Project 2. WENO reconstruction

Consider the one-dimensional *shallow water* equations

$$\begin{bmatrix} h \\ m \end{bmatrix}_t + \begin{bmatrix} m \\ \frac{m^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}_x = \mathbf{S}(x,t), \tag{1}$$

as in Project 1. Choose $g = 1$ and the spatial domain $\Omega = (0, 2)$.

Remark: Remember to discretize all data by cell averages.

2.1 (a) Explain qualitatively how the ENO method works, and why such a method is of interest.

(b) With respect to the ENO method, what is the fundamentally different approach of the WENO method in constructing approximations at cell-interfaces?

2.2 (a) Implement a component wise WENO method for constructing higher-order cell boundary approximations in the conserved variables h and m. Note that the mid-point rule is no longer sufficient for evaluating cell-averages, since we are interested in third or higher-order accuracy. Furthermore, it is recommended that you use the exact expression for evaluating the cell-averages of the source, in order to obtain a faster numerical code.

(b) The code should be capable of performing WENO reconstruction for at least $K = 2, 3$, where K corresponds to the size of the ENO stencils being using to obtain a $2K - 1$ order accurate reconstruction with the WENO algorithm.

(c) Implement a semi-discrete for (1) with the Lax-Friedrichs flux. Update the solution using SSP-RK3. Reconstruct the solution at the cell-interfaces using the WENO method. Implement both open and periodic boundary conditions. Note that you may need additional ghost cells on either side of the domain to implement the boundary conditions.

2.3 (a) Test your code on the problem with the initial condition

$$h(x,0) = h_0(x) = 1 + 0.5\sin(\pi x) \qquad m(x,0) = m_0(x) = uh_0(x)\ , \tag{2}$$

and

$$\mathbf{S}(x,t) = \begin{bmatrix} \frac{\pi}{2}\,(u-1)\cos\pi(x-t) \\ \frac{\pi}{2}\cos\pi(x-t)\left(-u + u^2 + gh_0(x-t)\right) \end{bmatrix}\ , \tag{3}$$

where $u = 0.25$. The exact solution to this problem is given by

$$h(x,t) = h_0(x-t) \qquad m = u\,h\ . \tag{4}$$

In your computations use periodic boundary conditions and evaluate the time-step as

$$k = \text{CFL}\frac{\Delta x}{\max_i\left(|u_i| + \sqrt{gh_i}\right)},$$

with CFL $= 0.5$. Plot the numerical solution at the final time $T = 1$. Measure the error of the schemes at $T = 1$ as a function of Δx and plot the results on a log-log graph, for $K = 2$.

2.4 (a) Set $\mathbf{S} = 0$ and run the program for each of the following initial conditions

$$h(x,0) = 1 - 0.1\sin(\pi x) \ , \qquad m(x,0) = 0, \tag{5}$$

$$h(x,0) = 1 - 0.2\sin(2\pi x) \ , \qquad m(x,0) = 0.5, \tag{6}$$

with periodic boundary conditions. Plot the solution at time $T = 1$.

(b) For each initial function (5) and (6), measure the error of the schemes at $T = 1$ as a function of Δx and plot the results on a log-log graph, for $K = 2$. To measure the error, compare each numerical solution with a reference numerical solution computed on a very fine mesh.