econ 202A; problem set 4; second half

1.1.a. Consider,

$$\lim_{\sigma \to 1} u(c) = \lim_{\sigma \to 1} \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}.$$

We see $\frac{\lim_{\sigma \to 1} (c^{1-\frac{1}{\sigma}} - 1)}{\lim_{\sigma \to 1} (1 - \frac{1}{\sigma})} = \frac{0}{0}$. Thus, we can apply L'hopital's rule to obtain,

$$\lim_{\sigma \to 1} \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} = \frac{\lim_{\sigma \to 1} [c^{1-\frac{1}{\sigma}} \ln c]}{\lim_{\sigma \to 1} [\sigma^{-2}]} = \frac{c^0 \ln c}{1} = \ln c,$$

as was to be shown. $\square$

1.1.b. Taking the first, second, & third derivative of $u(\cdot)$ w.r.t. c gives,

i. $u'(c) = \frac{(1 - \frac{1}{\sigma}) c^{-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}} = c^{-\frac{1}{\sigma}}$

ii. $u''(c) = -\frac{1}{\sigma} c^{-\frac{1}{\sigma} - 1}$

iii. $u'''(c) = \frac{1}{\sigma}(\frac{1}{\sigma} + 1) c^{-\frac{1}{\sigma} - 2}$. Then,

$$\underline{\text{coefficient of relative prudence}} = -\frac{(\frac{1}{\sigma}(\frac{1}{\sigma}+1) c^{-\frac{1}{\sigma}-2}) c}{-\frac{1}{\sigma} c^{-\frac{1}{\sigma}-1}}$$

$$= \underline{\frac{1}{\sigma} + 1}.$$

The coefficient of relative prudence is related to the coefficient of relative risk aversion (RRA), where $RRA = -\frac{u''(c) c}{u'(c)}$. While the RRA is a measure of aversion to risk, the coefficient of relative prudence captures an agent's willingness to save in response to future uncertainty.

1.2.a. We can write the maximization problem as follows,

$$\max_{c_0} \{u(c_0) + \beta \mathbb{E}[u(c_1)]\} \text{ s.t. } c_0 + i_0 = y_0 \ ; \ c_1 = (1+r)i_0 + \mathbb{E}[y_1].$$
$$\Leftrightarrow c_1 = (1+r)(y_0 - c_0) + \mathbb{E}[y_1]$$

Then, the F.O.C. w.r.t. c_0

$$u'(c_0) + \beta \mathbb{E}[u'(c_1) \cdot \tfrac{dc_1}{dc_0}] = 0$$

$\Leftrightarrow u'(c_0) = -\beta \mathbb{E}[u'(c_1) \cdot \frac{dc_1}{dc_0}]$. From $u(c_t) = \frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$,

$\underline{c_0^{-1/\sigma} = \beta(1+r)\mathbb{E}[c_1^{-1/\sigma}]}$, which is the consumption Euler equation.

1.2.b. As $\sigma \to \infty$, the Euler equation becomes,

$c_0^0 = \beta(1+r)\mathbb{E}[c_1^0] \Leftrightarrow 1 = \beta(1+r)$. The condition $\beta(1+r) < 1$ is necessary & sufficient for the agent to have positive consumption in period 0 & period 1. If $\beta(1+r) > 1$, the agent values future consumption more & thus saves all income in period 0, leading to 0 consumption. This can be seen from $\lim_{\sigma \to \infty} u(c_t) = c_t$ i.e., consumption in a given period is linear. With respect to question 1, we cannot recover an RRA or coefficient of relative prudence, since $u''(\cdot)$ & $u'''(\cdot)$ do not exist asymptotically. Namely, as $\sigma \to \infty$, the agent desires perfect consumption smoothing & is risk-neutral.

