econ 202A; problem set 3; second half

1.a. Consider $Cov(B_s, B_t)$. By definition,

$Cov(B_s, B_t) = \mathbb{E}[B_s B_t] - \mathbb{E}[B_s]\mathbb{E}[B_t]$. From $B_t \sim N(0,t)$,

$= \mathbb{E}[B_s B_t] - \cancel{\mathbb{E}[B_s](0)}^{0}$. Note $B_t = B_s + (B_t - B_s)$,

$= \mathbb{E}[B_s(B_s + B_t - B_s)]$

$= \mathbb{E}[B_s^2 + B_s(B_t - B_s)]$. Since B_t is Brownian, $B_s \perp\!\!\!\perp B_t - B_s$,

$= \mathbb{E}[B_s^2] + \mathbb{E}[B_s]\mathbb{E}[B_t - B_s]$. From $B_s \sim N(0,s)$,

$= s$

$= \min\{s,t\}$ since $0 \le s < t$. $\square$

1.b. Let $X_t = B_t^2$. Then, by definition of variance,

$\mathbb{E}[B_t^2] = Var[B_t] + \mathbb{E}[B_t]^2$. From $B_t \sim N(0,t)$,

$= t + 0 \quad = t$. $\square$

1.c. Given $X_t = B_{t+s} - B_s$ for $s>0$; $Y_t = \frac{1}{\sqrt{\lambda}} B_{\lambda t}$,

i. x. $X_0 = B_s - B_s = 0$; xx. $Y_0 = \frac{1}{\sqrt{\lambda}} B_0 = 0$ since $\{B_t\}_{t\ge 0}$ is Brownian.

ii. x. $\underline{X_t}$: $\underline{\mathbb{E}[X_t]} = \mathbb{E}[B_{t+s} - B_s] = 0 + 0 = \underline{0}$ by $B_t \sim N(0,t)$; $B_{t+s} \sim N(0, t+s)$.

$\underline{Var[X_t]} = Var[B_{t+s} - B_s]$. From $B_t \sim N(0,t)$; $B_{t+s} \sim N(0, t+s)$

$= t+s-s = \underline{t}$.

$\Rightarrow \underline{X_t \sim N(0,t)}$.

xx. $\underline{Y_t}$: $\mathbb{E}[Y_t] = \frac{1}{\sqrt{\lambda}} \mathbb{E}[B_{\lambda t}]$. From $B_{\lambda t} \sim (0, \lambda t)$,

$= 0$.

$Var[Y_t] = \left(\frac{1}{\sqrt{\lambda}}\right)^2 Var[B_{\lambda t}]$. From $B_{\lambda t} \sim (0, \lambda t)$,

$= \frac{1}{\lambda} \cdot \lambda t = t$

$\Rightarrow \underline{Y_t \sim N(0,t)}$.

iii. x. $\underline{X_t}$: Consider $X_t - X_h$ for $0 \le h < t$. By definition,

$$X_t - X_h = B_{t+s} - B_s - B_{h+s} + B_s$$

$$= B_{t+s} - B_{h+s}$$. Since $\{B_t\}_{t\ge 0}$ is Brawnian, $B_{t+s} \sim N(0, t+s)$ & $B_{h+s} \sim N(0, h+s)$. Then,

$$\mathbb{E}[X_t - X_h] = 0.$$

$$Var[X_t - X_h] = Var[B_{t+s}] - Var[B_{h+s}]$$

$$= t+s-h-s = t-h$$

Thus, $X_t - X_h \sim N(0, t-h) \Leftrightarrow X_t$ satisfies stationarity.

xx. $\underline{Y_t}$: Consider $Y_t - Y_h$ for some $0 \le h < t$. Similar to the above,

$$\mathbb{E}[Y_t - Y_h] = \frac{1}{\sqrt{\lambda}}\mathbb{E}[B_{\lambda t}] - \frac{1}{\sqrt{\lambda}}\mathbb{E}[B_{\lambda h}] = 0$$ from $B_{\lambda s} \sim N(0, \lambda s)$.

