_i \chi_{B_i}(x) + \epsilon.$$

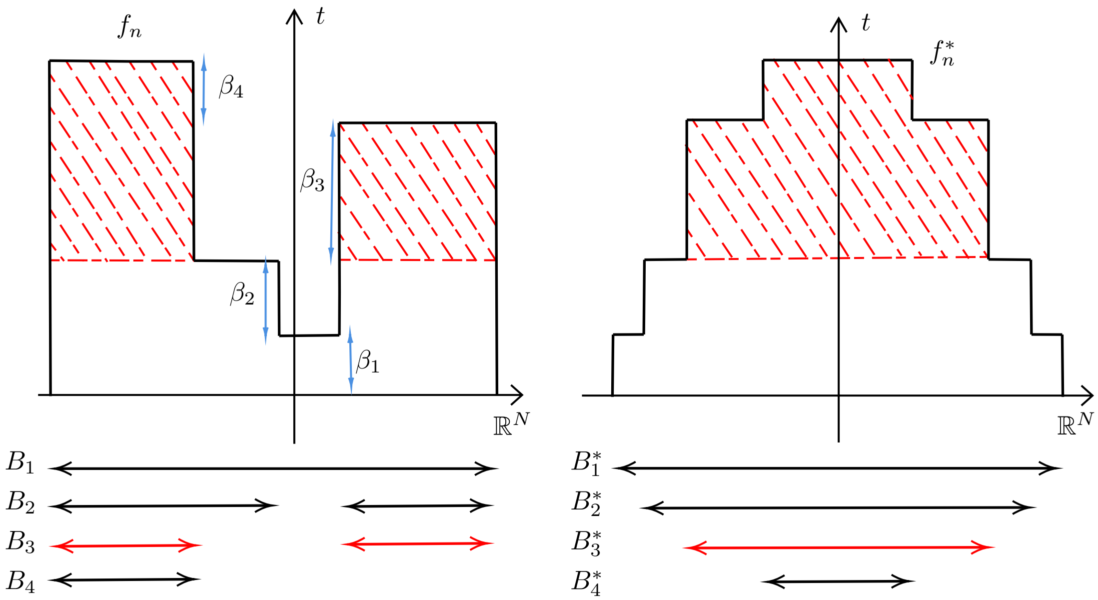



Figure 4: Illustration of Schwarz symmetrization for a simple function $f_n$. On the left, an illustration of the decomposition (22), and on the right of the decomposition (23).

What is important to note is that the right-hand side is still of the form (22), so its Schwarz symmetrization will be in the form (23). Then, Property I.27 gives that almost everywhere

$$f^*(x) \leq (\sum_{i=1}^{m} \beta_i \chi_{B_i} + \epsilon \chi_{\mathbb{R}})^*(x) = \sum_{i=1}^{m} \beta_i \chi_{B_i^*} + \epsilon \chi_{\mathbb{R}^*}(x).$$

Since $\chi_{\mathbb{R}^*}(x) = \chi_{\mathbb{R}}(x) = 1$, we finally obtain

$$f^*(x) - \sum_{i=1}^{m} \beta_i \chi_{B_i^*} \leq \epsilon.$$

Thus, we have the almost everywhere pointwise convergence of $(f_n^*)$ to $f^*$. Therefore, thanks to Beppo-Levi's theorem B.4, we only need to show the result for $f, g$ and $h$ simple functions. What is more, since integration is linear, it is sufficient to show this for indicator functions of measurable sets. So we consider $f = \chi_A$, $g = \chi_B$ and $h = \chi_C$ where $A$, $B$ and $C$ are measurable sets. Thanks to the outer regularity of Lebesgue measure B.2, we know the existence of a sequence of open measurable sets $A_k$ such that $A \subset A_{k+1} \subset A_k$ and the Lebesgue measure $\lambda(A_k \setminus A) < \frac{1}{2^k}$ for all $k$. Then, we have almost everywhere that $\chi_{A_k}(x)$ converges to $\chi_A(x)$ when $k$ goes to infinity. Similarly, we choose sets $B_k$ and $C_k$. Then, for $x$ and $y$ almost everywhere

$$\lim_{k \to \infty} \chi_{A_k}(x)\chi_{B_k}(x-y)\chi_{C_k}(y) = \chi_A(x)\chi_B(x-y)\chi_C(y).$$

Also,

$$|\chi_{A_k}(x)\chi_{B_k}(x-y)\chi_{C_k}(y)| \leq \chi_{A_0}(x)\chi_{B_0}(x-y)\chi_{C_0}(y).$$

Then, thanks to Theorem B.5, we have

$$\lim_{k \to \infty} I(\chi_{A_k}, \chi_{B_k}, \chi_{C_k}) = I(\chi_A, \chi_B, \chi_C).$$

Similarly, since $\chi_A^* = \chi_{A^*}$ we show similarly

$$\lim_{k \to \infty} I(\chi_{A_k^*}, \chi_{B_k^*}, \chi_{C_k^*}) = I(\chi_{A^*}, \chi_{B^*}, \chi_{C^*}).$$

Therefore, it is sufficient to prove the inequality for indicator functions of open measurable sets.

**Dimension 1.**

Now we set $A \subset \mathbb{R}$ as an open measurable set. Then $A$ is the disjoint union of countably many intervals $(I_j)$, i.e. $A = \bigcup_{\mathbb{N}} I_j$. First, we reduce again our problem, this time to a finite union of disjoint open intervals. We set

$$A_l = \bigcup_{j=1}^{l} I_j.$$

The sequence of functions $(\chi_{A_l})$ is an increasing sequence of non-negative measurable functions since $A_l \subset A_{l+1}$ and $A_l$ is measurable. In addition, we have almost everywhere the pointwise convergence of $\chi_{A_l}(x)$ to $\chi_A(x)$. We set again similarly $B_l$ and $C_l$. The monotone convergence theorem gives

$$\lim_{l\to\infty} I(\chi_{A_l}, \chi_{B_l}, \chi_{C_l}) = I(\chi_A, \chi_B, \chi_C),$$

and

$$\lim_{l\to\infty} I(\chi_{A_l^*}, \chi_{B_l^*}, \chi_{C_l^*}) = I(\chi_{A^*}, \chi_{B^*}, \chi_{C^*}).$$

These results allow us to work with $A, B$ and $C$ as finite unions of disjoint open intervals. Now $f$ is now an indicator function of a finite union of disjoint open intervals, then

$$f(x) = \sum_{i=1}^{k} \chi_{I_j}(x).$$

We set $a_k$ the center of the open interval $I_k$ and $I_k^*$ the interval $I_k$ centered. Therefore, we have $\chi_{I_k}(x) = \chi_{I_k^*}(x - a_k)$. If we set $f_k := \chi_{I_k^*}$, we can write

$$f(x) = \sum_{k=1}^{k'} f_k(x - a_k).$$

Similarly, we write

$$g(x) = \sum_{l=1}^{l'} g_l(x - b_l) \qquad \text{and} \qquad h(x) = \sum_{m=1}^{m'} h_m(x - c_m).$$

Then, $I(f, g, h)$ is a finite sum of terms of the form

$$I_{jlm} = \int_{\mathbb{R}^2} f_k(x - a_k) g_l(x - y - b_l) h_m(y - c_m) dxdy.$$

We start by dealing with the case $k' = l' = m' = 1$, i.e. $A, B$ and $C$ are one open interval, to get an idea of the element of the proof. Note that is not sufficient since the Schwarz symmetrization is not linear, so we will generalize our result for any $k', l'$ and $m'$. We define for $t$ in $[0, 1]$,

$$I_{f,g,h}(t) = \int_{\mathbb{R}^2} f_1\big(x - (1-t)a_1\big) g_1\big(x - y - (1-t)b_1\big) h_1\big(y - (1-t)c_1\big) dxdy.$$

Notice that $I_{f,g,h}(0) = I(f, g, h)$ and $I_{f,g,h}(1) = I(f^*, g^*, h^*)$. It is important to understand the influence of $t$ graphically. The closer $t$ gets to 1, the more we center our intervals, as shown in Figure 5.
By a first change of variable $x \to x + (1-t)a_1$ and a second $y \to y + (1-t)c_1$ we obtain

$$I_{f,g,h}(t) = \int_{\mathbb{R}^2} f_1\big(x\big) g_1\big(x - y + (1-t)(a_1 - b_1 - c_1)\big) h_1\big(y\big) dxdy = \int_{A^*}\int_{C^*} g_1\big(x - y + (1-t)(a_1 - b_1 - c_1)\big) dydx.$$

We can set $A^* = ]-\alpha, \alpha[$, $B^* = ]-\beta, \beta[$ and $C^* = ]-\gamma, \gamma[$ with $\alpha, \beta, \gamma$ positive numbers. It comes,

$$\begin{aligned} I_{f,g,h}(t) &= \lambda\left(]-\alpha, \alpha[\times]-\gamma, \gamma[ \cap \big\{(x, y) \in \mathbb{R}^2 \text{ s.t. } x - y + (1-t)(a_1 - b_1 - c_1) \in ]-\beta, \beta[\big\}\right) \\ &= \lambda\left(\big\{(x, y) \in ]-\alpha, \alpha[\times]-\gamma, \gamma[ \text{ s.t. } x - y + (1-t)(a_1 - b_1 - c_1) \in ]-\beta, \beta[\big\}\right) \end{aligned} \tag{24}$$

This quantity is maximum when the shift is null. Indeed, this can be seen from the symmetry of the problem, or from a drawing as in Figure 6. This happens when $t = 1$, then we obtain $I_{f,g,h}(0) \leq I_{f,g,h}(1)$,

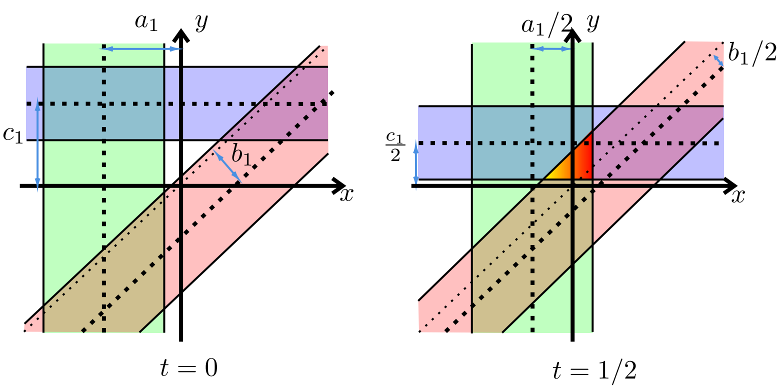



Figure 5: Illustration of the evolution of the integrated functions over $t$. The green area (resp. blue and red) represents where $f_1$ (resp. $h_1$, $g_1$) equals 1. Thus, the red shaded area represents the quantity $I_{f,g,h}(t)$.

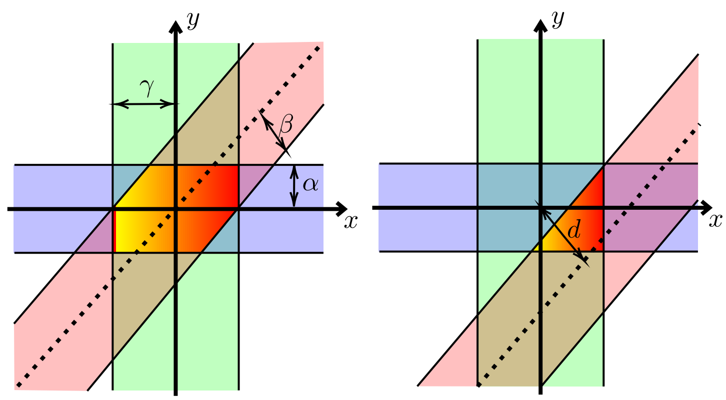



Figure 6: Illustration of the influence of the shift $d$ in Eq. (24). On the left figure $d = 0$, and on the right figure $d \neq 0$.

i.e. $I(f, g, h) \leq I(f^*, g^*, h^*)$. This shows the Riesz inequality for $f, g$ and $h$ being indicator functions of one interval. Now we consider $f, g$ and $h$ being indicator functions of finite unions of disjoint open intervals, i.e. $k', l'$ and $m'$ arbitrary in $\mathbb{N}$. The idea of proof is the same as before. To keep the reasoning readable, we will not detail the entire formalism. We will instead provide the idea of the proof and the intuition of why it is true. Indeed, we can apply the same procedure as previously; however, we will encounter an overlap problem with our intervals defining a function. To overcome this problem, as soon as two intervals associated with the same function touch, we stop the process and start again with these two intervals combined into one. This decision will enable us to find our Schwartz symmetries at the end. An illustration is provided in Figure 7. We understand that our final state maximizes the total volume of the intersection of our three finite unions of open intervals. Moreover, the final state is again the Schwarz rearrangement of our initial sets. We therefore find that $I(f, g, h) \leq I(f^*, g^*, h^*)$.

**Dimension $N$.**
We recall that thanks to the reduction of our problem, we only need to show the inequality for indicator functions of measurable open sets. The aim here is to generalize the result of dimension one, which we know to be true, at any dimension $N$. The idea is to introduce a kind of partial one-dimension Schwarz symmetrization operation for a measurable set relative to a hyperplane.

**Definition I.29:** Steiner's rearrangement
*Let $A$ be a measurable set and $H$ a hyperplane of $\mathbb{R}^N$. For $A'_H \subset \mathbb{R}^N$, if for any line $D$ orthogonal to $H$ the set $A'_H \cap D$ is a centered interval with the same measure as $A \cap D$, then $A'_H$ is a Steiner rearrangement of $A$.*

An illustration of such a rearrangement is given in figure Figure 8. First, let us ensure that this definition indeed refers to a unique class of sets for the relation of equality up to sets of measure zero. This amounts

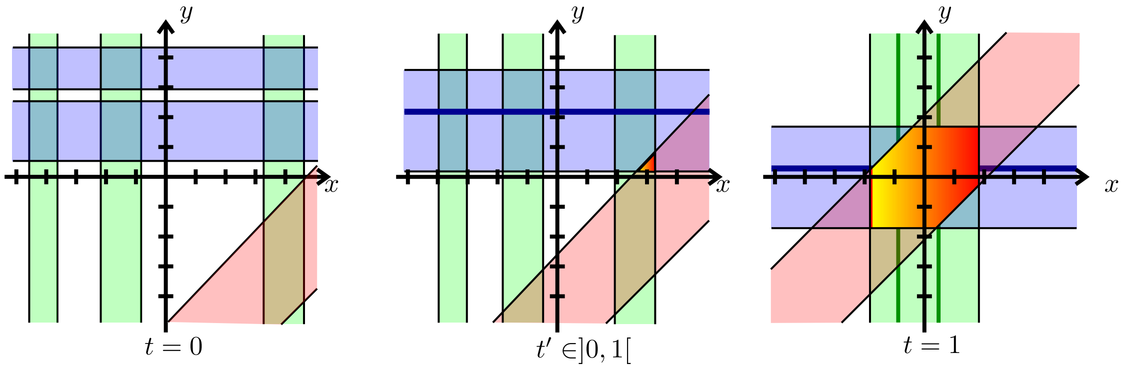



Figure 7: Illustration of the new process for several open intervals. The time $t'$ is the first moment when two intervals associated with the same function touch. So, the second figure illustrates the new start of the initial process with these two intervals combined into one.

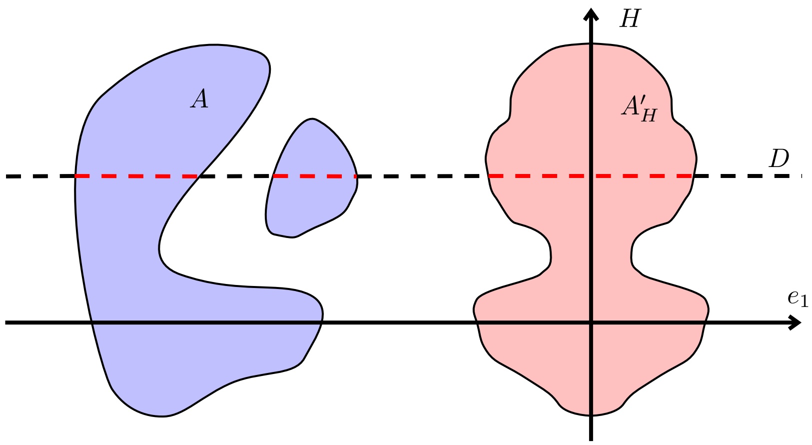



Figure 8: Example of Steiner rearrangement. The blue area is the initial set $A$, the green area is its rearrangement relative to $H$, i.e., $A'_H$. An arbitrary line $D$ is represented.

to having equality of indicator functions almost everywhere. Let $(e_1, e_2, \dots, e_n)$ be a basis of $\mathbb{R}^N$ such that $(e_2, \dots, e_n)$ is a basis of $H$ and $e_1$ is orthogonal to the others. We will note the coordinates $x = (x_1, x')$ with $x' \in H$ and $x_1 \in span(e_1) = H^\perp$. Moreover, we will write $A_H(x') = \{x_1 \in H^\perp \text{ s.t. } (x_1, x') \in A\}$. If $A'_H$ is a Steiner rearrangement of $A$ relatively to $H$ then for all $x' \in H$, $A'_H(x')$ is a centered interval, and we have

$$l_H(x') := \int_{\mathbb{R}} \chi_{A_H(x')}(x_1)dx_1 = \int_{\mathbb{R}} \chi_{A'_H(x')}(x_1)dx_1.$$

Therefore, $I_H(x') :=]-l_H(x')/2, l_H(x')/2[$ is the smallest representative of the class $A'_H(x')$ in terms of inclusion and thus $A'_H$ is in the class of $A'_s := \cup_{x' \in H}(I_H(x'), x')$. Indeed, if we choose another element $A'_o$ from the equivalence class of $A'_H$, by construction $A'_s$ is included in $A'_o$ and thanks to Theorem B.6 we have

$$\int_{\mathbb{R}^N} \chi_{A'_o}(x)dx = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \chi_{A'_o(x')}(x_1)dx_1dx' = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \chi_{A'_s(x')}(x_1)dx_1dx' = \int_{\mathbb{R}^N} \chi_{A'_s}(x)dx.$$

In the rest of the proof, we will omit to specify each time that our equalities between the sets are true up to a negligible set. Let us now highlight the connection between Steiner's rearrangement and Schwarz's rearrangement. We start by noting that for any $x' \in H$, the Schwarz rearrangement of $A_H(x')$, i.e. $A_H(x')^*$, is a centered interval of length $\int_{\mathbb{R}} \chi_{A(x')}(x_1), dx_1$. This corresponds to the description of an element of the class of $A'_H(x')$, i.e.

$$A_H(x')^* = A'_H(x').$$

We will be able to deduce a kind of partial Riesz inequality from this. To do so, we consider $A$, $B$, and $C$, three measurable open sets. We start by noting, thanks to Theorem B.6, that

$$
\begin{aligned}
I(\chi_A,\chi_B,\chi_C) &= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\chi_A(x_1,x')\chi_B(x_1-y_1,x'-y')\chi_C(y_1,y')dx_1dx'\right)dy_1dy'\\
&= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{A_H(x')}(x_1)\chi_{B_H(x'-y')}(x_1-y_1)\chi_{C_H(y')}(y_1)dx_1dy_1\right)dx'dy'.
\end{aligned}
$$

For any $x'$ and $y'$ in $H\times H$, the one-dimensional Riesz inequality applied to function $\chi_{A_H(x')}, \chi_{B_H(x'-y')}$, and $\chi_{C_H(y')}$ gives us

$$
\int_{\mathbb{R}^2}\chi_{A_H(x')}(x_1)\chi_{B_H(x'-y')}(x_1-y_1)\chi_{C_H(y')}(y_1)dx_1dy_1 \leq \int_{\mathbb{R}^2}\chi_{A'_H(x')}(x_1)\chi_{B'_H(x'-y')}(x_1-y_1)\chi_{C'_H(y')}(y_1)dx_1dy_1.
$$

Now, by integrating with respect to $x'$ and $y'$, we obtain

$$
\begin{aligned}
I(\chi_A,\chi_B,\chi_C) &\leq \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{A'_H(x')}(x_1)\chi_{B'_H(x'-y')}(x_1-y_1)\chi_{C'_H(y')}(y_1)dx_1dy_1\right)dx'dy'\\
&= I(\chi_{A'_H},\chi_{B'_H},\chi_{C'_H})
\end{aligned} \tag{25}
$$

It now suffices to show that iterating Steiner rearrangements for various hyperplanes allows us to converge to Schwarz rearrangement. Then, the iterative use of this inequality will therefore allow us to conclude. Let $A$, $B$ and $C$ measurable sets. We begin by reducing the problem to bounded sets. For that, we consider the sequence $A_k = A\cap B(0,k)$ with $B(0,k)$ the centered ball of radius $k$. It is easy to see that $A_k \subset A_{k+1}$ and that this sequence converges to $A$. Consequently, the sequence $(\chi_{A_n})$ is non-decreasing and converges pointwise to $\chi_A$. Similarly, we denote the sequences $(B_n)$ and $(C_n)$, and thus, by the monotone convergence theorem, we have

$$
\lim_{n\to\infty} I(A_n,B_n,C_n) = I(A,B,C).
$$

We now consider $A, B$ and $C$ to be bounded measurable sets. Then, we can use the following result taken from [BLL74]. We prove this result below after the current proof. We use the following notation for the symmetrical difference $A\Delta B = (A\cup B)\setminus(A\cap B)$.

