1 Model

$$y(t) = \sqrt{h(t)}u(t)$$
$$\ln(h(t)) = \alpha + \delta \ln(h(t-1)) + \sigma_\nu \nu(t)$$

$t = 1, ..., N$

$$u(t), \nu(t) \sim N(0, 1)$$

2 MCMC

From the model

$$(y(t)|h(t)) \sim N(0, h(t))$$
$$(h(t)|h(t-1), \omega) \sim N(\alpha + \delta \ln(h(t-1)), \sigma_\nu^2)$$

Let $\beta = (\alpha, \delta)$ and assume (β, σ) have prior Normal-Inverse Gamma distribution

$$(\beta|\sigma) \sim N(\bar{\beta}, \sigma^2 A^{-1})$$
$$\sigma \sim IG(\frac{\nu_0}{2}, \frac{s_0^2}{2})$$

We have the marginal distribution

$$p(y, h, \omega) \propto \frac{1}{\sigma_\nu^{N+3+\nu_0}} \exp\left(-\frac{(\beta - \bar{\beta})^T A(\beta - \bar{\beta})}{2\sigma_\nu^2} - \frac{s_0^2}{2\sigma_\nu^2}\right) \tag{1}$$
$$\times \prod_t \frac{1}{h(t)^{\frac{3}{2}}} \exp\left(-\frac{y(t)^2}{2h(t)} - \frac{(\ln h(t) - \delta \ln h(t-1) - \alpha)^2}{2\sigma_\nu^2}\right)$$

From which we can write the posterior distribution or marginal distribution

$$p(h(t)|h(t+1), h(t-1), \beta) \propto \frac{1}{\sqrt{h(t)}} \exp\left(-\frac{y(t)^2}{2h(t)}\right) \tag{2}$$
$$\times \frac{1}{h(t)} \exp\left(-\frac{(\ln h(t) - \mu(t))^2}{2\sigma_\delta^2}\right)$$

where $\sigma_\delta^2 = \frac{\sigma_\nu^2}{1+\delta^2}$ and $\mu(t) = \frac{\delta \ln h(t+1) + \delta \ln h(t-1) + (1-\delta)\alpha}{1+\delta^2}$

Possible samplings methods are

1. Gibbs sampling: Sufficient for updating ω but too tedious for h.
2. Rejection sampling: Approximate $p(h)$ by a blacket density

$$q(h)$$

   and choose a constant c such that $p(h) \neq cq(h)$ for any h. Consider a draw from q and accept it with probability $p(h)/cq(h)$. If rejected, draw again. Still not efficient enough for h as it has to compute the overall constant of $p(h)$ for every time point. Also, the choice of c can be tricky as it might leads to too high or low rejection rate.

3. Metropolis-Hastings algorithm: Using a transition kernel $f(h)$, make the $n+1$th draw from $h(t)$, accept it with probability

$$\min\left(\frac{p(h(t)^{n+1})/f(h(t)^{n+1})}{p(h(t)^{n})/f(h(t)^{n})},1\right)$$

Finally, the method we are going to use is a combination of all above: