[redacted]

- 数域[redacted]

[redacted]义1：[redacted]

> K [redacted]
>
> K [redacted]
>
> K [redacted]
>
> - $\mathbb{Z}$ [redacted]
> - $\mathbb{Q}$ [redacted]
> - $\mathbb{R}$ [redacted]
>
> [redacted]

2 [redacted]

> $st. f(\alpha)=0$ α $f(x)$

4 $S=\{a+b\sqrt{2}\mid a,b\in\mathbb{Z}\}$ 不 。

> S $\mathbb{Z}\subseteq S$
>
> $1\in S\Rightarrow 2\in S\Rightarrow \frac{1}{2}\in S$ $\frac{1}{2}=a+b\sqrt{2}$ $a,b\in\mathbb{Z}$
>
> $b\neq 0$ $\sqrt{2}=\frac{a-\frac{1}{2}}{b}\in\mathbb{Q}$

5 $S=\{a\sqrt{2}\mid a\in\mathbb{R}\}$ 不 。

> S $\sqrt{2}\in S\Rightarrow 2=\sqrt{2}\cdot\sqrt{2}\in S$
>
> $2=a\sqrt{2}$ $a\in\mathbb{Q}\Rightarrow a=\sqrt{2}\notin\mathbb{Q}$

理2 任一 K $\mathbb{Q}$ $\mathbb{Q}$ 。

> $\forall a\in K\Rightarrow 0=a-a\in K$
>
> K b $1=\frac{b}{b}\in K$
>
> $$\{\forall m\in\mathbb{Z}^+,\quad m=(1+\cdots+1)\in K,\quad -m=0-m\in K\}\Rightarrow\mathbb{Z}\subseteq K$$
>
> $\frac{m}{n}\in\mathbb{Q}$ $n\in\mathbb{Z}^+$ $m\in\mathbb{Z}^+$ $n\in K$ $m\in K\Rightarrow\frac{m}{n}\in K$ $\mathbb{Q}\subseteq K$

- ## 线性空间

义1

> K $a_1,a_2,\ldots,a_n\in K$
>
> $1\times n$ $[a_1,a_2,\ldots,a_n]$ K n $n\times 1$ $\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}$ K n
>
> $K_n=\{(a_1,\ldots,a_n)\mid a_i=k,i\in\mathbb{N}\}$ K n
>
> $K^n=\{\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}\mid a_i\in K,i\in\mathbb{N}\}$ K n

$$\alpha=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix},\quad \beta=\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}\in K^n,\quad c\in K$$

$$\alpha-\beta\overset{\text{def}}{=}\begin{pmatrix}a_1-b_1\\\vdots\\a_i-b_i\end{pmatrix},\quad\forall i\in\mathbb{N}\qquad \alpha+\beta\overset{\text{def}}{=}\begin{pmatrix}a_1+b_1\\\vdots\\a_n+b_n\end{pmatrix}\qquad c\cdot\alpha\overset{\text{def}}{=}\begin{pmatrix}ca_1\\\vdots\\ca_n\end{pmatrix}$$

> $\alpha,\beta,\gamma\in K^n(K_n),k\in K$
>
> $\alpha+\beta=\beta+\alpha$
>
> $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$
>
> $\alpha+0=\alpha$
>
> $\alpha+(-\alpha)=0$

[redacted] 代 [redacted]

[redacted] V [redacted]

[redacted] $K \times V \to V$ [redacted]

[redacted] V [redacted]

[redacted] 中 [redacted] 于 [redacted] 1+1 于 [redacted]
义 [redacted]

[redacted] 个 [redacted] 于 [redacted] 0. 们 [redacted] 以 [redacted]
。

[redacted] 个 [redacted]

$$f(\alpha+\beta)=f(\alpha)+f($$ [redacted] $$(k\alpha)=kf(\alpha)$$ [redacted]

[redacted] V [redacted] α [redacted] β [redacted] $k \in K$ [redacted]

[redacted] K_n [redacted] K 上 [redacted]

[redacted] $\{a_0x^n+\cdots+a_n \mid x$ [redacted] $, a_i \in K, i \in$ [redacted]

命[redacted] V[redacted]

0 [redacted]

k [redacted]

— [redacted]

Co[redacted]tio[redacted]

(1) [redacted]

(2) $\alpha +$ [redacted]

Part [redacted]

$_2, \ldots, \alpha_n)$ 为 中

$$\alpha_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in K^m$$

$A\mathbf{x} = (\alpha$

$\alpha_1,$ [redacted] $\ldots, \alpha_n$ 线性 [redacted]

Import [redacted]