**Property I.30:** ([BLL74], Lemma A.1)
*Let $K$ be a bounded measurable set in $\mathbb{R}^N$. Then there exists a sequence of sets $(K_n)$, where $K_0 = K$ and where $K_{n+1}$ is obtained from $K_n$ by Steiner symmetrization with respect to some hyperplane, such that*

$$
\lim_{n\to\infty}\lambda(K_n\Delta K^*) = 0.
$$

Then we have a sequence $A_n$ constructed from $A$ and Steiner rearrangements verifying

$$
\lim_{n\to\infty}\lambda(A_n\Delta A^*) = \lim_{n\to\infty}\int_{\mathbb{R}^N}|\chi_{A_n}(x)-\chi_{A^*}(x)|dx = 0.
$$

We do not have a monotonicity argument for the sequence $(|\chi_{A_n}-\chi_{A^*}|)$, so we will apply the dominated convergence theorem to show that $\chi_{A_n}$ converges pointwise almost everywhere to $\chi_{A^*}$. Given that $A$ is bounded, there exists $R<\infty$ such that $\chi_A \leq \chi_{B(0,R)}$. Thus, for any hyperplane $H$, for all $x = (x_1,x')$ with $x'\in H$ and $x_1\in H^\perp$, $\chi_{A_H(x')}(x_1) \leq \chi_{B(0,R)_H(x')}(x_1)$. Meanwhile, the function $\chi_{B(0,R)(x')} : H^\perp \ni x_1 \mapsto \chi_{B(0,R)}(x_1,x')$ is radial since it is one if and only if $||x||_2 = ||x_1||_2 + ||x'||_2 < R$ since $x_1$ is orthogonal to $x'$, i.e., $||x_1||_2 < R - ||x'||_2$, otherwise it is 0. Moreover, this function is non-increasing with respect to the radius, thus Property I.15 assures us that it is invariant under Schwarz symmetrization, meaning that the function $\chi_{B(0,R)}$ is invariant under Steiner symmetrization. Thanks to Property I.27, it follows that $\chi_{A'_H(x')}(x_1) \leq \chi_{B(0,R)'_H(x')}(x_1)$, i.e., $\chi_{A'_H}(x) \leq \chi_{B(0,R)}(x)$. Thus, since $A_n$ is constructed from $A$ and multiple Steiner symmetrizations, we have $\chi_{A_n} \leq \chi_{B(0,R)}$. Similarly, $\chi_{B(0,R)}$ is a radial non-increasing function, so it is invariant under Schwarz symmetrization, and by applying Property I.27 again, we have $\chi_{A^*} \leq \chi_{B(0,R)}$. It follows that for all $n$, $|\chi_{A_n} - \chi_{A^*}| \leq 2\chi_{B(0,R)}$ which is integrable. Thus, Theorem B.5 gives

$$
\int_{\mathbb{R}^N}\lim_{n\to\infty}|\chi_{A_n}(x)-\chi_{A^*}(x)|dx = 0.
$$

So, Property B.2 gives that $\lim_{n\to\infty} \chi_{A_n}(x) = \chi_{A^*}(x)$ for $x$ almost everywhere on $\mathbb{R}^N$. Similarly, we show the same thing for $B$ and $C$. We obtain $\lim_{n\to\infty} \chi_{A_n}(x)\chi_{B_n}(x-y)\chi_{C_n}(y) = \chi_{A^*}(x)\chi_{B^*}(x-y)\chi_{C^*}(y)$ almost everywhere on $\mathbb{R}^N \times \mathbb{R}^N$ and $\chi_{A_n}(x)\chi_{B_n}(x-y)\chi_{C_n}(y) \le \chi_{B(0,R_A)}(x)\chi_{B(0,R_B)}(x-y)\chi_{B(0,R_C)}(y)$ with $R_A$,$R_B$ and $R_C$, the radius of the balls in which $A$, $B$ and $C$ are respectively included. Given that $A$, $B$, and $C$ are bounded, we can choose finite $R_A$, $R_B$, and $R_C$, so the function $\chi_{B(0,R_A)}(x)\chi_{B(0,R_B)}(x-y)\chi_{B(0,R_C)}(y)$ is integrable on $\mathbb{R}^N$. The dominated convergence theorem gives

$$\lim_{n\to\infty} I(\chi_{A_n}, \chi_{B_n}, \chi_{C_n}) = I(\chi_{A^*}, \chi_{B^*}, \chi_{C^*}).$$

Now, thanks to Eq. (25), we have for all $n$ that $I(\chi_A, \chi_B, \chi_C) \le I(\chi_{A_n}, \chi_{B_n}, \chi_{C_n})$. Thus, we have proven the desired result, namely

$$I(\chi_A, \chi_B, \chi_C) \le I(\chi_{A^*}, \chi_{B^*}, \chi_{C^*}).$$

□

Before getting into the proof of Property I.30, we will show some properties of Steiner symmetrization.

**Property I.31:** Some properties of Steiner symmetrization
*Let $K$ and $M$ be bounded measurable sets in $\mathbb{R}^N$. We have for any hyperplane $H$,*

1. *Centered balls are invariant by Steiner symmetrization,*
2. $\lambda(K'_H) = \lambda(K)$,
3. $\lambda(K'_H \cap M'_H) \ge \lambda(K \cap M)$,
4. $\lambda(K'_H \Delta M'_H) \le \lambda(K \Delta M)$,

*Proof.* The first point has already been proven. We showed just above that $\chi_{B(0,R)}$ was invariant by Steiner symmetrization. For the second point, using the same notation as above and the Fubini-Tonelli theorem, we have

$$\lambda(K) = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \chi_{K_H(x')}(x_1) dx_1 dx' = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \chi_{K'_H(x')}(x_1) dx_1 dx' = \lambda(K'_H).$$

For the third point, we first notice that for all $x'$in $H$ the sets $K'_H(x')$ and $M'_H(x')$ are centered intervals, then

$$\lambda(K'_H(x') \cap M'_H(x')) = \min(\lambda(K'_H(x')), \lambda(M'_H(x'))).$$

Therefore, we have

$$\lambda(K_H(x') \cap M_H(x')) \le \min(\lambda(K'_H(x')), \lambda(M'_H(x'))) = \lambda(K'_H(x') \cap M'_H(x')).$$

By integrating over $H$, we get our result

$$\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \chi_{K_H(x')}(x_1)\chi_{M_H(x')}(x_1) dx_1 dx' \le \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \chi_{K'_H(x')}(x_1)\chi_{M'_H(x')}(x_1) dx_1 dx',$$

i.e. $\lambda(K \cap M) \le \lambda(K'_H \cap M'_H)$. For the last point, we have

$$\lambda(K\Delta M) = \lambda(K \cup M) - \lambda(K \cap M) = \lambda(K) + \lambda(M) - 2\lambda(K \cap M).$$

Our second and third points lead us to conclude that

$$\lambda(K\Delta M) \ge \lambda(K'_H) + \lambda(M'_H) - 2\lambda(K'_H \cap M'_H) = \lambda(K'_H \Delta M'_H).$$

□

These properties will allow us to prove the following result.

**Corollary I.32:**
*Let $K$ be a bounded measurable set in $\mathbb{R}^N$. We have for any hyperplane $H$,*

$$\lambda(K'_H \Delta K^*) \le \lambda(K \Delta K^*). \tag{26}$$

*Moreover, equality is verified for any hyperplane $H$ if and only if $K = K^*$.*

*Proof.* The last point of our previous property gives $\lambda(K'_H\Delta(K^*)'_H) \leq \lambda(K\Delta K^*)$. Meanwhile, by the definition of Schwarz symmetrization, $K^*$ is a centered ball. Thanks to the first point of the property, we know that $(K^*)'_H = K^*$. Then,

$$\lambda(K'_H\Delta K^*) \leq \lambda(K\Delta K^*).$$

Now, let's assume that $K$ is not a ball, i.e. $\lambda(K\Delta K^*) > 0$. We will show that there exists a hyperplane $W$ such that

$$\lambda(K'_W\Delta K^*) < \lambda(K\Delta K^*).$$

Let $H_1, \ldots, H_N$ be $N$ hyperplanes having two by two their normal orthogonal components. We set $K'_{H_{1,N}} := (((K'_{H_1})'_{H_2})'_{H_3}\ldots)'_{H_N}$. Let us do a case disjunction on whether $K'_{H_{1,N}}$ is a ball or not. We start with the simplest case, that is to say, $K'_{H_{1,N}}$ is in the class of $K^*$. Therefore,

$$\lambda(K'_{H_{1,N}}\Delta K^*) = 0 < \lambda(K\Delta K^*).$$

Thus, among the hyperplanes $(H_1, \ldots, H_N)$ there is at least one which has allowed the symmetrical difference to be strictly reduced. Now we assume that $K'_{H_{1,N}}$ is not in the class of $K^*$. We know that $K'_{H_{1,N}}$ and $K^*$ have the same measure, so $\lambda(K'_{H_{1,N}} \setminus K^*) = \lambda(K^* \setminus K'_{H_{1,N}})$, and by hypothesis we have $\lambda(K'_{H_{1,N}} \setminus K^*) > 0$. Let us start by showing that balls can be found in $K'_{H_{1,N}} \setminus K^*$ and $K^* \setminus K'_{H_{1,N}}$. Note that it is for this point that it is important to have made the disjunction of cases. Indeed, the property $\lambda(K \setminus K^*) > 0$ alone does not allow us to deduce that there is a non-empty ball included in it, we need an additional argument. So, to simplify notation, we now denote $K'_{H_{1,N}}$ by $K$ again, and we decompose $x = (x_1, \ldots, x_N)$ such that $x_i$ is the coordinate of $x$ with respect to the orthogonal component of $H_i$. We start by stating the following result. Let $A$ be a measurable set in $\mathbb{R}^N$. By the definition of Steiner symmetrization, for any $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$, the set of $x_j$ such that $x$ is in $A'_{H_j}$ is a centered interval. We note that for $i \neq j$, the set of $x_j$ such that $x$ is in $A''_{H_jH_i}$ is also a centered interval since the orthogonal complements of $H_j$ and $H_i$ are orthogonal. In order to make the proof easier to read, we will just use Figure 9 to convince ourselves. Therefore, if $x$ is in $K$ and $y$ verifies $|y_i| \leq |x_i|$ for all $i$, then $y$ is also in $K$. Let $x$ be in $K^* \setminus K$, we have the

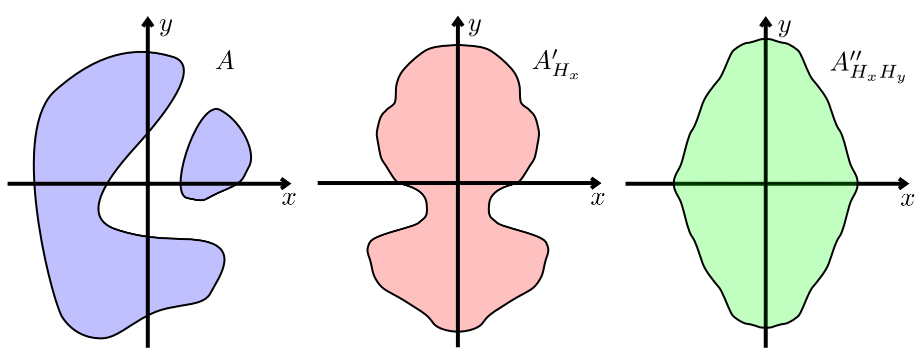



Figure 9: Example of Steiner symmetrization for hyperplanes with perpendicular orthogonal components. Note that $H_x$ is the hyperplane orthogonal to $x$.

inclusion

$$S(x) := K^* \cap \{y \in \mathbb{R}^N \text{ s.t. } \forall i, |y_i| > |x_i|\} \subset K^* \setminus K.$$

Since $\lambda(K'_{H_{1,N}} \setminus K^*) > 0$, we can find $x$ which is not included in hyperplanes $H_1, \ldots, H_N$. This implies that $S(x)$ has a positive measure. Moreover, $K^*$ and $\{y \in \mathbb{R}^N \text{ s.t. } \forall i, |y_i| > |x_i|\}$ are open sets, then $S(x)$ too. So we can find a nonempty ball included in and then in $K^* \setminus K$. We can do the same thing with $K \setminus K^*$. Let $z$ be in $K \setminus K^*$, we have the inclusion

$$T(z) := \{y \in \mathbb{R}^N \text{ s.t. } \forall i, |y_i| < |z_i|\} \setminus \overline{K^*} \subset K \setminus K^*,$$

Since $\lambda(K^* \setminus K'_{H_{1,N}}) > 0$, we can also find $z \in K \setminus K^*$ which is not included in hyperplanes $H_1, \ldots, H_N$. Such a $z$ implies that $T(z)$ has a positive measure. Moreover, $\{y \in \mathbb{R}^N \text{ s.t. } \forall i, |y_i| < |z_i|\}$ and the complementary of $\overline{K^*}$ are open sets, then $T(z)$ too. We deduce that there exists a ball in $K \setminus K^*$. These processes are illustrated in Figure 10. Then, there exists $r > 0$ such that there are $x_1 \in K \setminus K^*$ and $x_2 \in K^* \setminus K$ satisfying $B_N(x_1, r) \subset K \setminus K^*$ and $B_N(x_2, r) \subset K^* \setminus K$, where $B_N$ is the $N$-dimensional ball. Let $V$ be the hyperplane

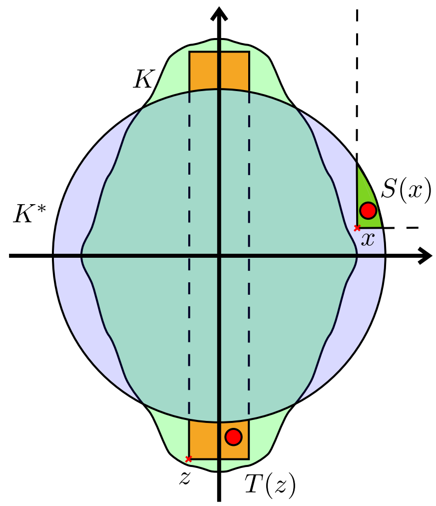



Figure 10: Illustration of the existence process for balls $B_N(x_1, r)$ and $B_N(x_2, r)$.

orthogonal to $x_1 - x_2$ and we denote by $p_V$ the orthogonal projection onto this hyperplane. We have that $p_V(B_N(x_1, r)) = B_{N-1}(p_V(x_1), r) \subset p_V(K \setminus K^*)$ and $B_{N-1}(p_V(x_2), r) \subset p_V(K^* \setminus K)$. By construction, $p_V(x_1) = p_V(x_2)$. Therefore, the measure of $P := p_V(K \setminus K^*) \cap p_V(K^* \setminus K)$ is positive since $p_V(B_N(x_1, r))$ is included in it and $r > 0$. All this construction is illustrated in Figure 11. By the definition of $P$, for all $x'$

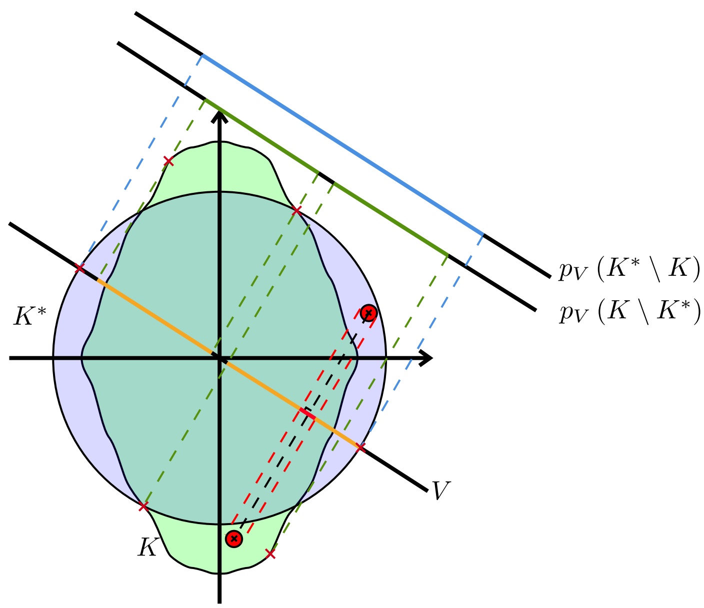



Figure 11: Illustration of the construction of $P$ (orange set) and the proof that it is not of negligible measure when $K \neq K^*$.

in $P \subset V$, we have neither $K_V(x') \subset K^*(x')$ nor $K^*(x') \subset K_V(x')$. Consequently, since by definition $K_V(x')$ and $K^*(x')$ are open intervals, we have for all $x'$ in $P$ that

$$\lambda(K_V(x') \cap K^*(x')) < \min(\lambda(K_V(x')), \lambda(K^*(x'))), \tag{27}$$

which leads to

$$\lambda(K_V(x') \Delta K^*(x')) = \lambda(K_V(x')) + \lambda(K^*(x')) - 2\lambda(K_V(x') \cap K^*(x')) > |\lambda(K_V(x')) - \lambda(K^*(x'))|.$$

However, we know that $K^*(x')$ and $K'_V(x')$ are centered intervals, so $\lambda(K^*(x') \Delta K'_V(x')) = |\lambda(K'_V(x')) - \lambda(K^*(x'))|$ and $\lambda(K'_V(x')) = \lambda(K_V(x'))$. We deduce that for all $x'$ in $P$,

$$\lambda(K'_V(x') \Delta K^*(x')) = |\lambda(K_V(x')) - \lambda(K^*(x'))| < \lambda(K_V(x') \Delta K^*(x')). \tag{28}$$

In general, for all $x'$ in $V$, we have a non-strict inequality in Eq. (27). Therefore, for all $x'$ in $V$

$$\lambda(K'_V(x')\Delta K^*(x')) \leq \lambda(K_V(x')\Delta K^*(x')). \tag{29}$$

Integrating (28) over $P$ and (29) over $V \subset P$ and then summing these results, we obtain

$$\lambda(K'_V \Delta K^*) < \lambda(K_V \Delta K^*). \tag{30}$$

We have proven that if $K$ is not in the same class as $K^*$, there exists a hyperplane that decreases the measure of their symmetric difference. □

We have all the results needed to prove the Property I.30

*Proof of Property I.30.* To prove the existence result, we will simply construct such a sequence $(K_n)$. Let $K_0 = K$ and we construct $K_{n+1}$ from $K_n$ by $N$ consecutive Steiner symmetrizations. Note that the notation $K'_{nV}$ symbolizes the Steiner symmetrization of the set $K_n$ for hyperplane $V$. To do this, we choose a hyperplane $H$ such that

$$\lambda(K'_{nH}\Delta K^*) < \inf_{V \text{ hyperplane}} \lambda(K^*_{nV}\Delta K^*) + \frac{1}{n}.$$

We select $N-1$ hyperplanes $H_2, \ldots, H_N$ such that $H, H_2, \ldots, H_N$ have their orthogonal complementary that are pairwise orthogonal. We obtain $K_{n+1}$ by applying $N$ consecutive Steiner symmetrizations to $K_n$ with respect to $H, H_2, \ldots, H_N$, i.e. $K_{n+1} = (((K'_{nH})'_{H_2})'_{H_3} \ldots)'_{H_N}$. By iterating Eq. (26) we get

$$\lambda(K_{n+1}\Delta K^*) \leq \lambda(K'_{nH}\Delta K^*),$$

and by construction of $H$, for any hyperplane $W$, we have

$$\lambda(K_{n+1}\Delta K^*) < \lambda(K'_{nW}\Delta K^*) + \frac{1}{n}. \tag{31}$$

We will now show that there exists a subsequence $(K_{n_j})$ of $(K_n)$ and a measurable set $M$ such that

$$\lambda(K_{n_j}\Delta M) \underset{j\to\infty}{\longrightarrow} 0. \tag{32}$$

In other words, $M$ would be a subsequential limit of the sequence $(K_n)$ in $L^1$. To show this, we will use the following theorem.

**Theorem I.33:** [Bre83], (IV.25) Riesz-Fréchet-Kolmogorov
*Let $\Omega$ be an open set of $\mathbb{R}^N$ and $\omega \subset \Omega$. We consider $\mathscr{F}$ a bounded subset of $L^p(\Omega)$ with $1 \leq p < \infty$. For all $f \in \mathscr{F}$, we suppose that for all positive $\epsilon$ there exists $\delta > 0$ (with $\delta < d(x, \mathbb{R}^N \setminus \Omega)$) such that for all $h$ in $\mathbb{R}^N$ satisfying $|h| < \delta$,*

$$||\tau_h f - f||_{L^p(\Omega)} < \epsilon.$$

*Then $\mathscr{F}_{|\omega}$ is a relatively compact subset of $L^p(\Omega)$.*

We recall that $\tau_y f = x \mapsto f(x+y)$. Our sequence of functions $(\chi_{K_n})$ is bounded in $L^1(\mathbb{R}^N)$ since our sets $K_n$ are so. Thus, it will be sufficient for us to show that $\tau_y f \to f$ in $L^1(\mathbb{R}^N)$ when $y$ goes to 0 to ensure that our sequence of functions $(\chi_{K_n})$ has a subsequential limit in $L^1(\mathbb{R}^N)$. For $x$ in $\mathbb{R}^N$, we adopt the notation $x = (x_1, \ldots, x_N)$ such that each coordinate is expressed according to the orthonormal components of the hyperplanes $H, H_2, \ldots, H_N$. We recall that if $A$ is a measurable set in $\mathbb{R}^N$, for $i \neq j$, the set of $x_j$ such that $x$ is in $A''_{H_j H_i}$ is also a centered interval since the orthogonal complements of $H_j$ and $H_i$ are orthogonal (see Figure 9). Therefore, the set of $x_j$ such that $x$ is in $K_n$ is a centered interval. If we translate the coordinate $x_j$ by $y_j$, then this same set is an interval of the same length but centered at $y_j$. We deduce

$$\int_{\mathbb{R}} |\chi_{K_n}(x_1, \ldots, x_j, \ldots, x_N) - \chi_{K_n}(x_1, \ldots, x_j + y_j, \ldots, x_N)|\, dx \leq 2\,|y_j|\,. \tag{33}$$

Since our domain $K$ is bounded, there exists a finite $R$ such that $K$ is included in the centered ball with radius $R$. We denote $y_{[\![1,j]\!]} = (y_1, \ldots, y_j, 0, \ldots, 0)$ with the convention that $y_{[\![1,0]\!]} = 0$. We observe that for all $y$ in $\mathbb{R}^N$

$$\begin{aligned}
|\chi_{K_n}(x+y) - \chi_{K_n}(x)| &\leq \left|\chi_{K_n}(x+y) - \chi_{K_n}(x + y_{[\![1,N-1]\!]})\right| + \left|\chi_{K_n}(x + y_{[\![1,N-1]\!]}) - \chi_{K_n}(x)\right| \\
&\leq \sum_{j=1}^{N} \left|\chi_{K_n}(x + y_{[\![1,j]\!]}) - \chi_{K_n}(x + y_{[\![1,j-1]\!]})\right|.
\end{aligned}$$

Now let $y$ be in $B(0,R)$. We have that if $x$ is in $K_n$, then $x$ is in the ball $B(0,R)$, and if $x+y$ is in $K_n$, then $x$ is in the ball $B(0,2R) \subset ]-2R,2R[^N$. Then

$$\begin{aligned} f(y) := \int_{\mathbb{R}^N} |\chi_{K_n}(x+y) - \chi_{K_n}(x)|\, dx &= \int_{]-2R,2R[^N} |\chi_{K_n}(x+y) - \chi_{K_n}(x)|\, dx \\ &\leq \sum_{j=1}^{N} \int_{]-2R,2R[^N} \left|\chi_{K_n}(x+y_{[\![1,j]\!]}) - \chi_{K_n}(x+y_{[\![1,j-1]\!]})\right| dx. \end{aligned}$$

The Fubini-Tonelli Theorem allows us to write, denoting $dx' = \prod_{k\in[\![1,N]\!]\setminus\{j\}} dx_k$,

$$f(y) \leq \sum_{j=1}^{N} \int_{]-2R,2R[^{N-1}} \int_{]-2R,2R[} \left|\chi_{K_n}(x+y_{[\![1,j]\!]}) - \chi_{K_n}(x+y_{[\![1,j-1]\!]})\right| dx_j dx'.$$

We can thus use Eq. (33), which finally gives us that

$$\int_{\mathbb{R}^N} |\chi_{K_n}(x+y) - \chi_{K_n}(x)|\, dx \leq \sum_{j=1}^{N} \int_{]-2R,2R[^{N-1}} 2\,|y_j|\, dx' = 2(4R)^{N-1} \sum_{j=1}^{N} |y_j|\,.$$

Therefore, we indeed have that $\tau_y \chi_{K_n} \to \chi_{K_n}$ in $L^1(\mathbb{R}^N)$ as $y$ goes to 0, ensuring the subsequential limit of $(\chi_{K_n})$ in $L^1(\mathbb{R}^N)$. Furthermore, the limit of this subsequence $(K_{n_j})_j$ must be an indicator function since the limit in $L^1$ of indicators is necessarily an indicator. Moreover, as this limit is in $L^1$, we can assure that it is the indicator function of a measurable set, denoted $M$. We have thus proven Eq. (32). Additionally, Eq. (26) ensures that the sequence $(\lambda(\chi_{K_{n_j}}\Delta K^*))$ is non-increasing. Furthermore, the dominated convergence theorem assures us that this sequence converges to $\lambda(M\Delta K^*)$. Hence, for all $j$,

$$\lambda(\chi_{K_{n_j}}\Delta K^*) \geq \lambda(M\Delta K^*). \tag{34}$$

We will now reason by contradiction, assuming that $M$ is not in the class of $K^*$. This implies that

$$\lambda(M\Delta K^*) = \delta > 0.$$

Thanks to Corollary I.32, we know the existence of a hyperplane $W$ and $\epsilon > 0$ such that

$$\lambda(M'_W \Delta K^*) = \delta - \epsilon > 0.$$

Otherwise, $M$ would be in the class of $K^*$. Therefore, we have that $\lambda(K'_{n_j W}\Delta K^*) \to \delta - \epsilon$ when $j$ goes to infinity. Note that the sequence $(\lambda(K'_{n_j W}\Delta K^*))$ is also non-increasing by using again Eq. (26). So, there exists $k$ such that $n_k > 2/\epsilon$ and

$$\lambda(K'_{n_k W}\Delta K^*) < \delta - \frac{\epsilon}{2}.$$

This implies that

$$\lambda(K'_{n_k W}\Delta K^*) + \frac{1}{n_k} < \delta.$$

Finally, by using Eq. (34) we have

$$\lambda(K'_{n_k W}\Delta K^*) + \frac{1}{n_k} < \lambda(\chi_{K_{n_j}}\Delta K^*),$$

which is impossible by the construction of our sequence $(K_n)$. In conclusion, $M$ is indeed in the class of $K^*$. □

### I.3.4 Pólya-Szegő inequality

The following result is the key point in the proof of Faber-Krahn inequality. More precisely, what is proved below is a special case of the Pólya-Szegő inequality (see [Tal76], [PS51]). Meanwhile, this is the one that will be useful to us.

