econ 2024; problem set 1

1. a. Given $P = \begin{array}{c|cc} & U & E \\ \hline U & \alpha & 1-\alpha \\ E & 1-\beta & \beta \end{array}$, the economic interpretation of α & β are as follows,

i. $\underline{\alpha = P_{11}}$ is the probability an individual who is unemployed in a given quarter remains unemployed in the next quarter

ii. $\underline{\beta = P_{22}}$ is the probability an individual who is employed in a given quarter remains employed in the next quarter

b. The rows of P sum to one since they constitute a probability measure (note that this also requires $\alpha, \beta \in [0,1]$, but this is not given). In other words, since there are two states in the state space, & time is discrete, the probabilities of transitioning to either state must sum to one.

c. If $\alpha = 1$, then state U (unemployed) is an absorbing state. Likewise, if $\beta = 1$, then state E (employed) is an absorbing state. However, this model does not necessarily have an absorbing state.

d. Note,

$$P^2 = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}\begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} = \begin{array}{c} \\ U \\ E \end{array}\begin{array}{cc} U & E \\ \end{array}\begin{bmatrix} \alpha^2 + (1-\alpha)(1-\beta) & \alpha(1-\alpha) + (1-\alpha)\beta \\ (1-\beta)\alpha + \beta(1-\beta) & (1-\beta)(1-\alpha) + \beta^2 \end{bmatrix}.$$

Then, $\underline{\mathbb{P}(U_{t+2} \mid E_t) = [P^2]_{EU} = (1-\beta)\alpha + \beta(1-\beta).}$

e. We can express the marginal probability of being in state y^L at time $t+1$ as,

$\psi_{t+1}(y^L) = \mathbb{P}(y_{t+1} = y^L)$. Since the state space Y is mutually exclusive & exhaustive, the Law of Total Probability gives,

$= \sum_{s} \mathbb{P}(y_{t+1} = y^L, y_t = s)$, $s \in Y = \{y^U, y^E\}$. Since $\{y_i\}$ follows a Markov chain,

$= \sum_{s} \mathbb{P}(y_{t+1} = y^L \mid y_t = s)\, \mathbb{P}(y_t = s)$.

i. If $y^L = y^U$, we have,

$\psi_{t+1}(y^U) = \mathbb{P}(y_{t+1} = y^U \mid y_t = y^U)\, \mathbb{P}(y_t = y^U) + \mathbb{P}(y_{t+1} = y^U \mid y_t = y^E)\, \mathbb{P}(y_t = y^E)$.

Using the transition matrix P, we can express this as,

$\psi_{t+1}(y^U) = \alpha\, \mathbb{P}(y_t = y^U) + (1-\beta)\, \mathbb{P}(y_t = y^E) = \alpha\, \psi_t(y^U) + (1-\beta)\, \psi_t(y^E)$

ii. If $y^L = y^E$, we similarly get,

$\psi_{t+1}(y^E) = (1-\alpha)\, \psi_t(y^U) + \beta\, \psi_t(y^E)$

f. Observe, $\psi_{t+1}(y^U) = \begin{bmatrix} \psi_t(y^U) & \psi_t(y^E) \end{bmatrix} \begin{bmatrix} \alpha \\ 1-\beta \end{bmatrix}$ &

$\psi_{t+1}(y^E) = \begin{bmatrix} \psi_t(y^U) & \psi_t(y^E) \end{bmatrix} \begin{bmatrix} 1-\alpha \\ \beta \end{bmatrix}$.

Then, $\begin{bmatrix} \psi_{t+1}(y^U) & \psi_{t+1}(y^E) \end{bmatrix} = \begin{bmatrix} \psi_t(y^U) & \psi_t(y^E) \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$

$\iff \psi_{t+1} = \psi_t P$. $\square$

g. We proceed with a proof by induction. Let $\{X_i\}_{i=1}^{t+1}$ be a random vector corresponding to the state of the stochastic process at time i.

<u>base step:</u> We are given $X_0 \sim \psi_0$. From part f, the marginal distribution of X_1 is given by, $\psi_1 = \psi_0 P$. As such, $X_1 \sim \psi_0 P$.

<u>inductive step:</u> Suppose the given property holds $\forall i \in \{1, \ldots, t-1\}$. As such, $X_{t-1} \sim \psi_0 P^{t-1} \iff \psi_{t-1} = \psi_0 P^{t-1}$. From part f,

$\psi_t = \psi_{t-1} P$. By the inductive assumption,

$$= [\psi_0 P^{t-1}] P$$

$$\iff \psi_t = \psi_0 P^t \implies X_t \sim \psi_0 P^t.$$

Taken collectively, the base case & inductive step gives,

$X_0 \sim \psi_0 \implies X_t \sim \psi_0 P^t$. $\square$

h. Let time t denote the initial quarter where the individual is employed. WLOG, since $\{y_i\}$ follows a Markov process, let $t=0$. By part g., we can express the marginal distribution of y_i at time $t+n=n$ as,

$\psi_n = \psi_0 P^n$, where we are given $\psi_0 = (0 \ 1)$. Evaluating,

$$= [0 \ 1]\begin{bmatrix} P_{11}^n & P_{12}^n \\ P_{21}^n & P_{22}^n \end{bmatrix} = [0 \cdot P_{11}^n + 1 \cdot P_{21}^n \quad 0P_{12}^n + 1 \cdot P_{22}^n]$$

$= [P_{21}^n \ P_{22}^n]$. Thus, the probability of being unemployed n quarters after being employed is $\psi_n(y^u) = P_{21}^n$.

To find a closed form expression for P_{21}^n, we diagonalize P s.t. $P = Q\Lambda Q^{-1}$ where $Q :=$ eigenvector matrix ; $\Lambda :=$ diagonal matrix of eigenvalues.