(1) [redacted] 价 义

$k_1, \ldots,$ [redacted] $\in K$

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = 0$$

k_1 [redacted] $= \cdots = k_n = 0$。

(2) 于 K $k_1, \ldots, k_n \in K$

[redacted] 组: 个 。

[redacted]

理

> $1 \le m \le n$
>
> $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$
>
> $$\{\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_n\}$$
>
> $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$
>
> $$\{\alpha_1, \alpha_2, \dots, \alpha_m\}$$
>
>> (1) K **不全为零** $k_1, \dots, k_m$
>>
>> $$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m = 0$$
>>
>> $$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m + 0 \cdot \alpha_{m+1} + \cdots + 0 \cdot \alpha_n = 0$$
>>
>> $\Rightarrow$
>>
>> (2)

理4 V_K $\alpha_1, \dots, \alpha_n \in V$. $\alpha_1, \dots, \alpha_n$ $\iff \exists\, 1 \le i \le n$ α_i $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n$.

> **证明：**
>
> $\Leftarrow$
>
> α_1 $\alpha_2, \dots, \alpha_n$
>
> $$\exists\, k_2, \dots, k_n \in K, \quad \alpha_1 = k_2\alpha_2 + \cdots + k_n\alpha_n$$
>
> $$(-1)\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = 0$$
>
> $\Rightarrow \alpha_1, \dots, \alpha_n$
>
> $\Rightarrow$
>
> $\alpha_1, \dots, \alpha_n$ $k_1, \dots, k_n \in K$
>
> $$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = 0$$
>
> $k_1 \neq 0$
>
> $$\alpha_1 = -\frac{k_2}{k_1}\alpha_2 - \cdots - \frac{k_n}{k_1}\alpha_n \quad (*)$$

理5

> V_K $\alpha_1, \dots, \alpha_n, \beta \in V$ $\alpha_1, \dots, \alpha_n$
>
> - $\alpha_1, \dots, \alpha_n, \beta$
> - β $\alpha_1, \dots, \alpha_n$

理6 [redacted]

> $\alpha_1, \ldots,$ [redacted]

理7 [redacted]

> [redacted]
>
> A [redacted]
>
> C [redacted]

/example/ [redacted]

$V=\mathbb{R}^3$ [redacted]

$\overrightarrow{OA}=(a_1, a$ $_3)$

- $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ [redacted] $\iff O, A$ $, C$
- $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ [redacted] $\iff OAB$

推广:

[redacted] $\alpha_1,\alpha_2,\ldots,\alpha_n$ [redacted]

$\iff$

[redacted] 组 秩

[redacted]

> S V [redacted] S
>
> $\alpha_1,\alpha_2,\cdots,\alpha_r$
>
> S [redacted] $\alpha_1,\alpha_2,\cdots,\alpha_r$
>
> $\{\alpha_1,\alpha_2,\cdots,\alpha_r\}$ S 极大线性无关组 [redacted]

Important

- $\{\alpha_1,\alpha_2,\cdots,\alpha_r\}$ 。
- $\forall\alpha\in S$ $\{\alpha_1,\alpha_2,\cdots,\alpha_r,\alpha\}$ 。

命题2 S 。

引理5 A, B 为两个 A 任一 B 中 且 B 任一 A 中 $\#A = \#B$。

推论6 A, B S $\#A = \#B$。

> 证明
>
> - A, B
> - $A \subseteq S \overset{\text{linear}}{\to} B$
> - $B \subseteq S \overset{\text{linear}}{\to} A$
>
> $\Rightarrow \#A = \#B$

义7 S 中 个 为 S 为 $\operatorname{rank}(S)$ $r(S)$。 6 义不 于 。 为 0.

义8 A, B 两个 A 中 个 以 B 中 且 B 中 个 也 以 A 中 A, B 为 价 。

推论9 价 。

> A B $r(A) = r(B) = 0$
>
> A, B
>
> A_1 A B_1 B
>
> $r(A) = \#A_1$ $r(B) = \#B_1$
>
> A_1, B_1
>
> $$A_1 \subseteq A \overset{\text{linear}}{\to} B \overset{\text{linear}}{\to} B_1$$
> $$B_1 \subseteq B \overset{\text{linear}}{\to} A \overset{\text{linear}}{\to} A_1$$
>
> $A_1 = B_1$ $\#A_1 = \#B_1$ $r(A) = r(B)$
>
> > 结论

$S = V_k$ $\to$ $\to$

义10 V_k 为 V 中 $\{e_1, e_2, \ldots, e_n\}$ V 中任一 $\{e_1, e_2, \ldots, e_n\}$ $\{e_1, e_2, \ldots, e_n\}$ 为 V 一 V 为 n 为 $\dim_k V = n$ V 为 k 上 n 。 不 个 V 一 V 为 。