**Theorem I.34:** PÓLYA-SZEGŐ INEQUALITY
*Let $u$ a non-negative function in $H_0^1(\Omega)$, then $u^* \in H_0^1(\Omega^*)$ and*

$$\int_{\Omega^*} |\nabla u^*|^2 \le \int_\Omega |\nabla u|^2.$$

*Proof.* Let $u$ be in $H_0^1(\Omega)$, and we extend $u$ by 0 outside $\Omega$. We have therefore also that $u$ is in $H^1(\mathbb{R}^N)$. Since $u$ is in $L^2(\mathbb{R}^N)$, Property I.14 gives us that $u^*$ is also in $L^2(\mathbb{R}^N)$ and

$$||u||_2^2 = ||u^*||_2^2. \tag{35}$$

Thanks to Theorem I.26, we can write

$$||\nabla u||_2^2 = \lim_{t\to 0} D_u(t) = \lim_{t\to 0} \frac{1}{t}\left(||u||_2^2 - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u(y)K_t(x-y)u(x)dxdy\right).$$

The Riesz rearrangement inequality I.28 and Eq. (35) gives

$$||\nabla u||_2^2 \ge \lim_{t\to 0} \frac{1}{t}\left(||u^*||_2^2 - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u^*(y)K_t^*(x-y)u^*(x)dxdy\right).$$

For any $t > 0$, $K_t$ satisfy the assumptions of Property I.15 then $K_t^* = K_t$. Therefore,

$$||\nabla u||_2^2 \ge \lim_{t\to 0} \frac{1}{t}\left(||u^*||_2^2 - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u^*(y)K_t(x-y)u^*(x)dxdy\right) = \lim_{t\to 0} D_{u^*}(t).$$

We deduce $\lim_{t\to 0} D_{u^*}(t) < \infty$. Thanks again to Theorem I.26, we now know that $u^*$ is in $H^1(\mathbb{R}^N)$ and also $\lim_{t\to 0} D_{u^*}(t) = ||\nabla u^*||_2^2$. So,

$$||\nabla u||_2^2 \ge ||\nabla u^*||_2^2.$$

We assume that $u^*$ is in $H_0^1(\Omega^*)$. □

### I.3.5 Non-negativity of the first eigenfunction

Most of the previous results call for non-negative functions, so it will be useful to have the non-negativity of the first eigenfunction. Before starting the proof, we will need some additional results. The following one and its proof come from [All07].

**Property I.35:**
*Let $G : \mathbb{R} \to \mathbb{R}$ be of class $C^1$ such that $G(0) = 0$ and $G'$ is bounded. Then, for all $u \in H_0^1(\Omega)$, $G(u)$ is also in $H_0^1(\Omega)$ and $\nabla G(u) = G'(u)\nabla u$.*

*Proof.* By definition of $H_0^1(\Omega)$, there exists a sequence $(u_n)$ of $C_c^1(\Omega)$ which converges to $u$ in $H^1(\Omega)$. By composition of functions of class $C^1$, we have for all $n$ that $G(u_n)$ is in $C^1(\Omega)$. In addition, given that $G(0) = 0$, we also have that $G(u_n)$ is compactly supported, so belongs to $C_c^1(\Omega)$. We have,

$$|G(u_n) - G(u)| \le ||G'||_\infty |u_n - u|, \tag{36}$$

then,

$$||G(u_n) - G(u)||_2 \le ||G'||_\infty ||u_n - u||_2 \underset{n\to\infty}{\longrightarrow} 0. \tag{37}$$

So $G(u_n)$ converges to $G(u)$ in $L^2$. Moreover, for $1 \le j \le N$ we have

$$|G'(u_n)\nabla u_n - G'(u)\nabla u| \le |G'(u_n)\nabla u_n - G'(u_n)\nabla u| + |G'(u_n)\nabla u - G'(u)\nabla u|,$$

then,

$$|G'(u_n)\nabla u_n - G'(u)\nabla u| \le ||G'||_\infty |\nabla u_n - \nabla u| + |G'(u_n) - G'(u)||\nabla u|.$$

Thanks to Theorem I.20, we can extract a subsequence of $(u_n)$ which converges pointwise almost everywhere to $u$. As a consequence, we have $|G'(u_n(x)) - G'(u(x))| \to 0$ when $n \to \infty$ almost everywhere on $\Omega$. Also, we have $|G'(u_n) - G'(u)||\nabla u| \le 2||G'||_\infty |\nabla u| \in L^2(\Omega)$. So, dominated convergence theorem B.5 gives us

$$\int_\Omega |G'(u_n) - G'(u)|^2 |\nabla u|^2 = ||(G'(u_n) - G'(u))\nabla u||_2 \underset{n\to\infty}{\longrightarrow} 0$$

Moreover, we have by definition $||\nabla u_n - \nabla u||_2 \to 0$ when $n \to 0$. We deduce that $||G'(u_n)\nabla u_n - G'(u)\nabla u||_2 \to 0$ when $n \to 0$, i.e. $G'(u_n)\nabla u_n$ converges to $G'(u)\nabla u$ in $L^2(\Omega)$. We obtain

$$||G(u_n) - G(u)||_{H^1} = ||G(u_n) - G(u)||_2 + ||\nabla G(u_n) - \nabla G(u)||_2 \underset{n\to\infty}{\longrightarrow} 0.$$

In conclusion, the sequence $(G(u_n))$ is in $C_c^1(\Omega)$ and converges to $G(u)$ in $H^1(\Omega)$, then $G(u)$ is in $H_0^1(\Omega)$. □

The following result will be very useful in our final proof.

**Theorem I.36:**
*Let $u \in H_0^1(\Omega)$, then $u^+ = max(u, 0) = 1_{u>0}u$ is also in $H_0^1(\Omega)$ and, almost everywhere on $\Omega$,*

$$\nabla u^+ = 1_{u>0}\nabla u,$$

*where $1_{u>0}(x)$ is 1 where $u(x) > 0$ and 0 otherwise.*

*Proof.* Since $t \mapsto \max(0, t)$ is not $C^1$, we will approximate it by a sequence of $C^1$ functions. Let $G : \mathbb{R} \to \mathbb{R}$ a function of $C^1$ such that $G(t) = 0$ if $t \leq 0$, $G(t) = t$ if $t \geq 1$ and $|G'(t)| \leq 1$ for all $t$. We define $G_n(t) = G(nt)/n$ for $n \geq 1$, a sequence of functions. Defined like that, $G_n$ verifies the Property I.35 assumption. In addition, we have for all $t$, $G_n(t) \to t^+$ and $G_n'(t) = G'(nt) \to 1_{t>0}$ when $n \to \infty$. From Property I.35, we know for all $u \in H_0^1(\Omega)$ that $G_n(u)$ is also in $H_0^1(\Omega)$ and $\nabla G_n(u) = G_n'(u)\nabla u$. We have,

$$|G_n(u) - u^+| \leq (||G_n'||_\infty + 1)|u| = (||G'||_\infty + 1)|u| \in L^2(\Omega).$$

Thanks to dominated convergence theorem B.5, we deduce that $G_n(u) \to u^+$ when $n \to \infty$ in $L^2(\Omega)$. Also,

$$|\nabla G_n(u) - 1_{u>0}\nabla u| = |(G_n'(u) - 1_{u>0})||\nabla u| \leq 2|\nabla u| \in L^2(\Omega).$$

Once again, from Theorem B.5 we have $\nabla G_n(u) \to 1_{u>0}\nabla u$ when $n \to \infty$ in $L^2(\Omega)$. Therefore, $G_n(u)$ converges to $u^+$ in $H^1(\Omega)$, with $\nabla u^+ = 1_{u>0}\nabla u$. We therefore obtain our result given that $(H_0^1(\Omega), ||\cdot||_{H^1})$ is closed. Indeed, let $(v_n)$ a sequence of $H_0^1(\Omega)$ which converges to $u$ with respect to the $H^1$ norm. Then, for all $\epsilon > 0$, there exists $N$ such that for all $n \geq N$,

$$||u_n - u||_{H_1} \leq \epsilon.$$

Since, for all $n$, $v_n$ is in $H_0^1(\Omega)$, there exists a sequence $(v_{n,m})_m$ of $C_c^1(\Omega)$ which converges to $v_n$ with respect to the $H^1$ norm. Therefore, there exists $M$ such that for all $m \geq M$,

$$||v_{n,m} - v_n||_{H^1} \leq \epsilon.$$

Then, for $n \geq N$ and $m \geq M$,

$$||v_{n,m} - v||_{H^1} \leq ||v_{n,m} - v_n||_{H^1} + ||v_n - v||_{H^1} \leq 2\epsilon.$$

So, $(v_{n,m})$ converges to $v$ in $H_1$, so $v$ is in $H_0^1(\Omega)$. □

We will also need the following inequality.

**Property I.37:**
*Let $u$ be of class $C^\infty$ on $\Omega$ and $\Delta u \leq 0$. We get for all $x_c$ in $\Omega$, that for any $r < d(x_c, \partial\Omega)$,*

$$u(x_c) \geq \frac{1}{|B(x_c, r)|}\int_{B(x_c,r)} u.$$

*Proof.* For $0 < r < d(x_c, \partial\Omega)$, we write

$$T(r) = \frac{1}{|B(x_c, r)|}\int_{B(x_c,r)} u(x)dx,$$

where $|B(x_c, r)|$ is the measure of the ball of radius $r$ and center $x_c$. We begin by noting that $T(r) \to u(x_c)$ when $r$ goes to 0. Indeed, since $u$ is continuous on $\Omega$, for any $\epsilon > 0$ there exists a small enough $r$ such that for $x$ in $B(x_c, r)$ we have $|u(x_c) - u(x)| \leq \epsilon$. So,

$$|T(r) - u(x_c)| \leq \frac{1}{|B(x_c,r)|} \int_{B(x_c,r)} |u(x) - u(x_c)| dx \leq \frac{\epsilon}{|B(x_c,r)|} \int_{B(x_c,r)} dx = \epsilon.$$

We can therefore extend by continuity $T$ in 0 by $u(x_c)$. So, to show our inequality, all we have to do is show that the function $T$ is non-increasing. We recall that $|B(x_c, r)| = r^N |B(x_c, 1)|$. The diffeomorphism $s \mapsto x_c - sr$ which sends $B(0, 1)$ onto $B(x_c, r)$ has Jacobian matrix $rI_N$ with determinant $r^N$. Then, the change of variables formula B.10 gives

$$T(r) = \frac{r^N}{|B(x_c,r)|} \int_{B(0,1)} u(x_c + rs) ds = \frac{1}{|B(x_c,1)|} \int_{B(0,1)} u(x_c + rs) ds.$$

The continuous image by $u$, or its derivatives, of the closed ball, then bounded, is a bounded closed space. Thus, we easily apply the theorem of differentiation under the integral sign B.9. So,

$$T'(r) = \frac{1}{|B(x_c,1)|} \int_{B(0,1)} \nabla u(x_c + rs) \cdot s \; ds.$$

By performing the inverse change of variable $x \mapsto \frac{x - x_c}{r}$ of Jacobian $r^{-1} I_N$, we obtain that

$$T'(r) = \frac{1}{r^N |B(x_c,1)|} \int_{B(x_c,r)} \nabla u(x) \cdot \frac{x - x_c}{r} dx.$$

Applying Green's formula A.26 gives

$$T'(r) = \frac{1}{|B(x_c,r)|} \left( \int_{\partial B(x_c,r)} u(x) \frac{x - x_c}{r} \cdot n d\sigma_x - \int_{B(x_c,r)} u(x) \nabla \cdot \left( \frac{x - x_c}{r} \right) dx \right).$$

We notice $\nabla \cdot \left( \frac{x - x_c}{r} \right) = \frac{N}{r}$ and for $x$ on the boundary of $B(x_c, r)$, the vector $\frac{x - x_c}{r}$ is the unitary outward normal vector $n$. Then,

$$T'(r) = \frac{1}{|B(x_c,r)|} \left( \int_{\partial B(x_c,r)} u(x) d\sigma - \frac{N}{r} \int_{B(x_c,r)} u(x) dx \right).$$

Let us show that this quantity is negative. The idea is to start by noticing that

$$\int_{B(x_c,r)} u = \frac{1}{2N} \int_{B(x_c,r)} u(x) \Delta(|x - x_c|^2 - r^2) dx.$$

Indeed, $\nabla(|x - x_c|^2 - r^2) = 2|x - x_c|$ and $\Delta(|x - x_c|^2 - r^2) = 2N$. Using Green's formula, we obtain

$$\int_{B(x_c,r)} u = \frac{1}{2N} \left( \int_{\partial B(x_c,r)} 2|x - x_c| u(x) d\sigma - \int_{B(x_c,r)} \nabla(|x - x_c|^2 - r^2) \nabla u(x) dx \right).$$

Noting that in the first term $|x - x_c| = r$, and by applying again Green's formula to the second, we get

$$\int_{B(x_c,r)} u = \frac{1}{2N} \left( 2r \int_{\partial B(x_c,r)} u \; d\sigma - \int_{\partial B(x_c,r)} (|x - x_c|^2 - r^2) \nabla u \cdot n \; d\sigma + \int_{B(x_c,r)} (|x - x_c|^2 - r^2) \Delta u \; dx \right).$$

However, on $\partial B(x_c, r)$ we have $|x - x_c|^2 = r^2$, so the second term is zero. So, given that for $x$ in $B(x_c, r)$, $|x - x_c|^2 \leq r$ and that $\Delta u \leq 0$, we obtain

$$\frac{r}{N} \int_{\partial B(x_c,r)} u(x) d\sigma - \int_{B(x_c,r)} u = \int_{B(x_c,r)} (r^2 - |x - x_c|^2) \Delta u(x) dx \leq 0.$$

For $0 < r < d(x_c, \partial\Omega)$, we therefore have that $T'(r) \leq 0$. In conclusion, $T$ is a non-increasing function and given that $T(0) = u(x_c)$, we have finally $u(x_c) \geq T(r)$. □

We finally arrived at our result.

**Theorem I.38:**
*Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$. On each connected part of $\Omega$, the eigenspace associated to $\lambda_1$, i.e. $ES(\lambda_1)$, is of dimension 1 and if $e_1$ span $ES(\lambda_1)$, then $e_1$ is of constant sign.*

*Proof.* Let $e$ be in $ES(\lambda_1)$, we can choose $e$ in $H_0^1(\Omega) \cap C^\infty(\Omega)$ thanks to Theorem A.1. We recall that $|e| = e^+ + e^-$, with $e^+ = max(e, 0)$ and $e^- = (-e)^+$. In addition, since $e$ is in $C^\infty(\Omega)$, so it can be evaluated for any $x$ on $\Omega$. From Theorem I.36, we have that $e^+$ and $e^-$ are in $H_0^1(\Omega)$ and almost everywhere on $\Omega$,

$$\nabla e^+ = 1_{e>0}\nabla e \qquad \text{and} \qquad \nabla e^- = 1_{-e>0}\nabla(-e) = -1_{e<0}\nabla u.$$

Then, $|e|$ is also in $H_0^1(\Omega)$ and almost everywhere, $\nabla|e| = (1_{e>0} - 1_{e<0})\nabla e$. Therefore, we have

$$||\nabla e||_2^2 = \int |\nabla e|^2 dx = \int |\nabla(|e|)|^2 dx = ||\nabla(|e|)||_2^2.$$

So,

$$\rho(e) = \frac{||\nabla e||_{L^2}^2}{||e||_{L^2}^2} = \frac{||\nabla(|e|)||_{L^2}^2}{||(|e|)||_{L^2}^2} = \rho(|e|) = \lambda_1.$$

Thanks to Corollary I.2, we have that $|e|$ is also in $ES(\lambda_1)$. Moreover, $\Delta|e| = -\lambda_1|e| \leq 0$, so Property I.37 gives for all $x_c$ in $\Omega$, that for any $r < d(x_c, \partial\Omega)$,

$$|e|(x_c) \geq \frac{1}{|B(x_c, r)|} \int_{B(x_c,r)} |e|.$$

If there exists $x$ in $\Omega$ such that $e(x) = 0$, $e$ is then zero on $B(x, d(x_c, \partial\Omega))$. Reiterating this process, we show that $e$ is then zero over the connected part associated with $x$ of $\Omega$. Let us work now on one connected part of $\Omega$. Let $x_c$ such that $e(x_c) \neq 0$ and $u$ be in $ES(\lambda_1)$ be another eigenfunction of $H_0^1 \cap C^\infty$ in $ES(\lambda_1)$. Then $u - \frac{u(x_c)}{e(x_c)}e$ is in $ES(\lambda_1)$, meanwhile is zero at $x_c$, then over all the connected part. So $u = \frac{u(x_c)}{e(x_c)}e$. We deduce the eigenspace $ES(\lambda_1)$ associated with each connected part is of dimension 1. We also conclude that the functions in this space have a constant sign. □

### I.3.6 Faber-Krahn inequality

The Faber-Krahn inequality addresses the problem of minimizing the first eigenvalue. Specifically, it asserts that the first eigenvalue is not less than the corresponding Dirichlet eigenvalue of a ball with the same volume.

**Theorem I.39:** Faber-Krahn inequality
*Let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue associated with the bounded open domains $\Omega$ of $\mathbb{R}^N$. We have,*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*).$$

*Proof.* Thanks to Theorem I.1, we know that

$$\lambda_1(\Omega) = \inf_{e \in H_0^1(\Omega)\setminus 0} \frac{||\nabla e||_{L^2}^2}{||e||_{L^2}^2}.$$

Theorem I.38 allows us to choose $e_1 \in H_0^1(\Omega)$ being a non-negative, non-zero and unitary eigenfunction with respect to eigenvalue $\lambda_1$. Moreover, from Corollary I.2 we have $\lambda_1(\Omega) = \rho(e_1)$. According to Property I.14 we have $||e_1||_2^2 = ||e_1^*||_2^2$ and according to Theorem I.34, $e_1^*$ is in $H_0^1(\Omega^*)$ and $||\nabla e_1||_2^2 \geq ||\nabla e_1^*||_2^2$. Therefore,

$$\lambda_1(\Omega) = \frac{||\nabla e_1||_{L^2}^2}{||e_1||_{L^2}^2} \geq \frac{||\nabla e_1^*||_{L^2}^2}{||e_1^*||_{L^2}^2} \geq \inf_{e \in H_0^1(\Omega^*)\setminus 0} \frac{||\nabla e||_{L^2}^2}{||e||_{L^2}^2} = \lambda_1(\Omega^*). \tag{38}$$

□

*Remark.* It is also possible to prove that the ball (up to translation and negligible set) is the unique minimizer of the first eigenvalue (see Example 2.11 of [Kaw85]). This addresses the question, "Can one hear whether a domain is a ball?" Indeed, if one knows all the eigenvalues of the Laplacian of a domain, then the volume and dimension of the domain can be determined by Weyl's law. If the first eigenvalue matches that one of a ball with the same volume and dimension, then by the Faber-Krahn theorem, we know that the domain is a ball of that volume.

# II Numerical approach

In the previous section, we presented a series of theoretical results on the spectral analysis of the Laplacian. To incorporate a modeling component into our study, we will here be interested in a numerical approach to the eigenvalues of the Laplacian. Here, our objective is to introduce a numerical method for addressing the problems previously discussed, such as Weil's inequality. We aim to propose a modeling approach to test the validity of the results. Our primary interest will be in the scientific methodology of this approach rather than the underlying theory. All the simulations and numerical results presented below come from my Python simulations. You can find my code on my Github page associated with this study.

## II.1 Estimation of Laplacian eigenfunctions and eigenvalues

Here we will develop a method to numerically determine the spectrum and eigenfunctions of $-\Delta$ with Dirichlet condition, associated with any domain $\Omega$ of $\mathbb{R}^2$. For this purpose, we will employ the finite difference method to discretize our Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \quad (39a) \\ u = 0 & \text{on } \Gamma := \partial\Omega. \quad (39b) \end{cases}$$

As shown previously, $-\Delta$ is a positive operator, so we consider $\lambda > 0$. We will proceed step by step, starting with the simplest case, the 1D interval. Once this is accomplished, we will move on to the 2D rectangle, and finally, we will address an arbitrary domain problem in $\mathbb{R}^2$.

### II.1.1 One-dimensional case

Without loss of generality, we will consider the open unit interval here, i.e. $\Omega_I = ]0,1[$. Before moving on to the discrete case, let us look at the exact solutions in our situation.

#### Solving the Laplace eigenvalue problem

This is similar to what we have done in the Section I.2.1. The Eq. (39a) admits as exact solution

$$u(x) = A\cos(\mu x) + B\sin(\mu x) \qquad \forall x \in \Omega_i,$$

with $\lambda = \mu^2$. Boundary conditions from (39b) are $u(0) = 0 = A$ and $u(1) = 0 = B\sin(\mu)$. This implies that $\mu_k = k\pi$ with $k \in \mathbb{N}^*$ (if $k = 0$, the eigenfunction considered is the null one). Then, for all $k > 0$, the eigenvalue associated with the eigenfunction $u_k : x \mapsto \sin(xk\pi)$ is $\lambda_k = k^2\pi^2$. Note that these eigenfunctions form a Hilbert basis of $L^2(\Omega_I)$, since $(\sin(nx))_{n\geq 1}$ forms a Hilbert basis of $L^2(]0,\pi[)$.

#### Finite Difference for the 1D Laplacian Operator

Now that we have solved the Dirichlet eigenvalue problem, we will proceed to its numerical resolution, using the finite-difference method. To numerically solve this equation, the idea is to use a finite difference scheme. For that, we introduce a subdivision of our initial domain $\Omega_i$ into $N \in \mathbb{N}$ intervals of equal length, using $N+1$ equidistant points $(x_i)_{i\in[\![0,N]\!]}$ with $x_0 = 0$ and $x_N = 1$. Utilizing the Taylor-Lagrange formula, a centered second-order approximation for the second derivative of a function $u$, with respect to a small variation $h$, is derived as

$$u_{xx}(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2).$$

By choosing $x = x_i (= \frac{i}{N})$ and $h = 1/N$, we obtain for $i \in [\![1, N-1]\!]$,

$$u_{xx}(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + \mathcal{O}(h^2).$$

Thus, for $i \in [\![1, N-1]\!]$, we have the discretization of the relation (39a) as

$$-\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} \approx \lambda u(x_i).$$

Let $\widetilde{u}$ now denote such an approximation of $u$. We have $N-1$ equations for $N+1$ unknowns, the sequence $(\widetilde{u}(x_i))_{i\in[\![0,N]\!]} \subset \mathbb{R}^{N+1}$. However, we have not yet considered our boundary conditions, they impose that

$\widetilde{u}(x_0) = \widetilde{u}(x_N) = 0$. Our system is thus well-defined. To summarize, our finite difference discretization of the Dirichlet eigenvalue problem is

$$M\widetilde{u} = \lambda\widetilde{u},$$

with

$$M = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix} \in \mathscr{M}_{N-1}(\mathbb{R}), \qquad \widetilde{u} = \begin{pmatrix} u(x_1) \\ \vdots \\ \vdots \\ u(x_{N-1}) \end{pmatrix}, \qquad \widetilde{u}(x_0) = \widetilde{u}(x_N) = 0.$$

The solutions to this problem are therefore the eigenvectors $(\widetilde{u}_j)_{j\in[\![1,N-1]\!]}$ and the eigenvalues $(\widetilde{\lambda}_j)_{j\in[\![1,N-1]\!]}$ of our real symmetric matrix $M$ completed with 0 at extremities.