The eigenvalues for P satisfy,

$$\det(P - \lambda I) = 0 \iff \det\begin{bmatrix} \alpha-\lambda & 1-\alpha \\ 1-\beta & \beta-\lambda \end{bmatrix} = 0$$

$$\iff (\alpha-\lambda)(\beta-\lambda) - (1-\alpha)(1-\beta) = 0$$

$$\iff \alpha\beta - \lambda\alpha - \lambda\beta + \lambda^2 - [1-\beta-\alpha+\alpha\beta] = 0$$

$$\iff \cancel{\alpha\beta} - \lambda\alpha - \lambda\beta + \lambda^2 - 1 + \beta + \alpha - \cancel{\alpha\beta} = 0$$

$$\iff \lambda^2 - \lambda\alpha - \lambda\beta + \beta + \alpha - 1 = 0$$

$$\iff \lambda^2 - (\alpha+\beta)\lambda + (\beta+\alpha-1) = 0. \text{ Then,}$$

$$\lambda = \frac{\alpha+\beta \pm \sqrt{(\alpha+\beta)^2 - 4(\beta+\alpha-1)}}{2} = \frac{\alpha+\beta \pm \sqrt{\alpha^2+2\alpha\beta+\beta^2-4\beta-4\alpha+4}}{2}$$

$$= \frac{\alpha+\beta \pm \sqrt{(\alpha^2-2\alpha+1)+2\alpha\beta+\beta^2-4\beta-2\alpha+3}}{2} = \frac{\alpha+\beta \pm \sqrt{(\alpha^2-2\alpha+1)+(\beta^2-2\beta+1)+2(\alpha-1)(\beta-1)}}{2}$$

$$= \frac{\alpha+\beta \pm \sqrt{(\alpha+\beta-2)^2}}{2}. \text{ Then, } \underline{\lambda_1} = \frac{\alpha+\beta-\alpha-\beta+2}{2} = \frac{2}{2} = \underline{1}$$

$$\underline{\lambda_2} = \frac{\alpha+\beta+\alpha+\beta-2}{2} = \frac{2[\alpha+\beta-1]}{2} = \underline{\alpha+\beta-1}$$

λ_1 ; λ_2 are distinct if $\alpha \neq 1$ or $\beta \neq 1$, ; thus P will be diagonalizable.

The eigenvectors associated with each of λ_1 ÷ λ_2 are,

$\underline{\lambda_1 = 1:}$ $(P - \lambda_1 I)\vec{v}_1 = \vec{0} \iff \left\{ \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} v_1^{(1)} \\ v_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\iff \begin{bmatrix} \alpha-1 & 1-\alpha \\ 1-\beta & \beta-1 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} (\alpha-1)v_1^{(1)} + (1-\alpha)v_1^{(2)} = 0 \\ (1-\beta)v_1^{(1)} + (\beta-1)v_1^{(2)} = 0 \end{array}$$

$\iff v_1^{(1)} = v_1^{(2)}$ for $\alpha \neq 1$, $\beta \neq 1$. Let $\underline{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

$\underline{\lambda_2 = \alpha + \beta - 1:}$ $(P - \lambda_2 I)\vec{v}_2 = \vec{0} \iff \begin{bmatrix} \alpha-\alpha-\beta+1 & 1-\alpha \\ 1-\beta & \beta-\alpha-\beta+1 \end{bmatrix} \begin{bmatrix} v_2^{(1)} \\ v_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\iff \begin{bmatrix} 1-\beta & 1-\alpha \\ 1-\beta & 1-\alpha \end{bmatrix} \begin{bmatrix} v_2^{(1)} \\ v_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} (1-\beta)v_2^{(1)} + (1-\alpha)v_2^{(2)} = 0 \\ (1-\beta)v_2^{(1)} + (1-\alpha)v_2^{(2)} = 0 \end{array}$$

$$\iff v_2^{(2)} = -\frac{1-\beta}{1-\alpha} v_2^{(1)}, \alpha \neq 1 \implies (1-\beta)v_2^{(1)} + (1-\alpha)\left[-\frac{1-\beta}{1-\alpha} v_2^{(1)}\right] = 0$$

$$\iff (1-\beta)v_2^{(1)} = (1-\beta)v_2^{(1)} \implies \text{choose } v_2^{(1)} = 1-\alpha$$

$$\implies v_2^{(2)} = \beta - 1.$$

Then, $\underline{\vec{v}_2 = \begin{bmatrix} 1-\alpha \\ \beta-1 \end{bmatrix}}$.

Thus, $Q = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1-\alpha \\ 1 & \beta-1 \end{bmatrix}$. The inverse is,

$Q^{-1} = \frac{1}{\beta+\alpha-2} \begin{bmatrix} \beta-1 & \alpha-1 \\ -1 & 1 \end{bmatrix}$. We proceed with finding P_{21}^n,

$$P^n = Q\Lambda^n Q^{-1} = \begin{bmatrix} 1 & 1-\alpha \\ 1 & \beta-1 \end{bmatrix} \begin{bmatrix} (1)^n & 0 \\ 0 & (\alpha+\beta-1)^n \end{bmatrix} Q^{-1}$$

$$= \begin{bmatrix} 1\cdot 1^n + (1-\alpha)(0) & 1\cdot 0 + (1-\alpha)(\alpha+\beta-1)^n \\ 1\cdot 1^n + (\beta-1)(0) & 1\cdot 0 + (\beta-1)(\alpha+\beta-1)^n \end{bmatrix} Q^{-1}$$

$$= \frac{1}{\alpha+\beta-2} \begin{bmatrix} 1 & (1-\alpha)(\alpha+\beta-1)^n \\ 1 & (\beta-1)(\alpha+\beta-1)^n \end{bmatrix} \begin{bmatrix} \beta-1 & \alpha-1 \\ -1 & 1 \end{bmatrix}$$

$$\implies P_{21}^n = \frac{(\beta-1) - (\beta-1)(\alpha+\beta-1)^n}{\alpha+\beta-2} = \frac{(\beta-1)\left[1-(\alpha+\beta-1)^n\right]}{\alpha+\beta-2}$$

As such, the probability of being unemployed n quarters after being employed is, $\psi_n(y^U) = P^n_{21} = \frac{(\beta-1)[1-(\alpha+\beta-1)^n]}{\alpha+\beta-2}$.

i. As $n\to\infty$, $\lim_{n\to\infty} P^n = \begin{bmatrix} 1 & 1-\alpha \\ 1 & \beta-1 \end{bmatrix} \begin{bmatrix} (1)^n & 0 \\ 0 & (\alpha+\beta-1)^n \end{bmatrix} Q^{-1}$.

