

1. We can rewrite the given expression in standard form,

$$\frac{\dot{C}(t)}{C(t)} = \frac{r_t - \rho}{\sigma} \Leftrightarrow \dot{C}(t) = \left[ \frac{r_t - \rho}{\sigma} \right] C(t) \Leftrightarrow \dot{C}(t) - \left( \frac{r_t - \rho}{\sigma} \right) C(t) = 0. (*)$$

$$\text{Let } \mu(t) = e^{-\int \frac{r_t - \rho}{\sigma} dt} = e^{-\frac{1}{\sigma} \int (r_0 + \alpha t - \rho) dt} = e^{-\frac{1}{\sigma} [(r_0 - \rho)t + \frac{\alpha}{2} t^2]}$$

Multiplying (\*) by  $\mu(t)$  gives,

$$\mu(t) \dot{C}(t) - \mu(t) C(t) = 0 \mu(t)$$

$\Rightarrow \int \frac{d}{dt} [\mu(t) C(t)] dt = 0 dt$ . Letting  $K$  denote the constant of integration,

$$\Rightarrow \mu(t) C(t) = K$$

$\Rightarrow e^{-\frac{1}{\sigma} [(r_0 - \rho)t + \frac{\alpha}{2} t^2]} C(t) = K$ . Using  $C(T) = C_T$  at  $T=T$ ,

$\Rightarrow K = C_T e^{-\frac{1}{\sigma} [(r_0 - \rho)T + \frac{\alpha}{2} T^2]}$ . The final solution is thus,

$$C(t) = K e^{\frac{1}{\sigma} [(r_0 - \rho)t + \frac{\alpha}{2} t^2]} = C_T e^{\frac{1}{\sigma} [(r_0 - \rho)t + \frac{\alpha}{2} t^2] - \frac{1}{\sigma} [(r_0 - \rho)T + \frac{\alpha}{2} T^2]}$$

$$\Rightarrow C(t) = C_T e^{\frac{1}{\sigma} [(r_0 - \rho)(t - T) + \frac{\alpha}{2} (t^2 - T^2)]}$$

$$\Rightarrow C(t) = \underline{C_T e^{\frac{1}{\sigma} [(r_0 - \rho)(t - T) + \frac{\alpha}{2} (t^2 - T^2)]}}$$

2. Given  $\frac{\dot{C}(t)}{C(t)} = \frac{r_t - \rho}{\theta}$ , we use the backwards difference method where  $\dot{C}(t) = \frac{C(t) - C(t-\Delta t)}{\Delta t}$ , which gives,

$$\frac{C(t) - C(t-\Delta t)}{\Delta t} \cdot \frac{1}{C(t)} = \frac{r_t - \rho}{\theta}$$

$$\Leftrightarrow C(t) - C(t-\Delta t) = \frac{\Delta t(r_t - \rho)}{\theta} C(t)$$

$$\Leftrightarrow C(t-\Delta t) = \underline{\left( \theta - \frac{\Delta t(r_t - \rho)}{\theta} \right) C(t)}$$

4. The numerical solution curve is entirely below the analytic solution curve in the 100 time-step case. For the 10 time-step case, they are mostly overlapping. This difference could be due to the finite difference method accumulating errors over a longer time step.

