econ 202A; problem set 2; second half

1.a. Given $V(x) = \max\{x, \beta \mathbb{E}[V(x')]\}$

i. $V(\cdot)$:= value function at current-period job offer x; denotes maximum expected net present value given x

ii. x := job offer today/in period t; reflects immediate value

iii. $\beta \mathbb{E}[V(x')]$:= discounted expected value of job offer tomorrow i.e., period $t+1$, x'

a. β := discount rate; $\rho = -\log(\beta)$; reflects the action of waiting has a cost

b. $\mathbb{E}[V(x')]$:= expected value of job offer in period $t+1$, x', which is a random variable w/ distribution $x' \sim U[0,1]$ s.t. $x' \perp\!\!\!\perp x \ \forall x, x'$

Intuitively, this Bellman equation indicates the DM will choose to take the current job offer if the discounted expected value of the job offer in the next period is less than that of the one in the current period, and if not, will draw another job offer. Note that there is some point of indifference where $x = \beta \mathbb{E}[V(x')]$.

1.b. My undergraduate university was in a small college town. As such, options for night life were constrained. Often, my friends & I would be at a given social gathering (i.e., a bar or party), & inevitably the question would arise, should we stay here or venture to somewhere else. The value of being at a location x is a random variable, as people do not know the state of the next location e.g., is the bar empty tonight. Under the assumption that the payoff from being at a location, $x \sim U[0,1]$, we can model this decision process using the above Bellman equation. Here, x reflects the value of staying in the current location, and x' reflects the value of the draw of the next location. There is also a cost to continual draws, as e.g., there are lines to get into establishment, & there is finite time in a night.

1.c. Note that $x \sim U[0,B] \Rightarrow X = [0,B] \subset \mathbb{R}^{\ell}$. Let $C(X)$ be the space of bounded functions $f: X \to \mathbb{R}$, $x, x' \in X$ arbitrary, & let $B: C(X) \to C(X)$ be the given operator.

↳ i. <u>monotonicity:</u> Let $f, g \in C(X)$ s.t. $f(x) \leq g(x)\ \forall x \in X$. By definition, $(Bf)(x) = \max\{x, \beta \mathbb{E}[f(x')]\}$ &
$(Bg)(x) = \max\{x, \beta \mathbb{E}[g(x')]\}$. Note,
$f(x) \leq g(x)\ \forall x \Rightarrow \beta \mathbb{E}[f(x')] \leq \beta \mathbb{E}[g(x')]\ \forall x$. Then,
$(Bf)(x) = \max\{x, \beta \mathbb{E}[f(x')]\} \leq \max\{x, \beta \mathbb{E}[g(x')]\} = (Bg)(x)$.
As such, $\forall x, x' \in X$, $(Bf)(x) \leq (Bg)(x)$. Therefore, B satisfies monotonicity.

ii. <u>discounting:</u> Let $a \geq 0$, $x \in X$, $f \in C(X)$ arbitrary. Then,
$[B(f+a)](x) = \max\{x, \beta \mathbb{E}[f(x') + a]\}$. By linearity of $\mathbb{E}$,

$$
\begin{aligned}
&= \max\{x, \beta \mathbb{E}[f(x')] + \beta \cdot a\} \leq \max\{x + \beta a, \beta \mathbb{E}[f(x')] + \beta a\} \\
&\qquad = \max\{x, \beta \mathbb{E}[f(x')]\} + \beta \cdot a \\
&\qquad = (Bf)(x) + \beta \cdot a.
\end{aligned}
$$

As such, $\exists \delta = \beta \in (0,1)$ s.t. $[B(f+a)](x) \leq (Bf)(x) + \delta a\ \forall a \geq 0$, $f \in C(X)$, & $x \in X$. Therefore, B satisfies discounting.