### Discrete Eigenvectors

Let us start by checking that such a solution agrees with the expected exact results. To do this, let us define for $n \in [\![0, N]\!]$, inspired from exact solution, the quantity $e_n = \sin(n\eta/N)$ with $\eta$ in $[0, 2\pi[$. By linearizing the sine, we obtain for all $n \in [\![1, N-1]\!]$

$$2e_n - e_{n-1} - e_{n+1} = \underbrace{2(1-\cos(\eta/N))}_{h^2\lambda} e_n.$$

Moreover, boundary condition $e_N = 0$ impose $\eta_j = j\pi$, for $j \in \mathbb{N}$. We notice $e_0 = 0$ is always satisfied. The vector $\widetilde{u}_j = (\sin(n\frac{j\pi}{N}))_{n\in[\![1,N-1]\!]}$ is then an eigenvector of $M$. Meanwhile, we notice $\widetilde{u}_{j+N} = -\widetilde{u}_j$ and $\widetilde{u}_0 = 0$, so we must restrict $j$ to $[\![1, N-1]\!]$ in order to maintain a linearly independent set of vectors. So, we finally end up with a set of $N-1$ linearly independent vectors, so we have found all the eigenvectors of $M$. Then, we have that the eigenvector $\widetilde{u}_j$ of our real symmetric matrix $M$ will be a discretization at points $(x_i)_{i\in[\![1,N-1]\!]}$ of the eigenfunction $u_j : x \mapsto \sin(xj\pi)$.

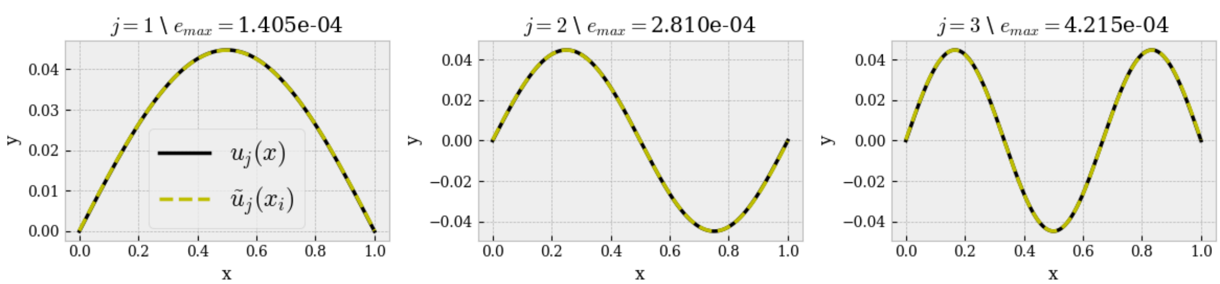



Figure 12: The first three eigenvectors of $M$, $\widetilde{u}_j$ (numerically computed with $N = 1000$), compared to eigenfunctions of $-\Delta$ with Dirichlet condition, $u_j$ (exact). Here, $e_{\max} = \max_i |\widetilde{u}_j(x_i) - u_j(x_i)|$.

Since these vectors are supposed to be identical, we deduce that the error $e_{\max}$ is due to the approximate numerical diagonalization of $M$.

### Discrete Eigenvalues

Let us now look at the eigenvalues $(\widetilde{\lambda}_j)$ of $M$ associated with the eigenvectors $(\widetilde{u}_j)$. Moreover, we proved earlier that for $j$ in $[\![1, N-1]\!]$ we have

$$\widetilde{\lambda}_j(N) = 2N^2\left(1 - \cos\left(\frac{j\pi}{N}\right)\right).$$

In what follows, for the sake of readability, we will note $\widetilde{\lambda}_j(N) = \widetilde{\lambda}_j$. In this way, we can examine the difference between the eigenvalues of $M$ and the exact eigenvalues of the continuous problem. Let us run a first numerical simulation to illustrate their evolution, this is illustrated in Figure 13. It makes sense to try

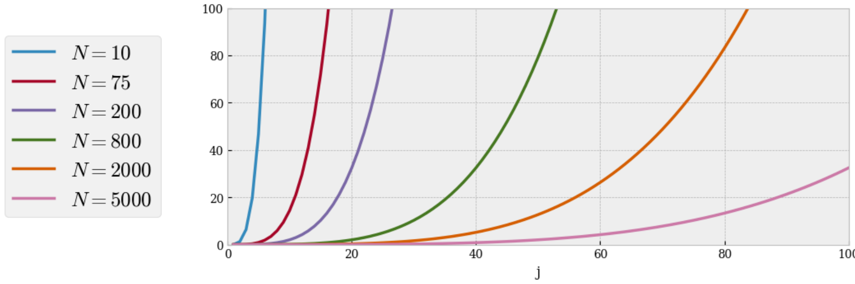



Figure 13: Evolution of $j \mapsto \lambda_j - \widetilde{\lambda}_j(N)$ for several $N$.

to quantify the error introduced by our discretization. For this purpose, by linearization of the cosine, we have $1 - \cos(a) = 2\left(\sin \frac{a}{2}\right)^2$ and then

$$\widetilde{\lambda}_j = 4N^2 \left(\sin \frac{j\pi}{2N}\right)^2.$$

This expression allows us to study the difference between the eigenvalues of $M$ and the exact eigenvalues of (39a). Indeed,

$$\lambda_j - \widetilde{\lambda}_j = j^2\pi^2 - \frac{4}{h^2}\left(\sin \frac{j\pi h}{2}\right)^2 = j^2\pi^2 \left(1 - \frac{2}{j\pi h}\sin(j\pi h/2)\right)\left(1 + \frac{2}{j\pi h}\sin(j\pi h/2)\right).$$

It is known that for $0 < \theta \leq \frac{\pi}{2}$ we have

$$\theta - \frac{\theta^3}{6} \leq \sin\theta \leq \theta, \tag{40}$$

thus we easily get $2 - \frac{\theta^2}{6} \leq 1 + \frac{\sin\theta}{\theta} \leq 2$ and $0 \leq 1 - \frac{\sin\theta}{\theta} \leq \frac{\theta^2}{6}$. Since for $j \in [\![1, N-1]\!]$ we have $0 < \frac{j\pi}{2N} \leq \frac{\pi}{2}$, we deduce

$$0 \leq \lambda_j - \widetilde{\lambda}_j \leq \frac{j^4\pi^4}{12N^2},$$

and therefore the relative error $e_{r,j}(N) := \frac{\lambda_j - \widetilde{\lambda}_j}{\lambda_j}$ is bounded such that

$$0 \leq e_{r,j}(N) \leq \frac{j^2\pi^2}{12N^2}. \tag{41}$$

A first observation, as suggested by Figure 13, is that the exact eigenvalue is always greater than its numerical estimation. Furthermore, this inequality states that to get a relative error $e_{r,j}(N)$ lower than $\epsilon$, it is sufficient to choose $N \geq \widetilde{N}_j(\epsilon)$ with

$$\widetilde{N}_j(\epsilon) := \frac{j\pi}{\sqrt{12\epsilon}}. \tag{42}$$

We can simulate the evolution of this inequality to determine its accuracy, as shown in Figure 14. It seems

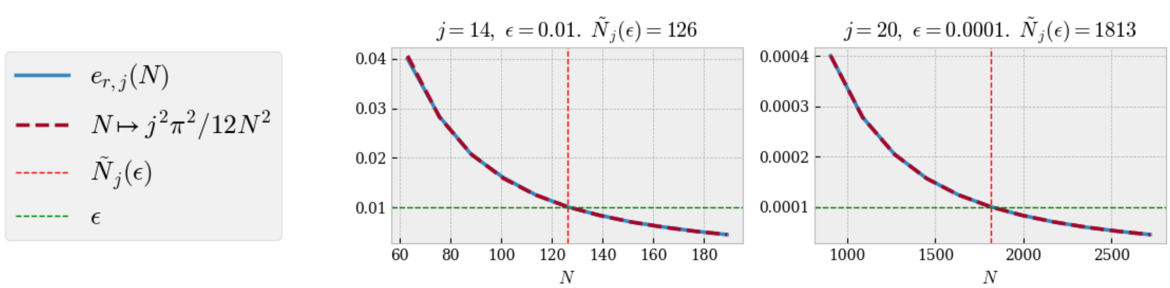



Figure 14: Evolution of relative error $e_{r,j}$ in function of $N$ for several $j$ and $\epsilon$.

that this bound closely approximates $N_{\text{opt}}$, the optimal value of $N$, which is the smallest value ensuring

$e_{r,j}(N) < \epsilon$. Indeed, taking the inequality (40) a step further, such that for $0 < \theta \leq \frac{\pi}{2}$,

$$\theta - \frac{\theta^3}{3!} \leq \sin\theta \leq \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!},$$

we obtain very easily that for $j \ll N$ that

$$\frac{j^2\pi^2}{12N^2} + \mathscr{O}\left(\frac{j^4}{N^4}\right) \leq e_{r,j} \leq \frac{j^2\pi^2}{12N^2} + \mathscr{O}\left(\frac{j^4}{N^4}\right).$$

In other words when $\frac{j}{N} \to 0$ we have

$$e_{r,j}(N) \sim \frac{j^2\pi^2}{12N^2},$$

and so $\widetilde{N}_j(\epsilon)$ is (very closely to) the optimal one. In addition, this result states the evolution of $\widetilde{N}$ is linear with respect to $j/\epsilon^{-1/2}$. We can deduce some very interesting properties regarding the complexity of solving this eigenvalue problem numerically. The usual numerical methods used for this kind of problem involve a divide-and-conquer eigenvalue algorithm, see the LAPACK library. Then, the complexity of diagonalizing the matrix $M$ is in $\mathscr{O}(N^3)$ ([Dem97], chap. V.3). Overall, the complexity of our problem becomes $\mathscr{O}(j^3\epsilon^{-3/2})$. Consequently, if we wish to study eigenvalues ten times further with the same precision, we would need to increase our complexity by roughly a factor of 1000. Now let us imagine that if we wanted to increase our precision by a factor of 100, we would need $N$ to be ten times larger, which would multiply the calculation time by a thousand.

### II.1.2 Extension to the rectangle of $\mathbb{R}^2$

To simplify our calculations and without loss of generality, we're going to work in $\Omega_R = ]0,1[\times]0,1[$. We have shown in the Section I.2.1 that the Laplacian with Dirichlet boundary conditions has in this rectangle the eigenvalues

$$\lambda_{l,m} = (l^2 + m^2)\pi^2,$$

with associated eigenfunctions

$$u_{l,m} : (x,y) \in \Omega_R \mapsto \sin(xl\pi)\sin(ym\pi), \qquad l,m \in \mathbb{N}^* \times \mathbb{N}^*. \tag{43}$$

To recover these results numerically, we will once again use the finite-difference method. Eigenfunctions are shown in Figure 15.

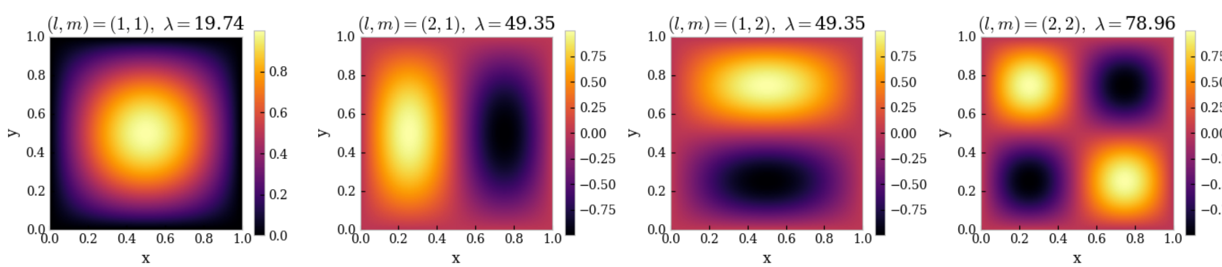



Figure 15: Exact eigenfunctions $u_{l,m}$ on the 2D rectangle, from Eq. (43).

#### Finite Difference for the 2D Laplacian Operator

For this we will consider the uniform grid, i.e. we will consider $N \in \mathbb{N}$ and points $(x_i, y_j) = (i/N, j/N)$ for $(i,j) \in [\![0,N]\!]$. As before, we will discretize the Laplacian operator, written in the canonical basis of $\mathbb{R}^2$ as $\Delta u = u_{xx} + u_{yy}$. To do this, we will consider a centered second-order scheme for each of its derivatives. We therefore obtain as a discretization of Eq. (39a),

$$\frac{4u(x_i,y_j) - u(x_{i-1},y_j) - u(x_{i+1},y_j) - u(x_i,y_{j-1}) - u(x_i,y_{j+1})}{h^2} + \mathscr{O}(h^2) = \lambda u(x_i,y_j),$$

with $h = 1/N$. Let us again denote $\widetilde{u}$ as the numerical solution of our problem such that $\widetilde{u}_{i,j} \approx u(x_i, y_j)$. The latter must therefore be in $\mathscr{M}_{N+1}(\mathbb{R})$ and satisfy

$$\frac{4\widetilde{u}_{i,j} - \widetilde{u}_{i-1,j} - \widetilde{u}_{i+1,j} - \widetilde{u}_{i,j-1} - \widetilde{u}_{i,j+1}}{h^2} = \widetilde{\lambda}\widetilde{u}_{i,j}, \qquad \forall (i,j) \in [\![1, N-1]\!], \tag{44}$$

and also

$$u_{i,0} = u_{i,N} = u_{0,j} = u_{N,j} = 0, \tag{45}$$

which is the Dirichlet boundary conditions. Note that we have $(N+1)^2$ equations and $(N+1)^2$ unknowns, so our problem is well-posed. As done in the previous section, we will formulate this problem in matrix form to evaluate numerically the eigenvectors and eigenfunctions of this system being an approximation of our initial problem. So we want to rewrite our problem as follows

$$M\widetilde{u} = \widetilde{\lambda}\widetilde{u}.$$

For this we will consider the vector version of $\widetilde{u}$ such that

$$\begin{aligned} \widetilde{u}^T &= (\widetilde{u}_{1,1}, \widetilde{u}_{2,1}, \ldots, \widetilde{u}_{N-1,1}; \widetilde{u}_{1,2}, \ldots, \widetilde{u}_{N-1,2}; \ldots; \widetilde{u}_{1,N-1}, \ldots, \widetilde{u}_{N-1,N-1}) \in \mathbb{R}^{(N-1)^2} \\ &= (\widetilde{u}_1, \ldots, \widetilde{u}_{(N-1)^2}). \end{aligned} \tag{46}$$

This is equivalent to putting the columns of the matrix version end-to-end. Specifically, the index $(i, j)$ in matrix form then becomes $i + (N-1)(j-1)$ in vector form. This defines a bijection between our objects. Note that there is no need to include values of $\widetilde{u}$ at the boundary of $\Omega_R$ in our system since they are zero. Naturally, when we use one (resp. two) index to $\widetilde{u}$, we will be referring to its vector (resp. matrix) version. We can then rewrite the Eq. (44) such that for all $(i, j) \in [\![1, N-1]\!]$,

$$\frac{4\widetilde{u}_{i+(N-1)(j-1)} - \widetilde{u}_{i-1+(N-1)(j-1)} - \widetilde{u}_{i+1+(N-1)(j-1)} - \widetilde{u}_{i+(N-1)(j-2)} - \widetilde{u}_{i+(N-1)j}}{h^2} = \widetilde{\lambda}\widetilde{u}_{i+(N-1)(j-1)},$$

where when the index of $\widetilde{u}$ is no longer in $[\![1, (N-1)^2]\!]$, the associated value is set to 0. Let us go into a little more detail on this point to highlight the construction of the rows and columns on the boundary. We do that in view to well understanding the construction of the matrix. Let us take the last line as an example, i.e. the $(u_{N-1,j})_{j\in[\![1,N-1]\!]}$. We want them to check

$$\frac{1}{h^2}(4\widetilde{u}_{N-1,j} - \widetilde{u}_{N-2,j} - 0 - \delta_{j\neq 1}\widetilde{u}_{N-1,j-1} - \delta_{j\neq N-1}\widetilde{u}_{N-1,j+1} = \widetilde{\lambda}\widetilde{u}_{N-1,j} \qquad \forall j \in [\![1, (N-1)]\!].$$

Although this precision may seem trivial, it highlights the fact that the sub-diagonal of our matrix $M$ has some coefficients at 0. For example, we have the coefficient $M_{N,N-1} = 0$. After a further disjunction of cases, we can write

$$M = \frac{1}{h^2}\begin{pmatrix} N & -I & 0 & \cdots & 0 \\ -I & N & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -I & N & -I \\ 0 & \cdots & 0 & -I & N \end{pmatrix} \in \mathscr{M}_{(N-1)^2}, \quad \text{with} \quad N = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 4 & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix} \in \mathscr{M}_{(N-1)}, \tag{47}$$

and $I$ the identity of $\mathscr{M}_{(N-1)}$. As in the previous case, the solutions to this problem are the eigenvectors $(\widetilde{u}_j)_{j\in[\![1,N-1]\!]}$ with the corresponding eigenvalues $(\widetilde{\lambda}_j)_{j\in[\![1,N-1]\!]}$ of our real symmetric matrix $M$.

**Discrete Eigenvectors**

Inspired by the exact solution (43), let us define $e_{i,j} = \sin(i\eta/N)\sin(j\theta/N)$. We will use the same method as above to show that this quantity satisfies (44) and (45). Then, thanks to the same linearization as before, we can easily obtain that

$$4e_{i,j} - e_{i-1,j} - e_{i+1,j} - e_{i,j-1} - e_{i,j+1} = \underbrace{2(2 - \cos(\eta/N) - \cos(\theta/N))}_{h^2\lambda} e_{i,j}.$$

The Dirichlet boundary condition $e_{N,j} = 0$ implies that $\sin(\eta) = 0$ and therefore that

$$\eta_l = l\pi, \qquad \text{for } l \in \mathbb{N}.$$

Similarly, $e_{i,N} = 0$ implies $sin(\theta) = 0$ and therefore

$$\theta_m = m\pi, \qquad \text{for } m \in \mathbb{N}.$$

We want our families generated by these results to be linearly independent and different from 0. We therefore impose that $(\eta_l/N, \theta_m/N) \in ]0, 2\pi[$, and thus that $(l, m) \in [\![1, N-1]\!]$. Furthermore, it is obvious that the conditions $e_{0,j} = e_{i,0} = 0$ are indeed satisfied. The eigenvectors of our matrix $M$ are thus

$$\widetilde{u}_{l,m} = \left(\sin\left(l\pi\frac{i}{N}\right)\sin\left(m\pi\frac{j}{N}\right)\right)_{i,j\in[\![1,N-1]\!]}, \tag{48}$$

for $(l, m) \in [\![1, N-1]\!]$. Consequently, we have $(N-1)^2$ linearly independent eigenvectors, thus, we have identified all the eigenvectors of $M$. The associated eigenvalues are

$$\widetilde{\lambda}_{l,m} = 2N^2\left(2 - \cos\left(\frac{l}{N}\pi\right) - \cos\left(\frac{m}{N}\pi\right)\right), \tag{49}$$

The numerical eigenvalues of $M$ are presented in Figure 16. At first glance, they don't look like the expected

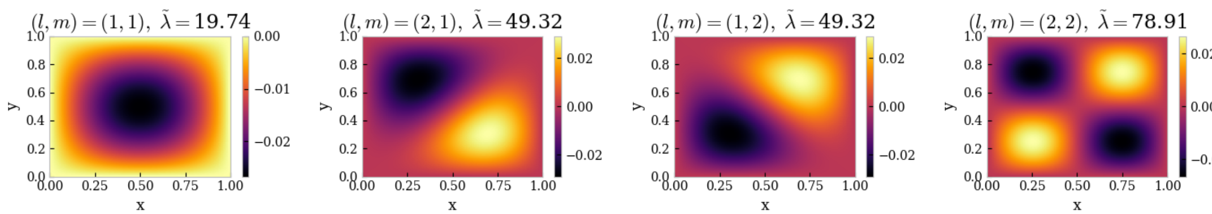



Figure 16: Numerically computed eigenfunctions $\widetilde{u}_{l,m}$ of $M$, from Eq. (47).

results of Figure 15. Looking a little closer, we notice that the eigenvectors obtained for $l = m$ are those expected. This situation corresponds to eigenfunctions associated with eigenvalues of multiplicity one. Indeed, the eigenvalues (49) for $(l, m)$ are the same as for $(m, l)$. More precisely, for $l$ different from $m$, the associated eigenvalues are exactly of multiplicity two. We can deduce that the numerical results in such a situation are a linear combination of the two eigenvectors $\widetilde{u}_{l,m}$ and $\widetilde{u}_{m,l}$ of (48). It is easy to verify that the vector (48) is an eigenvector of $M$. As a side note, if we take a rectangle $]0, a[\times]0, b[$ instead of a square, all our eigenvalues become singular. Indeed, for the same discretization cardinal $N$ for each coordinate and after small modification on $M$, they become

$$\widetilde{\gamma}_{l,m} = 2\left(\left(\frac{a}{N}\right)^2(1 - \cos(l\pi/N)) + \left(\frac{b}{N}\right)^2(1 - \cos(m\pi/N))\right).$$

Consequently, we would no longer encounter this problem in such a case, as illustrated in the Figure 17.

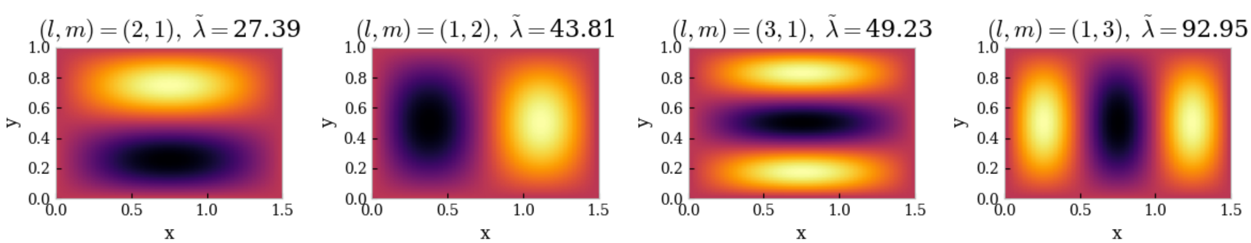



Figure 17: Numerically computed eigenfunctions of $M$ on $\Omega = ]0, 1.5[\times]0, 1[$.