1.2.c. From the budget constraint, $c_1 = (1+r)b_0 + \mathbb{E}[y_1]$. With $y_1 \in \{y_L, y_H\}$, $c_1 = (1+r)b_0 + y_1$. There are two cases,

i. if $\underline{y_1 = y_H}$, $c_1 = (1+r)b_0 + y_H := c_H$

ii. if $\underline{y_1 = y_L}$, $c_1 = (1+r)b_0 + y_L := c_L$.

Since $y_H > y_L$, $c_H > c_L$.

1.3. a. Observe,

$$\frac{\partial c_0}{\partial(1+r)} = \frac{\partial}{\partial(1+r)}\left\{\left[1+\beta^\sigma(1+r)^{\sigma-1}\right]^{-1} y_0\right\}$$

$$= -\left[1+\beta^\sigma(1+r)^{\sigma-1}\right]^{-2}\left[(\sigma-1)\beta^\sigma(1+r)^{\sigma-2}\right] y_0$$

$$= -\frac{(\sigma-1)\beta^\sigma(1+r)^{\sigma-2}}{(1+\beta^\sigma(1+r)^{\sigma-1})^2}\, y_0$$

1.3. b. Consider two cases,

i. if $\sigma>1$, $\sigma-1>0$, $\beta^\sigma>0$, $y_0>0$, $\therefore (1+r)>0$. Then, $\frac{\partial c_0}{\partial(1+r)}<0$, indicating that a decrease in the real interest rate results in an increase in consumption

ii. if $\sigma\leq 1$, $\sigma-1\leq 0$ $\therefore$ $\frac{\partial c_0}{\partial(1+r)}\geq 0$, indicating a decrease in the real interest rate results in either no change or a decrease in consumption.

$\therefore$ period 0 consumption responds positively to a decrease in the real interest rate iff $\sigma>1$.

1.3. c. From the budget constraint, $c_1=(1+r)(y_0-c_0)$. Substituting for c_0 given in 1.3. a.,

$$c_1=(1+r)y_0\left(1-\frac{1}{1+\beta^\sigma(1+r)^{\sigma-1}}\right)$$

$$=(1+r)y_0\left(\frac{\cancel{1}+\beta^\sigma(1+r)^{\sigma-\cancel{1}}\cancel{-1}}{1+\beta^\sigma(1+r)^{\sigma-1}}\right)$$

$$\Leftrightarrow c_1=\left[\frac{\beta^\sigma(1+r)^\sigma}{1+\beta^\sigma(1+r)^{\sigma-1}}\right]y_0 \;\square$$

1.3.d. Observe,

$$\frac{\partial c_1}{\partial(1+r)} = \left\{ \frac{(1+\beta^\sigma(1+r)^{\sigma-1})(\sigma\beta^\sigma(1+r)^{\sigma-1}) - \beta^\sigma(1+r)^\sigma(\sigma-1)\beta^\sigma(1+r)^{\sigma-2}}{\underbrace{(1+\beta^\sigma(1+r)^{\sigma-1})^2}_{:=D}} \right\} y_0$$

$$= \left\{ \frac{\sigma\beta^\sigma(1+r)^{\sigma-1} + \sigma\beta^{2\sigma}(1+r)^{2(\sigma-1)} - (\sigma-1)\beta^{2\sigma}(1+r)^{2(\sigma-1)}}{D} \right\} y_0$$

$$= \left\{ \frac{\sigma\beta^\sigma(1+r)^{\sigma-1} + \beta^{2\sigma}(1+r)^{2(\sigma-1)}}{D} \right\} y_0 .$$ Since $D>0$ & $\beta^{2\sigma}(1+r)^{2(\sigma-1)}>0$, if $\sigma>0$, this derivative is positive. $\square$

1.3.e. First, note that σ captures the willingness of the agent to substitute consumption between the two periods. Thus, σ modulates the substitution effect in period 0 (from a change in the real interest rate). If $\sigma<1$, $\frac{\partial c_0}{\partial(1+r)}>0$, suggesting the income effect of an increase in the real interest outweighs the substitution effect i.e., the ability of the agent to substitute c_0 for c_1. If $\sigma>1$, the converse is true. Since the agent decides on a consumption savings decision at time 0, c_1 adjusts mechanically, & thus is unaffected by σ i.e., the effect of σ is already internalized by period 1. Thus, under $\sigma>0$, c_1 is not affected by changes in $r+1$.

