1. We can rewrite the given expression in standard form,

$$\frac{\dot{C}(t)}{C(t)} = \frac{r_t - \rho}{\theta} \iff \dot{C}(t) = \left[\frac{r_t - \rho}{\theta}\right] C(t) \iff \dot{C}(t) - \left(\frac{r_t - \rho}{\theta}\right) C(t) = 0. \quad (*)$$

Let $\mu(t) = e^{-\int \frac{r_t - \rho}{\theta} dt} = e^{-\frac{1}{\theta}\int (r_0 + \alpha t - \rho) dt} = e^{-\frac{1}{\theta}\left[(r_0 - \rho)t + \frac{\alpha}{2}t^2\right]}$.

Multiplying (*) by $\mu(t)$ gives,

$$\mu(t)\dot{C}(t) - \mu(t)C(t) = 0\,\mu(t)$$

$\iff \int \frac{\partial}{\partial t}\left[\mu(t)C(t)\right]dt = \int 0\,dt$. Letting K denote the constant of integration,

$\iff \mu(t)C(t) = K$

$\iff e^{-\frac{1}{\theta}\left[(r_0 - \rho)t + \frac{\alpha}{2}t^2\right]} C(t) = K$. Using $C(T) = C_T$ at $T = T$,

$\iff K = C_T e^{-\frac{1}{\theta}\left[(r_0 - \rho)T + \frac{\alpha}{2}T^2\right]}$. The final solution is thus,

$$C(t) = K e^{\frac{1}{\theta}\left[(r_0 - \rho)t + \frac{\alpha}{2}t^2\right]}$$

$$\iff C(t) = C_T e^{\frac{1}{\theta}\left[(r_0 - \rho)t + \frac{\alpha}{2}t^2\right] - \frac{1}{\theta}\left[(r_0 - \rho)T + \frac{\alpha}{2}T^2\right]}$$

$$\iff \underline{C(t) = C_T e^{\frac{1}{\theta}\left[(r_0 - \rho)(t - T) + \frac{\alpha}{2}(t^2 - T^2)\right]}}.$$

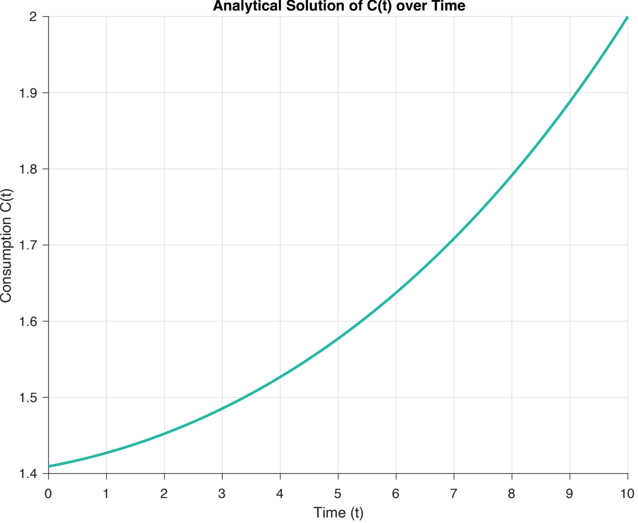
2. Given $\frac{\dot{C}(t)}{C(t)} = \frac{r_t - \rho}{\theta}$, we use the backwards difference method where $\dot{C}(t) = \frac{C(t) - C(t-\Delta t)}{\Delta t}$, which gives,

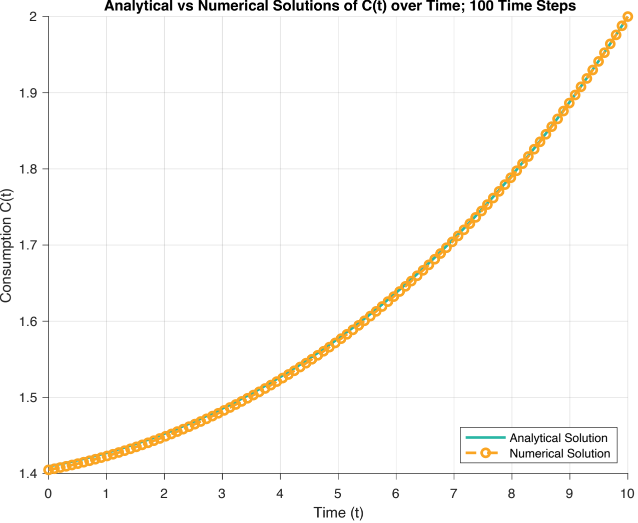
$$\frac{C(t) - C(t-\Delta t)}{\Delta t} \cdot \frac{1}{C(t)} = \frac{r_t - \rho}{\theta}$$