Putting i. & ii. together, by Blackwell's sufficiency conditions, B is a contraction mapping with modulus β. $\square$

1.d Since $\mathcal{B}$ is a contraction mapping, it has a single fixed point $V \in C(X)$, where V is a function s.t. $V: X \to \mathbb{R}$. Additionally, $\forall w \in C(X)$, $\lim_{n\to\infty} \mathcal{B}^n w = V$. $\square$

1.e. $(\mathcal{B}w)(x) = \max\{x, \beta\mathbb{E}[w(x')]\} = \max\{x, \beta\}$

$(\mathcal{B}^2 w)(x) = \max\{x, \beta\mathbb{E}[(\mathcal{B}w)(x')]\} = \max\{x, \beta\mathbb{E}[\max\{x, \beta\}]\}$

$= \max\{x, \beta[\beta\int_0^\beta dx' + \int_\beta^1 x' dx'] = \max\{x, \beta[\beta^2 + \frac{1}{2}[1-\beta^2]]$

$= \max\{x, \frac{\beta}{2}[\beta^2+1]\} = \max\{x, \frac{\beta^3+\beta}{2}\}$

$(\mathcal{B}^3 w)(x) = \max\{x, \beta\mathbb{E}[(\mathcal{B}^2 w)(x')]\} = \max\{x, \beta\mathbb{E}[\max\{x, \frac{\beta^3+\beta}{2}\}]\}$

$= \max\{x, \beta[\frac{\beta^3+\beta}{2}\int_0^{\frac{\beta^3+\beta}{2}} dx' + \int_{\frac{\beta^3+\beta}{2}}^1 x' dx']\} = \max\{x, \beta[(\frac{\beta^3+\beta}{2})^2 + \frac{1}{2}[1-(\frac{\beta^3+\beta}{2})^2]]\}$

$= \max\{x, \beta[\frac{1}{2}+\frac{1}{2}(\frac{\beta^3+\beta}{2})^2]\}$. Let $\beta_1 = \beta$ & $\beta_2 = \frac{\beta^3+\beta}{2}$.

Then,

$(\mathcal{B}w)(x) = \max\{x, \beta_1\}$, $(\mathcal{B}^2 w)(x) = \max\{x, \frac{\beta^3+\beta}{2}\} = \max\{x, \beta(\frac{\beta_1^2+1}{2})\}$, &

$(\mathcal{B}^3 w)(x) = \max\{x, \beta[\frac{1}{2}+\frac{1}{2}(\frac{\beta^3+\beta}{2})^2]\} = \max\{x, \beta[\frac{\beta_2^2+1}{2}]\}$.

⋮

$(\mathcal{B}^n w)(x) = \max\{x, \beta[\frac{\beta_{n-1}^2+1}{2}]\}$. Since $\beta_n < \beta_{n-1}$, the sequence β_n is monotonically decreasing & bounded below by 0, $\exists x^* \in [0,1]$ s.t. $x^* = \lim_{n\to\infty} s_n$ by the monotone convergence theorem. Then,

$x^* = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \beta[\frac{\beta_{n-1}^2+1}{2}] = \beta[\frac{x^{*2}+1}{2}]$

$\Leftrightarrow 2x^* = \beta x^{*2} + \beta \Leftrightarrow \beta x^{*2} - 2x^* + \beta = 0$

$\Rightarrow x^* = \frac{2 \pm \sqrt{4-4\beta^2}}{2\beta} \Leftrightarrow x^* = \frac{2 \pm 2\sqrt{1-\beta^2}}{2\beta} = \frac{1 \pm \sqrt{1-\beta^2}}{\beta}$.