2.a. Such an economic problem may arise for e.g., individuals considering how to allocate retirement income, which could be fixed after a certain age. Some implicit assumptions in the construction of the model are that agents have perfect foresight & no uncertainty. Discounting future utility & a dynamic budget constraint seem economically sensible. Another implicit assumption is that there is no borrowing, which does not seem economically sensible, along with no uncertainty. The name "eat-the-pie" might refer to an analagous problem where an agent has one pie, & needs to allocate its consumption over multiple time periods, without the ability to produce more pie.

2.b. The Bellman is as stated since at each time t, the agent chooses c to maximize the sum of instantaneous utility & the discounted continuation value $\beta V(w')$. Note,

i. $c \in [0, w]$ since the agent's consumption today is necessarily nonnegative & cannot exceed wealth today w

ii. $w' = R(w-c)$ i.e., wealth tomorrow is leftover wealth today, $w-c$, times the gross intest rate R.

Thus, the Bellman equation takes the given form. There is no expectation operator on the continuation value due to there being no uncertainty. There is no t subscript on $V(w)$ since the parameters β & R are stationary.

2.c. Assume u is bounded, & consider $C(X)$ be the space of bounded functions $f: \mathbb{R} \to \mathbb{R}$. Define $B: C(X) \to C(X)$ as the given Bellman operator. Note that u must be bounded for f to be bounded, which is necessary for applying Blackwell's theorem.

i. monotonicity: Let $f, g \in C(X)$ s.t. $f(x) \leq g(x)$ $\forall x \in \mathbb{R}$. Observe,

$$(Bf)(W) = \sup_{c \in [0,W]} \{ u(c) + \beta f[R(W-c)] \}.$$ Since $f(x) \leq g(x)$ $\forall x \in \mathbb{R}$, $f[R(W-c)] \leq g[R(W-c)]$. So,

$$\leq \sup_{c \in [0,W]} \{ u(c) + \beta g[R(W-c)] \}$$

$= (Bg)(W)$ $\forall W \in \mathbb{R}$. Thus, B satisfies monotonicity.

ii. Let $f \in C(X)$ arbitrary, $a \geq 0$, & $W \in \mathbb{R}$

$$[B(f+a)](W) = \sup_{c \in [0,W]} \{ u(c) + \beta [f[R(W-c)] + a] \}$$

$$= \sup_{c \in [0,W]} \{ u(c) + \beta f[R(W-c)] + \beta a \}$$

$$= \sup_{c \in [0,W]} \{ u(c) + \beta f[R(W-c)] \} + \beta a$$

$= (Bf)(W) + \delta a$ for $\delta = \beta \in (0,1)$. Thus B satisfies discounting.

Putting i. & ii. together, by Blackwell's theorem, B is a contraction mapping, which implies $\exists$ fixed point of B that the iterative solutions will converge to asymptotically, where the rate of convergence of $B^n v_0 \geq -\ln(\beta)$. $\square$

2.d. The F.O.C. w.r.t. c is,

$$u'(c) = -\beta v'(R(w-c))$$

i. if $r \in (0,\infty)$ & $r \neq 1$,

$$c^{-r} = -\beta \frac{d}{dc}\left[\psi \frac{[R(w-c)]^{1-r}}{1-r}\right]$$

$$\Leftrightarrow \quad c^{-r} = -\beta\psi[R(w-c)]^{-r} \cdot -R$$

$$\Leftrightarrow \quad c = [\beta R \psi]^{-\frac{1}{r}} R(w-c)$$

$$\Leftrightarrow \quad c\left[1+(\beta R^{1-r}\psi)^{-\frac{1}{r}}\right] = [\beta R^{1-r}]^{-\frac{1}{r}} \psi^{-\frac{1}{r}} w$$

$$\Leftrightarrow \quad c = \frac{[\beta R^{1-r}]^{-\frac{1}{r}}}{[1+(\beta R^{1-r})^{-\frac{1}{r}}\psi^{-\frac{1}{r}}]} \psi^{-\frac{1}{r}} w$$

