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Color-dual NLSM multi-loop integrands

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Abstract: For now, let's put the things we need to check/compute here in the abstract.

- (1) Does our construction manifest all the cuts for the two-loop n-gon?
- (2) Is there a natural extension to 3-loop
- (3) Is there a simple Feynman rule/ghost we can add to correct the one-loop n-gon of YZ theory
- (4) Can we make our two-loop numerators functionally symmetric
- (5) What is the physics story here? Perturbative corrections to special Galilon, DBI for inflation?
- (6) What is the rule for cuts we don't care about? Like the weird snaking diagrams that James identified?
- (7) How can bubble on external legs that are zero by IBP inform generalizations beyond pions?

Contents

1 Introduction

Some refs. [1–76]

2 Tree-level

We begin with a review of the construction of NLSM numerators at tree-level. For this work we extend the tree-level construction of YZ model of Cheung and Hsien [10]. In this construction, the generators of the kinematic algebra can be expressed as follows

$$T^a_{ij} = i\varepsilon_a(p_i - p_j) \tag{2.1}$$

where momentum conservation requires

$$p_a + p_i + p_j = 0 \tag{2.2}$$

The kinematic half-ladder diagrams then takes on the following concise definition:

$$n^{\text{NLSM}}_{(i|a_1a_2\ldots a_n|j)} = (T^{a_1}T^{a_2}\cdots T^{a_n})_{ij} \tag{2.3}$$

Since there are on pole cancelling factors of $s_{ij} = (p_i + p_j)^2$, this definition of the chiral current algebra is manifestly cubic. Thus, the kinematic structure constants defined in terms of these generators are invariant under generalized gauge freedom. They can be defined implicitly below:

$$[T^a, T^b]_{ij} = F^a_{b|c}T^c_{ij} \tag{2.4}$$

Given this definition, the Feynman rule associated with kinematic structure constant is simply:

$$iF^{a}_{b|c} = (\varepsilon_b p_{ab})(\varepsilon_a \bar{\varepsilon}_c) - (\varepsilon_a p_{ab})(\varepsilon_b \bar{\varepsilon}_c) \tag{2.5}$$

where $p_{ab} = p_a + p_b$ and ϵ and $\bar{\epsilon}$ are the polarizations of the Z-vectors particle and its conjugate field, respectively. The on-shell state sum for the polarization vectors is simply:

$$\sum_{\text{states}} \varepsilon^{\mu}_{(p)} \bar{\varepsilon}^{\nu}_{(-p)} = \eta^{\mu\nu} \tag{2.6}$$

Notice that the vector state sum is gauge fixed since the YZ model explicitly chooses Lorenz gauge for the Z particles, $\partial_\mu Z^\mu = 0$. To recover NLSM amplitudes from this these kinematic structure constants at tree-level, we simply plug in the following on-shell polarizations for the Z and $\bar{Z}$ particles:

$$\varepsilon^{\mu}_{(p)} = p^\mu \qquad \bar{\varepsilon}^{\mu}_{(p)} = \frac{q^\mu}{pq} \tag{2.7}$$

where $q^2 = 0$ is some null reference momentum. With this, we can define tree-level pion scattering in two equivalent ways:

$$A^{\text{NLSM}} = A(..., Y, ..., Y, ...) = A(..., \bar{Z}, ...) \tag{2.8}$$

where the ellipses denote additional on-shell Z-particles. Subject to a particular gauge choice, the kinematic numerators in latter definition for pion scattering is equivalent to those J-theory, first written down by Cheung and one of the authors [20]. In this paradigm, the on-shell $\bar{Z}$ state is equivalent to the root leg appearing in the color-dual J-theory numerators.

It is instructive to see how both of these constructions produce valid tree-level amplitudes for the pion. First we'll start with two Y particles on legs 1 and 4. Applying the Feynman rules above, and plugging in on-shell states for the Z-particles produces the following s- and t-channel numerators:

$$n^{YY}_s = (T^2 T^3)_{14} = s_{12}^2 \tag{2.9}$$

$$n^{YY}_t = F^3_{2|X} T^X_{14} = s_{14}(s_{13} - s_{12}) \tag{2.10}$$

Plugging these numerators into the ordered amplitudes $A(s,t)$ yields the desired result:

$$A^{YY}_{(s,t)} = \frac{n^{YY}_s}{s_{12}} + \frac{n^{YY}_t}{s_{14}} = s_{13} \tag{2.11}$$