Since $x^* \in [0,1]$, $x^* = \frac{1-\sqrt{1-\beta^2}}{\beta}$. Since $\rho = -\ln(\beta)$,

$x^* = \frac{1-\sqrt{1-e^{-2\rho}}}{e^{-\rho}} = e^{\rho}[1-(1-e^{-2\rho})^{1/2}]$. Therefore,

$$\lim_{n\to\infty}(B^n w)(x) = \max\{x, x^*\}$$

$$= V(x) = \begin{cases} x^* & \text{if } x \le x^* \\ x & \text{if } x > x^* \end{cases}$$ as was to be shown. $\square$

2.a. Assuming no discounting, the cost in period t for undertaking the project is c plus the additional cost in period $t+1$ if the project is not completed in period t. With probability of completing the project given by $1-p$, the probability of not completing the project is p. In the case of not completing the project in period t, we have a $t+1$ cost of $\mathbb{E}[V(c')]+\ell$, which reflects the expected value of the future draw plus the late fee. Therefore, the Bellman equation is,

$\underline{V(c) = \min\{c + p[\mathbb{E}(V(c')) + \ell], \mathbb{E}[V(c') + \ell]\}}$. Here, we do not need discounting, since we have a late fee $\ell > 0$ (assumption) & p. These elements of the model ensures the agent does not infinitely postpone the project in order to get a lower cost $c \sim U(0,1)$.

2.b. In the lemma from class, for optimal stopping problems, a policy is a best response to a continuation value function $V(x)$ iff the policy function is a threshold rule with cutoff $x^* \equiv \beta\mathbb{E}[V(x')]$. In this scenario, $V(c) = \begin{cases} \mathbb{E}[V(c')] + \ell & \text{if } c^* < c \\ c + p[\mathbb{E}[V(c')] + \ell] & \text{if } c \leq c^* \end{cases}$, where c^*

satisfies, $\quad c^* + p[\mathbb{E}[V(c')] + \ell] = \mathbb{E}[V(c')] + \ell \iff c^* = (1-p)\{\mathbb{E}[V(c')] + \ell\}$

$$\iff c^* = (1-p)\left\{\int_0^{c^*}\left(c + p[\mathbb{E}[V(c')] + \ell]\right)dc + \int_{c^*}^1 (\mathbb{E}[V(c')] + \ell)\,dc + \ell\right\}$$

$$= (1-p)\left\{\tfrac{1}{2}c^{*2} + c^* p\left[\mathbb{E}[V(c')] + \ell\right] + \left[\mathbb{E}[V(c')] + \ell\right] - c^*\left[\mathbb{E}[V(c')] + \ell\right] + \ell\right\}$$

$= (1-p)\left\{\tfrac{1}{2}c^{*2} + \mathbb{E}[V(c') + \ell]\left[c^* p + 1 - c^*\right] + \ell\right\}$. From $c^* = (1-p)\{\mathbb{E}[V(c') + \ell]\}$

$$\iff c^* = (1-p)\left\{\tfrac{1}{2}c^{*2} + \frac{c^*}{1-p}\left[1 + (p-1)c^*\right] + \ell\right\}$$

$$\iff c^* - \tfrac{1}{2}(1-p)c^{*2} - c^*\left[1 + (p-1)c^*\right] = (1-p)\ell$$

$$\iff -\tfrac{1}{2}\cancel{(1-p)}c^{*2} + \cancel{(1-p)}c^{*2} = \cancel{(1-p)}\ell$$

$\iff \tfrac{1}{2}c^{*2} = \ell \iff c^* = \pm\sqrt{2\ell}$. Since $c \in (0,1)$, $\underline{c^* = \sqrt{2\ell}}$. $\square$

Intuitively, c^* not depending on p indicates that at the threshold, a marginal change in the cost of attempting the project is offset by a corresponding opposite change in the late fees, in expected value terms.