$$Var[Y_t - Y_h] = \frac{1}{\lambda} Var[B_{\lambda t}] - \frac{1}{\lambda} Var[B_{\lambda h}] = t-h.$$

Thus, $Y_t - Y_h \sim N(0, t-h) \Leftrightarrow Y_t$ satisfies stationarity.

iv. Let $0 \le t_1 < t_2 < t_3 < t_4$.

x. $\underline{X_t}$: By definition,

$$\mathbb{P}(X_{t_4} - X_{t_3}, X_{t_2} - X_{t_1}) = \mathbb{P}(B_{t_4+s} - B_s - B_{t_3+s} + B_s, B_{t_2+s} - B_s - B_{t_1+s} + B_s)$$
$$= \mathbb{P}(B_{t_4+s} - B_{t_3+s}, B_{t_2+s} - B_{t_1+s}).$$

Since $\{B_t\}_{t\ge 0}$ is Brownian, it satisfies independent increments. With $0 \le t_1 < t_2 < t_3 < t_4$,

$$= \mathbb{P}(B_{t_4+s} - B_{t_3+s})\,\mathbb{P}(B_{t_2+s} - B_{t_1+s}).$$

Thus, X_t satisfies independent increments.

xx. $\underline{Y_t}$: $\mathbb{P}(Y_{t_4} - Y_{t_3}, Y_{t_2} - Y_{t_1}) = \mathbb{P}\left(\frac{1}{\sqrt{\lambda}}(B_{\lambda t_4} - B_{\lambda t_3}), \frac{1}{\sqrt{\lambda}}(B_{\lambda t_2} - B_{\lambda t_1})\right).$

Similarly, from $\{B_t\}_{t\ge 0}$ Brownian,

$$= \mathbb{P}\left(\frac{1}{\sqrt{\lambda}}(B_{\lambda t_4} - B_{\lambda t_3})\right)\mathbb{P}\left(\frac{1}{\sqrt{\lambda}}(B_{t_2} - B_{\lambda t_1})\right).$$

Thus, Y_t satisfies independent increments.

v. x. $\underline{X_t}$: Since $X_t = B_{t+s} - B_s$ & B_t is Brownian, X_t is continuous, since both B_{t+s}, B_s are continuous.

xx. $\underline{Y_t}$: Since $Y_t = \frac{1}{\sqrt{\lambda}} B_{\lambda t}$ & B_t is Brownian, Y_t is continuous, since it just scales the continuous path $B_{\lambda t}$.

1.d. Given,

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

$\Leftrightarrow$ $\dfrac{dX_t}{X_t} = \mu dt + \sigma dB_t$. From $\dfrac{d\ln(X_t)}{dt} = \dfrac{1}{X_t} \cdot \dfrac{dX_t}{dt}$

$\Leftrightarrow d\ln(X_t) = \dfrac{dX_t}{X_t}$. Then,

$\Leftrightarrow$ $d\ln(X_t) = \mu dt + \sigma dB_t$. By Ito's lemma,

$$d\ln(X_t) = [\ln X_t]' dB_t + \frac{1}{2}[\ln X_t]'' dt$$

$$= \frac{1}{X_t} dB_t - \frac{1}{2} X_t^{-2} X_t^2 \sigma^2 dt$$

$$= \frac{1}{X_t}[\sigma X_t dB_t + \mu X_t dt] - \frac{1}{2}\sigma^2 dt$$

$\Leftrightarrow$ $d\ln(X_t) = \sigma dB_t + (\mu - \frac{1}{2}\sigma^2) dt$

$\Leftrightarrow$ $\int d\ln(X_t) = \int \sigma dB_t + \int (\mu - \frac{1}{2}\sigma^2) dt$

$\ln(X_t) = \sigma B_t + (\mu - \frac{1}{2}\sigma^2)t + C$. Letting $t=0$,

$\ln X_0 = C$.