. From $\psi^{-\frac{1}{r}} = 1-(\beta R^{1-r})^{1/r}$,

$$\Leftrightarrow \quad c = \frac{[\beta R^{1-r}]^{-\frac{1}{r}}}{1+(\beta R^{1-r})^{-\frac{1}{r}}[1-(\beta R^{1-r})^{\frac{1}{r}}]} \psi^{-\frac{1}{r}} w$$

$$\Leftrightarrow \quad c = \frac{[\beta R^{1-r}]^{-\frac{1}{r}}}{\underbrace{\cancel{1}+(\beta R^{1-r})^{-\frac{1}{r}}\cancel{-1}}_{=1}} \psi^{-\frac{1}{r}} w$$

$$\Leftrightarrow \quad c = \psi^{-\frac{1}{r}} w$$

ii. if $\psi = 1$,

$$\frac{1}{c} = -\beta \frac{d}{dc}\left[\phi + \psi \ln[R(w-c)]\right] \Leftrightarrow \frac{1}{c} = \beta \cdot \frac{\psi}{\cancel{R}(w-c)} \cdot \cancel{R}$$

$$\Leftrightarrow c = \frac{1}{\beta\psi}(w-c) \Leftrightarrow c = \frac{w}{\beta\psi}\left[1+\frac{1}{\beta\psi}\right]^{-1} \Leftrightarrow c = \frac{w}{\beta\psi}\left[\frac{\beta\psi}{1+\beta\psi}\right]$$

$$\Leftrightarrow c = \frac{w}{1+\beta\psi}$$

. With $r=1$, $\psi^{-1} = 1-\beta$

$$\Leftrightarrow c = \frac{w}{1+\frac{\beta}{1-\beta}} \Leftrightarrow c = \frac{w}{\frac{1}{1-\beta}} = w(1-\beta)$$

$$\Leftrightarrow \underline{c = \psi^{-1} w = \psi^{-\frac{1}{r}} w}$$

Thus, $c = \Psi^{-\frac{1}{\gamma}} W \quad \forall \gamma$. Returning ~~to~~ the Bellman gives,

i. if $\underline{\gamma \in (0,\infty) \,\&\, \gamma \neq 1}$,

$$\Psi \frac{W^{1-\gamma}}{\cancel{1-\gamma}} = \frac{(\Psi^{-\frac{1}{\gamma}} W)^{1-\gamma}}{\cancel{1-\gamma}} + \beta \Psi \frac{[R(W - \Psi^{-\frac{1}{\gamma}} W)]^{1-\gamma}}{\cancel{1-\gamma}}$$

$$\Leftrightarrow \Psi \cancel{W^{1-\gamma}} = \Psi^{-\frac{1-\gamma}{\gamma}} \cancel{W^{1-\gamma}} + \beta \Psi \cancel{W^{1-\gamma}} R^{1-\gamma} (1 - \Psi^{-\frac{1}{\gamma}})^{1-\gamma}$$

$$\Leftrightarrow \Psi = \Psi^{-\frac{1-\gamma}{\gamma}} + \Psi (1 - \Psi^{-\frac{1}{\gamma}})^{\gamma} (1 - \Psi^{-\frac{1}{\gamma}})^{1-\gamma}$$

$$\Leftrightarrow \Psi = \Psi^{-\frac{1-\gamma}{\gamma}} + \Psi (1 - \Psi^{-\frac{1}{\gamma}})$$

