202A: Dynamic Programming and Applications

Homework #3

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Problem 1: Brownian Motion

This problem collects several exercises on Brownian motion and stochastic calculus. We denote standard Brownian motion by B_t.

(a) Show that $\mathbb{C}ov(B_s, B_t) = \min\{s, t\}$ for two times $0 \leq s < t$. Use the following tricks: Use the covariance formula $\mathbb{C}ov(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B)$. Use $B_t \sim \mathcal{N}(0, t)$ as well as $B_t - B_s \sim \mathcal{N}(0, t - s)$. And use $B_t = B_s + (B_t - B_s)$.

(b) Let $X_t = B_t^2$. What is $\mathbb{E}X_t$? What is $\mathbb{C}ov(X_t, X_s)$?

(c) Let $X_t = B_{t+s} - B_s$ for some fixed $s > 0$. Let $Y_t = \frac{1}{\sqrt{\lambda}} B_{\lambda t}$. Show that both X_t and Y_t are standard Brownian motion; that is, show the following 5 properties:

   (i) $X_0 = Y_0 = 0$

   (ii) $X_t, Y_t \sim \mathcal{N}(0, t)$

   (iii) Stationarity

   (iv) Independent increments

   (v) Continuous

(d) Geometric Brownian motion evolves as: $dX_t = \mu X_t dt + \sigma X_t dB_t$ given an initial value X_0. Show that

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2} t + \sigma B_t}.$$

(e) For Geometric Brownian motion as defined above, show that $\mathbb{E} = X_0 e^{\mu t}$.

(f) The Ornstein-Uhlenbeck (OU) process is like a continuous-time variant of the AR(1) process. It evolves as $dX_t = -\mu X_t dt + \sigma dB_t$ for drift parameter μ, diffusion parameter σ, and some X. Show that it solves

$$X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dB_s.$$

(g) Let $X_t = \int_0^t B_s ds$ and $Y_t = \int_0^t B_s^2 ds$. Compute $\mathbb{E}(X_t)$ and $\mathbb{E}(Y_t)$, as well as $\mathbb{V}ar(X_t)$ and $\mathbb{V}ar(Y_t)$.

Problem 2: Poisson Process

(a) Consider a two-state Markov chain in continuous time denoted Y_t, which can take on values in $\{Y^1, Y^2\}$. The transition rate is given by λ regardless of the current state. Suppose we are in state 1 at $t = 0$. Compute the expected time until the process switches to state 2.

(b) Now suppose that the transition rates differ depending on which state we are in. That is, if we're in state 1 we transition to rate 2 at rate λ_1, and vice versa at rate λ_2. Show that the fraction of time the process spends in states 1 and 2 converges in the long run to $\frac{\lambda_2}{\lambda_1+\lambda_2}$ and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

(c) What is $\mathbb{E}(Y_t \mid Y_0 = Y^1)$?

Problem 3: Ito's Lemma

(a) Let $dX_t = -\alpha X_t dt + \sigma dB_t$, and $f(t, X_t) = e^{\alpha t} X_t$. Show that $df = \sigma e^{\alpha t} dB$.

(b) Consider the capital accumulation equation $dK_t = (\iota - \delta) K_t dt + \sigma K_t dB_t$, where ι is the investment rate. Suppose our value function is $V(K_t)$. Use Ito's lemma to solve for dV_t.

(c) Suppose $X_t = f(t, B_t)$ for some function f. In this problem, we will solve for f. The only information we have is that

$$dX_t = X_t dB_t.$$

Use Ito's lemma to derive two conditions on the partial derivatives of $f(\cdot)$. (That is, group the dt and dB terms and reason via coefficient-matching.) Show that the function $f(t, x) = e^{x - \frac{1}{2}t}$ satisfies these conditions.

Problem 4: Generator

We defined the generator of a stochastic process in class. The generator is extremely useful when we want to quickly write down the HJB associated with a dynamic optimization problem.

(a) Consider the wealth and income processes

$$da_t = (ra_t + y_t - c_t)dt + \sigma dB_t$$
$$dy_t = \theta(\bar{y} - y_t)dt + \nu W_t$$

where B_t and W_t are independent standard Brownian motion. Write down the generator $\mathcal{A}$ for the two-dimensional process

$$\begin{pmatrix} da_t \\ dy_t \end{pmatrix}$$

That is, what is $\mathcal{A}V$ for a given (smooth) function $V(a_t, y_t)$?

(b) Consider the capital accumulation process

$$dk_t = (\iota - \delta)k_t dt.$$

Also suppose that firm technology A_t follows a two-state Markov chain (Poisson process) with transition rates λ. That is, A_t can take on values in $\{A^1, A^2\}$. Suppose that the enterprise value of the firm is given by some function $V(k_t, A_t)$. Characterize the generator of the process

$$\begin{pmatrix} dk_t \\ dA_t \end{pmatrix}$$

That is, what is $\mathcal{A}V$ for a given (smooth) function $V(k_t, A_t)$? (Recall that $\mathcal{A}V = \mathbb{E}[dV]$, so the expression you just solved for tells us how the firm's enterprise value evolves in expectation.)