$\Rightarrow$ $\ln(X_t) = \sigma B_t + (\mu - \frac{1}{2}\sigma^2)t + \ln X_0$. Exponentiating gives,

$\Leftrightarrow$ $X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$

$\Leftrightarrow$ $X_t = X_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t}$

. $\square$

1.e. By definition,

$\mathbb{E}[X_t] = X_0 e^{(\mu - \frac{\sigma^2}{2})t} \mathbb{E}[e^{\sigma B_t}]$. Since $B_t \sim N(0, t)$,

$\mathbb{E}[e^{\sigma B_t}] = e^{\frac{1}{2}\sigma^2 t}$ by MGF of B_t .

$$= X_0 e^{(\mu t - \frac{\sigma^2}{2}t + \frac{1}{2}\sigma^2 t)}$$

$$= X_0 e^{\mu t} .$$

$\square$

1.f. Converting the differential equation to standard form gives,

$dX_t + \mu X_t dt = \sigma dB_t$. Multiplying the integrating factor $e^{\int \mu dt} = e^{\mu t}$ gives,

$e^{\mu t} dX_t + \mu e^{\mu t} X_t dt = \sigma e^{\mu t} dB_t$. Integrating from 0-t gives,

$$\Leftrightarrow \int_0^t d(e^{\mu s} X_s) = \int_0^t \sigma e^{\mu s} dB_s$$

$$\Leftrightarrow e^{\mu t} X_t - X_0 = \int_0^t \sigma e^{\mu s} dB_s$$

$$\Leftrightarrow X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dB_s \ . \square$$

1.g. $\underline{\mathbb{E}[X_t]}$: By definition,

$\underline{\mathbb{E}[X_t]} = \mathbb{E}\left[\int_0^t B_s ds\right] = \int_0^t \mathbb{E}[B_s] ds$. Since $\{B_t\}_{t \geq 0}$ is Brownian, $B_t \sim (0,t)$

$$\underline{= 0.}$$

$\underline{Var[X_t]}$: Note $Var[X_t] = \mathbb{E}[X_t^2] - \cancel{\mathbb{E}[X_t]^2} 0$. By definition,

$\mathbb{E}[X_t^2] = \mathbb{E}\left[\left(\int_0^t B_s ds\right)^2\right] = \int_0^t \int_0^t \mathbb{E}[B_s B_r] ds dr$. From part a,

$\mathbb{E}[B_s B_r] = Cov(B_s, B_r) = \min\{s,r\}$

$$= \int_0^t \int_0^t \min\{s,r\} ds\, dr = 2 \int_0^t \int_0^r s ds dr$$

$$= 2 \int_0^t \frac{1}{2} r^2 dr = \frac{1}{3} r^3 \Big|_0^t = \frac{1}{3} t^3$$

$\Rightarrow \underline{Var[X_t] = \frac{1}{3} t^3}.$

$\underline{\mathbb{E}[Y_t]}$: By definition,

$\underline{\mathbb{E}[Y_t]} = \int_0^t \mathbb{E}[B_s^2]\,ds$. From $B_s \sim N(0, s)$,

$= \int_0^t s\,ds = \frac{1}{2}s^2\Big|_0^t = \underline{\frac{1}{2}t^2}$.

$\underline{Var[Y_t]}$: Note $Var[Y_t] = \mathbb{E}[(Y_t)^2] - \mathbb{E}[Y_t]^2$.

