Math 373, Nov 11, 2014

DFT

Discrete Fourier Transform

Recall least squares continuous approximation to $f(x)$ on $[a,b]$ from span$(\varphi_1 \cdots \varphi_n)$

$$A\vec{c} = \vec{f}$$

$$A_{ij} = \langle \varphi_i, \varphi_j \rangle \qquad \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} \langle f, \varphi_1 \rangle \\ \vdots \\ \langle f, \varphi_n \rangle \end{pmatrix}$$

$$Pf(x) = \sum_{i=1}^{n} c_i \varphi_i(x)$$

Example: $\{\varphi_i\} = \{x^0, x^1, x^2 \cdots x^n\}$

If $\{\varphi_i\}$ are orthogonal, A is diagonal

$$Pf = \sum_i \frac{\langle f, \varphi_i \rangle}{\langle \varphi_i, \varphi_i \rangle} \varphi_i$$

Example: Legendre polynomials

If $\{\varphi_i\}$ are orthonormal, $A = I$

$$Pf = \sum \langle f, \varphi_i \rangle \varphi_i$$

Example: φ_i = complex exponentials (Fourier series)

$$f(x) \sim \frac{1}{2\pi} \sum_{k=-n}^{n} \hat{f}_k e^{ikx}$$

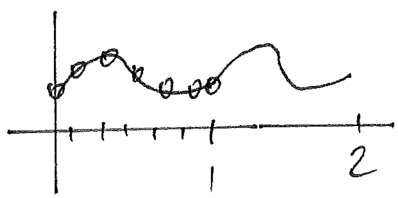
$$\hat{f}_k = \langle f, e^{ikx} \rangle = \int_0^{2\pi} f(x) e^{-ikx} dx$$

Slight modification: use $[0, 1]$ instead of $[0, 2\pi]$

$$\varphi_k(x) = e^{2\pi i k x} \quad , \quad \langle \varphi_k, \varphi_l \rangle = \delta_{kl} = \begin{cases} 1 : & k = l \\ 0 : & k \neq l \end{cases}$$

Kronecker delta

Now we want to discretize this



$f(x)$ 1-periodic

$f(0) = f(1)$

choose N, $h = \frac{1}{N}$

$$\vec{f} = \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f((N-1)h) \end{pmatrix}$$

$$\langle f, \varphi_k \rangle = \int_0^1 f(x)\, e^{-i2\pi k x}\, dx \approx \frac{1}{N} \cdot \sum_{m=0}^{N-1} f(mh) \cdot e^{-2\pi i k/N}$$

(trapezoidal rule)

This is the same as

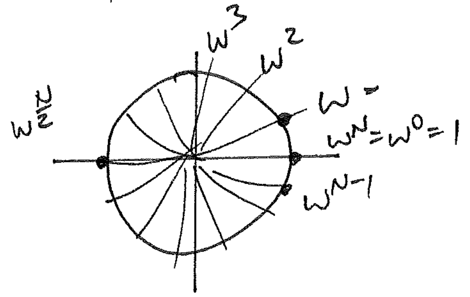
$$\frac{1}{N} \langle \vec{f}, \vec{\varphi_k} \rangle \qquad \vec{\varphi_k} = \begin{pmatrix} \varphi_k(0) \\ \varphi_k(h) \\ \vdots \\ \varphi_k((N-1)h) \end{pmatrix} = \begin{pmatrix} e^{2\pi i\, 0 \cdot k/N} \\ e^{2\pi i\, 1 \cdot k/N} \\ \vdots \\ e^{2\pi i\, (N-1) k/N} \end{pmatrix}$$

Nice coincidence

$$\langle \vec{\varphi_l}, \vec{\varphi_m} \rangle = \sum_{k=0}^{N-1} e^{2\pi i k l/N} \cdot e^{-2\pi i k m/N}$$

$$= \sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N}(l-m)} = \begin{cases} N : & l = m \\ 0 : & l \neq m \end{cases}$$

<u>proof</u> Let $w = e^{\frac{2\pi i}{N}} = \sqrt[N]{1}$ ("N^{th} root of unity")

