econ 2024; problem set 5

1.a. The household problem in sequence form is given by,

$$\max_{\{c_{i,t}\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_{i,t})\,dt \qquad \text{s.t.},$$

i. $\dot{k}_{i,t} = r_i k_{i,t} + z_{i,t} w_{i,t} - c_{i,t}$

ii. $z_{i,t} \in \{z^L, z^h\}$ Poisson w/ intensity λ

iii. $k_{i,t} \geq 0$

1.b. The state variables from which we can derive a recursive representation are k & z. The HJB is given by,

$$\rho V_t(k,z) = \max_c \left\{ u(c) + \mathbb{E}\frac{dV_t(k,z)}{dt} \right\}$$

$$\Leftrightarrow \rho V_t(k,z_j) = \max_c \Big\{ u(c) + [r_t k + z_j w_t - c]\,\partial_k V_t(k,z_j) + \lambda [V_t(k,z_{-j}) - V_t(k,z_j)] + \partial_t V_t(k,z_j) \Big\}$$

i. The HJB is not stationary since r_t & w_t vary over time & are not state variables.

ii. We drop the i subscripts, as all households face the same prices & transition probabilities in equilibrium; in other words, the above characterizes the household problem for all i, aside from differing states k & z_j

1.c. The short-sale constraint is embedded into the value problem through the choice of $\dot{k}$, where the short-sale constraint is $k_{i,t} \geq 0$. Specifically, it appears as a boundary condition in the value function, $\frac{\partial V_t(\underline{k}, z)}{\partial k} \geq 0$ at $\underline{k} = 0$

1.d. The first-order condition is given by,

$$\underline{u'(c) = \partial_k V_t(k, z_j)}$$

1.e. The Kolmogorov Forward equation is given by,

$$\underline{\partial_t g_t(k, z_j) = -\partial_k [s_t(k, z_j) g_t(k, z_j)] - \lambda g_t(k, z_j) + \lambda g_t(k, z_{-j})},$$

where $s_t(k, z_j) = r_t k + z_j w_t - c_t(k, z_j)$.

1.f. The firm problem is given by,

$$\max_{K_t, L_t} \Big\{ \underbrace{Y_t - w_t L_t - r_t^k K_t}_{= \Pi_t} \Big\} \text{ where } Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

i. $\underline{\text{F.O.C}_{L_t}}$: $\frac{\partial \Pi_t}{\partial L_t} = 0 \iff \underline{(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} = w_t}$

ii. $\underline{\text{F.O.C}_{K_t}}$: $\frac{\partial \Pi_t}{\partial K_t} = 0 \iff \underline{\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = r_t^k}$

There are zero profits since the firm's production function is characterized by constant returns to scale.

1.g. The following markets must clear,

i. <u>goods market</u>: $Y_t = \int_0^1 c_{i,t}\, di$

ii. <u>labor market</u>: $L_t = \int_0^1 z_{i,t}\, di$

iii. <u>capital market</u>: $K_t = \int_0^1 k_{i,t}\, di$

1.h. <u>recursive competitive equilibria definition</u>: Let an initial joint density $g_0(a,y)$ & an exogenous path of TFP $\{A_t\}$ be given. Then, a competitive equilibria is characterized by functions,

$$\{V_t(k,z),\ c_t(k,z),\ g_t(k,z)\} \ \&\ \{Y_t,\ L_t,\ K_t,\ r_t^k,\ w_t\} \text{ such that,}$$

i. households optimize

ii. firms optimize

iii. markets clear

iv. the joint density evolves consistently with household behavior

1.i. i. HJB & F.O.C.:

a. $\rho V_t(k,z_j) = \max_c \{ u(c) + [r_t^k k + z_j w_t - c]\partial_k V_t(k,z_j)$
$+ \lambda [V_t(k,z_{-j}) - V_t(k,z_j)] + \partial_t V_t(k,z_j) \}$

b. $\partial_k V_t(\underline{k}, z_j) \geq u'([\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}]\underline{k} + [(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}] z_j)$; $\underline{k} = 0$

c. $u'(c) = \partial_k V_t(k, z_j)$

ii. KF:

$\partial_t g_t(k,z_j) = -\partial_k [s_t(k,z_j) g_t(k,z_j)] - \lambda g_t(k,z_j) + \lambda g_t(k,z_{-j})$,

iii. capital markets clear: $K_t = \int_0^1 k_{i,t}\, di$

1.j. Let $A_t = A$. A stationary competitive equilibria is characterized by functions

$\{V(k,z), c(k,z), g(k,z)\}$ & $\{Y, L, K, r^k, w\}$ such

that,

i. households optimize

ii. firms optimize

iii. markets clear

iv. the joint density evolves consistently with household behavior