Stability Theory For Multivalue Methods

We assume m (EC) steps:

$$\vec{y}_n^{(0)} = B \cdot \vec{y}_{n-1} \qquad P$$

$$\left.\begin{aligned} \vec{y}_n^{(1)} &= \vec{y}_n^{(0)} + G(\vec{y}_n^{(0)}) \cdot \vec{c} \\ &\vdots \\ \vec{y}_n^{(m)} &= \vec{y}_n^{(m-1)} + G(\vec{y}_n^{(m-1)}) \cdot \vec{c} \end{aligned}\right] (EC)^m$$

$$\left(\quad \vec{y}_n^{(m+1)} = \vec{y}_n^{(m)} + G(\vec{y}_n^{(m)}) \cdot \vec{c}_2 \quad \right) \text{ optional final } E$$

$$\vec{y}_n = \vec{y}_n^{(m)} \text{ or } \vec{y}_n^{(m+1)}$$

To keep the notation simpler, ignore optional final E for now.

<u>Def:</u> $\vec{y}_n$ = numerical solution at x_n

$\vec{y}(x_n)$ = true solution at x_n

$\tilde{y}_n$ = numerical method applied to $\vec{y}(x_{n-1})$ instead of $\vec{y}_{n-1}$.

local error $\vec{d}_n = \tilde{y}_n - \vec{y}(x_n)$

global error $\vec{e}_n = \vec{y}_n - \vec{y}(x_n)$

intermediate steps $\vec{e}_n^{(k)} = \vec{y}_n^{(k)} - \tilde{y}_n^{(k)}$

<u>Note:</u> This is different from what we used to analyze the error in RK methods. The global error is the same, but

before: local error = true ODE applied to previous numerical result

now: local error = numerical method applied to previous true value

By definition

$$\vec{e}_n = \vec{y}_n - \vec{y}(x_n)$$

$$= \underbrace{(\vec{y}_n - \tilde{y}_n)}_{\vec{e}_n^{(m)}} + \underbrace{(\tilde{y}_n - \vec{y}(x_n))}_{\vec{d}_n}$$

$\vec{d}_n$ ← new local error

$\vec{e}_n^{(m)}$ ← magnification of previous error

$$\vec{e}_n^{(0)} = \vec{y}_n^{(0)} - \tilde{y}_n^{(0)} = B \cdot \vec{y}_{n-1} - B \cdot \vec{y}(x_{n-1}) = B\left(\vec{y}_{n-1} - \vec{y}(x_{n-1})\right) = B \cdot \vec{e}_{n-1}$$

$$\vec{e}_n^{(k+1)} = \vec{y}_n^{(k+1)} - \tilde{y}_n^{(k+1)}$$

$$= \left(\vec{y}_n^{(k)} + G(\vec{y}_n^{(k)}) \cdot \vec{c}\right) - \left(\tilde{y}_n^{(k)} + G(\tilde{y}_n^{(k)}) \cdot \vec{c}\right)$$

$$= \left(\vec{y}_n^{(k)} - \tilde{y}_n^{(k)}\right) + \left[G(\vec{y}_n^{(k)}) - G(\tilde{y}_n^{(k)})\right] \cdot \vec{c}$$

$$\approx \left(\vec{y}_n^{(k)} - \tilde{y}_n^{(k)}\right) + \left[DG(\tilde{y}_n^{(k)}) \cdot (\vec{y}_n^{(k)} - \tilde{y}_n^{(k)})\right] \cdot \vec{c}$$

$$= \left[I + \vec{c} \cdot DG(\tilde{y}_n^{(k)})\right] \cdot \vec{e}_n^{(k)}$$

now $\quad G(\vec{y}_n) = G(y_n, hf_n, \ldots) = h\left[f(x_n, y_n) - f_n\right]$

$$\frac{\partial G}{\partial y_n} = h \cdot \frac{\partial f}{\partial y} = h\lambda \quad \text{for test problem } y' = \lambda y$$

$$\frac{\partial G}{\partial (hf_n)} = -1 \qquad \text{rest are } 0.$$

in general we asume $\quad DG(\tilde{y}_n^{(k)}) \approx \text{constant}$

Then $\quad \vec{e}_n = \vec{e}_n^{(m)} + \vec{d}_n$

$$\approx \underbrace{(I + \vec{c} \cdot DG)^m B}_{S_n} \vec{e}_{n-1} + \vec{d}_n$$