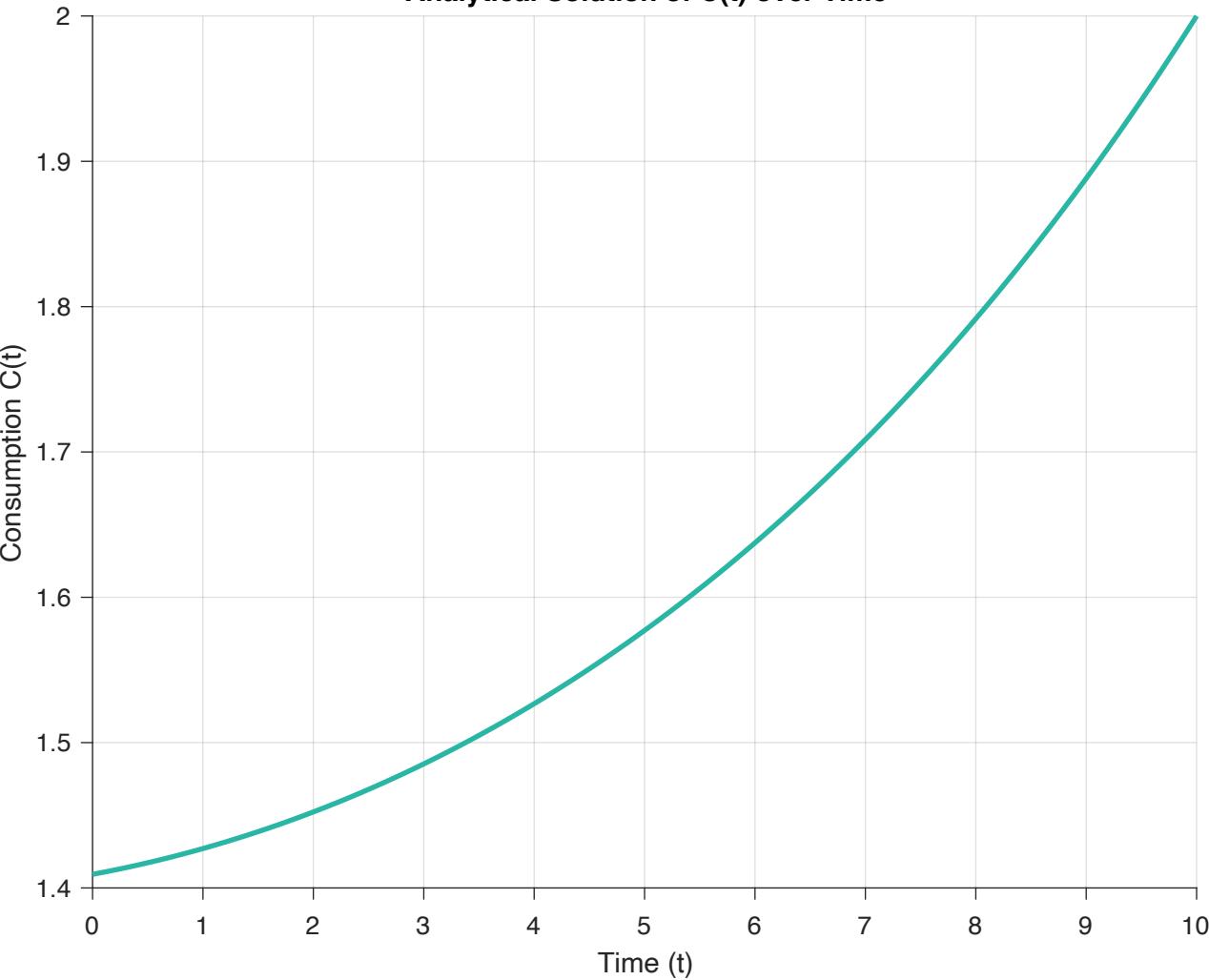
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~Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/problem_set_1_code.m

