problem_sets/problem_set_5/problem_set_5_question_1_2.m

```matlab
%{
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Purpose: Coding part of problem set 5
Created: Nico Rotundo 2024-12
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%}

%% Define parameters and initialize grids

%{
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Define parameters
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%}

clear all;

% Relative risk aversion coefficient
sigma = 2;

% Productivity
z_u = 1;
z_e = 2;
z = [z_u, z_e];

% Transition rates
lambda_u = 1/3;
lambda_e = 1/3;
lambda = [lambda_u, lambda_e];

% Discount rate
rho = 0.05;

% Capital depreciation rate
delta = 0.05;

% Capital share in production
alpha = 1/3;

% TFP
A = .1;

% Interest rate for partial equilibrium part of the problem set
r = 0.035;

%{
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Define capital grid
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%}

% Borrowing constraint minimum
k_min = 0;

% Borrowing constraint maximum
k_max = 20;

% Number of grid points
num_points = 1000;

k_grid = linspace(k_min, k_max, num_points)';
dk = (k_max-k_min)/(num_points-1);

kk = [k_grid, k_grid]; % I*2 matrix

%{
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Define utility function and its derivative
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%}

% Utility function (CRRA), handling vector inputs
U = @(c) (c.^(1 - sigma)) / (1 - sigma);

% Derivative of utility, handling vector inputs
U_prime = @(c) c.^(-sigma);

% Inverse of derivative of utility
U_prime_inv = @(Vp) (Vp).^(-1 / sigma);

%{
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Firm production technology and FOCs
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%}

% Production technology Y = A K^alpha L^{1-alpha}
Y = @(K,L) K.^alpha*L.^(1-alpha);

% F.O.C.:
w_foc = @(K,L) (1-alpha)*A*(K/L).^alpha;
r_foc = @(K,L) alpha*A*(L/K).^(1-alpha);
K_partial = @(L,r) L*((alpha*A/(r+delta)).^(1/(1-alpha))) ;

%{
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Tuning parameters (general)
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%}

% Step size: can be arbitrarily large in implicit method
Delta = 1000;

% The maximum number of value function iterations
max_iterations_vf = 100;

% Tolerance for value function iterations
tolerance = 10^(-6);

 %{
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Tuning parameters (interest rate iteration)
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%}

% The maximum number of interest rate iterations
num_iterations_r = 1000;

% Tolerance for interest rate iterations
tolerance_S = 10^(-5);

%% Initialize matrices before iterating

%{
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Define differential operator
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%}


% Forward; satisfies Df*V=dVf
Df = zeros(num_points, num_points);
for i = 1:num_points-1
    Df(i,i) = -1/dk; Df(i,i+1) = 1/dk;
end
Df = sparse(Df);

% Backward; satisfies Db*V=dV
Db = zeros(num_points, num_points);
for i = 2:num_points
    Db(i,i-1) = -1/dk; Db(i,i) = 1/dk;
end
Db = sparse(Db);

%{
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Define A-switch matrix
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%}

A_switch = [speye(num_points).*(-lambda(1)), speye(num_points).*lambda(1);
            speye(num_points).*lambda(2), speye(num_points).*(-lambda(2))];


%{
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Calculate labor, capital, and wage levels
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%}

% Labor
L = (z_e*lambda_u + z_u*lambda_e)/(lambda_e+lambda_u);

% Capital
K = K_partial(L,r);

% Wage
w = w_foc(K,L);

%{
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Define initial guesses
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%}

% Guess for initial value of the interest rate
r_0_guess = 0.03;

% Set bounds on interest rate
r_min = 0.01;
r_max = 0.04;

% Define the number of points * 2 matrix
z = ones(num_points, 1).*z;

% Initial guess for value function
V_0 = U(w.*z + r.*kk)./rho;
V = V_0;

%% Value function iteration

% Loop over number of interations for the value function
for n=1:max_iterations_vf

    % Derivative of the forward value function
    dVf = Df*V;

    % Derivative of the backward value function
    dVb = Db*V;

    % Boundary condition on backwards value function i.e., the borrowing constraint; a>=a_min
    dVb(1,:) = U_prime(w.*z(1,:) + r.*kk(1,:));

    % Boundary condition on forward value function; a<=a_max
    dVf(end,:) = U_prime(w.*z(end,:) + r.*kk(end,:));

    % Indicator whether value function is concave; for stability purposes
    I_concave = dVb > dVf;

    % Compute optimal consumption using forward derivative
    cf = U_prime_inv(dVf);

    % Compute optimal consumption using backward derivative
    cb = U_prime_inv(dVb);

    % Compute optimal savings using forward derivative
    sf = w.*z + r.*kk - cf;

    % Compute optimal savings using backward derivative
    sb = w.*z + r.*kk - cb;

    % Upwind scheme
    If = sf>0;
    Ib = sb<0;
    I0 = 1-If-Ib;
    dV0 = U_prime(w.*z + r.*kk); % If sf<=0<=sb, set s=0

    dV_upwind = If.*dVf + Ib.*dVb + I0.*dV0;

    c = U_prime_inv(dV_upwind);

    % Update value function
    V_stacked = V(:);

    % Update consumption function