Similarly we can do the same for the Z and $\bar{Z}$ configuration. Below we have plugged in the on-shell Z-particle states, but have left the $\bar{Z}$ index free. This produces the following numerators:

$$n_s^{\bar{Z}Z} = {}_4(F^3F^2)_1 = s_{12}(s_{14} - s_{13})p_2^{\mu_1} + s_{12}^2(p_3 - p_4)^{\mu_1} \tag{2.12}$$

$$n_t^{\bar{Z}Z} = {}_2(F^3F^4)_1 = s_{14}(s_{12} - s_{13})p_4^{\mu_1} + s_{14}^2(p_3 - p_2)^{\mu_1} \tag{2.13}$$

where we have defined the short hand notation:

$${}_x(F^{a_1}F^{a_2}\cdots F^{a_n})_y \equiv F^{a_1}_{x|b_2}F^{a_2}_{b_2|b_3}\cdots F^{a_n}_{b_n|y} \tag{2.14}$$

Similarly, we find the following ordered amplitude when plugging in these numerators:

$$A_{(s,t)}^{\bar{Z}Z} = \frac{n_s^{\bar{Z}Z}}{s_{12}} + \frac{n_t^{\bar{Z}Z}}{s_{14}} = -s_{13}(p_2 + p_3 + p_4)^{\mu_1} = s_{13}\, p_1^{\mu_1} \tag{2.15}$$

Plugging in the on-shell polarizatoin of the conjugate field in eq. (2.7) produces precisely the desired result of eq. (2.11). Indeed, this construction is valid to all orders at tree-level. There are only two possible factorization channels the contribute the each of these amplitudes, the YY cut and the $\bar{Z}Z$ cut :

$$\begin{aligned} A(...,Y,...,Y,...) &\to A(...,Y,...,Y)A(Y,...,Y,...) && (2.16)\\ &\to A(...,Y,...,Y,...,Z)A(\bar{Z},...) && (2.17) \end{aligned}$$

Now we are prepared to discuss the one-loop case.

3 One-loop

At one-loop, weight counting tells us that the n-gon master numerator must have n on-shell Z-particles. Unitarity requires that there are three distinct contributions from YZ theory – the first from an off-shell Y-loop particle and then two more from different orientations of a $\bar{Z}Z$-loop. Thus, YZ theory gives us the following one-loop n-gon numerator:

$$N_{(12\ldots n)}^{n\text{-gon},YZ} = (T^1T^2\cdots T^n) + 2\,(F^1F^2\cdots F^n) \tag{3.1}$$

where $(\cdots)$ indicates an internal contraction over the YY and $\bar{Z}Z$ loops. Plugging in the Feynman rule of eq. (2.1) and eq. (2.5), we can readily obtain expressions in terms of the internal loop factors l_i and the external momenta k_i (we define the l_i loop momentum as that flowing into k_i and out of k_{i-1}):

$$N_{(12\ldots n)}^{\text{NLSM}} = (T^1T^2\cdots T^n) = (l_1k_1)(l_2k_2)\cdots(l_nk_n) \tag{3.2}$$

and similarly so for the internal vector contribution:

$$(F^1F^2\cdots F^n) = (D-4)(l_1k_1)(l_2k_2)\cdots(l_nk_n) + \mathcal{O}(D^0) \tag{3.3}$$

The dimension dependent factor essentially counts that number of internal vector states. While this n-gon is manifestly color-dual, it does not produce the right cuts for NLSM. However, the scalar contribution, that comes dressed with an overall factor $(D-4)$ *does* manifest the duality globally, and satisfies all the desired pion cuts. In order for the Feynman rules for YZ theory compute one-loop color-dual numerators consistent with NLSM cuts, we must add some additional states to cancel off he spurious poles, while preserving color-kinematics duality. We leave this as a direction of future work.

While the $\bar{Z}Z$ vector loop spoils color-kinematics off-shell, the YZ loop alone gives us a desired expression for the n-gon. The important takeaway is that we now have a guess for the form of the off-shell three-point vertex that has a chance of manifesting the duality off-shell. The n-gon numerator above has scalar insertions of the following kinematic vertex:

$$T^a_{LR} = k_a(l_L - l_R) = (l_L + l_R)(l_L - l_R) = l_L^2 - l_R^2 \tag{3.4}$$

Given this structure, in the next section we will attempt to construct two-loop basis diagrams from these cubic vertex assignments and try to reverse engineer the particle content that produces these master numerators.