**Discrete Eigenvalues**

By linearizing Eq. (49), we obtain

$$\widetilde{\lambda}_{l,m} = 4N^2 \left( \sin\left(\frac{l\pi}{2N}\right)^2 + \sin\left(\frac{m\pi}{2N}\right)^2 \right).$$

Using exactly the same method as for the interval, we show

$$0 \leq \lambda_{l,m} - \widetilde{\lambda}_{l,m} \leq (l^4 + m^4)\frac{\pi^4}{12N^2}.$$

This allows us to write that

$$0 \leq \frac{\lambda_{l,m} - \widetilde{\lambda}_{l,m}}{\lambda_{l,m}} \leq \frac{\max(l^2, m^2)\pi^2}{12N^2}.$$

This result enables us once again to ensure that the numerical estimate is less than the exact eigenvalue of our problem. We also found an upper bound for our relative error. More precisely, we even show when $\max(l,m) \ll N$ that

$$\frac{\lambda_{l,m} - \widetilde{\lambda}_{l,m}}{\lambda_{l,m}} = \frac{\max(l,m)^2\pi^2}{12N^2} + \mathscr{O}\left(\frac{\max(l,m)^4}{N^4}\right).$$

We can therefore deduce some precision results similar to those shown in the previous section. Indeed, to have the eigenvalue $\lambda_{l^*,m^*}$ with a relative precision $\epsilon > 0$, we can take $N = \widetilde{N}_{l^*,m^*}(\epsilon)$ where

$$\widetilde{N}_{l,m}(\epsilon) := \frac{\max(l,m)\pi}{\sqrt{12\epsilon}},$$

which is very close to the optimal one. However, this time the complexity of diagonalizing our matrix $M$ is in $\mathscr{O}(N^6)$ since here $M$ is of size $(N-1)\times(N-1)$. Therefore, if we want to study eigenvalues ten times further in at least one direction, we would need to increase our complexity by roughly a factor of $10^6$. If we divide our precision by a factor of 100, we would need $N$ to be ten times larger, which would multiply the calculation time by a million.

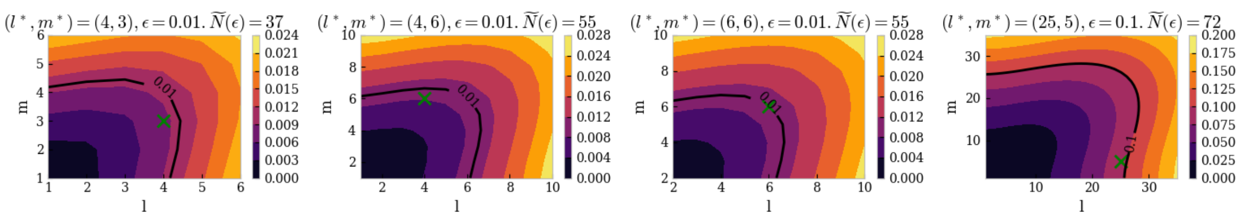



Figure 18: Illustration of accuracy result for $\widetilde{N}_{l^*,m^*}(\epsilon)$. Level map of relative error $(\lambda - \widetilde{\lambda})/\lambda$ as a function of $(l,m)$. The green cross is the position of $(l^*,m^*)$ and the black line is the level line of $\epsilon$.

### II.1.3 On any domain of $\mathbb{R}^2$

Let us now generalize the previous method to an arbitrary domain in the plane. The finite difference method with a regular grid can be used again, with the only difference being that there are no unknowns corresponding to points that are not in the arbitrary domain. Let us go into a little more detail so that we can come up with an easy-to-implement algorithm for evaluating the eigenfunctions and eigenvalues associated with this domain.

**Algorithm design**

We start by considering a discretization of our domain. More precisely, we will use a rectangular mesh of our domain. The idea is simple: to use a matrix, let us call it $D$, with dimensions $N_x \times N_y$, to represent the domain $\Omega$. Now $N_x$ and $N_y$ which are natural numbers, represent the number of discretization points, then we have $N_x - 1$ sub-interval for $x$ and $N_y - 1$ for $y$. Let $a$ and $b$ be real numbers such that the rectangle

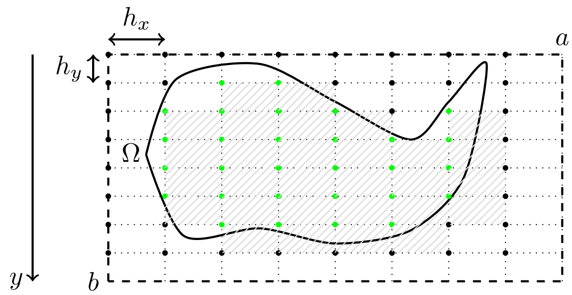



Figure 19: Illustration of the construction of the matrix $D$, rectangular mesh of $\Omega$. The domain represented by $D$ corresponds to the shaded region.

$\Omega_R = ]0, a[\times]0, b[$ contains $\Omega$. To represent the domain $\Omega$ using $D$, we will set the coefficient $D_{i,j}$ to 1 if the point $\left(a\frac{i}{N_x}, b\frac{j}{N_y}\right)$ is in $\Omega$, otherwise to 0. Now we denote $h_x = \frac{a}{N_x}$ and $h_y = \frac{b}{N_y}$. Building upon what has been done previously, we will require our numerical approximation $\widetilde{u}$ to satisfy

$$\frac{2\widetilde{u}_{i,j} - \widetilde{u}_{i-1,j} - \widetilde{u}_{i+1,j}}{h_x^2} + \frac{2\widetilde{u}_{i,j} - \widetilde{u}_{i,j-1} - \widetilde{u}_{i,j+1}}{h_y^2} = \widetilde{\lambda}\widetilde{u}_{i,j}, \tag{50}$$

where $D_{i,j} = 1$, i.e., $(x_i, y_j) \in \Omega$, otherwise

$$\widetilde{u}_{i,j} = 0. \tag{51}$$

In Figure 19, this corresponds to asking $\widetilde{u}$ to be zero at black intersection points, and at green points to satisfy relation (50). The only thing left is to automate the construction of $M$, the matrix such that (50) is equivalent to

$$M\widetilde{u}_c = \lambda\widetilde{u}_c,$$

with $\widetilde{u}_c$ being the vector constructed using the same procedure as (46), but without the variable $\widetilde{u}_{i,j}$ already set to 0 by (51). To construct such a matrix $M$, the idea is to start with the matrix representing (50) as if the domain were the complete rectangle $\Omega_R$, and then remove the rows and columns corresponding to $\widetilde{u}_{i,j}$ set to 0. The initial matrix can be expressed as

$$M_{\text{init}} = \frac{1}{h_x^2}\begin{pmatrix} N & 0 & \cdots & \cdots & 0 \\ 0 & N & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & N & 0 \\ 0 & \cdots & 0 & 0 & N \end{pmatrix} + \frac{1}{h_y^2}\begin{pmatrix} 2I & -I & 0 & \cdots & 0 \\ -I & 2I & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -I & 2I & -I \\ 0 & \cdots & 0 & -I & 2I \end{pmatrix} \in \mathscr{M}_{(N_x-1)(N_y-1)} \tag{52}$$

with, this time, $I$ the identity of $\mathscr{M}_{(N_x-1)}$ and

$$N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \in \mathscr{M}_{(N_x-1)},$$

Then, if $(ih_x, jh_y)$ is not in $\Omega$, then the row and column $i + (N_x - 1)(j - 1)$ are deleted from the initial complete matrix. This procedure is inspired by the way Dirichlet conditions were handled in the case of the square in the previous section. The process of deleting rows and columns of $M_i$ to obtain $M$ amounts to imposing the condition (51). To conclude our algorithm, let us make some conjectures. Inspired by the previous study, we assume that the eigenvalues $i$, $\widetilde{\lambda}_i$, of $M$ provide a good approximation of the exact eigenvalue $\lambda_i$ for $i \ll \min(N_x, N_y)$. In the same spirit, we can conjecture that the associated eigenvector $\widetilde{u}_i$ is indeed a discretization of the exact eigenfunction $u_i$. Without proof of convergence, we cannot generally assert the validity of these conjectures. However, we have shown that this holds for a square domain in the previous section, which ultimately turns out to be a special case of what we have accomplished here.

### Chladni figures

The first test we can carry out is to see if our algorithm allows us to find the well-known Chladni figures. Historically, these are the geometric patterns formed by powder on a vibrating surface. Named after physicist Ernst Chladni, these patterns depend on the shape of the surface and vibration frequency, i.e. the eigenmode. They highlight the nodal lines, which are the lines of vibration nodes. Let us take a moment for a brief pause to mention the inspiring story of a great mathematician who significantly advanced the customs and knowledge of our discipline, Madame Sophie Germain. Following Chladni's presentation of this experiment in Paris in 1808, the Académie des Sciences set up a competition to establish the mathematical theory underlying these results. After various attempts, the self-taught Sophie Germain, who had to pass herself off as a man in her early days, won the prize in 1816 for her contribution to the problem. Sophie Germain stood out by refusing to conform to the customs of her time and greatly contributing to improving the position of women in the scientific world. Let us go back to our algorithm, and more specifically to the Chladni figures for the plate. This is the case of a square that is fixed at its center. It is interesting to see physical experiments on this phenomenon, many of which are available on the internet. One example is the video of the experiment realized at the *Palais de la découverte* in Paris. The results of the algorithm are shown in the Figure 20, we indeed recognize several patterns observed in the video of the experiment. However, it is not easy to obtain

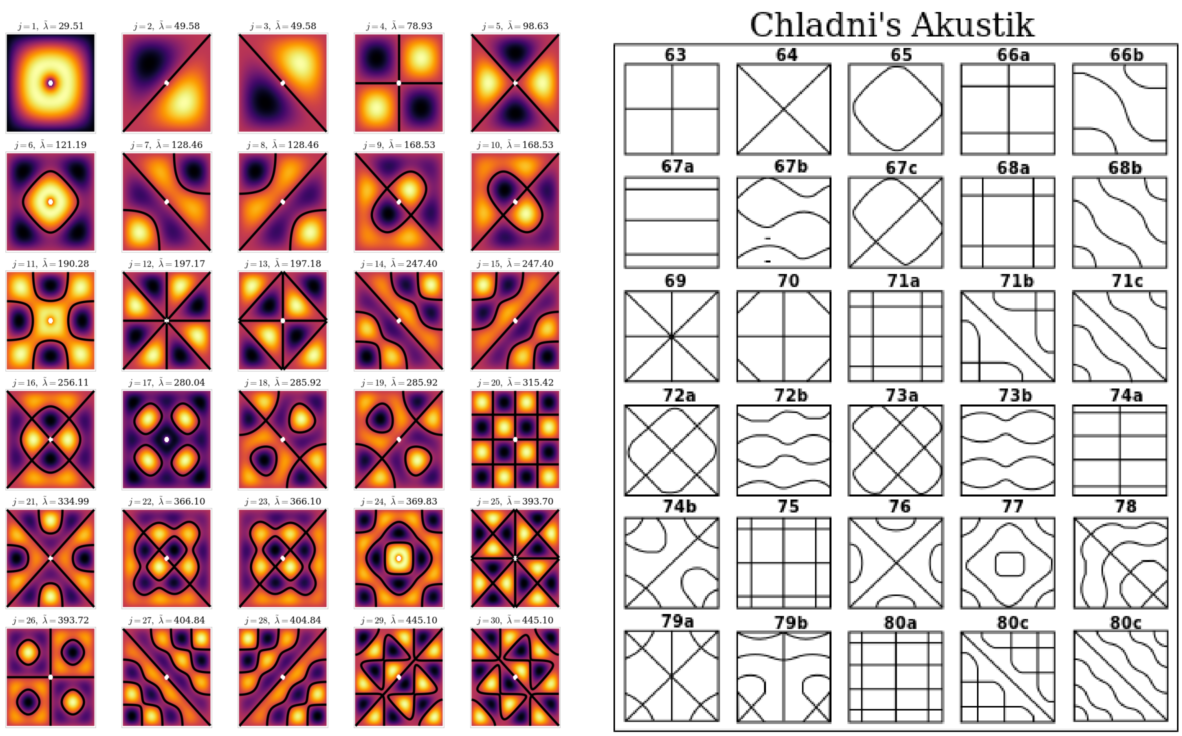



Figure 20: Chladni figures for the plate. On the left is the output of the algorithm for $N_x = N_y = 100$, the black lines are zero-level lines. On the right are some usual patterns. From the Wikipedia page of Chladni figures.

the exact results of this experiment, which is why we will turn to a simpler experiment, the disk. This first result has the merit of ensuring the consistency of our algorithm.

### Test on the disk

To convince ourselves that our algorithm works, let us look at the case of the disk, a form for which the exact solutions are known. We consider the Laplacian eigenvalue problem with homogeneous Dirichlet boundary condition on $\Omega = D(0, \alpha)$ the open disk of radius $\alpha$. In the polar coordinate system, it writes

$$\begin{cases} -\Delta u = -\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}\right) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{53}$$

We are going to present the main stages of the solution, but several points need to be explored in greater depth. We have decided to skip a lot of details to concentrate on our algorithm test and not get lost in the important theory of this well-known result. For a more detailed resolution, we invite you to look at [CH89]. Once again, we're going to use a variable separation $u(r,\theta) = R(r)\Theta(\theta)$ with $\Theta : [0, 2\pi[ \to \mathbb{R}$ and $\Theta : \mathbb{R}^+ \to \mathbb{R}$. The first equation becomes

$$-\frac{r^2 R''(r) + rR'(r) + \lambda r^2 R(r)}{R(r)} - \frac{\Theta''(\theta)}{\Theta(\theta)} = 0.$$

Then there exists $\Psi \in \mathbb{R}$ such that

$$\frac{r^2 R''(r) + rR'(r) + \lambda r^2 R(r)}{R(r)} = \Psi = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

We seek eigenfunctions of class $C^\infty$, so we ask to $\Theta$ and $R$ to be of class $C^\infty$. Let us start by looking at the angular part. We solve $-\Theta'' = \Psi\Theta$. By periodicity of the angular part, we need $\Theta(0) = \Theta(2\pi)$, therefore necessarily $\Psi \geq 0$. Otherwise, the solution cannot verify this condition without being zero. So $\Theta(\theta) = A_\psi \cos(\sqrt{\Psi}\theta) + B_\psi \sin(\sqrt{\Psi}\theta)$ with $A_\psi$ and $B_\psi$ in $\mathbb{R}$. Moreover, if we note $\psi = \sqrt{\Psi}$, this condition also implies $\psi \in \mathbb{N}$. Turning now to the amplitude, we have

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - \Psi)R(r) = 0.$$

From Theorem A.1, we know that $\lambda \neq 0$ and then we can perform the change of variable $s = \sqrt{\lambda}r$. So, $R$ satisfies

$$s^2 \frac{\partial^2}{\partial s^2} R(s) + s\frac{\partial}{\partial s} R(s) + (s^2 - \Psi)R(s) = 0. \tag{54}$$

The solutions to this equation are known as the Bessel function. A usual reference is [Wat22]. More precisely, we have

$$R(s) = aJ_\psi(s) + bY_\psi(s),$$

where $J_\psi$ is the Bessel function of the first kind and $Y_\psi$ is the Bessel function of the second kind, both of order $\psi$. The Bessel function of the second kind has an infinite limit at zero, so the continuity of our eigenfunction implies that $b = 0$. The solutions to our problem under variable separation are necessarily of the form

$$u_\psi(r,\theta) = J_\psi(\sqrt{\lambda}r)(A_\psi \cos(\psi\theta) + B_\psi \sin(\psi\theta))$$

The homogeneous Dirichlet condition, i.e. $u_\psi(\alpha,\theta) = 0$ implies $J_\psi(\sqrt{\lambda}\alpha) = 0$. So, $\lambda$ must be chosen as a root of $\mathbb{R}^+ \ni x \mapsto J_n(\sqrt{x}\alpha)$. For any order, a Bessel function has an infinite number of positive roots. We note for any order $\mu$ in $\mathbb{R}$, the root of the Bessel function of the first kind $J_\mu$ as

$$0 < \lambda^B_{\mu,1} \leq \lambda^B_{\mu,2} \leq \lambda^B_{\mu,3} \leq \dots$$

The eigenvalues of our problem are of the form $(\lambda^B_{\psi,k})^2/\alpha^2 := \lambda_{\psi,k}$ with associated eigenfunction

$$u_{\psi,k}(r,\theta) = J_\psi\left(\frac{\lambda^B_{\psi,k}}{\alpha} r\right)(A_\psi \cos(\psi\theta) + B_\psi \sin(\psi\theta)). \tag{55}$$

Let us now note $\psi$ which is in $\mathbb{N}$ as $n$. So, for $n = 0$, the associated eigenspace to $\lambda_{0,k}$ is the one spanned by $((r,\theta) \mapsto J_0(\lambda^B_{0,k}/r\alpha))$ for $k$ in $\mathbb{N}^*$. For $n \neq 0$, the associated eigenspace to $\lambda_{0,k}$ is the one spanned by $((r,\theta) \mapsto J_n(\lambda^B_{n,k}/r\alpha)\cos(n\theta), (r,\theta) \mapsto J_n(\lambda^B_{\psi,k}/r\alpha)\sin(n\theta))$ for $k$ in $\mathbb{N}^*$. We admit that the union of these sets of eigenspaces is dense in $L^2(\Omega)$ in order to consider that we have all the eigenvalues. Let us return to our algorithm. The eigenfunctions it provides are very satisfactory given that the relative error after normalization is quite reasonable, at around 0.1%. However, this does not seem to be the case for the eigenvalues. In order to carry out an in-depth study on the eigenvalues, we set

$$\Delta_r \widetilde{\lambda}_{n,k} = \frac{\lambda_{n,k} - \widetilde{\lambda}_{n,k}}{\lambda_{n,k}} \qquad \text{and} \qquad \Delta_r \widetilde{u}_{n,k} = \max(|u_{n,k} - \widetilde{u}_{n,k}|), \tag{56}$$

where matrices $u_{n,k}$ and $\widetilde{u}_{n,k}$ are normalized. By sorting the eigenvalues in ascending order, we will now index our values by $j$ instead of $n, k$. The evolution of these quantities is shown in Figure 21. If the discretization

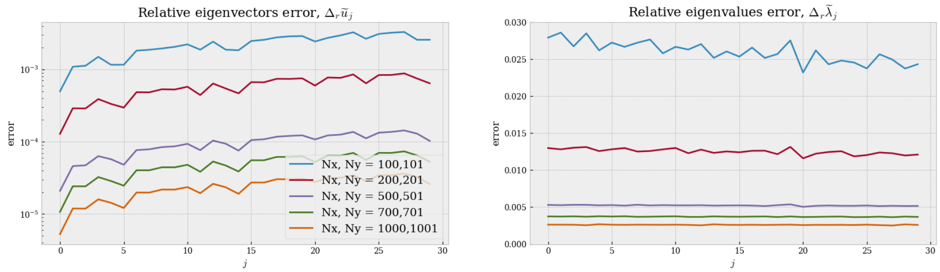



Figure 21: Relative errors of our algorithm for the disk. Evolution for different domain sizes.

grid is chosen to be too small, the results of our algorithm are very poor. Upon closer inspection, the error seems to be particularly sensitive to the smoothness of the discretized domain. It seems normal that our method is more adapted to straight domains than rounded ones given the discretization method. To strengthen this idea, let us consider the quarter-circle which still has a rounded area but is less significant than the full circle. We begin with the theoretical aspect, which is very similar to the previous analysis, except now $\theta$ belongs to $[0, \frac{\pi}{2}]$. For the angular part, solving the differential equation is the same, but instead of periodic constraints, we now have Dirichlet boundary conditions. In other words, we have the constraints $\Theta(0) = \Theta(\frac{\pi}{2}) = 0$, which implies that $A = 0$ and $\psi = 2n$ with $n \in \mathbb{N}$. We exclude $n = 0$, otherwise we have a null function. The radial part remains unchanged, so we can choose $\psi = 2n$ in Eq. (55). It follows that the eigenvalues of this new problem are of the form $(\lambda^B_{2n,k})^2/\alpha^2 := \lambda_{2n,k}$ with associated eigenfunction

$$u_{2n,k}(r, \theta) = J_{2n}\left(\frac{\lambda^B_{2n,k}}{\alpha} r\right) \sin(2n\theta), \tag{57}$$

with $n \in \mathbb{N}^*$. The Figure 22 presents the relative errors produced by our algorithm for the quarter-circle. Note that the magnitude of the relative error on the eigenfunctions is unchanged compared to the full circle result and that the relative error of the eigenvalues is greatly improved by a factor of 2. This allows us to strengthen the conjecture that our algorithm is particularly sensitive to the discretization of rounded parts. To be able to perform simulations with discretization grids of such sizes, it is favorable to use sparse matrices.

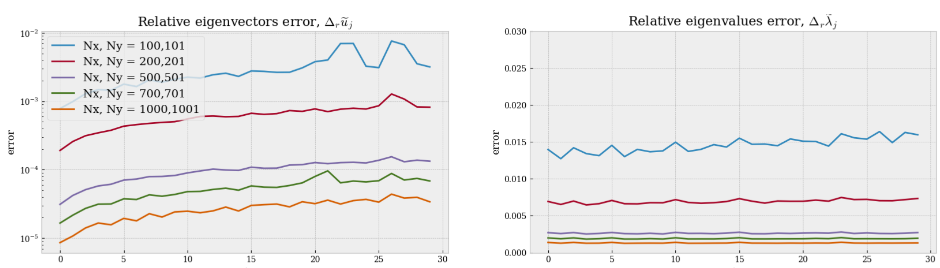



Figure 22: Relative errors in our algorithm for a quarter of a disk.

This allows optimizing both the memory space and the diagonalization calculation time. For instance, in previous Python simulations, the tools of the *scipy* library were used. Specifically, $M$ belongs to the *csr* class of *scipy.sparse* and is diagonalized using the *eigsh* function of *scipy.sparse.linalg*. This optimization has greatly reduced the simulation time and enabled the consideration of simulations with significantly higher precision. The diagonalization function used this time is from the ARPACK library, which is adapted to large, sparse matrices. They use the Implicitly Restarted Lanczos Method (IRLM) to find eigenvalues and eigenvectors (see [LSY98]).

## II.2 Numerical shape optimization

In this section, we will use the previously developed code to recover the two-dimensional Faber-Krahn inequality and then extend to the second eigenvalue. We will develop a method to numerically find that the domain minimizing the first eigenvalue is the circle. Our approach is based on the recent numerical optimization articles [Gar+23] and [HKL24]. The articles presents a general method to optimize linear functions of the eigenvalues of the Laplacian with Dirichlet or Neumann boundary conditions by adjusting the domain shape using a phase-field function. The shape and topology optimization problem can be reformulated as an optimal control problem with PDE constraints, where the phase-field function serves as the control variable. The article establishes first-order optimality conditions and derives the sharp interface limit, followed by numerical simulations to illustrate the optimization. Here, we will only focus on the numerical aspect, so we will not provide proof and refer to the article for that.

### II.2.1 Phase field approach

The idea here is not to directly manipulate the domain $\Omega$ but rather a scalar function $\phi_\epsilon : D \subset \mathbb{R}^N \to \mathbb{R}$, called phase field function of parameter $\epsilon > 0$. We will consider $D$ as the unit square. The domain $\Omega$ is described by this function, we consider $\Omega \approx \{\phi_\epsilon > 0\}$. The phase-field function takes values close to 1 within the specified domain $\Omega$, exhibits a transition in a tubular neighborhood around the boundary $\partial\Omega$, and approaches -1 outside of $\Omega$. Using this representation allows us to use a scalar function as the variable in

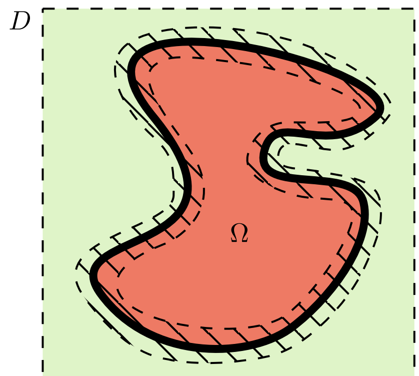



Figure 23: Illustration of phase field function. The red area is $\Omega = \{\phi_\epsilon > 0\}$, the green area is $D \subset \Omega = \{\phi_\epsilon \leq 0\}$ and the hatched area is the field phase transition zone between -1 and 1.

our optimization problem, rather than a domain in $\mathbb{R}^2$. The first challenge of this new representation is the computation of eigenvalues and eigenvectors.