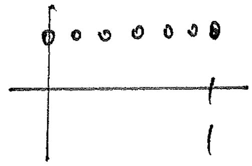


$$\boxed{\begin{aligned} w^N &= 1 \\ w^{\frac{N}{2}} &= -1 \end{aligned}}$$

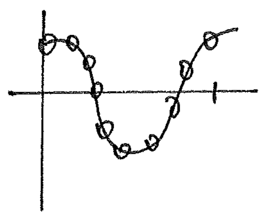
$$\sum_{k=0}^{N-1} w^{kl} = \frac{1 - w^{Nl}}{1 - w^l} = 0 \text{ by periodicity, unless } l = 0$$

<u>Interpretation</u>

$\vec{\varphi}_0 =$

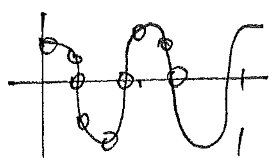


$\vec{\varphi}_1 =$



discretized sin or cos
one period in $[0,1]$

$\vec{\varphi}_2 =$



2 periods

$\boxed{\text{Discrete Fourier Transform}}$ (DFT)

$$\vec{f} = \begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix} = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}_k \cdot \vec{\varphi}_k$$

$$\hat{f}_k = \langle \vec{f}, \vec{\varphi}_k \rangle = \sum_{k=0}^{N-1} f_l \, e^{-\frac{2\pi i}{N} kl}$$

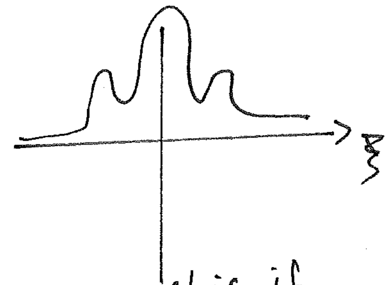
kl entry is w^{kl}

$k, l = 0 .. N-1$

matrix form: $\hat{f} = W \cdot \vec{f}$,
$$W = \begin{pmatrix} w^0 & w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & w^2 & \cdots & \\ w^0 & w^2 & w^4 & & \\ w^0 & w^3 & w^6 & & \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

There are 3 different Fourier transforms (F.T.)

1. Continuous F.T. $\quad f : \mathbb{R} \to \mathbb{C} \longrightarrow \hat{f} : \mathbb{R} \to \mathbb{C}$



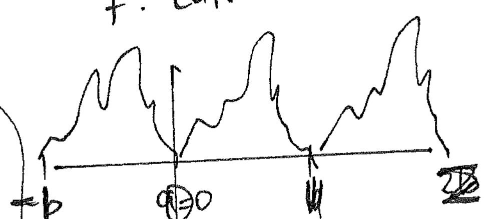
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\,dx$$

$|\hat{f}|$ is symmetric if f is real:

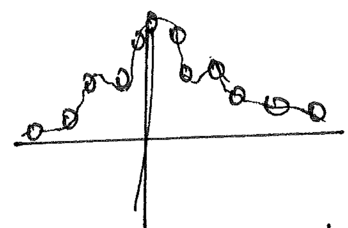
$$\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$$

2. Fourier series $\quad f : [0,1] \to \mathbb{C} \rightarrow \hat{f}$ = infinite sequence

$$\hat{f}_k = \int_0^1 f(x)\, e^{-2\pi i k x}\,dx$$



(or equivalently, f is periodic)

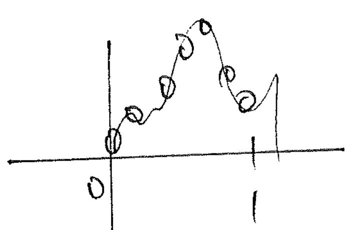


coefficients = samples of continuous transform

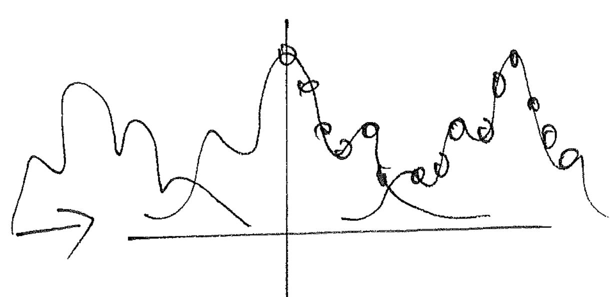
3. Discrete Fourier transform (DFT)

$\vec{f}$ = vector of length N $\longrightarrow$ $\hat{f}$ = vector of length N

$$\hat{f}_k = \sum_{k=0}^{N-1} f_l\, e^{-\frac{2\pi i}{N} k l}$$



discretized function



aliased + discretized