Under the assumption α or $\beta < 1$,

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 1 & 1-\alpha \\ 1 & \beta-1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

$$= \frac{1}{\alpha+\beta-2} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta-1 & \alpha-1 \\ -1 & 1 \end{bmatrix}$$

$$\Leftrightarrow \lim_{n\to\infty} P^n = \frac{1}{\alpha+\beta-2} \begin{bmatrix} \beta-1 & \alpha-1 \\ \beta-1 & \alpha-1 \end{bmatrix}.$$ Thus,

$\psi^* = \begin{bmatrix} \frac{\beta-1}{\alpha+\beta-2} & \frac{\alpha-1}{\alpha+\beta-2} \end{bmatrix}$ gives the stationary distribution of the markov chain. To verify,

$$\psi^* P = \begin{bmatrix} \frac{\beta-1}{\alpha+\beta-2} & \frac{\alpha-1}{\alpha+\beta-2} \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha(\beta-1)+(\alpha-1)(1-\beta)}{\alpha+\beta-2} & \frac{(\beta-1)(1-\alpha)+(\alpha-1)\beta}{\alpha+\beta-2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cancel{\alpha\beta}-\cancel{\alpha}+\cancel{\alpha}-\cancel{\alpha\beta}-1+\beta}{\alpha-\beta-2} & \frac{\cancel{\beta}-\cancel{\alpha\beta}-1+\alpha+\cancel{\alpha\beta}-\cancel{\beta}}{\alpha+\beta-2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\beta-1}{\alpha-\beta-2} & \frac{\alpha-1}{\alpha+\beta-2} \end{bmatrix} = \psi^*.$$

ii. The stationary distribution ψ^* satisfies,

$$\psi^* = \psi^* P = \psi^* \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}.$$

$$\Leftrightarrow \begin{bmatrix} \psi_1^* & \psi_2^* \end{bmatrix} = \begin{bmatrix} \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \psi_1^* & \psi_2^* \end{bmatrix} = \begin{bmatrix} \psi_1^*\alpha + \psi_2^*(1-\beta) & \psi_1^*(1-\alpha) + \psi_2^*\beta \end{bmatrix}.$$

This gives a system of 3 linear equations,

a. $\psi_1^*\alpha + \psi_2^*(1-\beta) = \psi_1^*$

c. $\psi_1^* + \psi_2^* = 1$, from definition of probability measure

b. $\psi_1^*(1-\alpha) + \psi_2^*\beta = \psi_2^*$

$$\Rightarrow \psi_1^*\alpha + (1-\psi_1^*)(1-\beta) = \psi_1^* \Leftrightarrow \psi_1^*\alpha + 1 - \beta - \psi_1^*(1-\beta) = \psi_1^*$$

$$\Leftrightarrow (1-\beta) = \psi_1^*[1 + 1 - \beta - \alpha] \Leftrightarrow \underline{\frac{\beta-1}{\alpha+\beta-2} = \psi_1^*}$$

$$\Rightarrow (1-\psi_2^*)(1-\alpha) + \psi_2^*\beta = \psi_2^* \Leftrightarrow (1-\alpha) = \psi_2^*[1-\beta+1-\alpha]$$

$$\Leftrightarrow \underline{\frac{\alpha-1}{\alpha+\beta-2} = \psi_2^*}$$

As such, $\psi^* = \begin{bmatrix} \frac{\beta-1}{\alpha+\beta-2} & \frac{\alpha-1}{\alpha+\beta-2} \end{bmatrix}.$

i. By the Law of Iterated expectations,

$$\mathbb{E}_0[y^t] = \mathbb{E}_{y_0}\left[\mathbb{E}_{0|y_0}[y^t \mid y_0]\right]. \text{ Since } y_0 = y^H,$$

$$= \mathbb{E}_0[\mathbb{E}_{0|y_0}[y^t \mid y_0 = y^H]]. \text{ By definition of } \mathbb{E}_{0|y_0},$$

$$\mathbb{E}_{0|y_0}[y^t \mid y_0 = y^H] = \sum_s s\,\mathbb{P}(y^t = s \mid y_0 = y^H),\ s \in \{y^U, y^E\}.$$

$$\Rightarrow \mathbb{E}_0[y^t] = \mathbb{E}_{y_0}\left[\sum_s s \cdot \mathbb{P}(y^t = s \mid y_0 = y^H)\right]$$

$$= \sum_{y_0} \mathbb{P}(y_0 = y^H) \cdot \left\{\sum_s s \cdot \mathbb{P}(y^t = s \mid y_0 = y^H)\right\}.$$

There are two cases,

i. <u>if $y_0 = y^H = y^U$:</u>

$$\mathbb{E}_0[y^t] = y^U\,\mathbb{P}(y^t = y^U \mid y_0 = y^U) + y^E\,\mathbb{P}(y^t = y^E \mid y_0 = y^U)$$

We know from above that the conditional probabilities can be rewritten,

$$\mathbb{E}_0[y^t] = y^U p_{11}^t + y^E p_{12}^t,$$

ii. <u>if $y_0 = y^H = y^E$:</u> Similarly, $E_0[y^t] = y^U p_{21}^t + y^E p_{22}^t$

Taken together,

$$\mathbb{E}_0[y^t] = \begin{cases} y^U p_{11}^t + y^E p_{12}^t & \text{if } y_0 = y^U \\ y^U p_{21}^t + y^E p_{22}^t & \text{if } y_0 = y^E \end{cases}$$

2. a. $\boxed{\text{WTS: } \lim_{n\to\infty} (B^n f)(x) = \frac{h(x)}{1-\alpha}}$ Let $f = w$.