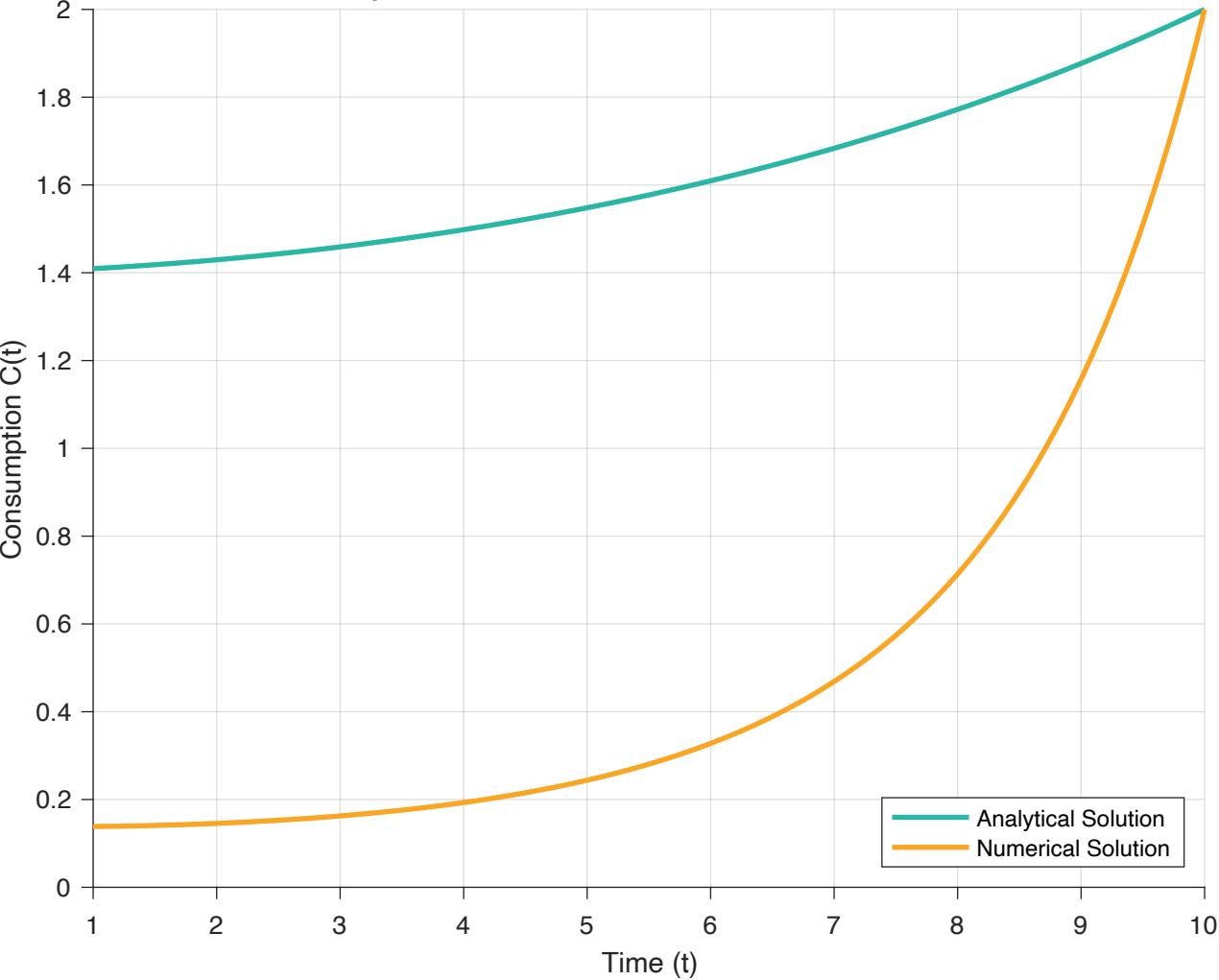
1 %
2 -----
3 Purpose: Coding part of problem set 1
4 Created: Nico Rotundo 2024-10-28
5 -----
6 %}
7 %
8 %
9 -----
10 Define parameters
11 -----
12 %}
13 %
14 % Terminal consumption at time T
15 C_T = 2.0;
16 %
17 % Initial interest rate
18 r_0 = 0.05;
19 %
20 % Growth rate of interest rate
21 alpha = 0.01;
22 %
23 %
24 % Interest rate function as a handle function of time t
25 function r = interest_rate_function(t, r_0, alpha)
26   r = r_0 + alpha * t;
27 end
28 %
29 %
30 % Inverse of the intertemporal elasticity of substitution
31 theta = 2.0;
32 %
33 % Time discount factor
34 rho = 0.03;
35 %
36 % Terminal time
37 T = 10;
38 %
39 % Display the result of interest rate function at specific time to check
40 %fprintf('The interest rate at t = %d is: %.3f\n', 20, interest_rate_function(20, r_0, alpha));
41 %
42 % Checking how assert statements work in this matlab
43 %assert(interest_rate_function(20, r_0, alpha) == .25);
44 %
45 %
46 %
47 -----
48 1. Solve the equation analytically using the integrating factor method. Then, plot the analytical solution as a function
49   of time t and consumption C(t)
50 %
51   - solved equation:  $C(t) = C_T \exp\left[\frac{1}{\theta}((r_0 - \rho)(t - T) + \frac{\alpha}{2}(t^2 - T^2))\right]$ 
52 -----
53 %
54 %
55 % Time vector from 0 to T with 100 points
56 t = linspace(0, T, 100);
57 %
58 % Define value for consumption at time t
59 C_t_analytical = C_T * exp((1 / theta) * ((r_0 - rho) * (t - T) + (alpha / 2) * (t.^2 - T.^2)));
60 %
61 % Generate figure for analytical solution
62 figure;
63 plot(t, C_t_analytical, 'LineWidth', 2, 'Color', [41/255, 182/255, 164/255]);
64 xlabel('Time (t)');
65 set(gca, 'TickDir', 'out');
66 box off;
67 ylabel('Consumption C(t)');
68 title('Analytical Solution of C(t) over Time');
69 grid on;
70 %
71 % Export
72 exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/output/question_1.pdf', 'ContentType', 'vector');
73 %
74 %
75 -----
76 2. Solve the equation numerically using the finite difference method with 100 time steps. Plot the numerical and analytical
77   solutions together in a single figure for comparison.
78 %
79   - equation:  $C(t-dt) = \frac{\theta}{\theta - dt(r_0 - \rho)} C(t)$ 
80 %
81 %
82 % Time step (\triangle t)
83 dt = 0.1;
84 %
85 % Number of time steps
86 N = T / dt;
87 %
88 % Vector to store consumption values
89 C_t_numeric = zeros(1, N);
90 %
91 % Set terminal consumption
92 C_t_numeric(end) = C_T;
93 %
94 % Time vector from 0 to T with 100 points
95 t = linspace(1, T, N);
96 %
97 % Time stepping using the finite difference method
98 for n = N:-1:2
99 %
100   % Compute consumption at the next time step using backward difference
101   C_t_numeric(n-1) = (theta - dt*(r_0 + alpha*n - rho))/theta * C_t_numeric(n);
102 %
103 end
104 %
105 % Generate figure for numeric solution
106 figure;