> $\Rightarrow V_k$
>
> $\Rightarrow V_k$

推论11 n V 中 n 个 。

> 📌 注

理12

> $e_1, e_2, \ldots, e_n$
>
> V $e_1, e_2, \ldots, e_n$

$, \ldots,$

di

$\{e_1,$

$\{e_1, \ldots$

V 为

$n \leq m$

v_m, e_i

理14 》 V 为 n

1. V 中任一 以 为 V 一

2. U 以 》为 V 一 。

$\varphi: V \to U$ 为 **线性同构** [redacted] V [redacted]

理 义 $\varphi: V \to$ [redacted]

> 证明 $\alpha, \beta \in V$
>
> [redacted] e_i
>
> [redacted]
>
> $$\alpha + \beta = \sum_{i=1}^{n}(a_i$$ [redacted] $$e_i =$$ [redacted] $$+ \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \varphi(\alpha) + \varphi(\beta)$$
>
> $k \in K$
>
> $$k$$ [redacted] $$\sum_{i=1}^{n}$$ [redacted] $$= k\varphi(\alpha)$$
>
> $\Rightarrow \varphi$ [redacted]

> **Important**
>
> - [redacted] 为 [redacted]
> - [redacted] 为 [redacted]
> - [redacted] 为 [redacted]

理4

(1) $\varphi: V \to U$

- [redacted] 为 [redacted]
- [redacted] 为 [redacted]
- [redacted] 为 [redacted]

(2) 3 $\varphi: V$ [redacted] K^n [redacted] 。[redacted] 令 $\tilde{\alpha}_i = \varphi(\alpha_i)$

[redacted] $\{\alpha$ [redacted]

[redacted]

$\{0\}$

$\cdots + \lambda_m \varphi(\alpha_m)$

$\alpha_1 \quad +$

$\Rightarrow \varphi($

$\{\alpha$

$\{\varphi$

理由

-
-
-

$n \times n$ 为

$$A = (a_{ij})_{n\times n}$$

为 **从基** $\mathcal{E}$ **到基** $\mathcal{F}$ **过渡矩阵**

注意 A 为 且

> $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ $\alpha_i \in V$
>
> $$(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\beta_1, \beta_2, \cdots, \beta_n) \overset{\text{def}}{\Longleftrightarrow} \alpha_i = \beta_i,\ \forall 1 \leq i \leq n$$
>
> $$(\alpha_1, \alpha_2, \cdots, \alpha_n) + (\beta_1, \beta_2, \cdots, \beta_n) \overset{\text{def}}{=} (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n)$$
>
> $$k \in K, \quad k \cdot (\alpha_1, \alpha_2, \cdots, \alpha_n) \overset{\text{def}}{=} (k\alpha_1, k\alpha_2, \cdots, k\alpha_n)$$
>
> $$(\alpha_1, \alpha_2, \cdots, \alpha_n) A_{m\times n} = \left(\sum_{i=1}^{n} \alpha_i a_{i1}, \cdots, \sum_{i=1}^{n} \alpha_i a_{im}\right)$$