Before proceeding, we note a strange property of the n-gon numerator for the pions. In this form, Jacobi relations produce *non-vanishing* values for bubbles on external legs (BELs). However, it is easy to see that the contribution integrates to zero after applying IBP relations/tensor reduction on the bubble. The BEL diagram can be reconstructed from the n-gon as follows:

$$N^{\text{BEL}}_{1|2,34} = N^{\text{NLSM}}_{(1234,l)} - N^{\text{NLSM}}_{(1243,l)} - N^{\text{NLSM}}_{(1342,l)} + N^{\text{NLSM}}_{(1432,l)} \tag{3.5}$$

where we define the loop momentum to be in between the left most and right most leg on the box. Plugging in particular values for l_i, we obtain the following expression for the BEL

$$N^{\text{BEL}}_{1|2,34} = s_{12}(l+k_1)^2 l^\mu k_1^\nu k_2^{[\mu} k_{[34]}^{\nu]} \tag{3.6}$$

Notice there is an overall factor that cancels one of the propagators. Plugging this in, produces an integral of the following form

$$\mathcal{I}^{\text{BEL}}_{1|2,34} = s_{12} k_1^\nu k_2^{[\mu} k_{[34]}^{\nu]} \int \frac{d^D l}{i\pi^{D/2}} \frac{l^\mu}{l^2 - \mu^2} \sim s_{12}(s_{13} - s_{14})(\mu^2)^{D/2} \tag{3.7}$$

where we have introduced a mass regulator that will be proportional the the on-shell momentum inside the BEL, $\mu^2 \equiv k_1^2$. Thus, in large enough dimension, this integral suppresses the μ^{-2} divergence appearing in the denominator of the BEL diagram.

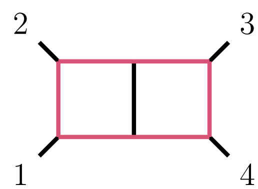
4 Two-loop

Let's start with a two-loop four-point example. At this order, the basis diagrams for color-dual representations are any two of the Jacobi triple, the double box, the penta-triangle or the cross-box. First we introduce a bit of shorthand notion:

$$[LR] \equiv l_L^2 - l_R^2 \qquad [M] \equiv l_M^2 \tag{4.1}$$

Now let's start with the double-box. Below we treat the color legs as Y particles that will get dressed with factors of $[LR]$. Thus, we wish to manifest the following cut:

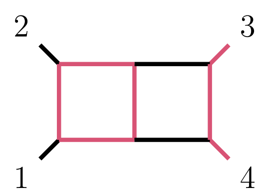
$N_M^{\text{2box}} \equiv$



$= [12][23][34][45][56][61] \quad (4.2)$

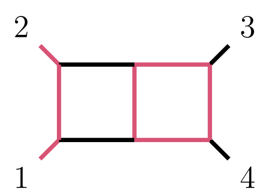
In addition, we want the following cuts to be simultaneously satisfied:

$N_R^{\text{2box}} \equiv$



$= [61][12][27][76][4]^2 \quad (4.3)$

$N_L^{\text{2box}} \equiv$



$= [45][57][73][34][1]^2 \quad (4.4)$

Where the middle cut takes $l_7^2 \to 0$, the right cut takes $l_3^2, l_5^2 \to 0$ and the left cut takes $l_2^2, l_6^2 \to 0$. Applying these cuts to all the diagrams above, we obtain the following expressions:

$$\text{Cut}_R(N_M^{\text{2box}}) = \Delta_R + N_R^{\text{2box}} \qquad \text{Cut}_L(N_M^{\text{2box}}) = \Delta_L + N_L^{\text{2box}} \tag{4.5}$$

However, we find that the remainder terms $\Delta_{L/R}$ are proportional to l_7^2, and thus vanish on the middle cut:

$$\text{Cut}_M(\Delta_{L/R}) = 0 \tag{4.6}$$

Similarly, applying the complement left/right cuts to the right/left remainder terms, we obtain a single kinematic function:

$$\text{Cut}_L(\Delta_R) = \text{Cut}_R(\Delta_L) = \Delta_{LR} \tag{4.7}$$

Thus, the double box numerator that simultaneously manifests all of the cut contributing to NLSM are as follows:

$$\boxed{N_{\text{2box}}^{\text{NLSM}} = N_M^{\text{2box}} - (\Delta_L + \Delta_R) + \Delta_{LR}} \tag{4.8}$$

Applying the cut relations above, we find that this double box numerator factorizes to the desired expressions:

$$\text{Cut}_X(N_{\text{2box}}^{\text{NLSM}}) = N_X^{\text{2box}} \tag{4.9}$$

The kinematic expressions obtained for the remainder functions are provided below:

$$\Delta_L = \frac{1}{2}[61][12]([27] + [67])[4]^2[7] \tag{4.10}$$

$$\Delta_R = \frac{1}{2}[34][45]([37] + [57])[1]^2[7] \tag{4.11}$$

$$\Delta_{LR} = [1]^2[4]^2[7]^2 \tag{4.12}$$

While these numerators are not functionally automorphic like the one-loop n-gon numerators, they do obey color-kinematics when evaluated on-shell and thus should be valid representations for double copy construction.

5 Discussion

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