i. Consider $(Bw)(x) \equiv h(x) + \alpha w(x)$, $\alpha \in (0,1)$. Iterating,

$$(Bw)(x) = h(x) + \alpha w(x)$$

$$(B(Bw))(x) = h(x) + \alpha(h(x) + \alpha w(x)) = h(x) + \alpha h(x) + \alpha^2 w(x)$$

$$(B(B^2 w))(x) = h(x) + \alpha h(x) + \alpha^2 h(x) + \alpha^3 w(x)$$

$$\vdots$$

$$(B(B^{n-1}w))(x) = h(x)\sum_{i=0}^{n-1} \alpha^i + \alpha^n w(x).$$ Then,

$$\lim_{n\to\infty} (B^n w)(x) = \lim_{n\to\infty}\left[h(x)\sum_{i=0}^{n-1}\alpha^i + \alpha^n w(x)\right]$$

$$= h(x)\sum_{i=0}^{\infty}\alpha^i + 0$$ since $\alpha \in (0,1)$. The sum of the geometric sequence $\sum_{i=0}^{\infty}\alpha^i = \frac{1}{1-\alpha}$. Thus,

$$\lim_{n\to\infty} (B^n w)(x) = \frac{h(x)}{1-\alpha}. \quad \square$$

ii. A fixed point for B satisfies $B(v(x)) = v(x)$, where $v(x)$ is the fixed point. Let $v(x) = \frac{h(x)}{1-\alpha}$.

Then, $B(v(x)) = h(x) + \alpha\left[\frac{h(x)}{1-\alpha}\right]$

$$= \frac{(1-\alpha)h(x) + \alpha h(x)}{1-\alpha} = \frac{h(x)}{(1-\alpha)}.$$

Thus, $v(x) = \frac{h(x)}{1-\alpha}$ is a fixed point for B. $\square$

3\. a. Let $d(f,g) = \sup_{x\in\mathbb{R}^L} |f(x) - g(x)|$. Let $x \in \mathbb{R}^L$. By definition,

$-d(f,g) \le f(x) - g(x) \le d(f,g) \Rightarrow f(x) \le g(x) + d(f,g) \ \forall x.$ □

b. Let $f, g \in C(X)$ arbitrary & $d(f,g) = \sup_{x\in\mathbb{R}^L} |f(x) - g(x)| \ge 0$.

i. From part a, $f(x) \le g(x) + d(f,g)$. By monotonicity,

$B(f(x)) \le B(g(x) + d(f,g))$. By discounting, $\exists \delta \in (0,1)$ s.t.,

$\le B(g(x)) + \delta d(f,g)$. As such,

$(Bf)(x) \le (Bg)(X) + \delta d(f,g)$.

ii. From part a, $g(x) \le f(x) + d(f,g)$. By monotonicity,

$(Bg)(x) \le B(f(x) + d(f,g))$. By discounting,

$\exists \delta \in (0,1)$ s.t. $\le (Bf)(x) + \delta d(f,g)$. As such,

$(Bg)(x) \le (Bf)(x) + \delta d(f,g)$.

Therefore, $\forall f, g \in C(X)$, $\exists \delta \in (0,1)$ s.t. $(Bf)(x) \le (Bg)(x) + \delta d(f,g)$ & $(Bg)(x) \le (Bf)(x) + \delta d(f,g)$. □

c. Consider $d(Bf, Bg) = \sup_{x \in \mathbb{R}^L} |(Bf)(x) - (Bg)(x)|$. Let $x \in \mathbb{R}^L$.

From part b, $\exists \delta \in (0,1)$ s.t.

$$(Bf)(x) \leq (Bg)(x) + \delta d(f,g) \iff (Bf)(x) - (Bg)(x) \leq \delta d(f,g)$$

Likewise, $(Bg)(x) - (Bf)(x) \leq \delta d(f,g)$

$$\iff (Bf)(x) - (Bg)(x) \geq -\delta d(f,g).$$

Putting the above ~~together~~,

$$-\delta d(f,g) \leq (Bf)(x) - (Bg)(x) \leq \delta d(f,g) \quad \forall x$$

$$\iff \sup_{x \in \mathbb{R}^L} |(Bf)(x) - (Bg)(x)| \leq \delta d(f,g)$$

$$\iff d(Bf, Bg) \leq \delta d(f,g). \square$$

4.a. Given $(\mathcal{B}f)(x) = \sup_{c \in [0,x]} \{u(c) + \delta \mathbb{E}[f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})]\} \; \forall x$,

i. $\mathcal{B}$:= Bellman operator, which operates on the function f, mapping it to a new function $\mathcal{B}f$

ii. f := value function, which gives the value attained by the objective function at its solution

iii. $\sup_{c \in [0,x]}$:= denotes we are maximizing total utility with respect to consumption, c

a. $c \in [0,x]$:= a constraint that indicates consumption can take any nonnegative value up to the amount x, which ~~likely~~ denotes the total wealth, which the DM can allocate to consumption in a given period; the liquidity constraint is implicit here since the agent cannot borrowing beyond available wealth

iv. $u(c)$:= instantaneous utility function, which gives utility from consuming c units in the current period

v. $\delta \mathbb{E}[f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})]$:= discounted expected utility from the next period

a. $\delta \in (0,1)$:= discount factor, which indicates how much the DM values future utility relative to current utility

b. $\mathbb{E}$:= expectation, which reflects the fact $\tilde{R}_{+1}$ & $\tilde{y}_{+1}$ are random variables

c. $\tilde{R}_{+1}(x-c)$:= random var. capturing the returns on assets from the current period to the next, where $x-c$ is the leftover liquidity of the agent after consuming c units in the current period

d. $\tilde{y}_{+1}$:= random variable capturing labor income in the next period

e. $f(\cdot)$:= value function evaluated at wealth available in the next period

b. The current period budget constraint is $x = s + c$, where s denotes savings. Wealth in the next period is,