$A = (a_{ij})_{m\times n}$

$$A = (\alpha_1, \alpha_2, \cdots, \alpha_n), \quad \alpha_i \in K^m$$

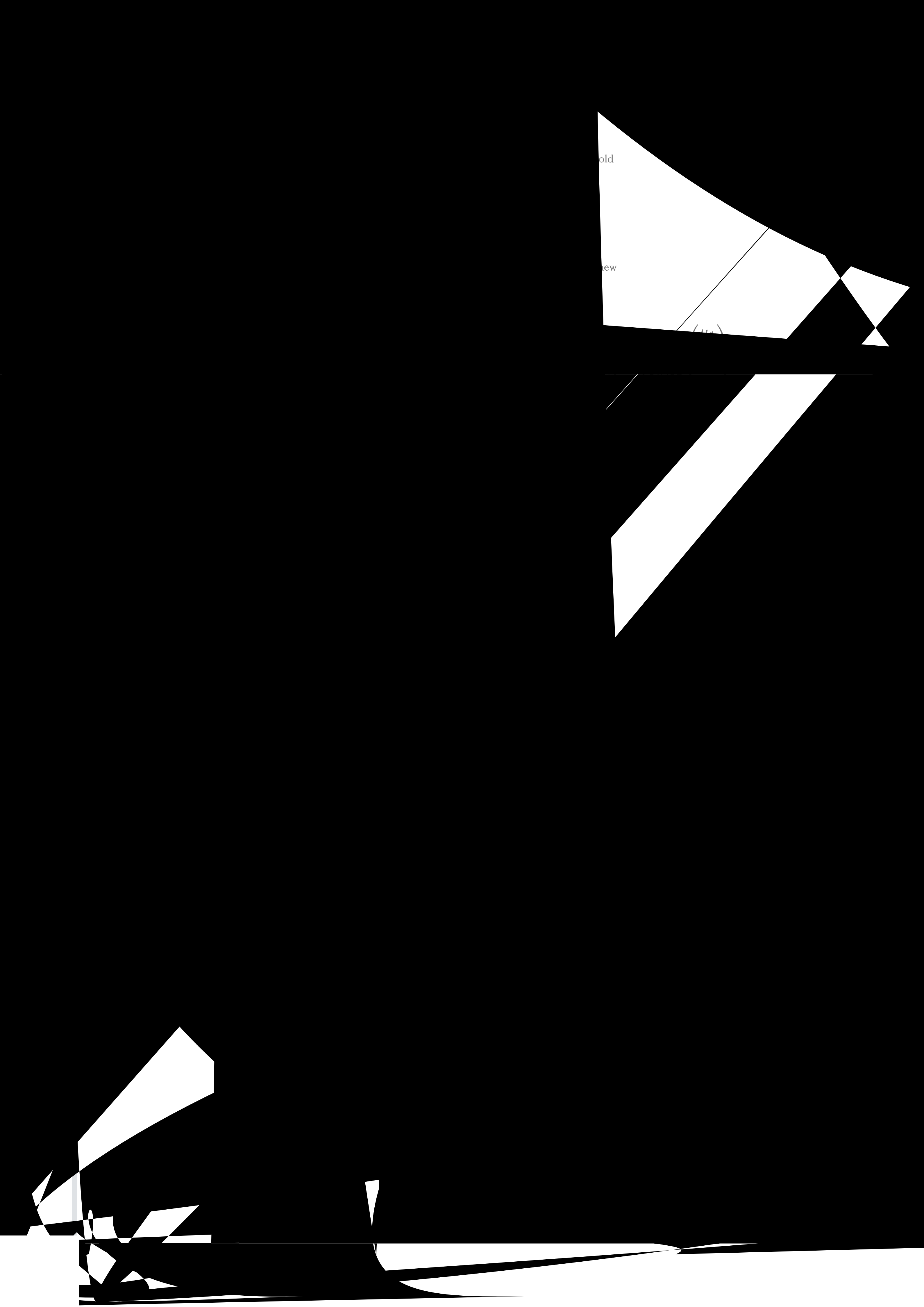
$$(f_1, f_2, \cdots, f_m) = (e_1, e_2, \cdots, e_m) A \cdots\cdots (*)$$

A 为**过渡矩阵**

引理2 $\{e_1, e_2, \cdots, e_n\}$ 为 V $A = (a_{ij})_{m\times n}$ 令 $B = (b_{ij})_{m\times n}$

$$(e_1, e_2, \cdots, e_n) A = (e_1, e_2, \cdots, e_n) B \Rightarrow A = B$$

> **证明**
>
> $$(e_1, e_2, \cdots, e_n) A = \left(\sum_{i=1}^{n} a_{i1} e_i, \cdots, \sum_{i=1}^{n} a_{im} e_i\right)$$
>
> $$(e_1, e_2, \cdots, e_n) B = \left(\sum_{i=1}^{n} b_{i1} e_i, \cdots, \sum_{i=1}^{n} b_{im} e_i\right)$$
>
> $$\Rightarrow \sum_{i=1}^{n} a_{i1} e_i = \sum_{i=1}^{n} b_{i1} e_i,\ \cdots,\ \sum_{i=1}^{n} a_{im} e_i = \sum_{i=1}^{n} b_{im} e_i$$
>
> $$\Rightarrow a_{ij} = b_{ij}, \quad \forall 1 \leq i \leq n,\ 1 \leq j \leq m.$$
>
> $\{e_1, e_2, \cdots, e_n\}$ $\{f_1, f_2, \cdots, f_n\}$
>
> $\alpha \in V$


old
new

> $$\Rightarrow (e_1, e_2, \cdots, e_n)I_n$$ [redacted]
>
> 引理2 $\Rightarrow AP = I_n$ A
>
> (2) e g C
>
> $$(f_1, f_2, \cdots, f_n) =$$ [redacted]
>
> $$(g_1, g_2, \cdots, g_n) =$$ [redacted]
>
> $$(g_1, g_2, \cdots, g_n) =$$ [redacted]
>
> $$(g_1, g_2, \cdots, g_n) =$$ [redacted]
>
> $$C =$$ [redacted]

$(\lambda_1, \lambda_2, \cdots, \lambda_n)$ [redacted]

$\{e_1, e_2, \cdots, e_n\}$、$\{f_1, \cdots$ [redacted]

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$ [redacted]

(*