107 plot(t, C_t_numeric, 'LineWidth', 2, 'Color', [250/255, 165/255, 35/255]);
108 xlabel('Time (t)');
109 set(gca, 'TickDir', 'out');
110 box off;
111 ylabel('Consumption C(t)');
112 title('Numerical Solution of C(t) over Time');
113 grid on;
114 %
115 % Generate both analytical and numeric sultions on the same plot
116 figure;
117 %
118   % Plot analytical solution
119   plot(t, C_t_analytical, 'LineWidth', 2, 'Color', [41/255, 182/255, 164/255]);
120 %
121   hold on;
122 %
123   % Plot numerical solution
124   plot(t, C_t_numeric, 'LineWidth', 2, 'Color', [250/255, 165/255, 35/255]);
125 %
126   % Add labels and title
127   xlabel('Time (t)');
128   ylabel('Consumption C(t)');
129   title('Analytical vs Numerical Solutions of C(t) over Time');
130 %
131   % Add a legend
132   legend({'Analytical Solution', 'Numerical Solution'}, 'Location', 'southeast');
133 %
134   % Adjust plot settings
135   set(gca, 'TickDir', 'out');
136   box off;
137   grid on;
138 %
139   hold off;
140 %
141 % Export
142 exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/output/question_2.pdf', 'ContentType', 'vector');
143 %
144 %
145 -----
146 3. Repeat the numerical solution using 10 time steps. Plot the numerical and analytical solutions together in a single
147   figure.
148 %
149 %
150 %
151 % Time step (\triangle t)
152 dt = 1;
153 %
154 % Number of time steps
155 N = T / dt;
156 %
157 % Vector to store consumption values
158 C_t_numeric = zeros(1, N);
159 %
160 % Set terminal consumption
161 C_t_numeric(end) = C_T;
162 %
163 % Time vector from 0 to T with 100 points
164 t_10 = linspace(1, T, N);
165 %
166 % Time stepping using the finite difference method
167 for n = N:-1:2
168 %
169   % Compute consumption at the next time step using backward difference
170   C_t_numeric(n-1) = (theta - dt*(r_0 + alpha*n - rho))/theta * C_t_numeric(n);
171 %
172 end
173 %
174 % Generate both analytical and numeric sultions on the same plot
175 figure;
176 %
177   % Plot analytical solution
178   plot(t, C_t_analytical, 'LineWidth', 2, 'Color', [41/255, 182/255, 164/255]);
179 %
180   hold on;
181 %
182   % Plot numerical solution
183   plot(t_10, C_t_numeric, 'LineWidth', 2, 'Color', [250/255, 165/255, 35/255]);
184 %
185   % Add labels and title
186   xlabel('Time (t)');
187   ylabel('Consumption C(t)');
188   title('Analytical vs Numerical Solutions of C(t) over Time');
189 %
190   % Add a legend
191   legend({'Analytical Solution', 'Numerical Solution'}, 'Location', 'southeast');
192 %
193   % Adjust plot settings
194   set(gca, 'TickDir', 'out');
195   box off;
196   grid on;
197 %
198   hold off;
199 %
200 % Export
201 exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/output/question_3.pdf', 'ContentType', 'vector');
202

```

## Analytical Solution of $C(t)$ over Time



## Analytical vs Numerical Solutions of $C(t)$ over Time



## Analytical vs Numerical Solutions of $C(t)$ over Time