$$\Leftrightarrow \Psi = \cancel{\Psi^{-\frac{1-\gamma}{\gamma}}} + \Psi - \cancel{\Psi^{\frac{\gamma-1}{\gamma}}}$$

$$\Leftrightarrow \Psi = \Psi. \checkmark$$

ii. if $\underline{\gamma = 1}$, $\quad W - c = W - (1-\beta) W$

$$\phi + \Psi \ln W = \ln[\Psi^{-1} W] + \beta[\phi + \Psi \ln[R(W - \Psi^{-1} W)]]$$

$$\Leftrightarrow \phi + \Psi \ln W = \ln[(1-\beta) W] + \beta[\phi + \Psi \ln[R \beta W]]$$

$$\Leftrightarrow \phi + \Psi \ln W = \ln[1-\beta] + \ln W + \beta\phi + \beta\Psi \ln R + \beta\Psi \ln\beta + \beta\Psi \ln W$$

$$\Leftrightarrow \phi + \Psi \ln W = \ln[1-\beta] + \underbrace{(1+\beta\Psi)}_{=\Psi} \ln W + \beta\phi + \beta\Psi \ln R + \beta\Psi \ln\beta$$

$$\Leftrightarrow \phi + \Psi \ln W = \ln[1-\beta] + \beta\phi + \beta\Psi \ln R + \beta\Psi \ln\beta + \Psi \ln W$$

$$\Rightarrow \underline{\phi = \frac{1}{1-\beta}\left[\ln[1-\beta] + \beta\Psi \ln R + \beta\Psi \ln\beta\right]} \checkmark$$

Thus, we confirmed the guess for the Bellman equation.

2.e. When $r=1$, the income & substitution effects from a change in the interest rate offset one another exactly such that optimal consumption in a given period is independent of the interest rate. $\square$

3.a. Since the household must pay back debt in the long run, it must be the case that the household cannot borrow more than the present value of their future income. Assuming there is a bound on future income, it must be that $\exists$ lower bound $\underline{a}^{nat}$ s.t. $a_t \geq \underline{a}^{nat}$. The present value of future income at time t is given by,

$\int_t^\infty e^{-\int_t^s r_u\,du} y_s\,ds$, where $e^{-\int_t^s r_u du}$ is the discount factor from t to s. Then, the natural borrowing limit is the negative of this present value of future income,

$$\underline{\underline{a}^{nat} = -\int_t^\infty e^{-\int_t^s r_u\,du} y_s\,ds}.$$

3.b. To derive the household's lifetime budget constraint, we first integrate the dynamic budget constraint over all possible time periods,

$$\int_0^\infty da_t\,dt = \int_0^\infty (r_t a_t + y_t - c_t)\,dt.$$

$\Leftrightarrow \lim_{t\to\infty} a_t - a_0 = \int_0^\infty (r_t a_t + y_t - c_t)\,dt$. Applying the transversality condition,

$$\lim_{t\to\infty} e^{-\int_0^t r_s ds} a_t = 0 \Rightarrow a_t \to 0,$$

& converting the RHS to present value gives,

$\Leftrightarrow 0 = a_0 + \int_0^\infty e^{-\int_0^t r_s ds}(y_t - c_t)\,dt$. Rearranging gives the lifetime budget constraint,

$$\underbrace{a_0 + \int_0^\infty e^{-\int_0^t r_s ds} y_t\,dt}_{= W = \omega(a_0, \{r_t\}, \{a_t\})} = \int_0^\infty e^{-\int_0^t r_s ds} c_t\,d_t.$$

3. c. Taking derivatives gives,

i. $\underline{W_{a_0}} = 1$; indicates a one-unit change in initial wealth changes lifetime wealth by one unit (in the same direction)

ii. $\underline{W_{r_t}} = \int_t^\infty \frac{\partial}{\partial r_t}\left[e^{-\int_0^s r_u du} y_s ds\right] = -\int_t^\infty e^{-\int_0^s r_u du} y_s ds$; captures the effect of a one unit change in interest rate at time t on lifetime wealth

iii. $\underline{W_{y_t}} = \int_t^\infty \frac{\partial}{y_t}\left[e^{-\int_0^s r_u du} y_s ds\right] = e^{-\int_0^t r_s ds}$; captures the effect of a one unit change in labor income at time t on lifetime wealth