$\Rightarrow Var[Y_t] = \mathbb{E}[(\int_0^t B_s^2\,ds)^2] - \frac{1}{4}t^4$

$= \mathbb{E}[\int_0^t\int_0^t B_s^2 B_r^2\,ds\,dr] - \frac{1}{4}t^4$

$= \int_0^t\int_0^t \mathbb{E}[B_s^2 B_r^2]\,ds\,dr - \frac{1}{4}t^4$. Since $B_s, B_r \sim N$,

$= \int_0^t\int_0^t (\mathbb{E}[B_s^2]\mathbb{E}[B_r^2] + 2\mathbb{E}[B_s B_r]^2)\,ds\,dr - \frac{1}{4}t^4$

$= \int_0^t\int_0^t (sr + 2\min\{s^2, r^2\})\,ds\,dr - \frac{1}{4}t^4$

$= 2\int_0^t\int_0^r (sr + 2s^2)\,ds\,dr - \frac{1}{4}t^4$

$= 2\int_0^t (r[\frac{1}{2}r^2] + \frac{2}{3}r^3)\,dr - \frac{1}{4}t^4 = \frac{7}{3}\int_0^t r^3\,dr - \frac{1}{4}t^4$

$= \frac{7}{12}t^4 - \frac{3}{12}t^4$

$= \underline{\frac{1}{3}t^4}$. □

2.a. Let T_1 denote the amount of time the process spends in state 1. From class, the waiting time $T_1 \sim Exp(\lambda)$. The pdf of T_1 is given by,

$f_{T_1}(t) = \lambda e^{-\lambda t}$, $t \geq 0$. Then, the expected value of T_1 is,

$u = t$; $du = dt$
$dv = e^{-\lambda t} dt$; $v = -\frac{1}{\lambda} e^{-\lambda t}$

$$\mathbb{E}[T_1] = \int_{-\infty}^{\infty} \lambda e^{-\lambda t} t \, dt = \lambda \int_0^{\infty} t e^{-\lambda t} dt$$

$$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda t} t \Big|_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} dt \right]$$

$$= \lambda \left[0 - 0 - \frac{1}{\lambda^2} [0 - 1] \right]$$

$$= \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}.$$

$\therefore$ The expected time until the process switches to state 2 is $\frac{1}{\lambda}$. $\square$

2.b. In the long run, the flow out of state 1 is equal to the flow out of state 1, where the former is the probability of being in state 1 times the transition rate from 1 to 2 ; the latter is the contrapositive. Let Π_i denote the probability of being in state i. Then,

$\Pi_1 \lambda_1 = \Pi_2 \lambda_2$; $\Pi_1 + \Pi_2 = 1$ by definition of a probability measure

$\iff \Pi_2 = \Pi_1 \frac{\lambda_1}{\lambda_2}$; $\iff \Pi_1 + \Pi_1 \left(\frac{\lambda_1}{\lambda_2} \right) = 1$

$\iff \Pi_1 \left(\frac{\lambda_2}{\lambda_2} + \frac{\lambda_1}{\lambda_2} \right) = 1$

$\iff \underline{\Pi_1 = \frac{\lambda_2}{\lambda_1 + \lambda_2}}$. Substituting into $\Pi_1 + \Pi_2 = 1$ gives,

$\underline{\Pi_2 = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}}$.

2.c. By definition,

$$\mathbb{E}[Y_t \mid Y_0 = Y^1] = \underbrace{\mathbb{P}(Y_t = Y^1 \mid Y_0 = Y^1)}_{\pi_1} Y^1 + \underbrace{\mathbb{P}(Y_t = Y^2 \mid Y_0 = Y^1)}_{\pi_2} Y^2$$

From part b,

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2} Y^1 + \frac{\lambda_1}{\lambda_1 + \lambda_2} Y^2 .$$

3.a. By Ito's lemma,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}dt$$

$$= \alpha e^{\alpha t}X_t\,dt + e^{\alpha t}dX_t \text{ . From } dX_t = -\alpha X_t\,dt + \sigma dB_t,$$

$$= \cancel{\alpha e^{\alpha t}X_t\,dt} + e^{\alpha t}[-\cancel{\alpha X_t\,dt} + \sigma dB_t]$$

$$= e^{\alpha t}\sigma dB_t$$, as was to be shown. □

3.b. By Ito's lemma,

$$dV_t = \frac{\partial V(K_t)}{\partial K_t}dK_t + \frac{1}{2}\frac{\partial^2 V(K_t)}{\partial K^2}(dK_t)^2 \text{. From } dK_t = (\iota-\delta)K_t\,dt + \sigma K_t\,dB_t,$$