2.c. Let $\beta \in (0,1)$. The Bellman equation in the given scenario is,

$V(c) = \min\{c, \beta[\mathbb{E}(V(c')) + \ell]\}$. The threshold in this case is,

$$V(c) = \begin{cases} \beta\{\mathbb{E}[V(c')] + \ell\} & \text{if } c^* < c \\ c & \text{if } c \le c^* \end{cases} \iff c^* = \beta\{\mathbb{E}[V(c')] + \ell\}$$

$$\iff c^* = \beta\left\{\int_0^{c^*} c \, dc + \beta \int_{c^*}^1 \mathbb{E}[V(c') + \ell] \, dc + \ell\right\}$$

$$= \beta\left\{\tfrac{1}{2}c^{*2} + \beta\mathbb{E}[V(c') + \ell][1 - c^*] + \ell\right\}.$$ From $\frac{c^*}{\beta} = \mathbb{E}[V(c')] + \ell$,

$$\iff c^* = \beta\left\{\tfrac{1}{2}c^{*2} + c^*[1 - c] + \ell\right\}$$

$$\iff c^* - \tfrac{\beta}{2}c^{*2} + \beta c^*[c - 1] - \beta\ell = 0$$

$$\iff \tfrac{\beta}{2}c^{*2} + (1-\beta)c^* - \beta\ell = 0$$

$$\Rightarrow c^* = \frac{-(1-\beta) \pm \sqrt{(1-\beta)^2 + 2\beta^2\ell}}{\beta}.$$ Since $-(1-\beta) < 0$ & $c^* \in (0,1)$,

$$c^* = \frac{\beta - 1 + \sqrt{(1-\beta)^2 + 2\beta^2\ell}}{\beta},$$ as was to be shown. $\square$

2.d. The Bellman equation here is,

$V(c) = \min\left\{c + \rho\beta[\mathbb{E}[V(c')] + \ell], \beta[\mathbb{E}[V(c')] + \ell]\right\}$. Similarly to 2.a.-2.c.,

$$\Rightarrow c^* = (1-\rho)\beta[\mathbb{E}[V(c')] + \ell]$$

$$\iff c^* = (1-\rho)\beta\left\{\int_0^{c^*} c + \rho\beta[\mathbb{E}[V(c')] + \ell] \, dc + \int_{c^*}^1 \beta\mathbb{E}[V(c') + \ell] \, dc + \ell\right\}$$

$$= (1-\rho)\beta\left\{\tfrac{1}{2}c^{*2} + \rho\beta c^*[\mathbb{E}[V(c')] + \ell] + \beta\mathbb{E}[V(c') + \ell](1 - c^*) + \ell\right\}$$

$$= (1-\rho)\beta\left\{\tfrac{1}{2}c^{*2} + \tfrac{\rho}{1-\rho}c^{*2} + \tfrac{1}{1-\rho}c^* - \tfrac{1}{1-\rho}c^{*2} + \ell\right\}$$

$$\iff c^* = (1-\rho)\beta\left\{-\tfrac{1}{2}c^{*2} + \tfrac{1}{1-\rho}c^* + \ell\right\}$$

$$\iff \frac{(1-\rho)\beta}{2}c^{*2} + c^* - \beta c^* - (1-\rho)\beta\ell = 0$$

$$\iff \frac{(1-\rho)\beta}{2}c^{*2} + (1-\beta)c^* - (1-\rho)\beta\ell = 0$$

$$\Rightarrow c^* = \frac{-(1-\beta) \pm \sqrt{(1-\beta)^2 + 2(1-\rho)^2\beta^2 \ell}}{(1-\rho)\beta}$$. Since $-(1-\beta) < 0$,

$$\Leftrightarrow c^* = \frac{-(1-\beta) + \sqrt{(1-\beta)^2 + 2(1-\rho)^2\beta^2 \ell}}{(1-\rho)\beta}$$, which depends on ρ.

Then, $$\frac{\partial c^*}{\partial \rho} = \frac{(1-\rho)\beta\left[\frac{1}{2}\left[(1-\beta)^2 + 2(1-\rho)^2\beta^2\ell\right]^{-1/2}\left[-4(1-\rho)\beta^2\ell\right] - (1-\beta) - \sqrt{(1-\beta)^2 + 2(1-\rho)^2\beta^2\ell}\right.}{(1-\rho)\beta} < 0.$$

Thus, c^* varies inversely with ρ. Intuitively, this reflects that if project incompletion probability is higher i.e., ρ, the threshold cost where the DM is indifferent between attempting to complete the project ; postponing is lower, since the DM is more likely to postpone when the chance of success is smaller.