3. d. The HJB is given by,

$$\rho V_t(a) = \max_c \left\{u(c) + \frac{dV}{dt}\right\}, \text{ where } \frac{dV}{dt} = V_a\left[\frac{da}{dt}\right]$$

$$\Leftrightarrow \rho V_t(a) = \max_c \left\{u(c) + [r_t a + y_t - c] V_a(a)\right\},$$ where the dependence on calendar time t captures the non-stationarity of y_t ; r_t.

3.e. The F.O.C of the HJB w.r.t. c is given by,

i. $u'(c) = V_a(a) \Rightarrow \frac{d}{dt} u'(c) = u_{c,t} \frac{dc}{dt} = \frac{\partial}{\partial t} V_a(a)$

$$= V_{aa}(a) \frac{da}{dt}$$

$$= V_{aa}(a) [r_t a + y_t - c]$$

Taking the derivative of the HJB w.r.t. a gives,

$\rho V_a(a) = V_{aa}(a)[r_t a + y_t - c] + V_a(a) r_t$. Using i.,

$\rho u_c = u_{c,t} \frac{dc_t}{dt} + u_c r_t$. Noting $u_{cc}(c_t)\dot{c}_t = \frac{du_c(c_t)}{dt}$,

$\Leftrightarrow \underline{\frac{du_{c,t}}{u_{c,t}} = (\rho - r_t)dt}$, as was to be shown.

3.f. Integrating both sides of the Euler equation w.r.t. time from s to t gives,

$$\int_s^t \frac{du_{c,u}}{u_{c,u}} du = \int_s^t (r_u - \rho) du$$

$$\Leftrightarrow \ln\left(\frac{u_{c,t}}{u_{c,s}}\right) = \int_s^t r_u du - \rho(t-s)$$

$$\Leftrightarrow \frac{u_{c,t}}{u_{c,s}} = e^{\int_s^t r_u du - \rho(t-s)}$$

$$\iff \frac{u_{c,t}}{u_{c,s}} = R_{s,t}^{-1} e^{-\rho(t-s)}$$

$$\iff u_{c,s} = R_{s,t}\, u_{c,t}\, e^{-\rho(t-s)}.$$ □

3.g. From part f, setting $s = 0$,

$u_{c,0} = R_{0,t}\, u_{c,t}\, e^{-\rho t}$. With CRRA utility, $u_c(c) = c^{-\gamma}$. So,

$$c_0^{-\gamma} = R_{0,t}\, c_t^{-\gamma} e^{-\rho t}$$

$$\iff \left(\frac{c_t}{c_0}\right)^{\gamma} = R_{0,t}\, e^{-\rho t}$$

$$\iff c_t = c_0 \left[R_{0,t}\, e^{-\rho t} \right]^{\frac{1}{\gamma}}.$$ This equation captures optimal consumption at time t as a function of initial consumption, the accumulated interest rate, & the discount factor on utility.

3.h. Combining parts b & g,

$$W = \int_0^\infty e^{-\int_0^t r_s \, ds} c_t \, dt$$

$$\Leftrightarrow W = \int_0^\infty \underbrace{e^{-\int_0^t r_s \, ds}}_{R_{0,t}^{-1}} c_0 \left[R_{0,t} e^{-\rho t} \right]^{\frac{1}{\gamma}} dt$$

$$\Leftrightarrow W = c_0 \int_0^\infty R_{0,t}^{\frac{1}{\gamma} - 1} e^{-\frac{\rho}{\gamma} t} dt$$

$$\Leftrightarrow \underline{W = c_0 \int_0^\infty R_{0,t}^{\frac{1-\gamma}{\gamma}} e^{-\frac{\rho}{\gamma} t} dt} .$$

This equation captures that present value lifetime wealth must be equal to the present value of future consumption, in terms of initial consumption c_0.