$$= \partial_K V_t[(\iota-\delta)K_t\,dt + \sigma K_t\,dB_t] + \frac{1}{2}\partial_{KK}V_t\Big[(\iota-\delta)^2K_t^2\underbrace{dt^2}_{=0} + \underbrace{2(\iota-\delta)K_t\,dt\,\sigma K_t\,dB_t}_{=0 \text{ since } dt\,dB_t=0} + \sigma^2K_t^2\underbrace{dB_t^2}_{=dt}\Big]$$

$$= \partial_K V_t[(\iota-\delta)K_t\,dt + \sigma K_t\,dB_t] + \frac{1}{2}\partial_{KK}V_t[(\sigma K_t)^2dt]$$

$$= \underline{\Big[(\iota-\delta)K_t\,\partial_K V_t + \frac{1}{2}(\sigma K_t)^2\partial_{KK}V_t\Big]dt + \sigma K_t\,dB_t\,\partial_K V_t} \text{ .}$$

3. c. By Ito's lemma,

$$dX_t = \frac{\partial X_t}{\partial t} dt + \frac{\partial X_t}{\partial b} dB_t + \frac{1}{2} \frac{\partial^2 X_t}{\partial b^2} dt$$

(1) $\Rightarrow \left[\frac{\partial X_t}{\partial t} + \frac{1}{2} \frac{\partial^2 X_t}{\partial b^2} \right] dt = 0$ since $dX_t = X_t \, dB_t$. Then,

(2) $\Rightarrow X_t = \frac{\partial X_t}{\partial b}$, where (1) & (2) are the two conditions.

Consider $f(t, x) = f(t, B_t) = e^{x - \frac{1}{2}t}$,

(1) $-\frac{1}{2} e^{x - \frac{1}{2}t} + \frac{1}{2} e^{x - \frac{1}{2}t} = 0$ ✓

(2) $X_t = e^{x - \frac{1}{2}t} = \frac{\partial X_t}{\partial b}$ ✓

4.a. The generator is given by,

$$\mathcal{A}V(a_t, y_t) = (ra_t + y_t - c_t)\,\partial_a V(a_t, y_t) + \theta(\bar{y} - y_t)\,\partial_y V(a_t, y_t) + \frac{\sigma^2}{2}\partial_{aa} V(a_t, y_t) + \frac{V^2}{2}\partial_{yy} V(a_t, y_t),$$

based on slide 34 from lecture

4.b. The generator is given by,

$$\mathcal{A}V(k_t, a_t) = (L-\delta)k_t\,\partial_k V(k_t, A_t) + \lambda\left[V(k_t, \bar{A}_t) - V(k_t, A_t)\right],$$

based on slide 34 from lecture

5.a. The HJB is given by,

$$\rho V(a,y) = \max_c \Big\{ u(c) + (ra + y - c)\,\partial_a V(a,y) + \theta(\bar{y} - y_t)\,\partial_y V(a,y) + \frac{\sigma^2}{2}\,\partial_{yy} V(a,y) \Big\},$$

based on slide 34 from lecture

5.b. The HJB is given by,

$$\rho V(a,y) = \max_c \Big\{ u(c) + (ra + y - c)\,\partial_a V(a,y) + \lambda^L [V(a, y^H) - V(a, y^L)] \Big\},$$

based on slide 34 from lecture

5.c. V is not stationary since the parameter r varies over time. The HJB is given by,

$$\rho V(a,y,t) = \max_c \Big\{ u(c) + [ra + y - c]\,\partial_a V(a,y,t) + \partial_t V(a,y,t) + \lambda^L [V(a, y^H, t) - V(a, y^L, t)] \Big\}$$

6. a. Observe,

$n = \theta n + (1-\theta)n = Qk + Pb$. Then,

$dn = d(Qk) + d(Pb)$.