2.e. We can model this scenario as finding the expected number of independent Bernoulli trials required to get one success. The probability of project completion in a given period t is

$\mathbb{P}(\text{attempt project}) \cdot \mathbb{P}(\text{project is completed}) = c^*(1-\rho)$.

The probability the project is completed in period k is then

$\mathbb{P}(t=k) = (1-c^*(1-\rho))^{k-1} c^*(1-\rho)$ from the pmf of the geometric distribution for the number of independent Bernoulli trials requiring one success.

Then, $$\mathbb{E}[t] = c^*(1-\rho)\sum_{t=0}^{\infty} t \cdot [1-c^*(1-\rho)]^{t-1} = c^*(1-\rho)\frac{d}{d\rho}\left[\frac{1}{c^*}\sum_{t=0}^{\infty}[1-c^*(1-\rho)]^t\right]$$

$$= c^*(1-\rho)\left[\frac{d}{d\rho}\left[\frac{1}{c^*}\cdot\frac{1}{c^*(1-\rho)}\right]\right] = c^*(1-\rho)\left[\frac{1}{c^{*2}(1-\rho)^2}\right] = \frac{1}{c^*(1-\rho)}.$$ To obtain the number of periods preceding completion, we subtract 1, which gives,

$$\frac{1}{c^*(1-\rho)} - 1 = \mathbb{E}[\text{delay}]$$, as was to be shown. $\square$

3.a. The flow budget constraint is given by,

$$\frac{d}{dt}(P_t a_t) = i_t(P_t a_t) - P_t c_t + P_t y_t \iff P_t \dot{a}_t + \dot{P}_t a_t = i_t(P_t a_t) - P_t c_t + P_t y_t$$

Substituting $i_t = r_t + \Pi_t$ gives,

$$\iff P_t \dot{a}_t + \dot{P}_t a_t = (r_t + \Pi_t)(P_t a_t) - P_t c_t + P_t y_t$$

$$\iff \dot{a}_t = -\frac{\dot{P}_t}{P_t} a_t + (r_t + \Pi_t) a_t - c_t + y_t$$

$$\iff \dot{a}_t = (r_t + \Pi_t - \Pi_t) a_t - c_t + y_t$$

$$\iff \underline{\dot{a}_t = r_t a_t - c_t + y_t.}$$

3.b. We can obtain the lifetime budget constraint by solving the above differential equation. The above is a linear ODE with standard form,

$\dot{a}_t - r_t a_t = y_t - c_t$. Let the integrating factor be $e^{-\int_0^t r_s \, ds} = e^{-R(t)}$.

$$\iff \int_0^\infty \frac{d}{dt}\left[a_t e^{-R(t)}\right] dt = \int_0^\infty e^{-R(t)} [y_t - c_t] \, dt$$

$$\iff a_t e^{-R(t)} \Big|_{t=0}^{\infty} dt = \int_0^\infty e^{-R(t)} [y_t - c_t] \, dt$$

$$\iff \lim_{t \to \infty} \cancel{a_t e^{-\int_0^t r_s ds}} - a_0 e^0 = \int_0^\infty e^{-R(t)} [y_t - c_t]$$

$$\iff \underline{\int_0^\infty e^{-R(t)} c_t \, dt = a_0 + \int_0^\infty e^{-R(t)} y_t \, dt, \text{ where } R(t) = \int_0^t r_s \, ds}$$

3.c. The Lagrangian is given by,

$$\mathcal{L} = \int_0^\infty e^{-\rho t} u(c_t) \, dt + \lambda \left[a_0 + \int_0^\infty e^{-R(t)} y_t \, dt - \int_0^\infty e^{-R(t)} c_t \, dt\right]. \text{ Then,}$$