Stability depends on S_n. If all eigenvalues of S_n are ≤ 1 for all n, the method is stable.

For test problem, $DG = (h\lambda, -1, 0 \ldots 0)$

$$S = (I + \vec{c} \cdot DG)^m B \qquad \text{(independent of } n\text{)}$$

or

$$S = (I + \vec{c_2} \cdot DG)(I + \vec{c} \cdot DG)^m B$$

<u>Example:</u> AB 2-step predictor, AM 1-step corrector

$$\vec{y_n} = \begin{pmatrix} y_n \\ hf_n \\ hf_{n-1} \end{pmatrix}$$

(it does not matter which representation of $\vec{y_n}$ you choose; a basis change does not affect eigenvalues)

$$B = \begin{pmatrix} 1 & 3/2 & -\frac{1}{2} \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \vec{c} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{c_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{c} \cdot DG = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot (h\lambda, -1, 0) = \begin{pmatrix} h\lambda/2 & -1/2 & 0 \\ h\lambda & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>PEC:</u>

$$S = [I + \vec{c} \cdot DG] B = \begin{pmatrix} 1 + \frac{h\lambda}{2}, & \frac{1}{2} + \frac{3}{4}h\lambda, & -\frac{1}{4}h\lambda \\ h\lambda & \frac{3}{2}h\lambda & -1/2 h\lambda \\ 0 & 1 & 0 \end{pmatrix}$$

to check zero stability set $h\lambda = 0$:

$$S = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{eigenvalues } 1, 0, 0$$

<u>stable</u>

For $h\lambda = -\frac{1}{2}$, eigenvalues are $-1, \frac{1}{2}, \frac{1}{2}$

can check numerically <u>stable on $[-\frac{1}{2}, 0]$</u>

$P(EC)^2$: $S = (I + \vec{c} \cdot DG)^2 B = \begin{pmatrix} 1 + \frac{h\lambda}{2} + \frac{(h\lambda)^2}{4} & \frac{1}{2} + \frac{h\lambda}{4} + \frac{3}{8}(h\lambda)^2 & -\frac{1}{8}(h\lambda)^2 \\ h\lambda + \frac{1}{2}(h\lambda)^2 & \frac{1}{2}h\lambda + \frac{3}{4}(h\lambda)^2 & -\frac{1}{4}(h\lambda)^2 \\ 0 & 1 & 0 \end{pmatrix}$

same result for $h\lambda = 0$: zero stable

region of stability for real λ $\approx [-1.45, 0]$

$P(EC)^3$: $\approx [-1.15, 0]$

$P(EC)E$: $S = (I + \vec{c_2} \cdot DG)(I + \vec{c} \cdot DG) B$

$$= \begin{pmatrix} 1 + \frac{h\lambda}{2} & \frac{1}{2} + \frac{3}{4}h\lambda & -\frac{1}{4}h\lambda \\ h\lambda + \frac{1}{2}(h\lambda)^2 & \frac{1}{2}h\lambda + \frac{3}{4}(h\lambda)^2 & -\frac{1}{4}(h\lambda)^2 \\ 0 & 1 & 0 \end{pmatrix}$$

region $= [-2, 0]$

Question: The region of stability of AM 1-step includes the entire negative real axis. Why is the region for multivalue methods so small?

Answer: see homework

Note: Results for AB-3, AM-2 and Gear's method are similar.