

# 1 Hierarchical models: data-analysis problems

## 1.1 Math tests

- 1.
2. Rewrite the distributions in terms of precision

$$(y_{ij}|\theta_i, \omega) \sim N(\theta_i, (\omega)^{-1})$$

$$(\theta_i|\omega, \lambda) \sim N(\mu, (\omega\lambda)^{-1})$$

Choose prior for parameters

$$\omega \sim \Gamma(\frac{d}{2}, \frac{\eta}{2})$$

$$\lambda \sim \Gamma(\frac{h}{2}, \frac{h}{2})$$

The joint distribution of everything is, suppose  $n$  data are grouped into  $m$  groups:

$$\text{constant} \times \omega^{n+m} \lambda^m e^{-\frac{\omega \sum_{ij} (y_{ij} - \theta_i)^2}{2} - \frac{\omega \lambda \sum_i (\theta_i - \mu)^2}{2} - \frac{\omega \eta}{2} - \frac{h \lambda}{2}} \omega^{\frac{d}{2}-1} \lambda^{\frac{h}{2}-1}$$

From which we have, suppose each group have  $g_i$  elements

$$(\omega|y, \lambda, \theta, \mu) \sim \Gamma(\frac{d+n+m}{2}, \frac{\eta}{2} + \frac{\sum_{ij} (y_{ij} - \theta_i)^2}{2} + \frac{\lambda \sum_i (\theta_i - \mu)^2}{2})$$

$$(\lambda|y, \omega, \theta, \mu) \sim \Gamma(\frac{h+m}{2}, \frac{h}{2} + \frac{\omega \sum_i (\theta_i - \mu)^2}{2})$$

$$(\theta_i|y, \omega, \lambda, \mu) \sim N(\frac{\lambda \mu + \sum_j y_{ij}}{\lambda + g_i}, \frac{1}{\omega(\lambda + g_i)})$$

$$(\mu|y, \omega, \lambda, \theta) \sim N(\frac{\sum_i \theta_i}{m}, \frac{1}{m \omega \lambda})$$

We will update the parameter according to these distribution. For the code, see mathtest.r.

3. See "mathtest\mathtest.r"

## 1.2 Price elasticity of demand

Let  $y = \log V$ ,  $x = (1, \log P, d, d \log P)$ . Where  $d = 0, 1$  is an indicator whether a particular store display ads or not. Each data point are indexed by the store indices  $i = 1, \dots, n$  and time index  $t = 1, \dots, p_i$ . Each  $x$  and  $\beta$  are four vectors whose vector index will be suppressed. We want to build the model:

$$y_{it} = x_{it} \beta_i + \epsilon_i$$

we have the prior distribution

$$\begin{aligned}\epsilon_i &\sim N(0, \sigma_i^2) \\ \beta_i &\sim N(\mu, \Sigma) \\ (\mu, \Sigma) &\sim NIW(m, \lambda, \psi, \nu) \\ \sigma^2 &\sim U(0, \infty)\end{aligned}$$

from which we have

$$(y_{it}, \beta_i, \Sigma, \mu, \sigma^2) \sim N(X_i \beta_i, \sigma^2 I) N(\mu, \Sigma) NIW(m, \lambda, \Psi, \nu)$$

The conditional distributions are:

$$(\beta|y_{it}, \mu, \Sigma, \sigma_i^2) \sim N(\mu', \Sigma')$$

where

$$\begin{aligned}\mu' &= \Sigma' \left( \sum_i \frac{X_i^T Y_i}{\sigma_i^2} + \Sigma^{-1} \mu \right) \\ \Sigma'^{-1} &= \sum_i \frac{X_i^T X_i}{\sigma_i^2} + \Sigma^{-1} \\ (\mu, \Sigma | \beta_i) &\sim NIW(m', \lambda', \Psi', \nu')\end{aligned}$$

where

$$\begin{aligned}m' &= \frac{\lambda m + \sum \beta_i}{\lambda + n} \\ \lambda' &= \lambda + n \\ \Psi' &= \Psi + \sum_i (\beta_i - \bar{\beta})^T (\beta_i - \bar{\beta}) + \frac{\lambda n}{\lambda + n} \sum_i (\beta_i - \mu)^T (\beta_i - \mu) \\ \nu' &= \nu + n \\ (\sigma_i^{-2} | y_{it}, \beta_i) &\sim \Gamma\left(\frac{p_i}{2} + 1, \frac{(Y_i - X_i \beta_i)^T (Y_i - X_i \beta_i)}{2}\right)\end{aligned}$$