$\underline{\text{F.O.C:}}$ $\frac{\partial \mathcal{L}}{\partial c_t} = 0 \iff e^{-\rho t} u'(c_t) = \lambda e^{-R(t)}$

$$\iff u'(c_t) = \lambda e^{-R(t) + \rho t} = \lambda e^{-\int_0^t r_s ds + \rho t}$$

Differentiating w.r.t. time gives,

$$u''(c_t)\,\dot{c}_t = \lambda e^{-R(t)+\rho t}(\rho - r_t)$$

$$\Leftrightarrow \frac{u''(c_t)\,\dot{c}_t}{u'(c_t)} = (\rho - r_t)$$

$$\Leftrightarrow \dot{c}_t = \frac{u'(c_t)}{u''(c_t)}(\rho - r_t)$$

$$\Leftrightarrow \underline{\frac{\dot{c}_t}{c_t} = \frac{u'(c_t)}{u''(c_t)\,c_t}(\rho - r_t)}.$$

We can use the lifetime budget constraint since this model is deterministic i.e., both y_t & r_t are known. If this were not the case, we would not be able to use the lifetime budget constraint.

3.d.

i. $\underline{u(c) = \frac{1}{1-\gamma} c_t^{1-\gamma}:}$ $u'(c_t) = c_t^{-\gamma}$, $u''(c_t) = -\gamma c_t^{-\gamma-1}$

$$\Rightarrow \frac{u'(c_t)}{u''(c_t)\,c_t} = \frac{c_t^{-\gamma}}{-\gamma c_t^{-\gamma-1} c_t} = \underline{-\frac{1}{\gamma}} \Rightarrow \underline{\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\gamma}}.$$

ii. $\underline{u(c) = \ln c_t:}$ $u'(c_t) = \frac{1}{c_t}$, $u''(c_t) = -\frac{1}{c_t^2}$

$$\Rightarrow \frac{u'(c_t)}{u''(c_t)\,c_t} = \frac{c_t^{-1}}{-\frac{1}{c_t^2} c_t} = -\frac{1}{\frac{c_t^2}{c_t^2}} = \underline{-1} \Rightarrow \underline{\frac{\dot{c}_t}{c_t} = r_t - \rho}.$$

3.e. $V(a_0) = \max_{\{c_t\}_{t\geq 0}} \left\{ \int_0^\infty e^{-\rho t} u(c_t)\,dt \right\}$ subject to,

$\dot{a}_t = r_t a_t + y_t - c_t$. Here, a_t is the state variable & c_t is the control variable.

3.f. $H(c_t, a_t, \underbrace{\mu_t}_{\text{multiplier}}) = u(c_t) + \mu_t[r_t a_t + y_t - c_t]$

3.g. i. optimality condition: $\frac{\partial}{\partial c_t} H = 0$

$$\Leftrightarrow \quad \underline{u'(c_t) = \mu_t} \quad \Rightarrow \frac{\partial \mu_t}{\partial t} = \dot{\mu}_t = u''(c_t)\, \dot{c}(t)$$

ii. multiplier condition: $\rho \mu_t - \dot{\mu}_t = \frac{\partial}{\partial a_t} \mathcal{H}$

$$\Leftrightarrow \quad \rho \mu_t - \dot{\mu}_t = \mu_t r_t$$

$$\Leftrightarrow \quad \underline{\dot{\mu}_t = (\rho - r_t)\mu_t}$$

iii. state condition: $\underline{\dot{a}_t = r_t a_t + y_t - c_t}$. From the multiplier & optimality conditions,

$$u''(c_t)\,\dot{c}_t = (\rho - r_t)\, u'(c_t) \Leftrightarrow \underline{\frac{\dot{c}_t}{c_t} = \frac{u'(c_t)}{u''(c_t)\, c(t)} (\rho - r_t)},$$ which

corresponds to part 3.c.

$\square$

4.a. V depends explicitly on calendar time since r_t & y_t are exogenous & change over time. The only state variable is a, since it is the only endogenous variable that affects future utility.