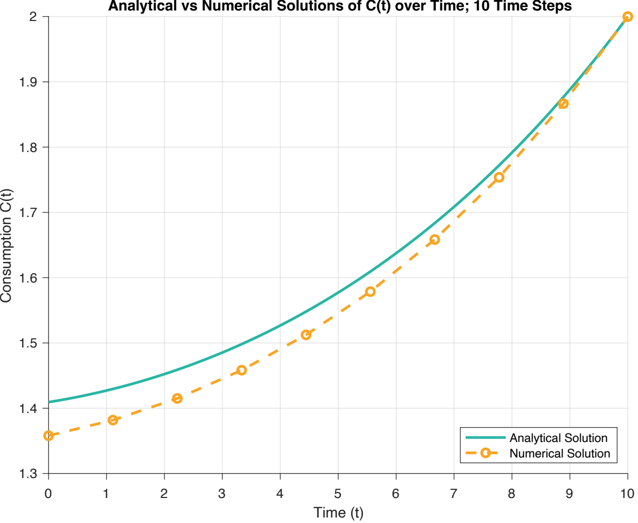
$$\Leftrightarrow C(t) - C(t-\Delta t) = \frac{\Delta t (r_t - \rho)}{\theta} C(t)$$

$$\Leftrightarrow \underline{C(t-\Delta t) = \left(\frac{\theta - \Delta t (r_t - \rho)}{\theta}\right) C(t)}.$$

4. The numerical solution curve is much closer to the analytic solution curve in the 100 time-step case. For the 10 time-step case, they are noticeably different.


Analytical Solution of C(t) over Time
Consumption C(t)
Time (t)
2
1.9
1.8
1.7
1.6
1.5
1.4
0
1
2
3
4
5
6
7
8
9
10


Analytical vs Numerical Solutions of C(t) over Time; 100 Time Steps
Consumption C(t)
Time (t)
Analytical Solution
Numerical Solution
2
1.9
1.8
1.7
1.6
1.5
1.4
0
1
2
3
4
5
6
7
8
9
10


Analytical vs Numerical Solutions of C(t) over Time; 10 Time Steps
Consumption C(t)
Time (t)
Analytical Solution
Numerical Solution
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
0
1
2
3
4
5
6
7
8
9
10