$= [Qdk + kdQ + \underbrace{dkdQ}_{=0}] + [Pdb + bdP + \underbrace{dbdP}_{=0}]$. From $Qdk + Pdb + cdt = Dkdt$

$= [Dk\,dt - Pdb - cdt + kdQ] + [Pdb + bdP]$. From $dR = \frac{Ddt + dQ}{Q} = \mu dt + \sigma dB$,

$= \cancel{Dkdt} - \cancel{Pdb} - cdt + k[Q(\mu dt + \sigma dB) - \cancel{Ddt}] + \cancel{Pdb} + bdP$

$= -cdt + Qk(\mu dt + \sigma dB) + bdP$. From $\frac{dP}{P} = rdt$,

$= -cdt + Qk(\mu dt + \sigma dB) + bPrdt$. From $Qk = \theta n$ & $Pb = (1-\theta)n$,

$= -cdt + \theta n(\mu dt + \sigma dB) + (1-\theta)nrdt$

$= [-c + \theta n\mu + nr - \theta nr]dt + \theta n\sigma dB$

$= \underline{[rn + \theta n(\mu - r) - c]dt + \theta n\sigma dB}$, as was to be shown. □

b. The HJB is given by,

$$\rho V(n) = \max_c \left\{ u(c) + [rn + \theta n(\mu - r) - c]\partial_n V(n) + \frac{(\theta n\sigma)^2}{2}\partial_{nn}V(n) \right\}.$$

It is useful to rewrite the household problem as a function of n since we reduce the dimensionality of the problem i.e., from k & b to just n. It is stationary since the parameters $r, \theta, \mu,$ & ρ are constant over time.

c. The F.O.C are given by,

i. $\underline{c}$: $\frac{\partial \rho V}{\partial c} = 0 \iff \underline{u'(c) = \partial_n V(n)}$

ii. $\underline{\theta}$: $\frac{\partial \rho V}{\partial \theta} = 0 \iff n(\mu - r)\partial_n V(n) = \theta (n\sigma)^2 \partial_{nn} V(n)$

$$\iff \underline{\theta = -\frac{(\mu - r)\partial_n V(n)}{n\sigma^2 \partial_{nn} V(n)}}.$$

d. From part c,

$$\left.\begin{aligned} &\text{i. } \theta n \sigma dB = -\frac{(\mu - r)}{\sigma} dB \frac{\partial_n V(n)}{\partial_{nn} V(n)} \\ &\text{ii. } \left[rn - \left(\frac{\mu - r}{\sigma}\right)^2 \frac{\partial_n V(n)}{\partial_{nn} V(n)} - c\right] dt \end{aligned}\right\} dn = \text{i.} + \text{ii.}$$

iii. $u'(c) = \partial_n V(n) \iff du'(c) = dn \cdot \partial_n V(n)$. Then,

$$\rho V(n) = u(c) + [rn + \theta n(\mu - r) - c]\partial_n V(n) + \frac{(\theta n \sigma)^2}{2} \partial_{nn} V(n)$$

$$\iff \rho V(n) - [rn + \theta n(\mu - r) - c]\partial_n V(n) - \frac{(\theta n \sigma)^2}{2} \partial_{nn} V(n) = u(c)$$

$$\iff \rho V(n) - \left[rn - \left(\frac{\mu - r}{\sigma}\right)^2 \frac{\partial_n V(n)}{\partial_{nn} V(n)} - c\right] - \frac{1}{2}\left(\frac{\mu - r}{\sigma^2}\right)^2 n \,\partial_{nn} V(n) = u(c).$$

Putting the above together,

$$\underline{\frac{du_c}{u_c} = (\rho - r)dt - \frac{\mu - r}{\sigma} dB}.$$

e. Given $V(n) = \frac{1}{1-\gamma}\left\{\underbrace{\frac{1}{\gamma}\left[\rho-(1-\gamma)r-\frac{1-\gamma}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]}_{k}\right\}^{-\gamma} n^{1-\gamma}$