$\tilde{x}_{+1} = \tilde{R}_{+1}(x-c) + \tilde{y}_{+1} \geq c_{+1}$.

c. Let $f, g \in C(X)$ with $f(x) \leq g(x)$ $\forall x \in X$. By definition,

$(Bf)(x) = \sup_{c\in[0,x]} \{u(c) + \delta \mathbb{E}[f(\tilde{R}_{+1}(x-c)+\tilde{y}_{+1})]\}$ $\forall x$. Since $f(x) \leq g(x)$ $\forall x$,

$\leq \sup_{c\in[0,x]} \{u(c) + \delta \mathbb{E}[g(\tilde{R}_{+1}(x-c)+\tilde{y}_{+1})]\}$

$= (Bg)(x)$. Thus, monotonicity is satisfied.

d. Let $f \in C(X)$, $a \geq 0$, $x \in X$ arbitrary. Consider

$[B(f+a)](x) = \sup_{c\in[0,x]} \{u(c) + \delta \mathbb{E}[f(\tilde{R}_{+1}(x-c)+\tilde{y}_{+1})+a]\}$.

Since $\mathbb{E}[a] = a$, $= \sup_{c\in[0,x]} \{u(c) + \delta \mathbb{E}[f(\tilde{R}_{+1}(x-c)+\tilde{y}_{+1})] + \delta a\}$

$$= \sup_{c\in[0,x]} \{u(c) + \delta \mathbb{E}[f(\tilde{R}_{+1}(x-c)+\tilde{y}_{+1})]\} + \delta a$$

$$= (Bf)(x) + \delta a.$$

As such, discounting is satisfied. □

5. a. Consider the guess $V(k) = \psi + \phi \ln k$. Then,

$$V(k) = \max_{k' \in \Gamma(k)} \{ \ln(k^\alpha - k') + \beta V(k') \}$$

$$= \max_{k' \in \Gamma(k)} \{ \ln(k^\alpha - k') + \beta [\psi + \phi \ln k'] \}.$$ The first-order condition w.r.t. k' is,

$$\frac{\partial}{\partial k'} \{ \ln(k^\alpha - k') \} + \frac{\partial}{\partial k'} \{ \beta [\psi + \phi \ln k'] \} = 0$$

$$\iff -\frac{1}{k^\alpha - k'} + \frac{\beta\phi}{k'} = 0$$

$$\iff \frac{\beta\phi}{k'} = \frac{1}{k^\alpha - k'} \iff \beta[\phi k^\alpha - \phi k'] = k' \iff \underline{\frac{\beta\phi k^\alpha}{1+\beta\phi} = k'}.$$

Separately, the Envelope theorem gives,

$$\frac{\partial u(k^\alpha - k')}{\partial k} = \frac{\partial V(k)}{\partial k} \iff \frac{\alpha k^{\alpha-1}}{k^\alpha - k'} = \frac{\phi}{k} \iff \alpha k^\alpha = \phi k^\alpha - \phi k'$$

$$\iff \underline{\frac{(\phi - \alpha) k^\alpha}{\phi}} = k'.$$

Putting these together gives,

$$\frac{\beta\phi k^\alpha}{1+\beta\phi} = \underline{\frac{(\phi-\alpha)}{\phi}} k^\alpha \iff \frac{\beta\phi}{1+\beta\phi} = \frac{\phi - \alpha}{\phi}$$

$$\iff \beta\phi^2 = \phi - \alpha + \beta\phi^2 - \beta\phi\alpha \iff \alpha = \phi(1-\beta\alpha) \iff \underline{\phi = \frac{\alpha}{1-\beta\alpha}}.$$

Returning to the F.O.C,

$$k'(k) = \frac{\beta\phi k^\alpha}{1+\beta\phi} = \frac{\beta\left[\frac{\alpha}{1-\beta\alpha}\right] k^\alpha}{1+\beta\left[\frac{\alpha}{1-\beta\alpha}\right]} = \frac{\beta\alpha k^\alpha}{1 \cancel{-\beta\alpha} \cancel{+\beta\alpha}}$$

$$\iff \underline{k'(k) = \beta\alpha k^\alpha}.$$

Returning to the Bellman equation, we have,

$$V(k) = \max_{k' \in \Gamma(x)} \{ \ln(k^\alpha - k') + \beta V(k') \}.$$ Substituting $V(k)$,

$$\Leftrightarrow \psi + \phi \ln(k) = \max_{k' \in \Gamma(x)} \{ \ln(k^\alpha - k') + \beta[\psi + \phi \ln(k')] \}.$$ Substituting $k'(k)$,

$$\Leftrightarrow \psi + \phi \ln(k) = \ln(k^\alpha - \beta\alpha k^\alpha) + \beta\psi + \beta\phi \ln[\beta\alpha k^\alpha].$$

$$\Leftrightarrow \psi + \phi \ln k = \alpha \ln(k) + \ln(1-\beta\alpha) + \beta\psi + \beta\phi[\ln(\beta\alpha) + \alpha \ln(k)]$$

$$\Leftrightarrow \psi + \phi \ln k = \ln(1-\beta\alpha) + \beta\psi + \beta\phi \ln(\beta\alpha) + (\beta\phi+1)\alpha \ln(k)$$

$$\Leftrightarrow \psi - \beta\psi = \ln(1-\beta\alpha) + \beta\phi \ln(\beta\alpha) + (\beta\phi+1)\alpha \ln(k) - \phi \ln(k)$$

$$\Leftrightarrow \psi = \frac{1}{1-\beta}\left[\ln(1-\beta\alpha) + \beta\phi \ln(\beta\alpha) + [(\beta\phi+1)\alpha - \phi]\ln(k)\right]$$

$$= \frac{1}{1-\beta}\left[\ln(1-\beta\alpha) + \beta\phi \ln(\beta\alpha) + [(\beta\alpha-1)\phi + \alpha]\ln(k)\right]$$

$$= \frac{1}{1-\beta}\left[\ln(1-\beta\alpha) + \beta\phi \ln(\beta\alpha) + \underbrace{\cancel{\left[\alpha - \frac{(1-\beta\alpha)\alpha}{(1-\beta\alpha)}\right]}}_{\to 0}\ln(k)\right]$$

$$\Leftrightarrow \psi = \frac{1}{1-\beta}\left[\ln(1-\beta\alpha) + \frac{\beta\alpha}{1-\beta\alpha}\ln(\beta\alpha)\right]. \quad (*)$$