3.i. With a constant r, $R_{0,t} = e^{rt}$, $c_t = c_0 e^{\left(\frac{r-\rho}{\gamma}\right)t}$, &

$$W = c_0 \int_0^\infty e^{\left(\frac{1-\gamma}{\gamma}\right) r} e^{-\frac{\rho}{\gamma} t} \Leftrightarrow c_0 = W \left(\frac{\rho - r(1-\gamma)}{\gamma} \right).$$

Then,

$$mpc_{s,t} = \frac{\partial c_t}{\partial y_s} = \frac{\partial}{\partial y_s} \left[W \underbrace{e^{\left(\frac{r-\rho}{\gamma}\right)t} \left(\frac{\rho - r(1-\gamma)}{\gamma} \right)}_{C} \right]$$

$$= \frac{\partial W}{\partial y_s} \cdot C$$

$$\underline{= e^{-rs} e^{\left(\frac{r-\rho}{\gamma}\right)t} \left(\frac{\rho - r(1-\gamma)}{\gamma} \right).}$$

i. if $s < t$, the mpc decreases with time if $r < \rho$

ii. if $s > t$, the mpc increases with time.

4. a. The generator is given by,

$$\mathcal{A} v(a) = [ra - c] v'(a) + \frac{\sigma^2}{2} a^2 v''(a).$$

4. b. The HJB is given by,

$$\rho v(a) = \max_c \left\{ u(c) + \underbrace{\mathbb{E}\left[\frac{d v(a)}{dt} \right]}_{= \mathcal{A} v(a)} \right\}$$

$$\Leftrightarrow \rho v(a) = \max_c \left\{ u(c) + [ra - c] v'(a) + \frac{\sigma^2}{2} a^2 v''(a) \right\}.$$

There is no t subscript on $v(a)$, since all parameters do not depend explizitly on calendar time. This is a second order non-linear ODE. It is not a PDE, since it only involves derivatives of one variable, a.

4.c. Letting $u(c) = \log c$, $c(a) = \rho a$, & $v(a) = \frac{1}{\rho}\log(\rho a) + \frac{r-\rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}$, the HJB is,

$$\rho v(a) = \max_c \left\{ u(c) + v'(a)[ra - c] + \frac{\sigma^2}{2} a^2 v''(a) \right\}$$

$$\Leftrightarrow \rho v(a) = \log(\rho a) + \frac{1}{\rho a}[ra - \rho a] - \frac{\sigma^2}{2} \cancel{a^2} \frac{1}{\rho} \cancel{a^{-2}}$$

$$\Leftrightarrow \rho v(a) = \log(\rho a) + \frac{r-\rho}{\rho} - \frac{\sigma^2}{2\rho}$$

$$= \rho v(a). \quad \square$$

Regarding interpretation,

$$v(a) = \underbrace{\frac{1}{\rho}\log(\rho a)}_{\text{i.}} + \underbrace{\frac{r-\rho}{\rho^2}}_{\text{ii.}} - \underbrace{\frac{\sigma^2}{2\rho^2}}_{\text{iii.}}$$

i. captures the value from wealth, scaled by patience parameter ρ

ii. captures the additional value from the gap between interest rate & the patience parameter

iii. captures the penalty associated with the volatility of wealth, σ^2

The MPC is given by,

$$\text{mpc} = \frac{\partial c(a)}{\partial a} = \rho, \text{ from } c(a) = \rho a.$$