Problem 5: Consumption with Income Uncertainty

Consider a household whose lifetime utility is given by

$$V_0 = \max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

The household makes consumption-savings decisions facing the budget constraint

$$da_t = ra_t + y_t - c_t.$$

Here, y_t is the household's income process. We cannot perfectly forecast our future income, so we will assume that y_t is a stochastic process. Below, you will be asked to write down the HJB associated with this problem for the two canonical models of income uncertainty.

(a) Suppose that income follows the diffusion (Ornstein-Uhlenbeck / AR1) process

$$dy_t = \theta(\bar{y} - y_t)dt + \sigma dB_t.$$

Write down the HJB for this problem that characterizes the household value function $V(a, y)$.

(b) Now suppose that income follows a two-state Markov chain (Poisson process). Income can be high or low, $\in \{y^L, y^H\}$. The transition rate from high to low is λ^H and the transition rate from low to high is λ^L. Write down the HJB for this problem that characterizes the household value function $V(a, y)$.

(c) Suppose that the interest rate is not constant but varies over time. We know with certainty, however, how the interest rate evolves. So $\{r_t\}$ is a deterministic sequence that is exogenously given to us. (In other words, we are characterizing the household problem in partial equilibrium; the household takes the interest rate as given.) Write down the HJB for this problem that characterizes the household value function $V(t, a, y)$. Why is this value function no longer stationary?

Problem 6: Consumption and Portfolio Choice

Consider a household that can trade two assets. The first asset is a stock. Stocks trade at price Q_t and they pay the holder dividends at rate D_t. That is, when you hold the stock, your "return" comprises both the dividend payouts and the change in price until you sell the stock, which may be positive or negative. Formally, the rate of return on the stock is

$$dR = \frac{Ddt + dQ}{Q},$$

where $\frac{D}{Q}$ is the dividend-price ratio and $\frac{dQ}{Q}$ is capital gains (change in price). We will now assume that the return process takes the form

$$dR = \mu dt + \sigma dB.$$

for some μ and σ, where B is standard Brownian motion.

The second asset the household can trade is a bond, which trades at price P. And we assume that the bond price evolves simply according to

$$\frac{dP}{P} = rdt,$$

which is the same as saying that holding the bond earns a riskfree rate of return rdt. Crucially, the household can both buy and sell these assets. Assume there are no borrowing (or short-sale) constraints.

The household's lifetime value is given by

$$V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

Let's denote by b_t and k_t the household's bond and stock holdings. Then the household budget constraint is

$$Qdk + Pdb + cdt = Dkdt.$$

Why does the budget constraint take this form? On the RHS, households earn dividends at rate D for each unit of stock they hold. On the LHS, households can spend on consumption, or they can choose to purchase stocks or bonds. If $dk > 0$, the household purchases new stocks at price Q on top of the current stock holdings. If $dk < 0$, the household sells stocks.

We now define *net worth* n and the *risky portfolio share* θ implicity via

$$\theta n = Qk$$

$$(1-\theta)n = Pb$$

In other words, n is a notion of total wealth of the household. And we say that the household invests fraction θ of total wealth in stocks, and fraction $1-\theta$ in bonds.

(a) Derive the law of motion for household net worth and show that it satisfies:

$$dn = \left[rn + \theta n \Big(\mu - r \Big) - c \right] dt + \theta n \sigma dB$$

(b) Derive the recursive representation of the households optimization problem. That is, write down the HJB equation that characterizes the household value function $V(n)$. Why was it useful to rewrite the household problem in terms of net worth (rather than stock and bond holdings)? Why is this value function stationary?

(c) Derive the first-order conditions for c and the portfolio share θ.

(d) Derive the Euler equation for marginal utility and show that it satisfies

$$\frac{du_c}{u_c} = (\rho - r)dt - \frac{\mu - r}{\sigma} dB.$$

Notice that with CRRA utility, this implies a consumption Euler equation

$$\frac{dc}{c} = \frac{r-\rho}{\gamma} dt + \frac{1+\gamma}{2} \left(\frac{\mu - r}{\gamma \sigma} \right)^2 dt + \frac{\mu - r}{\gamma \sigma} dB.$$

You don't have to derive this (but you can try).

(e) Guess and verify that

$$V(n) = \frac{1}{1-\gamma} \kappa^{-\gamma} n^{1-\gamma}$$

where $\kappa = \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{1-\gamma}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right]$.

(f) Given the solution for $V(n)$, use the FOCs to also solve for $c(n)$ and $\theta(n)$. You have now solved the household portfolio choice problem in closed form!!

(g) What does this model tell us? Consider the interesting special case of log utility with $\gamma = 1$. Show that consumption collapses to $c = \rho n$. This implies that you want to consume a constant fraction ρ of lifetime net worth. But how much should you invest in the stock market according to this model? Take the formula for θ and plug in log utility. Let's assume an equity premium of 6%, so $\mu - r = 0.0.06$. And let's assume volatility of stock returns of 16%, so $\sigma = 0.16$. Solve for the numeric value of θ, i.e., the fraction of your total wealth that this model tells you to invest in the stock market. Is this high or low? What if "total wealth" also includes a notion of your human capital?