As such, the functional equation is satisfied $\forall k_0 \in \Gamma(k)$ for $\psi = (*)$. $\square$

b. We proceed with iterating the policy function,

$$k_1 = \alpha\beta k_0^{\alpha}, \quad k_2 = \alpha\beta k_1^{\alpha} = \alpha\beta[\alpha\beta k_0^{\alpha}]^{\alpha} = (\alpha\beta)^{1+\alpha} k_0^{\alpha^2},$$

$$k_3 = \alpha\beta k_2^{\alpha} = \alpha\beta\left[(\alpha\beta)^{1+\alpha} k_0^{\alpha^2}\right]^{\alpha} = (\alpha\beta)^{1+\alpha+\alpha^2} k_0^{\alpha^3}$$

$\Rightarrow k_t^* = (\alpha\beta)^{\sum_{i=0}^{t-1}\alpha^i} k_0^{\alpha^t}$. Notice that $\sum_{i=0}^{t-1} a^i = \frac{1-\alpha^t}{1-\alpha}$,

$\Leftrightarrow k_t^* = (\alpha\beta)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t}$. Returning to,

$V(k_0) = \sum_{t=0}^{\infty} \beta^t \ln[(k_t^*)^{\alpha} - k_{t+1}^*]$ and substituting gives,

$$= \sum_{t=0}^{\infty} \beta^t \ln\left[(\alpha\beta)^{\frac{\alpha(1-\alpha^t)}{1-\alpha}} k_0^{\alpha^{t+1}} - (\alpha\beta)^{\frac{1-\alpha^{t+1}}{1-\alpha}} k_0^{\alpha^{t+1}}\right]$$

$$= \sum_{t=0}^{\infty} \beta^t \left\{\alpha^{t+1} \ln(k_0) + \frac{-\alpha^{t+1}}{1-\alpha} \ln(\alpha\beta) + \ln\left[(\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}\right]\right\}$$

$$= \sum_{t=0}^{\infty} (\beta\alpha)^t \alpha \left\{\ln(k_0) + \frac{-1}{1-\alpha} \ln(\alpha\beta)\right\} + \sum_{t=0}^{\infty} \beta^t \ln\left[(\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}\right]$$

$$= \frac{\alpha}{1-\alpha\beta}\left[\ln(k_0) - \frac{1}{1-\alpha}\ln(\alpha\beta)\right] + \frac{1}{1-\beta} \ln\left[(\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}\right]$$

$$= \frac{\alpha}{1-\alpha\beta}\ln(k_0) - \frac{\alpha}{(1-\alpha\beta)(1-\alpha)}\ln(\alpha\beta) + \frac{1}{1-\beta}\ln\left[(\alpha\beta)^{\frac{1}{1-\alpha}}\left[(\alpha\beta)^{\frac{\alpha-1}{1-\alpha}} - 1\right]\right]$$

$$= \frac{\alpha}{1-\alpha\beta}\ln(k_0) - \frac{\alpha\ln(\alpha\beta)}{(1-\alpha\beta)(1-\alpha)} + \frac{1}{(1-\alpha)(1-\beta)}\ln(\alpha\beta) \quad \frac{1}{1-\beta}\ln\left[\frac{1}{\alpha\beta} - 1\right]$$

$$= \frac{\alpha}{1-\alpha\beta}\ln(k_0) + \frac{-\alpha(1-\beta) + 1-\alpha\beta}{(1-\alpha\beta)(1-\alpha)(1-\beta)}\ln(\alpha\beta) + \frac{1}{1-\beta}\left\{\ln[1-\alpha\beta] - \ln(\alpha\beta)\right\}$$

$$= \frac{\alpha}{1-\alpha\beta}\ln(k_0) + \frac{1}{1-\beta}\left[\frac{1}{1-\alpha\beta}\ln(\alpha\beta) - \ln(\alpha\beta) + \ln[1-\alpha\beta]\right]$$

$$= \frac{\alpha}{1-\alpha\beta}\ln(k_0) + \frac{1}{1-\beta}\left\{\ln(\alpha\beta)\left[\frac{\alpha\beta}{1-\alpha\beta}\right] + \ln[1-\alpha\beta]\right\}$$

$= V(k_0)$ from our guess. Thus, the policy function is optimal. $\square$

C. We have,

$$(B\hat{V})(k) = \sup_{k' \in \Gamma(k)} \left\{ \ln(k^\alpha - k') + \beta\left[\frac{\alpha \ln k'}{1-\alpha\beta}\right]\right\}.$$

<u>F.O.C:</u> $\frac{1}{k^\alpha - k'} = \frac{\beta\alpha}{1-\alpha\beta} \cdot \frac{1}{k'} \iff (1-\alpha\beta)k' = \alpha\beta(k^\alpha - k')$