### II.2.2 The approximate eigenvalue problem

We consider a phase-field approximation of the classical Dirichlet eigenvalue problem on the shape represented by the set $\{\phi = 1\}$. For function $\phi \in L^\infty(\Omega, [-1, 1])$, the eigenvalue problem becomes

$$\begin{cases} -\Delta u + b^\epsilon(\phi^\epsilon)u = \lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{58}$$

where $b^\epsilon : [-1, 1] \to \mathbb{R}^+$. We denote the solutions of this problem $(u_k^\epsilon)$ with associated eigenvalues $(\lambda_k^\epsilon)$. It is important to keep in mind that these quantities are functions of $\phi^\epsilon$. In the sharp interface limit, i.e. when $\epsilon \to 0$, we want to have this solution converge in some sense to the solution of the initial problem (39a). Let us denote by $u_k$ a solution to the initial problem associated with the $\lambda_k$ eigenvalues. We can ensure under some strong conditions on $b^\epsilon$ and $\phi$ that when $\epsilon \to 0$ we have

$$\lambda_k^\epsilon \to \lambda_k \qquad \text{and} \qquad ||u_k^\epsilon - u_k||_{H^1} \to 0.$$

For more details and proofs, we invite the reader to read Section 4, more precisely Theorem 4.6 of [Gar+23]. Highly vulgarised, these conditions are strong enough to ensure that

$$b^\epsilon(\phi) \xrightarrow[\epsilon \to 0]{} \begin{cases} 0 & \text{in } \Omega, \\ +\infty & \text{on } \Omega^C. \end{cases}$$

and, in a sense, recover the original problem at the sharp interface. Also, Theorem 3.5 of [HKL24] gives us a type of phase-field Faber–Krahn inequality, accurately $\lambda_1^\epsilon(\phi) \geq \lambda_1^\epsilon(\phi^*)$. As far as we are concerned, for the simulation it is sufficient to choose $b^\epsilon(\phi) = b\frac{1-\phi}{2\epsilon^{4/3}}$, with $b > 0$ and $|\phi| \leq 1$.

**Numerical resolution**

By the same discretization as in the previous section, we can approximate the problem by

$$M\widetilde{u} + B_\epsilon(\phi^\epsilon)\widetilde{u} = \lambda\widetilde{u},$$

with $\widetilde{u}$ the vector version of $u$, $M$ complete as in Eq. (52) and

$$B_\epsilon(\phi^\epsilon) = \text{diag}(b^\epsilon(\phi_i^\epsilon)).$$

We obtain a numerical approximation of the solutions of problem (58) by the diagonalization of $M + B_\epsilon(\phi^\epsilon)$. These approximations are denoted by $(\widetilde{u_k^\epsilon})$ and $(\widetilde{\lambda_k^\epsilon})$. Next, we verify that these quantities converge, when $\epsilon \to 0$, to the numerical results of the method presented in Section II.1.3, which solves the initial problem (39a) numerically. We denote these quantities by $(\widetilde{u_k})$ and $(\widetilde{\lambda_k})$. We set

$$\Delta\widetilde{\lambda_k} = \frac{\widetilde{\lambda_k} - \widetilde{\lambda_k^\epsilon}}{\widetilde{\lambda_k}} \qquad \text{and} \qquad \Delta\widetilde{u_k} = \max|\widetilde{u_k} - \widetilde{u_k^\epsilon}|.$$

We can take the example of the unit square. We need to consider a discretization of $\phi^\epsilon$ on a mesh of $D$. For example, we can take a mesh of dimension $N \times N$ for $D$ and denote $\widetilde{\phi^\epsilon}$ the discretization of $\phi^\epsilon$ on this mesh. For the sharp square, we set $\widetilde{\phi^\epsilon}$ with 1 everywhere, except on the boundary with a coefficient at -1. Numerical simulation gives us Figure 24. These results are quite satisfactory and allow us to calculate

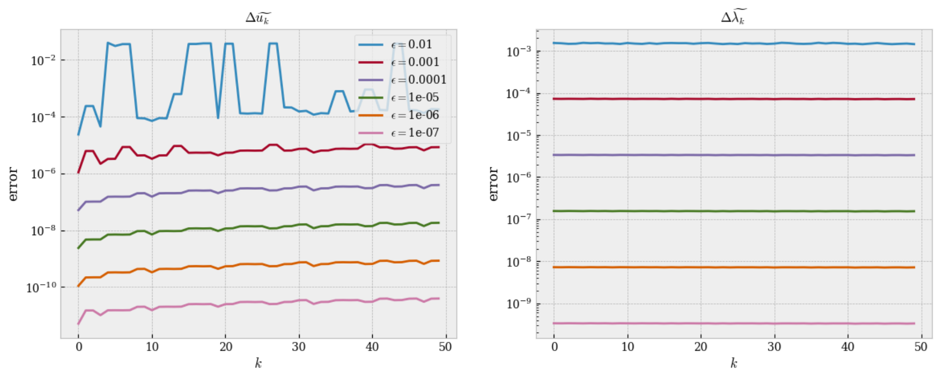



Figure 24: Convergence of solutions of the approximate eigenvalue problem. Simulation for $N = 100$.

numerically the eigenfunctions and eigenvalues accurately from a phase field representation. This example proves the efficiency of the term added to our problem. In the next section, we examine the importance of introducing this phase field function.

### II.2.3 Optimization processes

We minimize the principal eigenvalue of the Dirichlet Laplacian under a constant volume constraint. For that, we minimize the cost function

$$J_\beta^\epsilon(\phi^\epsilon) = \lambda_1^\epsilon + \beta E_{GL}^\epsilon(\phi^\epsilon) \tag{59}$$

over the class of admissible functions $A_\phi := \{\phi^\epsilon \in H^1(D) \text{ s.t. } Vol(\phi^\epsilon) = c\}$. Here, volume means

$$v(\phi^\epsilon) = \int_D \frac{1}{2}(\phi^\epsilon + 1)d\lambda.$$

The term $E_{GL}^\epsilon(\phi^\epsilon)$ is added as a regularization term in order to make the optimization problem well-posed. Here, $\beta$ is a positive constant and $E_{GL}^\epsilon(\phi^\epsilon)$ is the Ginzburg–Landau energy defined as

$$E_{GL}^\epsilon(\phi^\epsilon) := \int_D \frac{\epsilon}{2}|\nabla\phi^\epsilon|^2 + \frac{1}{\epsilon}\psi(\phi^\epsilon)d\lambda.$$

One of classical choice for $\psi$ is the smooth potential $\psi(x) = (1-x^2)^2$. The first term in this energy represents a form of perimeter regularization, while the second term represents a form of transition regularization. The following result allows us to state that the objective function $J_\beta^\epsilon$ admits the same minimizer as the fundamental eigenvalue, namely the circle.

**Theorem II.1:** ([HKL24], Theorem 3.8)
*Let $\phi^\epsilon$ be any minimizer of $J^\epsilon_\beta$ over the class of admissible functions $A_\phi$. Then $\phi^\epsilon = (\phi^\epsilon)^*$ almost everywhere in $D$ and the first positive-normalised eigenfunction $u^\epsilon_1$ also verifies $u^\epsilon_1 = (u^\epsilon_1)^*$ almost everywhere in $D$.*

The Ginzburg-Landau energy is of purely theoretical interest. This term allows the optimization problem to be well-posed and prove this result. Numerically, as we shall see later, it has no interest. The volume constraint we are going to introduce will overtake its interest.

### Numerical processes

To solve numerically this optimization problem, we will use a gradient descent algorithm. In our case, we consider admissible functions to be those verifying $|\phi^\epsilon| \leq 1$ and a constant volume constraint. We denote the results of its iterations by $(\phi^\epsilon_k)_k$. The first issue we face is how to work within the set of admissible functions. For the first constraint $|\phi^\epsilon_k| \leq 1$, we will handle it by simply truncating $\phi^\epsilon_k$ between -1 and 1. Regarding the fixed volume constraint, we will relax it by adding a penalization term to our objective function. Thus, the objective function becomes

$$J^\epsilon_{\alpha,\beta}(\phi^\epsilon) = \lambda^\epsilon_1 + \alpha V(\phi^\epsilon) + \beta E^\epsilon_{GL}(\phi^\epsilon)$$

with $\alpha > 0$ and

$$V(\phi^\epsilon) = (v(\phi^\epsilon) - v(\phi^\epsilon_0))^2. \tag{60}$$

To use the gradient descent algorithm, we need the gradient of $J^\epsilon_{\alpha,\beta}$. To achieve this, we consider its discretized version, namely

$$\widetilde{J}^\epsilon_{\alpha,\beta}(\widetilde{\phi}^\epsilon) = \widetilde{\lambda^\epsilon_1} + \alpha\widetilde{V}(\widetilde{\phi}^\epsilon) + \beta\widetilde{E}^\epsilon_{GL}(\widetilde{\phi}^\epsilon).$$

where $\widetilde{V}$ and $\widetilde{E}^\epsilon_{GL}$ are the numerical approximations of $V$ and $E^\epsilon_{GL}$. For example $\widetilde{V}(\widetilde{\phi}^\epsilon) = (\widetilde{v}(\widetilde{\phi}^\epsilon) - \widetilde{v}(\widetilde{\phi}^\epsilon))^2$ with $\widetilde{v} = \frac{1}{N^2}\sum_i \frac{1}{2}(\widetilde{\phi}^\epsilon_i + 1)$. It is therefore very easy to obtain explicit expressions for $\nabla\widetilde{V}$ and $\nabla\widetilde{E}^\epsilon_{GL}$. We will not detail the calculations as they are simple but tedious. We would not go into detail about the calculations, which are simple but tedious. Regarding $\widetilde{\lambda^\epsilon_1}$, by definition it satisfies the relation

$$(M + B_\epsilon(\widetilde{\phi}^\epsilon))\widetilde{u^\epsilon_1} = \widetilde{\lambda^\epsilon_1}\widetilde{u^\epsilon_1}.$$

For readability, we will omit the numerical notation, that is, we now denote $\widetilde{\lambda^\epsilon_1}$ as $\lambda^\epsilon_1$. Let's assume that this quantity is differentiable regarding the phase field function. Differentiating with respect to $\phi^\epsilon_i$ and multiplying on the left by ${u^\epsilon_1}^T$, we obtain

$${u^\epsilon_1}^T\frac{\partial B_\epsilon}{\partial\phi^\epsilon_i}u^\epsilon_1 + {u^\epsilon_1}^T(M + B_\epsilon)\frac{\partial u^\epsilon_1}{\partial\phi^\epsilon_i} = {u^\epsilon_1}^T\frac{\partial\lambda^\epsilon_1}{\partial\phi^\epsilon_i}u^\epsilon_1 + {u^\epsilon_1}^T\lambda^\epsilon_1\frac{\partial u^\epsilon_1}{\partial\phi^\epsilon_i}. \tag{61}$$

However, $M$ and $B_\epsilon$ are symmetric, so

$${u^\epsilon_1}^T(M + B_\epsilon) = ((M + B_\epsilon)u^\epsilon_1)^T = (\lambda^\epsilon_1 u^\epsilon_1)^T = \lambda^\epsilon_1{u^\epsilon_1}^T.$$

Thus, Eq. (61) becomes

$${u^\epsilon_1}^T\frac{\partial B_\epsilon}{\partial\phi^\epsilon_i}u^\epsilon_1 = {u^\epsilon_1}^T u^\epsilon_1\frac{\partial\lambda^\epsilon_1}{\partial\phi^\epsilon_i}.$$

So, we deduce that

$$\frac{\partial\lambda^\epsilon_1}{\partial\phi^\epsilon_i} = \frac{{u^\epsilon_1}^T\frac{\partial B_\epsilon}{\partial\phi^\epsilon_i}u^\epsilon_1}{||u^\epsilon_1||^2_2}.$$

We note that $\frac{\partial B_\epsilon}{\partial\phi^\epsilon_i} = \mathrm{diag}((0,\dots,0,b^{\epsilon\prime}(\phi^\epsilon_i),0,\dots,0))$. In our case, we have $b^{\epsilon\prime}(x) = -b\frac{1}{2\epsilon^{4/3}}$, so

$$\frac{\partial\lambda^\epsilon_1}{\partial\phi^\epsilon_i} = -b\frac{((u^\epsilon_1)_i)^2}{2\epsilon^{4/3}||u^\epsilon_1||^2_2}.$$

Note that in the case of a one-dimensional eigenspace, this expression is indeed independent of the amplitude of the eigenvector and is thus unique. This is no longer valid for eigenspaces of higher dimension, where the expression will depend on the choice of the eigenvector within the eigenspace. In fact, the eigenvalue associated with an eigenspace of dimension greater than one is not differentiable. However, thanks to Theorem I.38, we know that for a connected domain, the eigenspace associated with the first eigenvalue is of dimension one. Consequently, we have all the necessary elements to explicitly determine the expression of $\nabla \widetilde{J}^{\epsilon}_{\alpha,\beta}$ for $k = 1$. We can then perform a gradient descent on this function, starting from an initial connected condition and assuming that the iterated results will also be connected. We iterate, starting from a given initial value $\widetilde{\phi}^{\epsilon}_0 \in \mathbb{R}^{N^2}$,

$$\widetilde{\phi}^{\epsilon}_{k+1} = \widetilde{\phi}^{\epsilon}_k - \gamma \nabla \widetilde{J}^{\epsilon}_{\alpha,\beta}(\widetilde{\phi}^{\epsilon}_k), \tag{62}$$

with $\gamma > 0$. Note that the result of this algorithm depends on the volume of $\widetilde{\phi}^{\epsilon}_0$, due to Eq. (60).

### II.2.4 Numerical result

Let us focus on the selection of our parameters. An analysis is carried out to choose the parameter $b$ in [Gar+23]. Meanwhile, the simulation they carry out uses the finite element method; let us assume that the results are the same for the finite difference method. Their study suggests choosing $b = 550$. For the parameters $\alpha, \beta, \gamma$, and $\epsilon$, we simply select them by trying various configurations, analyzing the orders of magnitude of each term and by trial and error. Note that as expected, the thickness of the transition region is proportional to $\epsilon$. As a consequence, as the value of $\epsilon$ decreases, the simulation results become more sharp, but the computational time increases. You can find the animation of the evolution of $\widetilde{\phi}^{\epsilon}_k$ and each components of $\nabla \widetilde{J}^{\epsilon}_{\alpha,\beta}$ here in the code directory associated with the project.

**First eigenvalue**

According to the Faber-Krahn inequality, we expect here to find a ball of the same volume as the initial condition $\widetilde{\phi}^{\epsilon}_0$. The evolution of Eq. (62), shown in Figure 25, results in a domain where the eigenvalue deviates from the exact solution by approximately $10^{-1}$ and the volume differs by about $10^{-2}$. This simulation enables

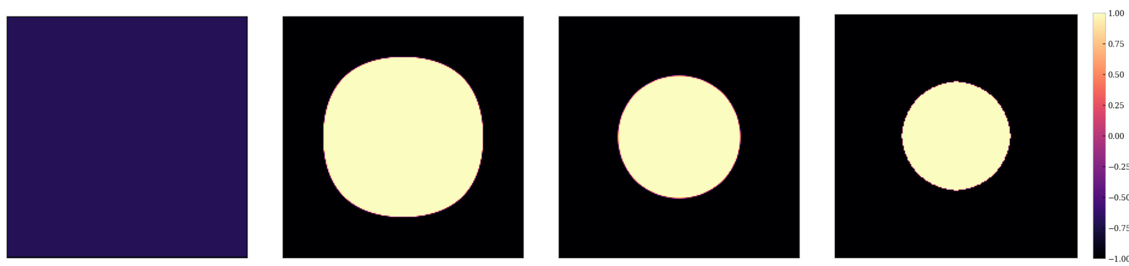



Figure 25: Minimistaion of first eigenvalue. Evolution of $\widetilde{\phi}^{\epsilon}_k$, from left to right $k = 0, 5, 10, 25$. Parameter: $N = 200$, $\alpha = 10^8$, $\beta = 0$, $\gamma = 0.025$, $\epsilon = 5 \cdot 10^{-5}$.

us to numerically verify the Faber-Krahn inequality. By successfully replicating this result, we proved the robustness and reliability of our simulation method. Additionally, it is important to note that the coefficient of the Ginzburg-Landau energy term was set to zero. The volume constraint replaces the transition area and perimeter constraints. This term is not relevant in our simplified context, this is numerically verified.

**Second eigenvalue**

The global optimal solution is stated in Theorem II.2.

**Theorem II.2:** Krahn-Szegö ([Hen04], Theorem 3)
*The minimum of $\lambda_2(\Omega)$ among bounded open sets of $\mathbb{R}^N$ with a given volume is achieved by the union of two identical balls.*

What we have designed for the first eigenvalue can be similarly applied to the second eigenvalue, with the notable difference that we encounter eigen spaces not of dimension one. Indeed, we expect to converge towards two spheres, a domain that admits an eigenspace associated with the second eigenvalue of dimension two. Therefore, as explained above, $\lambda_2$ will not be differentiable throughout our iterations. In practice, if we keep our existing algorithm, the outcome will be influenced by the randomness of the numerically computed eigenfunctions within the eigenspace corresponding to $\lambda_2$. To address this issue, we consider the arithmetic mean of the orthogonal eigenvectors associated with the eigenspace obtained from the simulation instead of just $\lambda_2$. The underlying result is that the sum of all eigenvalues within an eigenspace is differentiable. Consequently, this approach involves modifying the objective function to minimize the arithmetic mean of the numerical eigenvalues corresponding to the eigenspace of $\lambda_2$. For example, if we denote $\widetilde{\lambda^{\epsilon}_{2,1}}$ and $\widetilde{\lambda^{\epsilon}_{2,2}}$ as the two numerical eigenvalues associated with $\lambda_2$, we minimize

$$\widetilde{J}^{\epsilon}_{\alpha,\beta}(\widetilde{\phi}^{\epsilon}) = \frac{\widetilde{\lambda^{\epsilon}_{2,1}} + \widetilde{\lambda^{\epsilon}_{2,2}}}{2} + \alpha\widetilde{V}(\widetilde{\phi}^{\epsilon}) + \beta\widetilde{E}^{\epsilon}_{GL}(\widetilde{\phi}^{\epsilon}).$$

This does not change the problem being studied and enables us to address the issue of associated eigenspace of dimension greater than one. The result of this method is shown in Figure 26. We find an eigenvalue that deviates from the exact solution by about $10^{-1}$ and a volume that varies by approximately $10^{-2}$.

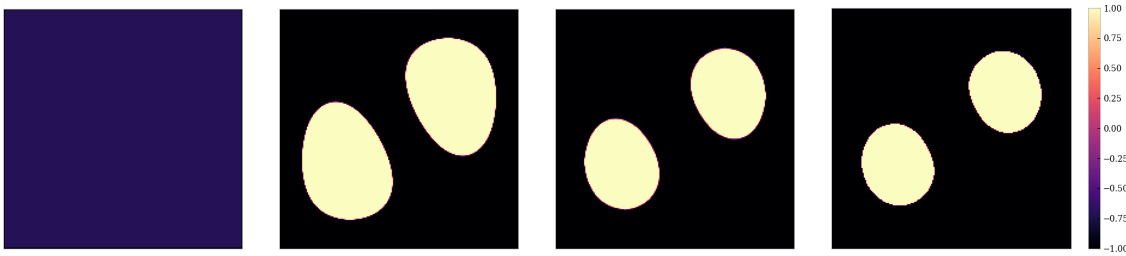
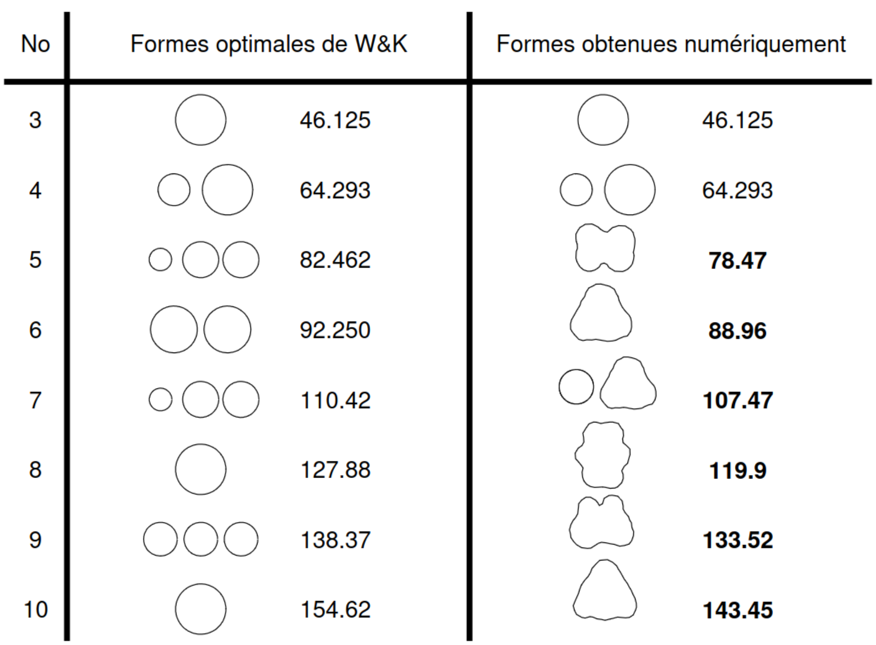



Figure 26: Minimisation of second eigenvalue. Evolution of $\widetilde{\phi}^{\epsilon}_{k}$, from left to right $k = 0, 5, 10, 25$. Parameter: $N = 200$, $\alpha = 10^8$, $\beta = 0$, $\gamma = 0.025$, $\epsilon = 5 \cdot 10^{-5}$.

**Higher eigenvalue**

We can continue to improve our algorithm to optimise the eigenvalues for $k \geq 3$. Meanwhile, we do not know the exact form of their minimizers, or at least we only have numerical estimates. These are still open problems. Numerical simulations for $N = 2$, as shown in Figure 27, suggest that their form is not always as simple as in previous cases, indicating complexities that require further investigation. The shape optimization



| No | Formes optimales de W&K | Formes obtenues numériquement |
|---|---|---|
| 3 | 46.125 | 46.125 |
| 4 | 64.293 | 64.293 |
| 5 | 82.462 | **78.47** |
| 6 | 92.250 | **88.96** |
| 7 | 110.42 | **107.47** |
| 8 | 127.88 | **119.9** |
| 9 | 138.37 | **133.52** |
| 10 | 154.62 | **143.45** |

Figure 27: ([Oud02], Figure 4.5): Outcomes of numerical optimisation of $\lambda_k$ for $k \geq 3$ in dimension two. The left column shows the results obtained for unions of balls and the right column shows the results without shape constraints.

problem for eigenvalues is a very complex problem and remains an active area of research. To illustrate the challenges these problems can present, we can take the example of $\lambda_3$. According to Table 1, in dimension 4, a union of balls does not minimize $\lambda_3$, whereas for $N = 2$, it does.

| $i$ | Multiplicity | $\lambda_i^*$ | $\lambda_i(B)$ |
|---|---|---|---|
| 3 | 2 | **53.95** | 56.50 |
| 4 | 3 | **57.06** | 58.59 |
| 5 | 4 | **58.59** | **58.59** |
| 6 | 5 | **67.06** | **67.06** |
| 7 | 4 | **76.28** | 74.57 |
| 8 | 4 | **79.17** | 81.39 |

Table 1: ([OA19], Table 1): Presents the optimal values for $\lambda_i$ and the corresponding multiplicities in dimension 4. The third column shows the minimal eigenvalue obtained without shape constraint, and the last column shows the best results achieved for unions of balls.