4.b. The sequence problem is given by,

$$V_0(a_0) = \max_{\{c_t\}_{t>0}} \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\}$$

$$= \max_{\{c_t\}_{t>0}} \left\{ \int_0^{\Delta t} e^{-\rho t} u(c_t) dt + \int_{\Delta t}^\infty e^{-\rho t} u(c_t) dt \right\}$$

$$= \max_{\{c_t\}_{t>0}} \left\{ u(c_0)\Delta t + e^{-\rho \Delta t} \int_0^\infty e^{-\rho t} u(c_{t+\Delta t}) dt \right\}$$

$$= \max_{c_0} \left\{ u(c_0)\Delta t + e^{-\rho \Delta t} \max_{\{c_t\}_{t>0}} \left\{ \int_0^\infty e^{-\rho t} u(c_{\Delta t}) dt \right\} \right\}$$

$$= \max_{c_0} \left\{ u(c_0)\Delta t + e^{-\rho \Delta t} V_{\Delta t}(a_{\Delta t}) \right\}, \text{ where } a_{\Delta t} = a_0 + \dot{a}_0 \Delta t$$

$$= a_0 + [r_0 a_0 + y_0 - c_0] \Delta t$$

Taylor expanding $V_{\Delta t}(a_{\Delta t})$ gives,

$$= \max_{c_0} \left\{ u(c_0)\Delta t + \underbrace{e^{-\rho \Delta t}}_{\approx 1-\rho \Delta t} \left[V_0(a_0) + \partial_t V_0(a_0) + \partial_a V_0(a_0) [r_0 a_0 + y_0 - c_0] \Delta t \right] \right\}$$

$$= \max_{c_0} \left\{ u(c_0)\Delta t + (1-\rho\Delta t) V_0(a_0) + (1-\rho\Delta t)\partial_t V_0(a_0) + (1-\rho\Delta t)\partial_a V_0(a_0)[r_0 a_0 + y_0 - c_0]\Delta t \right\}$$

$$\Leftrightarrow 0 = \max_{c_0} \left\{ u(c_0) - \rho V_0(a_0) + \partial_t V_0(a_0) + \partial_a V_0(a_0)[r_0 a_0 + y_0 - c_0] \right\}$$

$$\Leftrightarrow \underline{\rho V_t(a) = \partial_t V_t(a) + \max_c \left\{ u(c) + [r_t a + y_t - c] \partial_a V_t(a) \right\}} \; . \square$$

4.c. Since V_t is dependent on calendar time, changes in t (; thus y_t ; r_t) affects the value function directly. Hence, we have a $\partial_t V_t(a)$ term.

4.d. $\frac{\partial}{\partial c}\left\{u(c) + [r_t a + y_t - c]\partial_a V_t(a)\right\} = 0$

$\Leftrightarrow u'(c) = \partial_a V_t(a)$, where $u'(c) :=$ marginal utility from a one-unit increase in consumption ; $\partial_a V_t(a)$ is the marginal value of a one-unit increase in wealth at time t.

4.e. Substituting gives,

$\rho V_t(a) = \partial_t V_t(a) + \max_c \left\{u(c) + [r_t a + y_t - c]u'(c)\right\}$, which is nonlinear since we are trying to solve for V ; $u'(c)$ is dependent on V.

4.f. Differentiating HLB w.r.t. a gives,

$$\rho \partial_a V_t(a) = \partial_t(\partial_a V_t(a)) + r_t \partial_a V_t(a) + [r_t a + y_t - c_t]\partial_{aa} V_t(a)$$

$$\Leftrightarrow (\rho - r_t)\partial_a V_t(a) = \partial_t(\partial_a V_t(a)) + [r_t a + y_t - c_t]\partial_{aa} V_t(a)$$

$$(\rho - r_t)\partial_a V_t(a) = \partial_t(\partial_a V_t(a)) + \dot{a}_t \partial_{aa} V_t(a)$$

4.g. $$\frac{dV_t(a_t)}{dt} = \partial_t V_t(a) + \partial_a V_t(a) \cdot \dot{a}_t .$$

4.h. From the F.O.C., $u'(c) = \partial_a V_t(a)$, $u''(c)\dot{c}_t = \partial_t \partial_a V_t(a)$.
Substituting into the envelope condition gives,

$(\rho - r_t) u'(c_t) = u''(c)\dot{c}_t + \dot{a}_t \partial_a \partial_a V_t(a)$. From part g,

$\partial_a V_t(a)\dot{a}_t = 0 \Leftrightarrow \partial_a \partial_a V_t(a)\dot{a}_t = 0$

$$\Leftrightarrow (\rho - r_t) u'(c_t) = u''(c)\dot{c}_t \Leftrightarrow \frac{\dot{c}_t}{c_t} = \frac{u'(c_t)}{u''(c_t)c_t}(\rho - r_t). \quad \square$$

5.a. Given $x_t = x_0 - \int_0^t c_s ds$, we see that, since oil is non-renewable, oil at time t (x_t) is the initial amount of oil x_0 less oil consumed from period $s=0$ to $s=t$, $\int_0^t c_s ds$.

5.b. From $x_t = x_0 - \int_0^t c_s ds$, $\dot{x}_t = -c_t$. The Hamiltonian is then,

$$H(c_t, \mu_t) = u(c_t) + \mu_t[-c_t], \text{ where}$$

$c_t :=$ control variable, $\mu_t :=$ multiplier, $x_t :=$ state variable (implicit)

5.c. i. <u>optimality condition:</u> $\frac{\partial H}{\partial c} = 0 \iff u'(c_t) = \mu_t$

ii. <u>multiplier condition:</u> $\rho\mu_t - \dot{\mu}_t = \frac{\partial}{\partial x_t} H$

$$\iff \rho\mu_t = \dot{\mu}_t \iff \int \rho\mu_t dt = \int \dot{\mu}_t dt \iff \mu_t = \mu_0 e^{\rho t}$$

Putting the above together, $\mu_0 e^{\rho t} = u'(c_t)$. From $u(c_t) = \ln(c_t)$,

$$\mu_0 e^{\rho t} = \frac{1}{c_t}$$

$$\iff c_t^* = \frac{1}{\mu_0 e^{\rho t}}. \text{ From } \int_0^\infty c_t dt = x_0,$$

$$\int_0^\infty \frac{1}{\mu_0 e^{\rho t}} dt = x_0 \iff x_0 = \frac{1}{\mu_0}\frac{1}{\rho} \iff \mu_0 = \frac{1}{\rho x_0}$$

$$\Rightarrow c_t^* = \frac{\rho x_0}{e^{\rho t}}$$

5.d. HJB is given by,

$$\rho V(x) = \max_c \{u(c) + V'(x)\dot{x}_t\}$$

$$\iff \rho V(x) = \max_c \{u(c) - c_t V'(x)\}$$

$$\Leftrightarrow \underline{\rho V(x) = \max_{c} \{ \ln(c_t) - c_t V'(x) \}}.$$

S.e., $V(x) = a + b\ln x \Rightarrow V'(x) = \frac{b}{x}$. $\frac{\partial \rho V(x)}{\partial c} = 0$

$\Leftrightarrow c_t^* = \frac{x}{b}$.

Substituting,

$$\rho[a + b\ln x] = \ln\left(\frac{x}{b}\right) - 1$$

$$\rho[a + b\ln x] = \ln x - \ln b - 1$$

$$\Rightarrow \rho b = 1 \Leftrightarrow \underline{b = \frac{1}{\rho}}.$$

$$\Rightarrow \rho a = -\ln\left(\frac{1}{\rho}\right) - 1 \Leftrightarrow \underline{a = \frac{\ln(\rho) - 1}{\rho}}.$$