    c_stacked = c(:);

    % A = SD
    SD_u = spdiags(If(:,1).*sf(:,1), 0, num_points, num_points)*Df + spdiags(Ib(:,1).*sb(:,1), 0, num_points, num_points)*Db;
    SD_e = spdiags(If(:,2).*sf(:,2), 0, num_points, num_points)*Df + spdiags(Ib(:,2).*sb(:,2), 0, num_points, num_points)*Db;
    SD = [SD_u, sparse(num_points, num_points);
        sparse(num_points, num_points), SD_e];

    % P = A + A_switch
    P = SD + A_switch;

    % B = [(rho + 1/Delta)*I - P]
    B = (rho + 1/Delta)*speye(2*num_points) - P;

    % b = u(c) + 1/Delta*V
    b = U(c_stacked) + (1/Delta)*V_stacked;

    % V = B\b;
    V_update = B\b;
    V_change = V_update - V_stacked;
    V = reshape(V_update, num_points, 2);

    % Convergence criterion
    dist(n) = max(abs(V_change));

    if dist(n)<tolerance
        disp('Value function converged. Iteration = ')
        disp(n)
    break

    end
end

%% KF Equation

% Solve for 0=gdot=P'*g
PT = P';
PT_eigs = PT;
gdot_stacked = zeros(2*num_points,1);

% Fix one value to obtain a non-singular matrix, otherwise matrix is singular
i_fix = 1;
gdot_stacked(i_fix)=.1;

row_fix = [zeros(1,i_fix-1),1,zeros(1,2*num_points-i_fix)];
PT(i_fix,:) = row_fix;

g_stacked = PT\gdot_stacked;

% Normalization
g_sum = g_stacked'*ones(2*num_points,1)*dk;
g_stacked = g_stacked./g_sum;

% Reshape
gg = reshape(g_stacked, num_points, 2);

% Solve KF equation
[g_stacked_eigs, eigval] = eigs(PT_eigs, 1, 0);
g_sum_eigs = g_stacked_eigs'*ones(2*num_points,1)*dk;
g_stacked_eigs = g_stacked_eigs./g_sum_eigs;

%% Plots

%{
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Question 1: Optimal consumption
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%}

% Initialize plot
set(gca, 'FontSize', 18)

    % Employed
    plot(k_grid, c(:,2), 'LineWidth', 2, 'LineStyle', '-', 'Color', [41/255, 182/255, 164/255])
    hold on

    % Unmployed
    plot(k_grid, c(:,1), 'LineWidth', 2, 'LineStyle', '-', 'Color', [250/255, 165/255, 35/255])

    hold off

    grid
        xlabel('Capital, k','FontSize', 14)

        set(gca, 'TickDir', 'out');
            box off;

        ylabel('Consumption, c_j(k)','FontSize', 14)

        xlim([k_min k_max])

        legend(sprintf('Employed'), ...
            sprintf('Unemployed'), ...
            sprintf('r=%.4f', r), 'Location', 'best', 'FontSize', 14)

% Export graph
exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_5/output/question_1_consumption.pdf', 'ContentType', 'vector');

%{
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Question 1: Optimal savings
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%}

% Calculate savings
kdot = w.*z + r.*kk - c;

% Initialize plot
set(gca, 'FontSize', 18)

    % Employed
    plot(k_grid, kdot(:,2), 'LineWidth', 2, 'LineStyle', '-', 'Color', [41/255, 182/255, 164/255])
    hold on

    % Unemployed
    plot(k_grid, kdot(:,1), 'LineWidth', 2, 'LineStyle', '-', 'Color', [250/255, 165/255, 35/255])

    hold off

    grid
        xlabel('Capital, k', 'FontSize', 14)

        ylabel('Saving, s_j(k)', 'FontSize', 14)

        set(gca, 'TickDir', 'out');
            box off;

        xlim([k_min k_max])

        legend(sprintf('Employed'), ...
            sprintf('Unemployed'), ...
            sprintf('r=%.4f', r), 'Location', 'best', 'FontSize', 14)

% Export graph
exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_5/output/question_1_savings.pdf', 'ContentType', 'vector');

%{
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Question 1: Value function
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%}

% Initialize plot
set(gca, 'FontSize', 18)

    % Employed
    plot(k_grid, V(:,2), 'LineWidth', 2, 'LineStyle', '-', 'Color', [41/255, 182/255, 164/255])
    hold on

    % Unemployed
    plot(k_grid, V(:,1), 'LineWidth', 2, 'LineStyle', '-', 'Color', [250/255, 165/255, 35/255])

    hold off

    grid
        xlabel('Capital, k', 'FontSize', 14)

        ylabel('Value Function, V_j(k)', 'FontSize', 14)

        set(gca, 'TickDir', 'out');
            box off;

        xlim([k_min k_max])

        legend(sprintf('Employed'), ...
            sprintf('Unemployed'), ...
            sprintf('r=%.4f', r), 'Location', 'best', 'FontSize', 14)

% Export graph
exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_5/output/question_1_value_function.pdf', 'ContentType', 'vector');

%{
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Question 2: Stationary distribution
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%}

% Initialize plot
set(gca, 'FontSize', 18)

    % Employed
    plot(k_grid, gg(:,2), 'LineWidth', 2, 'LineStyle', '-', 'Color', [41/255, 182/255, 164/255])
    hold on

    % Unemployed
    plot(k_grid, gg(:,1), 'LineWidth', 2, 'LineStyle', '-', 'Color', [250/255, 165/255, 35/255])

    hold off

    grid
        xlabel('Capital, k', 'FontSize', 14)

        ylabel('Densities, g_j(k)', 'FontSize', 14)

        set(gca, 'TickDir', 'out');
            box off;

        xlim([-.01 1])

        legend(sprintf('Employed'), ...
            sprintf('Unemployed'), ...
            sprintf('r=%.4f', r), 'Location', 'best', 'FontSize', 14)

% Export graph
exportgraphics(gcf, '/Users/nicorotundo/Documents/GitHub/DynamicProgramming2024/problem_sets/problem_set_5/output/question_2.pdf', 'ContentType', 'vector');
```