~/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/problem_set_1_code.m

```matlab
%{
------------------------------------------------------------------------------------------------------------------------
Purpose: Coding part of problem set 1
Created: Nico Rotundo 2024-10-28
------------------------------------------------------------------------------------------------------------------------
%}

%{
------------------------------------------------------------------------------------------------------------------------
Define parameters
------------------------------------------------------------------------------------------------------------------------
%}

% Terminal consumption at time T
C_T = 2.0;

% Initial interest rate
r_0 = 0.05;

% Growth rate of interest rate
alpha = 0.01;

%{
% Interest rate function as a handle function of time t
function r = interest_rate_function(t, r_0, alpha)
    r = r_0 + alpha * t;
end
%}

% Inverse of the intertemporal elasticity of substitution
theta = 2.0;

% Time discount factor
rho = 0.03;

% Terminal time
T = 10;

% Display the result of interest rate function at specific time to check
%fprintf('The interest rate at t = %d is: %.3f\n', 20, interest_rate_function(20, r_0, alpha));

% Checking how assert statements work in this matlab
%assert(interest_rate_function(20, r_0, alpha) == .25);


%{
------------------------------------------------------------------------------------------------------------------------
1. Solve the equation analytically using the integrating factor method. Then, plot the analytical solution as a function
    of time t and consumption C(t)

    - solved equation: C(t) = C_T*exp[(1/theta)*[(r_0-rho)(t-T)+(alpha/2)(t^2-T^2)]]
------------------------------------------------------------------------------------------------------------------------
%}

% Time vector from 0 to T with 100 points
t = linspace(0, T, 100);

% Define value for consumption at time t
C_t_analytical = C_T * exp((1 / theta) * ((r_0 - rho) * (t - T) + (alpha / 2) * (t.^2 - T^2)));

% Generate figure for analytical solution
figure;
plot(t, C_t_analytical, 'LineWidth', 2, 'Color', [41/255, 182/255, 164/255]);
    xlabel('Time (t)');
    set(gca, 'TickDir', 'out');
    box off;
    ylabel('Consumption C(t)');
    title('Analytical Solution of C(t) over Time');
    grid on;

% Export
exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/output/question_1.pdf', 'ContentType', 'vector');

%{
------------------------------------------------------------------------------------------------------------------------
2. Solve the equation numerically using the finite difference method with 100 time steps. Plot the numerical and analytical
    solutions together in a single figure for comparison.

    - equation: C(t-dt) = (theta-dt(r_t-rho))/theta * C(t)
------------------------------------------------------------------------------------------------------------------------
%}

% Time step (\triangle t)
dt = 10/99;

% Number of time steps
N = 100;

% Vector to store consumption values
C_t_numeric = zeros(N, 1);

% Set terminal consumption
C_t_numeric(end) = C_T;

% Time vector from 0 to T with 100 points
t = linspace(0, T, N);

% Time stepping using the finite difference method
for n = N:-1:2

    % Compute consumption at the next time step using backward difference
   C_t_numeric(n-1) = (theta - dt*(r_0 + alpha*t(n)-rho))/theta * C_t_numeric(n);

end

% Generate figure for numeric solution
figure;
    plot(t, C_t_numeric, 'LineWidth', 2, 'Color', [250/255, 165/255, 35/255]);
    xlabel('Time (t)');
    set(gca, 'TickDir', 'out');
    box off;
    ylabel('Consumption C(t)');
    title('Numerical Solution of C(t) over Time');
    grid on;

% Generate both analytical and numeric sultions on the same plot
figure;

    % Plot analytical solution
    plot(t, C_t_analytical, 'LineWidth', 2, 'Color', [41/255, 182/255, 164/255]);

    hold on;

    % Plot numerical solution
    plot(t, C_t_numeric, 'bo--', 'LineWidth', 2, 'Color', [250/255, 165/255, 35/255]);

    % Add labels and title
    xlabel('Time (t)');
    ylabel('Consumption C(t)');
    title('Analytical vs Numerical Solutions of C(t) over Time; 100 Time Steps');

    % Add a legend
    legend({'Analytical Solution', 'Numerical Solution'}, 'Location', 'southeast');

    % Adjust plot settings
    set(gca, 'TickDir', 'out');
    box off;
    grid on;

    hold off;

% Export
exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/output/question_2.pdf', 'ContentType', 'vector');

%{
------------------------------------------------------------------------------------------------------------------------
3. Repeat the numerical solution using 10 time steps. Plot the numerical and analytical solutions together in a single
    figure.
------------------------------------------------------------------------------------------------------------------------
%}

% Time step (\triangle t)
dt = 10/9;

% Number of time steps
N = 10;

% Vector to store consumption values
C_t_numeric = zeros(N, 1);

% Set terminal consumption
C_t_numeric(end) = C_T;

% Time vector from 0 to T with 10 points
t_10 = linspace(0, T, N);

% Time stepping using the finite difference method
for n = N:-1:2

    % Compute consumption at the next time step using backward difference
    C_t_numeric(n-1) = (theta - dt*(r_0 + alpha*t_10(n)-rho))/theta * C_t_numeric(n);

end

% Generate both analytical and numeric sultions on the same plot
figure;

    % Plot analytical solution
    plot(t, C_t_analytical, 'LineWidth', 2, 'Color', [41/255, 182/255, 164/255]);

    hold on;

    % Plot numerical solution
    plot(t_10, C_t_numeric, 'bo--','LineWidth', 2, 'Color', [250/255, 165/255, 35/255]);

    % Add labels and title
    xlabel('Time (t)');
    ylabel('Consumption C(t)');
    title('Analytical vs Numerical Solutions of C(t) over Time; 10 Time Steps');

    % Add a legend
    legend({'Analytical Solution', 'Numerical Solution'}, 'Location', 'southeast');

    % Adjust plot settings
    set(gca, 'TickDir', 'out');
    box off;
    grid on;

    hold off;

% Export
exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_1/output/question_3.pdf', 'ContentType', 'vector');
```