5.a. Letting $u(C_t) = \frac{1}{1-\gamma} C_t^{1-\gamma}$; substituting into the Euler equation,

$$C_t^{-\gamma} = e^{-\rho}\mathbb{E}\left[R_{t+1}^j \, C_{t+1}^{-\gamma}\right]$$

$$\Leftrightarrow \quad 1 = e^{-\rho}\mathbb{E}\left[R_{t+1}^j \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]. \text{ Note, } \frac{C_{t+1}}{C_t} = e^{\ln C_{t+1} - \ln C_t} = e^{\Delta \ln C_{t+1}}$$

$$\Leftrightarrow \quad 1 = e^{-\rho}\mathbb{E}\left[e^{\ln R_{t+1}^j - \gamma \Delta \ln C_{t+1}}\right]. \text{ From } r_{t+1}^j = \ln R_{t+1}^j,$$

$$\Leftrightarrow \quad 1 = \mathbb{E}\left[e^{r_{t+1}^j - \rho - \gamma \Delta \ln C_{t+1}}\right]. \;\square$$

This equation must hold $\forall j$, as the consumer optimizes their choice of portfolio across all assets to maximize lifetime utility. If this equation did not hold for some j, there would be some degree of arbitrage such that the household could obtain higher utility.

5.b. Noting $R_{t+1}^j = e^{r_{t+1}^j + \sigma^j \varepsilon_{t+1}^j - \frac{1}{2}(\sigma^j)^2}$, the result from a becomes,

$$1 = \mathbb{E}\left[e^{\overbrace{r_{t+1}^j + \sigma^j \varepsilon_{t+1}^j - \frac{1}{2}(\sigma^j)^2 - \rho - \gamma \Delta \ln C_{t+1}}^{= X_t}}\right]$$

$$\Leftrightarrow \quad 1 = \mathbb{E}\left[e^{X_t}\right]. \;\square$$

5. c. Taking logs of both sides of b gives,

$$1 = \mathbb{E}[e^{X_t}]$$

$\Leftrightarrow$ $0 = \mathbb{E}[X_t]$. From the distribution of X_t given in b,

$$\Leftrightarrow \; 0 = -\rho + r^j_{t+1} - \tfrac{1}{2}(\sigma^j)^2 - \gamma \underbrace{\mu_{c,t}}_{=\mathbb{E}_t[\Delta \ln C_{t+1}]} + \tfrac{1}{2}\underbrace{\left[(\sigma^j)^2 + \gamma^2\sigma^2_{c,t} - 2\rho_{j,c}\gamma\sigma^j\sigma_{c,t}\right]}_{=\mathrm{Var}_t(\sigma^j\varepsilon_{t+1} - \gamma\Delta \ln C_{t+1})}$$

$$\Leftrightarrow \; 0 = -\rho + r^j_{t+1} - \tfrac{1}{2}(\sigma^j)^2 - \gamma\,\mathbb{E}_t[\Delta \ln C_{t+1}] + \tfrac{1}{2}\mathrm{Var}_t(\sigma^j\varepsilon_{t+1} - \gamma\Delta \ln C_{t+1})$$

5. d. Letting $\sigma^f = 0$ & $j = f$, the equation from c becomes,

$$0 = -\rho + r^f_{t+1} - \gamma\,\mathbb{E}_t[\Delta \ln C_{t+1}] + \frac{\gamma^2}{2}\mathrm{Var}_t(\Delta \ln C_{t+1})$$

$$\Leftrightarrow \; \underline{r^f_{t+1} = \rho + \gamma\mu_{c,t} - \frac{\gamma^2}{2}\sigma^2_{c,t}}\,.$$

5.e. From r^E_{t+1} given by the result in c & r^f_{t+1} given by the equation r^f_{t+1},

$$\pi^E_{t+1} = r^E_{t+1} - r^f_{t+1}$$

$$= \gamma \underbrace{\sigma^E \sigma_C \rho_{E,C}}_{= Cov(\Delta \ln C_{t+1},\, r^E_{t+1}) = \sigma_{C,E}}$$

$$= \underline{\gamma \sigma_{C,E}}.$$

6. optional