$\iff k' = \alpha\beta k^\alpha$. ✓

Then,
$$\begin{aligned}(B\hat{V})(k) &= \ln(k^\alpha - \alpha\beta k^\alpha) + \beta\left[\frac{\alpha \ln(\alpha\beta k^\alpha)}{1-\alpha\beta}\right] \\
&= \alpha\ln(k) + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\alpha\ln(k) \\
&= \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \left[1 + \frac{\alpha\beta}{1-\alpha\beta}\right]\alpha\ln(k)\end{aligned}$$

$$\begin{aligned}(B(B\hat{V}))(k) &= \ln(k^\alpha - \alpha\beta k^\alpha) + \beta\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \left[1+\frac{\alpha\beta}{1-\alpha\beta}\right]\alpha\ln(k')\right] \\
&= \alpha\ln(k) + \ln(1-\alpha\beta) + \beta\ln(1-\alpha\beta) + \frac{\alpha\beta^2}{1-\alpha\beta}\ln(\alpha\beta) + \left[\beta + \frac{\alpha\beta^2}{1-\alpha\beta}\right]\alpha\ln(\alpha\beta k^\alpha) \\
&= (1+\beta)\ln(1-\alpha\beta) + \frac{\alpha\beta^2}{1-\alpha\beta}\ln(\alpha\beta) + \left[\alpha\beta + \frac{\alpha^2\beta^2}{1-\alpha\beta}\right]\left[\alpha\ln(k) + \ln(\alpha\beta)\right] \\
&= (1+\beta)\ln(1-\alpha\beta) + \left[\alpha\beta + \frac{\alpha\beta^2}{1-\alpha\beta} + \frac{\alpha^2\beta^2}{1-\alpha\beta}\right]\ln(\alpha\beta) + \left[1 + \alpha\beta + \frac{\alpha^2\beta^2}{1-\alpha\beta}\right]\alpha\ln(k) \\
&= (1+\beta)\ln(1-\alpha\beta) + \left[1+\beta\right]\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \left[\frac{1 \cancel{-\alpha\beta} \cancel{+\alpha\beta} - \alpha^2\beta^2 + \alpha^2\beta^2}{1-\alpha\beta}\right]\alpha\ln(k) \\
&= (1+\beta)\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right] + \frac{\alpha\ln(k)}{1-\alpha\beta}\end{aligned}$$

$\vdots$

$$(B^n V)(k) = \sum_{i=1}^{n-1}\beta^i\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right] + \frac{\alpha\ln(k)}{1-\alpha\beta}.$$

Notice, $\sum_{i=0}^{n-1}\beta^i = \frac{1-\beta^n}{1-\beta}$

$$= \underline{\frac{1-\beta^n}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right] + \frac{\alpha\ln(k)}{1-\alpha\beta}}.$$

Then, $\lim_{n\to\infty}(B^n V)(k) = \frac{1-0}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right] + \frac{\alpha\ln(k)}{1-\alpha\beta}$, since $0<\beta<1$. As such,

$$\lim_{n\to\infty}(B^n V)(k) = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right] + \frac{\alpha\ln(k)}{1-\alpha\beta}$$

$= V(k)$ from part a. $\square$

6. a. Given $V(x) = \max_{x' \in [0,x]} \left\{ u(x-x') + \mathbb{E}\left[e^{-\rho} V\left(e^{r+\sigma u - \frac{\sigma^2}{2}} x' \right) \right] \right\}$,

i. $V(\cdot)$:= value function capturing the maximum utility with respect to wealth in the next period x'

ii. $\max_{x' \in [0,x]}$:= indicates we are maximizing the interior function w.r.t. x'

a. $x' \in [0,x]$ reflects choice of wealth in period $t+1$ cannot exceed wealth in period t, and must also be nonnegative

iii. $u(x-x')$:= instantaneous utility function capturing utility from consuming $c = x - x'$ units today i.e., wealth today less choice of wealth tomorrow; not a state variable since it only affects $V(\cdot)$ through current period decisions

iv. $\mathbb{E}\left[e^{-\rho} V\left(e^{r+\sigma u - \frac{\sigma^2}{2}} x' \right) \right]$:= collectively captures discounted value of consumption in the next period $t+1$ relative to period t

a. $e^{-\rho}$:= discount rate w.r.t. to impatience parameter ρ

b. $V(\cdot)$:= value function evaluated at wealth in the next period

c. $e^{r+\sigma u - \frac{\sigma^2}{2}} x'$:= captures the return on assets, which is a random variable

x. r := captures return on assets for period t

xx. σu := captures effect of i.i.d. random shock u, modulated by parameter σ, on returns in period t

xxx. $-\frac{\sigma^2}{2}$:= scaling parameter ensuring effects of schocks are unbiased

b. Given $V(x) = \max_{x' \in [0,x]} \left\{ u(x-x') + \mathbb{E}\left[e^{-\rho} V\left(e^{r+\sigma u - \frac{\sigma^2}{2}} x'\right)\right]\right\}$,

the Envelope theorem gives,

$$\frac{\partial V(x)}{\partial x} = \frac{\partial u(x-x')}{\partial x} \iff \frac{\partial}{\partial x}\left[\phi \frac{x^{1-r}}{1-r}\right] = \frac{\partial}{\partial x}\left[\frac{1}{1-r}(x-x')^{1-r}\right]$$

$$\iff \phi x^{-r} = (x-x')^{-r} \iff \phi^{-\frac{1}{r}} x = x - x' \iff \underline{c = \phi^{-\frac{1}{r}} x}$$

$$\Rightarrow x - x' = \phi^{-\frac{1}{r}} x \iff \underline{x' = x(1-\phi^{-\frac{1}{r}})}.$$

Returning to the Bellman equation,

$$V(x) = \max_{x' \in [0,x]} \left\{ u(x-x') + \mathbb{E}\left[e^{-\rho} V\left(e^{r+\sigma u - \frac{\sigma^2}{2}} x'\right)\right]\right\} \text{ ;}$$

substituting the guess for $V(x)$ gives,

$$\phi \frac{x^{1-r}}{1-r} = \max_{x' [0,x]} \left\{ u(x-x') + \mathbb{E}\left[e^{-\rho}\left[\phi \frac{(e^{r+\sigma u - \frac{\sigma^2}{2}} x')^{1-r}}{1-r}\right]\right]\right\}$$

Substituting for optimal x' ; c gives,

$$\phi \frac{x^{1-r}}{\cancel{1-r}} = \frac{1}{\cancel{1-r}}(\phi^{-\frac{1}{r}} x)^{1-r} + \mathbb{E}\left[e^{-\rho}\left[\phi \frac{(e^{r+\sigma u - \frac{\sigma^2}{2}} x(1-\phi^{-\frac{1}{r}}))^{1-r}}{\cancel{1-r}}\right]\right]$$

$$\iff \phi x^{1-r} = \phi^{-\frac{1-r}{r}} x^{1-r} + x^{(1-r)}(1-\phi^{-\frac{1}{r}})^{(1-r)} e^{-\rho}\phi e^{(1-r)r} e^{(r-1)\frac{\sigma^2}{2}} \mathbb{E}\left[e^{\sigma(1-r)u}\right].$$