# Appendix

## A Spectral analysis of the Laplacian

Let us start by discussing some results regarding the Laplacian to familiarize ourselves with some of its properties. Most of the outcomes presented here are taken from the book [Bre83]. The purpose of this section is to organize and present the arguments useful to establish the important theorem presented just below.

### A.1 Quick presentation of the problem

This study aims to look at the diagonalization in $L^2(\Omega)$ of the Laplacian operator (denoted $\Delta$) considering homogeneous Dirichlet boundary condition for $\Omega$ an open bounded subset of $\mathbb{R}^N$, with $N \in \mathbb{N}$.

**Theorem A.1:** `([Bre83], IX.31)` Spectral decomposition of $-\Delta$ for homogeneous Dirichlet boundary condition
*There exists a Hilbertian basis $(e_n)_{n\geq 1}$ of $L^2(\Omega)$ and a real sequence $(\lambda_n)_{n\geq 1}$ such that $\lambda_n > 0$ and $\lambda_n \to \infty$, satisfying*

$$e_n \in H_0^1(\Omega) \cap C^\infty(\Omega) \qquad \textit{and} \qquad -\Delta e_n = \lambda_n e_n \quad \textit{on } \Omega.$$

*We refer to $(\lambda_n)_{n\geq 1}$ as the eigenvalues of $-\Delta$ and $(e_n)_{n\geq 1}$ as the associated eigenfunctions.*

### A.2 A reminder on functional analysis

Here we will present the fundamental concepts and results of the functional analysis without proof. These will be crucial later to prove the result presented above. Some concepts presented in this section, although widely known, are included here for clarity. Additionally, some notions are defined more extensively than I'm used to. That is why we thought it would be interesting to present them here. The presented concepts are superficially covered; they all deserve further discussion to be properly defined.

#### A.2.1 Some definition about spaces

Let us start by recalling the fundamental notion of the topological dual of a space.

**Definition A.2:** Topological dual
*Let $E$ be a vector space, we define its topological dual noted $E'$ as the space of continuous linear forms on $E$.*

We can briefly recall that a form of E is an application from $E$ into $\mathbb{R}$. We endow $E'$ with the dual norm, defined as

$$||f||_{E'} = \sup_{x\in\overline{B_E}} f(x), \qquad \forall f \in E'$$

where $B_E$ denotes the open unit ball of $E$. In addition, when $\phi \in E'$ and $x \in E$ we will note $\langle \phi, x\rangle$ instead of $\phi(x)$. This is called the scalar product of the duality. An important property of spaces is reflexivity.

**Definition A.3:** Reflective space
*Let $E$ a Banach space and $J$ be the canonical injection of $E$ in $E''$. $E$ is said to be reflexive if $J(E) = E''$.*

By canonical injection we mean the natural continuous linear application

$$J(x)(\phi) = \langle \phi, x\rangle, \qquad \forall x \in E,\ \forall \phi \in E'.$$

So for $E$ to be reflexive, we need to ensure that $J$ is bijective. The last concept we're going to introduce is separability, which, roughly speaking, provides a characteristic related to the size of the space.

**Definition A.4:** Separable space
*A metric space $E$ is said to be separable if there is a countable and dense subset $D$ of $E$ in $E$.*

As one might imagine, this property offers a first intuition of the notion of decomposition.

### A.2.2 $L^p$ and $\mathscr{L}^p$ spaces

Now, let us recall the fundamental definition and essential results regarding $L^p$ spaces.

**Definition A.5:** $\mathscr{L}^p$ SPACES
*For $1 \le p < \infty$, we define*

$$\mathscr{L}^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ measurable and } \int_\Omega |f|^p \text{ finite}\},$$

*equipped with*

$$||f||_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}.$$

*For $p = \infty$, we set*

$$\mathscr{L}^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ measurable and there exist } C \in \mathbb{R} \text{ s.t. } |f(x)| \le C \text{ a.e. in } \Omega\},$$

*equipped with*

$$||f||_\infty = \inf\{C \text{ s.t. } |f(x)| \le C \text{ a.e. in } \Omega\}.$$

Defined as it is, for all p in $[1, +\infty]$, $\mathscr{L}^p(\Omega)$ is a semi-normed space since $||f||_p = 0 \iff f = 0$ amost everywhere on $\Omega$, which is not necessarily the null function. To answer this, we consider the equivalence relation $\mathscr{R}$ on $\mathscr{L}^p(\Omega)$ defined as

$$f \mathscr{R} g \qquad \iff \qquad f(x) = g(x) \text{ almost everywhere on } \Omega, \text{ i.e. } ||f - g||_p = 0.$$

Let $[f]$ be the equivalence class of $f \in \mathscr{L}^p(\Omega)$ by the relation $\mathscr{R}$.

**Definition A.6:** $L^p$ SPACES
*Let $p$ in $[1, +\infty]$. We define the functional spaces $L^p(\Omega)$ as the quotient of $\mathscr{L}^p(\Omega)$ by the relation $\mathscr{R}$. In other words, we define*

$$L^p(\Omega) = \{[f],\ f \in \mathscr{L}(\Omega)\}.$$

We will use this space instead of $\mathscr{L}$ in order to have a normed space. This leads to the following fundamental result.

**Theorem A.7:** `([Bre83], IV.8)` FISCHER-RIESZ
*For all $1 \le p \le \infty$, $L^p$ is a Banach space.*

Note that $L^2(\Omega)$ is a Hilbert space for which the norm follows from the scalar product

$$(f, g)_{L^2} = \int_\Omega fg.$$

We have the following important result.

**Theorem A.8:** `([Bre83], IV.23)`
*For all $p$ in $[1, +\infty[$, we have that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

It is worth remembering Hölder's inequality, it often proves to be very useful.

**Property A.9:** `([Bre83], IV.6)` HÖLDER INEQUALITY
*Let $p, q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \le p \le \infty$, then for $f \in L^p$ and $g \in L^q$,*

$$fg \in L^1 \qquad \text{and} \qquad \int |fg| \le ||f||_{L^p} ||g||_{L^q}.$$

This inequality is very useful whenever you want to manipulate the $L^p$-norms. Indeed, this result allows us to easily prove the subadditivity property of the $L^p$ norm.

**Proposition A.10:** MINKOWSKI INEQUALITY
*Let $p$ in $[1, +\infty]$. If $f$ and $g$ are in $L^p(\Omega)$, we have*

$$||f + g||_p \le ||f||_p + ||g||_p.$$

Let us quickly detail this proof because it contains some fundamental reasoning that will be used a lot. For $p = 1$ and $p = \infty$, the result is immediate since $|f + g| \leq |f| + |g|$, so we consider $1 < p < \infty$. Moreover, we can suppose $f$ and $g$ non-negative since $|f + g| \leq ||f| + |g||$. The application $x \mapsto x^p$ being convex on $\mathbb{R}^+$, we have for any $a, b$ in $\mathbb{R}^+$ that $\left(\frac{1}{2}a + \frac{1}{2}b\right)^p \leq \frac{1}{2}a^p + \frac{1}{2}a^p$. Then by taking $a = 2f(x)$ and $b = 2g(x)$, we get $(f(x) + g(x))^p \leq 2^{p-1}(f(x)^p + g(x)^p)$. Note that $x$ is taken almost everywhere on $\Omega$ since $f$ and $g$ are defined almost everywhere and the countable union of negligible sets is negligible. Therefore, $f + g$ is in $L^p(\Omega)$. Also,

$$||f + g||_p^p = \int_\Omega |f + g|^{p-1}|f + g| \leq \int_\Omega |f + g|^{p-1}|f| + \int_\Omega |f + g|^{p-1}|g|.$$

Since $|f + g|^{p-1}$ is in $L^{\frac{p}{p-1}}(\Omega)$, we deduce thanks to Hölder inequality,

$$||f + g||_p^p \leq ||f + g||_p^{p-1}||f||_p + ||f + g||_p^{p-1}||g||_p.$$

If $||f + g||_p = 0$, thanks to *Property B.3* we have that $f + g = 0$, then $f$ and $g$ are null and the inequality is verified. Otherwise, we obtain our result by dividing by $||f + g||_p^{p-1}$.

### A.2.3 Sobolev spaces

Let us remember the concepts and results associated with Sobolev spaces.

**Definition A.11:** Sobolev spaces
*Let $\Omega \subset \mathbb{R}^N$ be open and $p \in \mathbb{R}$ such that $1 \leq p \leq \infty$,*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \setminus \exists (g_i)_{i \in [\![1,N]\!]} \subset L^p(\Omega) \ s.t. \ \int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i \phi, \ \forall \phi \in C_C^\infty(\Omega), \ \forall i \in [\![1, N]\!] \right\}.$$

*Moreover, we define for $m \in \mathbb{N}^*$,*

$$\begin{aligned} W^{m,p}(\Omega) &= \left\{ u \in L^p(\Omega) \setminus \forall \alpha, |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \ s.t. \ \int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi, \ \forall \phi \in C_C^\infty(\Omega) \right\} \\ &= \left\{ u \in W^{m-1,p}(\Omega); \ \frac{\partial u}{\partial x_i} := g_i \in W^{m-1,p}(\Omega), \ \forall i \in [\![1, N]\!] \right\}, \end{aligned}$$

*with $\alpha \in \mathbb{N}^N$ the usual multi-index, with $|\alpha| = \sum \alpha_i$.*

We equip the space $W^{m,p}$ with

$$||u||_{W^{m,p}} = \sum_{0 \leq |\alpha| \leq m} ||D_\alpha u||_{L^p}.$$

This gives it the status of a Banach space, for any $p$ and $m$. Let us write $H^m(\Omega) = W^{m,2}(\Omega)$, they are Hilbert spaces. Indeed, the associated norms are derived from the following scalar products,

$$(u, v)_{H^m} = \sum_{0 \leq |\alpha| \leq m} (D_\alpha u, D_\alpha v)_{L^2}.$$

The spaces we will mainly use are $W^{1,p}$, whose norms are

$$||u||_{W^{1,p}} = ||u||_{L^p} + \sum_{i=1}^N ||\frac{\partial u}{\partial x_i}||_{L^p},$$

which is equivalent to

$$||u||_{W^{1,p}} = \left( ||u||_{L^p}^p + \sum_{i=1}^N ||\frac{\partial u}{\partial x_i}||_{L^p}^p \right)^{1/p}.$$

In the case $p = 2$, the associated scalar product is

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^N \left( \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2}.$$

**Definition A.12:** $W_0^{1,p}(\Omega)$ SPACES
*For $1 \leq p < \infty$, we denote $W_0^{1,p}(\Omega)$ the closure of $C_C^1(\Omega)$ in $W^{1,p}(\Omega)$.*

Equipped with the same norms as $W^{1,p}(\Omega)$, the spaces $W_0^{1,p}(\Omega)$ are also Banach spaces. In much the same way, we note $H_0^1(\Omega) := W_0^{1,2}(\Omega)$ which is a Hilbert space with respect to the $H^1$ scalar product. What particularly interests us about these spaces is their following characterization.

**Theorem A.13:** ([Bre83], IX.17) A CHARACTERIZATION OF $W_0^{1,p}(\Omega)$
*If $\Omega$ is of class $C^1$ and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $1 \leq p < \infty$, then we have the equivalence*

$$u = 0 \quad on \quad \Gamma \qquad \Longleftrightarrow \qquad u \in W_0^{1,p}(\Omega).$$

Intuitively, this space can be seen as a restriction that encodes the homogeneous Dirichlet condition. All these spaces enable us to introduce the notion of derivative in a weak sense, often less restrictive than the derivative in the usual sense. It is a change in the interpretation of derivatives, although they are not entirely independent. This notion will allow us to introduce two interpretations of the Laplacian which, as we will see, turn out to be closely related.

### A.2.4 Representation theorems

The interest of representation theorems is to provide a new interpretation of dual spaces.

**Theorem A.14:** ([Bre83], V.5) DUAL REPRESENTATION OF RIESZ-FRECHET
*Let $H$ be any Hilbert space and $\phi$ be in $H'$, then there exist a unique $y$ in $H$ such that*

$$\langle \phi, x \rangle = (y, x)_H, \qquad \forall x \in H.$$

*The application $\phi \mapsto y$ is an isometric isomorphism, i.e.*

$$||\phi||_{H'} = |y|_H.$$

This theorem allows us to represent continuous linear forms on $H$ thanks to the scalar product. Now we consider $a : H \times H \to \mathbb{R}$ a bilinear form. It is said to be continuous if

$$\exists C \in \mathbb{R} \qquad \text{s.t.} \qquad |a(u,v)| \leq C|u||v| \qquad \forall u, v \in H, \tag{63}$$

and is said to be coercive if

$$\exists \alpha > 0 \qquad \text{s.t.} \qquad |a(u,u)| \geq \alpha |u|^2 \qquad \forall v \in H. \tag{64}$$

These notions allow us to introduce the following fundamental theorem.

**Theorem A.15:** ([Bre83], V.8) LAX-MILGRAM
*Let $H$ be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a bilinear continuous and coercive form. For any $\phi$ on $H'$ there exists a unique $u$ in $H$ such that*

$$a(u, v) = \langle \phi, v \rangle, \qquad \forall v \in H.$$

*If $a$ is symmetric, $u \in H$ is characterized by*

$$\frac{1}{2} a(u,u) - \langle \phi, u \rangle = \min_{v \in H} \left( \frac{1}{2} a(v,v) - \langle \phi, v \rangle \right).$$

### A.2.5 Spectral decomposition of self-adjoint compact operators

An essential concept for the notion of decomposition on Hilbert spaces is, of course, the one of basis.

**Definition A.16:** HILBERTIAN BASIS
*Let $H$ be a Hilbert space. A Hilbertian basis of $H$ is a sequence $(e_n)$ of elements of $H$ such that*

1. $\forall m, n \qquad (e_n, e_m)_H = \delta_{m,n}$,

2. *The vector space spanned by $(e_n)$ is dense in $H$.*

*$\delta$ is the Kronecker symbol.*

This implies that one can express any element of the Hilbert space in terms of such a basis,

$$u = \sum_{i=1}^{\infty} (u, e_n) e_n \qquad \forall u \in H. \tag{65}$$

Let us also recall the following fundamental concepts of spectral analysis.

**Definition A.17:**
*Let $T \in \mathscr{L}(E)$.*

- *Resolving set: $\rho(T) = \{\lambda \in \mathbb{R} \mid (T - \lambda I)$ is bijective from $E$ to $E\}$.*
- *Spectrum: $\sigma(T) = \mathbb{R} \setminus \rho(T)$.*
- *Eigenvalues: $EV(T) = \{\lambda \in \mathbb{R} \mid Ker(T - \lambda I) \neq \{0\}\}$.*
- *Eigenspaces: $ES(\lambda) = Ker(T - \lambda I)$.*

*Where $I$ is the identity operator and $Ker$ the kernel.*

Let us now introduce two properties of operators that will enable us to consider the spectral decomposition of an operator.

**Definition A.18:** COMPACT OPERATOR $\mathscr{K}(E, F)$
*Let $E, F$ be two vector spaces. We say that $T$ a linear and continuous map from $E$ to $F$ is compact if $T(B_E)$ is relatively compact. In other words, if $T(B_E)$ is included in a compact part of $F$.*

This notion, roughly speaking, speaks about the stability of the operator. The second notion is that of the self-adjoint operator, but before defining it, let us recall what the adjoint of an operator is.

**Definition A.19:** ADJOINT OPERATOR
*Let $E, F$ be two open spaces, $T : D(T) \subset E \to F$, and $D(T)$ be the domain of definition of $T$, i.e. the vector subspace on which $T$ is defined. The domain $D(T^*)$ of the adjoint operator of $T$ is the following subset of $F'$,*

$$D(T^*) = \{u \in F' \mid \exists C > 0 \;\; s.t. \; |\langle u, Tv \rangle_F| \leq C||v||, \; \forall v \in D(T)\}.$$

*This definition allows us to write that the adjoint operator $T^*$ of $T$ is the operator of $D(T^*)$ in $E'$ satisfying*

$$\langle u, Tv \rangle_{F',F} = \langle T^* u, v \rangle_{E',E}, \qquad \forall v \in D(A), \; \forall u \in D(T^*).$$

It is interesting to note that this definition does not require the explicit notion of scalar product on $E$ and $F$, as it is the duality brackets that are considered here. So this definition is valid in any Banach space. Intuitively, we define the notion of self-adjoint operator as follows.

**Definition A.20:** SELF-ADJOINT OPERATOR
*Let $E$ a Banach space. An operator $T : E \to E$ is said to be self-adjoint if $D(T) = D(T^*)$ and*

$$\langle u, Tv \rangle_E = \langle Tu, v \rangle_E, \qquad \forall u, v \in E$$

Note that we ask $E$ to be identifiable with his dual. In the study, we will use this notion with $E$ a Hilbert space, and as explained before, this condition is satisfied. Now, we have all the necessary tools to understand the following theorem, which will be extremely important.

**Theorem A.21:** `([BRE83], VI.11)` SPECTRAL DECOMPOSITION OF COMPACT SELF-ADJOINT OPERATORS
*Let $H$ be a separable Hilbert space and $T : H \to H$ be a compact self-adjoint operator. Then $H$ admits a Hilbertian basis of eigenvectors of $T$.*

In other words, this theorem implies that any compact self-adjoint operator defined on a separable Hilbert space is diagonalizable, in the sense that there is an orthonormal basis composed of eigenvectors of $T$.

## A.3 Spectral decomposition of the Laplacian

This section aims to prove Theorem A.1. We will denote by $\Omega$ an open bounded set of $\mathbb{R}^N$ and by $f$ an arbitrary function in $L^2(\Omega)$. The starting point for the spectral analysis of the Laplacian is the following second-order elliptic partial differential equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma := \partial\Omega = \overline{\Omega} \setminus \Omega. \end{cases} \tag{66}$$

To solve this problem, i.e. to prove the existence and uniqueness of a solution, we're going to use a variational method. More precisely, we will use the Lax-Milgram theorem. After that, we will consider the operator which maps the parameter $f$ to its associated solutions concerning the Dirichlet problem. To address the problem of the eigenvalues of the Laplacian, we will show that this operator is diagonalizable using Theorem A.21.

### A.3.1 Variational formulation of the homogeneous Dirichlet problem

We recall that a classical solution of Eq. (66) is a function $u \in C^2(\Omega)$ which satisfies, in the usual sense

$$\begin{cases} -\Delta u = f & \text{in } \Omega, & (67a) \\ u = 0 & \text{on } \Gamma := \partial\Omega. & (67b) \end{cases}$$

A weak solution of this problem is a function $u \in H_0^1(\Omega)$ that satisfies

$$\int_\Omega \nabla u \nabla v = \int_\Omega fv, \qquad \forall v \in H_0^1(\Omega). \tag{68}$$

The idea of such a formulation is to relax the regularity requirement necessary to consider the second derivative of $u$. We now require $u$ to be a function of $H^1(\Omega)$, which is less restrictive in some sense. Indeed, we only ask now for $u$ to be square-integrable and to admit a weak first-order derivative that must be square-integrable. Let us note that the weak first-order derivative of $u$ is also denoted by $\nabla u$. Thanks to Theorem A.13 and some regularity of the weak solution, the fact that $u$ belongs to $H_0^1(\Omega)$ ensures that it also satisfies the homogeneous Dirichlet constraint (67b). This point will be clarified shortly afterward. To prove the existence of a solution $u \in H_0^1(\Omega)$ to Eq. (68), we aim to apply the Lax-Milgram Theorem. Indeed, we observe that this equation can be rewritten as

$$a(u, v) = \langle f, v\rangle_{L^2}, \qquad \forall v \in H_0^1(\Omega) \tag{69}$$

with

$$a(u, v) = \int_\Omega \nabla u \nabla v \qquad \text{and} \qquad \langle f, v\rangle_{L^2} = \int_\Omega fv.$$

It is easy to check that $a$ is bilinear and is continuous. Indeed, thanks to the Hölder inequality, for all $u, v$ in $H^1(\Omega)$,

$$|a(u, v)| \leq \int_\Omega |\nabla u \nabla v| \leq ||\nabla u||_{L^2} ||\nabla v||_{L^2} \leq ||u||_{H^1} ||v||_{H^1}.$$

Therefore, $a$ is continuous. Let us now check that $a$ is coercive. For this, we need the following result.

**Theorem A.22:** ([Bre83], IX.19) Poincaré inequality
*Let $1 \leq p < \infty$ and $\Omega$ be a bounded open space of $\mathbb{R}^N$. Their exists a $C$, only depending on $\Omega$ and $p$, such that*

$$||u||_{L^p} \leq C ||\nabla u||_{L^p}, \qquad \forall u \in W_0^{1,p}(\Omega).$$

*In particular, $||\nabla u||_{L^p}$ is a norm on $W_0^{1,p}(\Omega)$, which is equivalent to the norm $||u||_{W^{1,p}}$. On $H_0^1(\Omega)$, $(u, v) \mapsto \int_\Omega \nabla u \nabla v$ is a scalar product, this means that $||\nabla u||_{L^2}$ is equivalent to $||u||_{H^1}$.*

Then, for all $u$ in $H^1(\Omega)$, we have

$$||u||_{H^1}^2 = ||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 \leq (C^2 + 1) ||\nabla u||_{L^2}^2, \tag{70}$$

and then

$$a(u, u) = ||\nabla u||_{L^2}^2 \geq \frac{1}{C^2 + 1} ||u||_{H^1}^2.$$

We now have everything we need to apply the Lax-Milgram Theorem to Eq. (69). Thus, for all $\phi \in H^{-1}(\Omega) := H_0^1(\Omega)'$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \langle \phi, v\rangle, \qquad \forall v \in H_0^1(\Omega). \tag{71}$$

In addition, since $a$ is symmetric, $u \in H_0^1(\Omega)$ is characterized by

$$\frac{1}{2}a(u,u) - \langle \phi, u \rangle = \min_{v \in H_0^1(\Omega)} \left( \frac{1}{2}a(v,v) - \langle \phi, v \rangle \right).$$

Given that $L^2(\Omega)$ is a Hilbert space, considering its scalar product allows us to identify it with its dual. For $H^1(\Omega)$, if we consider its inner product, its dual space can be identified with itself. However, considering Eq. (71) and specifically $a$, we must consider the scalar product of $L^2(\Omega)$. In this configuration, its dual space is larger than $L^2(\Omega)$. Therefore, by considering the most general form of the dual $H_0^1(\Omega)$, $H^{-1}(\Omega)$, we have

$$H_0^1(\Omega) \subset L^2(\omega) \subset H^{-1}(\Omega),$$

with continuous and dense embeddings. This subject deserves further explanation, for which we redirect you to Remark 1 in [Bre83]. The slight difference here is that the Riesz-Frechet representation theorem depends on the scalar product under consideration. We deduce the following statement from this result.