Consider the HJB from part c,

$$\rho V(n) = \max_c \left\{u(c) + [rn + \theta n(\mu-r) - c]\partial_n V(n) + \frac{(\theta n\sigma)^2}{2}\partial_{nn}V(n)\right\}.$$

i. $\partial_n V(n) = k^{-\gamma} n^{-\gamma}$

ii. $\partial_{nn} V(n) = -\gamma k^{-\gamma} n^{-\gamma-1} = \frac{-\gamma}{n}\partial_n V(n)$. Assuming $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$,

iii. $u'(c) = \partial_n V(n) = c^{-\gamma} \iff c^{-\gamma} = k^{-\gamma}n^{-\gamma} \iff c^* = kn$.

iv. $\theta = \frac{(\mu-r)\partial_n V(n)}{n\sigma^2\partial_{nn}V(n)} = \frac{(\mu-r)}{n\sigma^2}\frac{n}{\gamma} = \frac{\mu-r}{\gamma\sigma^2}$. Substituting into the HJB gives,

$$\rho V(n) = u(c^*) + \left\{rn + \frac{\mu-r}{\gamma\sigma^2}\cdot n(\mu-r) - c^*\right\}k^{-\gamma}n^{-\gamma} - \frac{(\mu-r)^2 n^{\cancel{2}}\cancel{\sigma^2}}{2\gamma^{\cancel{2}}\sigma^{\cancel{4}2}}\frac{\cancel{\gamma}}{\cancel{n}}k^{-\gamma}n^{-\gamma}$$

$$= \frac{1}{1-\gamma}k^{1-\gamma}n^{1-\gamma} + \left\{r\cancel{n} + \frac{(\mu-r)^2}{\gamma\sigma^2}\cdot\cancel{n(\mu-r)} - k\cancel{n}\right\}k^{-\gamma}n^{1-\gamma} - \frac{(\mu-r)^2}{2\gamma\sigma^2}k^{-\gamma}n^{1-\gamma}$$

$$= k^{-\gamma}n^{1-\gamma}\left[\frac{k}{1-\gamma} + r + \frac{(\mu-r)^2}{\gamma\sigma^2} - k - \frac{(\mu-r)^2}{2\gamma\sigma^2}\right]$$

$$= k^{-\gamma}n^{1-\gamma}\left[\frac{k}{1-\gamma} + r + \frac{(\mu-r)^2}{2\gamma\sigma^2} - k\right]$$

$$= k^{-\gamma}n^{1-\gamma}\left[\frac{\gamma}{1-\gamma}k + r + \frac{1}{2\gamma}\frac{(\mu-r)^2}{\sigma^2}\right].$$

$$= k^{-\gamma}n^{1-\gamma}\left\{\frac{1}{1-\gamma}\left[\rho-(1-\gamma)r-\frac{1-\gamma}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right] + r + \frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right\}$$

$$= k^{-\gamma}n^{1-\gamma}\left\{\frac{\rho}{1-\gamma} - \cancel{r} - \cancel{\frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2} + \cancel{r} + \cancel{\frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2}\right\}$$

$$= \rho\frac{1}{1-\gamma}k^{-\gamma}n^{1-\gamma} = \rho V(n). \quad \square$$

f. From above,

$$c^*(n) = kn \quad \text{\&} \quad \theta = \frac{\mu - r}{\gamma\sigma^2}. \square$$

g. From parts e & f,

$$c^*(n) = kn = \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{1-\gamma}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right]n. \text{ With } \gamma = 1,$$

$$= 1[\rho - 0 - 0]n = \rho n.$$

$$\theta = \frac{\mu - r}{\gamma\sigma^2} = \frac{.06}{(.16)^2} \approx 2.34.$$ In other words, the model suggests placing 234% of net worth into stocks i.e., borrowing money to invest — this number is very high. If n also captures human capital, then this number is less high in practice since numerical net worth is not the entirety of n.