Since $u \sim N(0,1)$, the MGF of u gives,

$$\phi x^{1-r} = \phi^{-\frac{1-r}{r}} x^{1-r} + x^{(1-r)}(1-\phi^{-\frac{1}{r}})^{(1-r)} e^{-\rho}\phi e^{(1-r)r} e^{(r-1)\frac{\sigma^2}{2}} e^{\frac{((1-r)\sigma)^2}{2}}. \text{ With } x>0,$$

$$\iff (\phi - \phi^{-\frac{1-r}{r}}) = (1-\phi^{-\frac{1}{r}})^{1-r} e^{-\rho}\phi e^{(1-r)(r-\frac{\sigma^2}{2})} e^{\frac{(1-r)^2\sigma^2}{2}}$$

$$\iff \cancel{\phi}(1-\phi^{(\frac{r-1}{r})-1}) = (1-\phi^{-\frac{1}{r}})^{1-r} e^{-\rho}\cancel{\phi} e^{(1-r)(r-\frac{\sigma^2}{2})} e^{\frac{(1-r)^2\sigma^2}{2}}$$

$$\iff (1-\phi^{-\frac{1}{r}}) = (1-\phi^{-\frac{1}{r}})^{(1-r)} e^{-\rho} e^{(1-r)(r-\frac{\sigma^2}{2})} e^{\frac{(1-r)^2\sigma^2}{2}}$$

$$\iff (1-\phi^{-\frac{1}{r}})^{r} = e^{-\rho} e^{(1-r)(r-\frac{\sigma^2}{2})} e^{\frac{(1-r)^2\sigma^2}{2}}$$

$$\Leftrightarrow \gamma \ln(1-\phi^{-\frac{1}{\gamma}}) = -\rho + (1-\gamma)(r-\frac{\sigma^2}{2}) + \frac{(1-\gamma)^2\sigma^2}{2}$$

$$\ln(1-\phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma}\left[(1-\gamma)r-\rho + (\gamma-1+(1-\gamma)^2)\frac{\sigma^2}{2}\right]$$

$$= \frac{1}{\gamma}\left[(1-\gamma)r-\rho+(\gamma^2-\gamma)\frac{\sigma^2}{2}\right.$$

$$\Leftrightarrow \ln(1-\phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma}[(1-\gamma)r-\rho] + (\gamma-1)\frac{\sigma^2}{2}.$$ As such, the Bellman equation is satisfied for a particular value of ϕ. $\square$

c. Consider $\mathbb{E}[\ln(\frac{c_{t+1}}{c_t})] = \mathbb{E}\left\{\ln\left[\frac{\cancel{\phi^{-\frac{1}{\gamma}}}x'}{\cancel{\phi^{-\frac{1}{\gamma}}}x}\right]\right\} = \mathbb{E}\left\{\ln\left[\frac{x'}{x}\right]\right\}$

From $x' = e^{r+\sigma u_t - \frac{\sigma^2}{2}}(x-c)$

$$\Leftrightarrow = \mathbb{E}\left\{\ln\left[\frac{e^{r+\sigma u-\frac{\sigma^2}{2}}\cancel{x}(1-\phi^{-\frac{1}{\gamma}})}{\cancel{x}}\right]\right\} = \ln(1-\phi^{-\frac{1}{\gamma}}) + \mathbb{E}\left[r+\sigma u - \frac{\sigma^2}{2}\right].$$ Since $\mathbb{E}[u]=0$,

$$= \ln(1-\phi^{-\frac{1}{\gamma}}) + r - \frac{\sigma^2}{2}$$. From part b,

$$= \frac{1}{\gamma}[(1-\gamma)r-\rho] + \frac{1}{2}(\gamma-1)\sigma^2 + r - \frac{\sigma^2}{2}$$

$$= \frac{1-\gamma}{\gamma}r - \frac{\rho}{\gamma} + \frac{\gamma\sigma^2}{2} - \frac{\sigma^2}{2} + r - \frac{\sigma^2}{2}$$

$$= \left(\frac{1-\gamma}{\gamma}+1\right)r - \frac{\rho}{\gamma} + \frac{\gamma}{2}\sigma^2 - \sigma^2$$

$$\boxed{\frac{1-\gamma+\gamma}{\gamma} = \frac{1}{\gamma}}$$

$$= \frac{1}{\gamma}[r-\rho] + \frac{\gamma}{2}\sigma^2 - \sigma^2. \ \square$$

d. Given $\sigma^2 = 0$, $\mathbb{E}[\ln(\frac{c_{t+1}}{c_t})] = \frac{1}{\gamma}(r-\rho)$. In words, the expected value of the growth rate of consumption in the $\sigma^2 = 0$ case is $\frac{1}{\gamma}(r-\rho)$.

i. the expected value of the growth rate of consumption is increasing in r, since a higher return on the asset would encourage less consumption in the current period i.e., a lower $\ln(c_t)$ value, which increases $\ln(c_{t+1}) - \ln(c_t)$. Conversely, if ρ increases, the consumer places more value on consumption in the current period, hence a larger $\ln(c_t)$ value $\Rightarrow$ a lower growth rate.

ii-iii. γ in the denominator reflects the fact that if a consumer is particularly risk averse (high γ), they consume more in the current period, leading to a lower growth rate. To illustrate, $\lim_{\gamma\to\infty} \mathbb{E}[\ln(c_{t+1}) - \ln(c_t)] = 0$ indicates that if the consumer is infinitely risk averse, they consume their entire endowment in the current period leading to 0 consumption growth. As such, this parameter regulates the consumer's willingness to intertemporally substitute consumption.