**Corollary A.23:** Existence and uniqueness of weak solution
*For any element $f$ of $L^2(\Omega)$ there exists a unique solution $u$ of $H_0^1(\Omega)$ to problem (68), i.e.*

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega).$$

### A.3.2 Decomposition of the inverse Dirichlet Laplacian

In the following, we will denote by $S : f \in L^2(\Omega) \mapsto u \in H_0^1(\Omega)$ the operator which associates to $f$ its corresponding weak solution according to Eq. (68). As before, given that $H_0^1(\Omega) \subset L^2(\omega)$ with continuous and dense embeddings, we can consider $T : f \in L^2(\Omega) \mapsto u \in L^2(\Omega)$ the extension of $S$. Note that Corollary A.23 ensures that these operators are indeed well-defined. Given the way these operators have been constructed, they can be identified with the inverse Laplacian, in the weak sense. Indeed, they map $f$ the parameter to the weak solution of problem Eq. (68). This section will aim to apply Theorem A.21 to $T$. Thanks to the following results and the fact that $L^2(\Omega)$ is a Hilbert space, all it left to show to apply Theorem A.21 is that $T$ is a compact self-adjoint operator.

**Theorem A.24:** [Bre83], (IV.13) $L^p$ separable
*$L^p(\Omega)$ is separable for $1 \leq p < \infty$.*

We will start by establishing that $T$ is a compact operator. To do so, let us state the following proposition.

**Proposition A.25:** [Bre83], (VI.3) Composition on $\mathscr{K}$
*Let $E, F$ and $G$ be three Banach spaces. If $T \in \mathscr{L}(E,F)$ and $S : F \to G$ is a compact operator, i.e. is in $\mathscr{K}(F,G)$, then $S \circ T \in \mathscr{K}(E,F)$.*

Since the injection $I$ of $H_0^1(\Omega)$ into $L^2(\Omega)$ is dense and noticing $T = I \circ S$, all that remains is to show that $S$ belongs to $\mathscr{L}(L^2(\Omega), H_0^1(\Omega))$. By choosing $v = Sf$ in Eq. (68), we obtain

$$\int_\Omega (\nabla u)^2 = \int_\Omega f u, \tag{72}$$

by using (70) and Hölder inequality A.9 we obtain

$$||u||_{H_0^1}^2 \leq (C^2+1)||\nabla u||_{L^2}^2 \leq (C^2+1)||f||_{L^2}||u||_{L^2} \leq (C^2+1)||f||_{L^2}||u||_{H_0^1},$$

and finally by dividing by $||u||_{H_0^1}$ (trivially $\neq 0$ if $f \neq 0$),

$$||Sf||_{H_0^1} \leq (C^2+1)||f||_{L^2}.$$

Therefore, we have that $S$ is in $\mathscr{L}(L^2(\Omega), H_0^1(\Omega))$ and therefore that $T$ is in $\mathscr{K}(L^2(\Omega))$. Let us now show that $T$ is self-adjoint. To do this, we consider $f$ and $g$ in $L^2(\Omega)$, we have by definition of $S$

$$(f, Tg)_{L^2} = (f, Sg)_{L^2} \overset{(69)}{:=} a(Sf, Sg),$$

and similarly by symmetry of $a$ (or of scalar product of $L^2$) we get

$$(Tf, g)_{L^2} = a(Sf, Sg).$$

Moreover, we deduce, thanks to the Hölder inequality, that for all $u \in L^2(\Omega)$

$$|\langle u, Tv\rangle| = |\langle Tu, v\rangle| \leq \int_\Omega |vTu| \leq ||Tu||_{L^2}||v||_{L^2}, \qquad \forall v \in L^2(\Omega).$$

We just proved that $D(T^*) = \{u \in L^2(\Omega) | \exists C > 0, \forall v \in L^2(\Omega), |\langle u, Tv\rangle| \leq C||v||_{L^2}\} = L^2(\Omega)$. Then $D(T^*) = D(T)$, we therefore proved that $T$ is self-adjoint. Our operator $T$, therefore, satisfies all the conditions for applying Theorem A.21. In other words, we have proved that $L^2(\Omega)$ admits a Hilbertian basis of eigenfunctions of $T$.

### A.3.3 Back to the classical solution

We are going to prove that a weak solution is also a classical solution, but also that a classical solution is a weak solution. The key point is Green's formula.

**Proposition A.26:** Green's formula
*Let $d\sigma$ be the measure on $\Gamma$ and $\frac{\partial u}{\partial n} = (\nabla u, n)$ be the normal derivative, where $n$ is the normal vector pointing outwards. Green's formula is*

$$\int_\Omega (\Delta u)v = \int_\Gamma \frac{\partial u}{\partial n} v d\sigma - \int_\Omega \nabla u \nabla v, \qquad \forall u \in C^2(\overline{\Omega}),\ \forall v \in C^1(\overline{\Omega}).$$

In what follows, we will ask $\Omega$ to be of class $C^1$. Let us begin by assuming that $u$ is a solution in the classical sense. Then $u$ is in $C^2(\overline{\Omega})$ and therefore in $H^1(\Omega)$ since $\Omega$ is bounded. Thanks to Theorem A.13 and homogeneous Dirichlet conditions we can ensure $u$ is in $H_0^1(\Omega)$.
Now, we show that $u$ satisfies (68). Using Green formula A.26, we obtain

$$\int_\Omega (\Delta u)v = -\int_\Omega \nabla u \nabla v, \qquad \forall v \in C^1(\Omega) \cap H_0^1(\Omega),$$

and by multiplying the classical initial problem by $v$ and integrating it over $\Omega$, we deduce that

$$-\int_\Omega (\Delta u)v = \int_\Omega \nabla u \nabla v = \int_\Omega fv, \qquad \forall v \in C^1(\Omega) \cap H_0^1(\Omega).$$

Now, to show that this result extends to all functions $v$ in $H_0^1(\Omega)$, we want to use the density of $C^1(\Omega)$ in $H_0^1(\Omega)$. To do this, as the following theorem puts in evidence, we must have $\Omega$ of class $C^1$.

**Proposition A.27:** ([Bre83], IX.8) Density
*Assume $\Omega$ is of class $C^1$. Let $u$ be in $W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence $(u_n)$ of $C_C^\infty(\mathbb{R}^N)$ such that $u_{n|\Omega} \to u$ in $W^{1,p}(\Omega)$. In other words, the restrictions to $\Omega$ of the functions of $C_C^\infty(\mathbb{R}^N)$ form a dense subspace of $W^{1,p}(\Omega)$.*

This result allows us to conclude that $u$, a classical solution, is also a weak solution, i.e. satisfy $u$ is in $H_0^1(\Omega)$ and

$$\int_\Omega \nabla u \nabla v = \int_\Omega fv, \qquad \forall v \in H_0^1(\Omega).$$

Let us also note that

$$\int_\Omega \nabla u \nabla v = -\int_\Omega (\Delta u)v, \qquad \forall v \in H_0^1(\Omega). \tag{73}$$

To determine the reciprocal, we need some regularity results for the weak solution.

**Theorem A.28:** ([Bre83], IX.25) Regularity for the weak solution of the Dirichlet problem
*Let $u$ be the weak solution of the Dirichlet problem. Then, $u$ is in $H^2(\Omega)$ and $||u||_{H^2} \leq C||f||_{L^2}$ where $C$ is a constant depending only on $\Omega$. Moreover, if $\Omega$ is of class $C^{m+2}(\Omega)$ and if $f \in H^m(\Omega)$, then*

$$u \in H^{m+2}(\Omega) \qquad \textit{with} \qquad ||u||_{H^{m+2}} \leq C||f||_{H^m}.$$

*In particular, if $m > N/2$, then $u \in C^2(\Omega)$.*

Let us assume that the conditions for $u$, the weak solution, to appear at $C^2$ are met. Given that, $u$ is also in $H_0^1(\Omega)$, we can ensure thanks to Theorem A.13 that $u$ satisfy the homogeneous Dirichlet conditions. On the other hand, thanks to Green's formula and (68), we obtain by restricting $C^1(\Omega)$ to $C_C^1(\Omega)$ that

$$\int_\Omega (-\Delta u)v = \int_\Omega fv, \qquad \forall v \in C_C^1(\Omega).$$

Therefore, $\Delta u + f$ belongs to $C_C^1(\Omega)^\perp$, meanwhile $C_C^1(\Omega)$ is dense in $L^2(\Omega)$, then $C_C^1(\Omega)^\perp = \{0\}$. We conclude that $-\Delta u = f$ is almost everywhere on $\Omega$. Nevertheless, $u$ is $C^2(\Omega)$, then

$$-\Delta u = f \qquad \text{in } \Omega.$$

In conclusion, we have established the equivalence between the classical and the weak solutions.

### A.3.4 Conclusion of the proof

Let us briefly recap what we have shown previously. We proved the existence and uniqueness of a weak solution to the main problem (66). Then we proved that the operator $T$, which maps the parameter $f \in L^2(\Omega)$ to its corresponding weak solution $u \in L^2(\Omega)$, is diagonalizable in $L^2(\Omega)$. Finally, in the last section, we established the equivalence between classical and weak solutions. We will gather these results to conclude the proof of Theorem A.1. We know that there exists a Hilbertian basis $(e_n)_{n\geq 1}$ of $H_0^1(\Omega)$ and a sequence $(\nu_n)_{n\geq 1}$ such that $Te_n = \nu_n e_n$. Indeed, $e_n$ are images of $T$, then they are in $H_0^1(\Omega)$. Thanks to the equivalence between classical and weak solutions, we can ensure that for any $n \geq 1$ we have

$$-\nu_n \Delta e_n = e_n.$$

We can go further by proving $\nu_n > 0$. To achieve this, let us remember that for any $f \in L^2(\Omega)$, result (72) asserts that

$$\int_\Omega fTf = \int_\Omega (\nabla(Tf))^2 \geq 0,$$

from which we deduce that $T$ is a non-negative operator. So, for all $n \geq 1$ we have $\nu_n \geq 0$. Moreover, if we suppose that $Tf = 0$, we have

$$-\Delta Tf = f = 0.$$

As a consequence, the kernel of $T$ is reduced to $\{0\}$, then $\nu_n \neq 0$. We can therefore set $\lambda_n = \frac{1}{\nu_n} > 0$ and write for all $n \geq 1$

$$-\Delta e_n = \lambda_n e_n. \tag{74}$$

In addition, we also note that thanks to Theorem A.28, since $e_n$ is a solution to the Dirichlet problem, it belongs to $H^2(\Omega)$. Applying the regularity results of the same theorem to Eq. (74), we deduce that $e_n$ is in $H^4(\Omega)$, and consequently is also in $H^6(\Omega)$ ... Iterating this procedure, we perform a bootstrap on $e_n$ and show that $e_n \in \bigcap_{m\geq 1} H^m(\Omega)$ and thus $e_n \in C^\infty(\Omega)$. We have now proved all the points of Theorem A.1.
The sketch of the proof we have just carried out is based on two interpretations of the Laplacian operator. The first interpretation is in the usual sense, where the operator's domain of definition consists of functions that admit a second derivative in the classical sense. The second approach uses the notion of weak derivatives. The inverse operator is represented thanks to the Lax-Milgram theorem, offering a new interpretation of the operator. This Laplacian operator admits as the domain of definition $H^2(\Omega) \cap H_0^1(\Omega)$. What allowed us to link these two visions is the fundamental Theorem A.28.

## A.4 Digression on the Neumann problem

In the following parts, we will require the result analogous to Theorem A.1 but for the homogeneous Neumann boundary condition, i.e. $\frac{\partial u}{\partial n} = 0$ on $\Gamma$. We can immediately see that any non-zero constant function $e$ is an eigenfunction of this operator. Indeed, such a function verifies the homogeneous Neumann boundary condition and $\Delta e = 0e$. So, 0 is a eigenvalue of the Neumann Laplacian operator. We deduce that the operator which maps the parameter $f \in L^2(\Omega)$ to its corresponding weak solution $u \in L^2(\Omega)$, will not be diagonalizable as done in the previous section. To overcome this issue, we will work with the operator $-\Delta + Id$. The corresponding map to the weak solution will be invertible, allowing the same process as above to be applied. For $f \in L^2(\Omega)$, we call $u \in C^2(\Omega)$ a solution to the Neumann problem in the classical sense if $u$ satisfies

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

We denote $u \in H^1(\Omega)$ as a weak solution of the Neumann problem if it satisfies

$$\int_\Omega \nabla u \nabla v + \int_\Omega uv = \int_\Omega fv, \qquad \forall v \in H^1(\Omega). \tag{75}$$

With a very close analogy to the approach adopted in Appendix A.3, we will admit the following results, acknowledging that certain aspects require clarification. These intricacies are meticulously explained in [Bre83]. By applying the Lax-Milgram theorem, exactly as in Appendix A.3.1 but with $H^1(\Omega)$ instead of $H_0^1(\Omega)$, we obtain the equivalent of Corollary A.23.

**Corollary A.29:** ([BRE83], IX.24) EXISTENCE AND UNIQUENESS OF WEAK SOLUTION FOR NEUMANN PROBLEM
*For any element $f$ in $L^2(\Omega)$, there exists a unique solution $u$ of $H^1(\Omega)$ to problem (75).*

Let us expose the connection between the weak and the classical solution to the Neumann problem. This is significant because it highlights the main difference between the two boundary conditions under consideration. It also allows us to justify the choice of $H^1(\Omega)$ and to present results that will be useful later on. To begin, we have the fundamental result analogous to Theorem A.28.

**Theorem A.30:** ([BRE83], IX.26) REGULARITY OF WEAK SOLUTION OF THE NEUMANN PROBLEM
*Let $u$ be the weak solution of the Neumann problem (75). Then, $u$ is in $H^2(\Omega)$ and $||u||_{H^2} \le C||f||_{L^2}$ where $C$ is a constant depending only on $\Omega$. Moreover, if $\Omega$ is of class $C^{m+2}(\Omega)$ and if $f \in H^m(\Omega)$, then*

$$u \in H^{m+2}(\Omega) \qquad \textit{with} \qquad ||u||_{H^{m+2}} \le C||f||_{H^m}.$$

*In particular, if $m > N/2$, then $u \in C^2(\Omega)$.*

This enables us to ensure that such a weak solution $u$ satisfies, through the employment of Green formula A.26,

$$\int_\Omega \nabla u \nabla v + \int_\Omega uv = \int_\Gamma \frac{\partial u}{\partial n} v d\sigma + \int_\Omega (-\Delta u + u)v = \int_\Omega fv, \qquad \forall v \in C^1(\overline{\Omega}). \tag{76}$$

By first choosing $v \in C_C^1(\Omega) \cap L^2(\Omega) \subset C^1(\Omega) \cap L^2(\Omega)$, we get

$$\int_\Omega (\Delta u - u + f)v = 0, \qquad \forall v \in C_C^1(\overline{\Omega}),$$

i.e. $\Delta u - u + f$ belongs to $(C_C^1(\Omega) \cap L^2(\Omega))^\perp$ which is 0, then $-\Delta u + u = f$ almost everywhere. Eq. (76) becomes

$$\int_\Gamma \frac{\partial u}{\partial n} v d\sigma = 0, \qquad \forall v \in C^1(\overline{\Omega}).$$

Consequently, by the same reasoning on $C_C^1(\Omega)$, we obtain

$$\frac{\partial u}{\partial n} = 0, \qquad \text{on } \Gamma.$$

We have shown that the weak solution is also a classical solution. Conversely, if $u$ is the classical solution of the Neumann problem, we have with Green formula

$$\int_\Omega fv = \int_\Omega (-\Delta u + u)v = \int_\Omega \nabla u \nabla v + \int_\Omega uv, \qquad \forall v \in C^1(\overline{\Omega}),$$

and thus, thanks to the density results A.27, we get our point,

$$\int_\Omega fv = \int_\Omega \nabla u \nabla v + \int_\Omega uv, \qquad \forall v \in H^1(\Omega).$$

A classical solution is also a weak solution. □

In addition, for $u$ the solution to our Neumann problem, we have from Green's formula and density of $C_c^1(\Omega)$ in $H^1(\Omega)$

$$-\int_\Omega (\Delta u)v = \int_\Omega \nabla u \nabla v, \qquad \forall v \in H^1(\Omega). \tag{77}$$

As previously done, we deduce that there exists a Hilbertian basis $(a_n)_{n\geq 1}$ of $L^2(\Omega)$ satisfying Neumann boundary condition and a real sequence $(\eta_n)_{n\geq 1}$ such that

$$a_n \in H^1(\Omega) \cap C^\infty(\Omega) \qquad (-\Delta + Id)a_n = \eta_n a_n \quad \text{on } \Omega.$$

So, $(a_n)_{n\geq 1}$ also satisfy $\Delta a_n = (\eta_n - 1)a_n$. Finally, we conclude with the main theorem.

**Theorem A.31:** Spectral decomposition of $-\Delta$ for homogeneous Neumann boundary condition

*There exists a Hilbertian basis $(a_n)_{n\geq 1}$ of $L^2(\Omega)$ and a real sequence $(\mu_n)_{n\geq 1}$ such that $\mu_n \geq 0$ and $\mu_n \to +\infty$, satisfying*

$$a_n \in H^1(\Omega) \cap C^\infty(\Omega) \qquad - \Delta a_n = \mu_n a_n \quad \textit{on } \Omega.$$

*Note that $a_n$ satisfies the homogeneous Neumann conditions.*

*Remark.* The non-negativity of the eigenvalues result is immediate with Theorem I.1.

# B Some notes on integration theory

In this section, we set out some results from the theory of measurement and integration. This allows us to establish the statements of the theorems used. Statements below are taken from the "Théorie de la mesure et de l'intégration" course given by Thierry Gallay at Université Joseph Fourier, Grenoble. These results are proven in his document, which can be easily found online.

*Remark.* For the sake of completeness, the theorems are stated in their most general form. Meanwhile, in this manuscript, we exclusively work with $\mathbb{R}^N$ equipped with the Borel sigma-algebra and Lebesgue measure $\lambda$. Hence, we will omit to specify that it is a working space satisfying our needs (see. Thierry Gallay course). Therefore, when using these theorems, we don't specify in which measured space we are working. The only aspect we will focus on when using these theorems will be the properties regarding the concerned function $f$.

## Integrations of non-negative measurable functions

We recall that a simple function is a function that only takes a finite number of values.

**Theorem B.1:**
*Let $f$ be a non-negative measurable function. Then, there exists an increasing sequence of non-negative and simple functions that converge to $f$ pointwise.*

**Property B.2:**
*Let $A$ be Lebesgue measurable, we have*

- *Outer regularity:* $\lambda(A) = \inf\{\lambda(U) \mid U \text{ open }, A \subset U\}$
- *Inner regularity:* $\lambda(A) = \sup\{\lambda(K) \mid K \text{ compact }, K \subset A\}$

**Property B.3:**
*Let $f$ and $g$ be non-negative measurable functions.*

- *If $\int f d\mu < \infty$, then $f < \infty$ almost everywhere.*
- *$\int f d\mu = 0$ if and only if $f = 0$ almost everywhere.*
- *If $f = g$ almost everywhere, then $\int f d\mu = \int g d\mu$.*

**Theorem B.4:** Beppo-Levi - monotone convergence
*Let $(X, \mathscr{M}, \mu)$ a measured space and $(f_n)$ an increasing sequence of non-negative measurable functions on $X$ and $f = \lim_{n\to\infty} f_n$ the pointwise limit of $f_n$. Then, $f$ is measurable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

**Theorem B.5:** Dominated convergence
*Let $(X, \mathscr{M}, \mu)$ a measured space and $f_n : x \to \mathbb{C}$ a sequence of measurable functions. We suppose*

1. *limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for almost all $x$ in $X$,*
2. *there exists a non-negative and integrable function $g$ such that for $x$ almost everywhere in $\Omega$ and for all $n$ in $\mathbb{N}$,*
$$|f_n(x)| \leq g(x).$$

*Then, $f$ is integrable,*

$$\lim_{n\to\infty} \int |f - f_n| d\mu = 0 \qquad \text{and then} \qquad \lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

## Integration on product spaces

**Theorem B.6:** FUBINI-TONELLI
*Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ two $\sigma$-finite measured spaces. Let $f : X \times Y \to \overline{\mathbb{R}^+}$ a $\mathscr{M} \otimes \mathscr{N}$-measurable function. Then,*

1. *Functions $f_y : x \mapsto \int_Y f(x,y)d\nu(y) \in \overline{\mathbb{R}^+}$ and $f_x : y \mapsto \int_X f(x,y)d\mu(x) \in \overline{\mathbb{R}^+}$ are measurable for almost all $x$ and $y$.*
2. *We have the equalities*

$$\int_{X\times Y} fd(\mu \otimes \nu) = \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y) \quad \in [0, +\infty].$$

**Theorem B.7:** FUBINI-LEBESGUE
*Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ two $\sigma$-finite measured spaces. Let $f : X \times Y \to \mathbb{R}$ or $\mathbb{C}$ an integrable function, i.e. in $L^1(X \times Y, \mathscr{M} \otimes \mathscr{N}, \mu \otimes \nu)$. Then,*

1. *For almost all $x \in X$, the function $y \mapsto f(x,y)$ is in $L^1(Y, \mathscr{N}, \nu)$.*
   *For almost all $y \in Y$, the function $x \mapsto f(x,y)$ is in $L^1(X, \mathscr{M}, \mu)$.*
2. *The almost defined everywhere (w.r.t. $\mu$) function $x \mapsto \int_Y f(x,y)d\nu(y)$ is in $L^1(X, \mathscr{M}, \mu)$.*
   *The almost defined everywhere (w.r.t. $\nu$) function $y \mapsto \int_X f(x,y)d\mu(x)$ is in $L^1(Y, \mathscr{N}, \nu)$.*
3. *We have,*

$$\int_{X\times Y} fd(\mu \otimes \nu) = \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y).$$

## Integrals depending on a parameter

**Theorem B.8:** CONTINUITY THEOREM
*We suppose,*

1. *For all $\lambda$ in $\Lambda$, the function $x \mapsto f(x, \lambda)$ is integrable on $X$.*
2. *For almost all $x$ in $X$, the function $\lambda \mapsto f(x, \lambda)$ is continuous on $\Lambda$.*
3. *There exists $g$ integrable such that for all $\lambda$ in $\Lambda$, we have for almost all $x$ in $X$,*

$$|f(x, \lambda)| \leq g(x).$$

*Then, the function $\lambda \mapsto \int_X f(x, \lambda)dx$ is continuous on $\Lambda$.*

**Theorem B.9:** DERIVABILITY THEOREM
*We suppose,*

1. *For all $\lambda$ in $\Lambda$, the function $x \mapsto f(x, \lambda)$ is integrable on $X$.*
2. *For almost all $x$ in $X$, the function $\lambda \mapsto f(x, \lambda)$ is derivable on $\Lambda$.*
3. *There exists $g$ integrable such that for all $\lambda$ in $\Lambda$, we have for almost all $x$ in $X$,*

$$\left|\frac{\partial}{\partial \lambda} f(x, \lambda)\right| \leq g(x).$$

*Then, the function $F : \lambda \mapsto \int_X f(x, \lambda)dx$ is derivable on $\Lambda$ and*

$$F'(\lambda) = \int_X \frac{\partial}{\partial \lambda} f(x, \lambda)dx.$$

## Integration by substitution

**Theorem B.10:** Change of variables in $\mathbb{R}^N$
*Let $U$ and $V$ be open spaces of $\mathbb{R}^N$ and $\phi_U \to V$ be a diffeomorphism of class $C^1$. We note $J_\phi$ the determinant of the Jacobian of $\phi$. If $f : V \to \mathbb{R}$ is integrable, then $(f \circ \phi)|J_\phi| : U \to \mathbb{R}$ is integrable and*

$$\int_V f(v)dv = \int_U f(\phi(u))|J_\phi|(u)du.$$

# List of Figures

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