

1 A simple Gaussian location model

- (A) This is equivalent with determine the distribution x under a gaussian distribution with precision ω and mean μ where $\omega \sim \text{Gamma}(\frac{d}{2}, \frac{\eta}{2\kappa})$, the result should be

$$p(\theta) \sim (1 + \frac{\kappa(\theta - \mu)^2}{\eta})^{-\frac{d+1}{2}} \quad (1)$$

Compare to the problem we have $\nu = d$, $s^2 = \frac{\eta}{d\kappa}$ and $m = \mu$

- (B)

$$\begin{aligned} p(\theta, \omega | \vec{y}) &= \frac{p(\vec{y} | \theta, \omega) p(\theta, \omega)}{\int p(\vec{y} | \theta', \omega') p(\theta', \omega') d\theta' d\omega'} \\ &\propto \omega^{\frac{n}{2}} e^{-\frac{\omega \sum_i (y_i - \theta)^2}{2}} \omega^{\frac{d+1}{2} - 1} e^{-\omega \frac{\kappa(\theta - \mu)^2}{2}} e^{-\omega \frac{\eta}{2}} \\ &\propto \omega^{\frac{d+n+1}{2} - 1} e^{-\frac{\omega(n+\kappa)(\theta - \frac{\mu\kappa + \sum_i y_i}{n+\kappa})^2}{2}} e^{-\frac{\omega(\eta + \sum_i y_i^2 + \kappa\mu^2 - \frac{(\mu\kappa + \sum_i y_i)^2}{n+\kappa})}{2}} \end{aligned} \quad (2)$$

After rearranging parameters

$$\mu^* = \frac{\mu\kappa + \sum_i y_i}{n + \kappa} \quad (3)$$

$$\kappa^* = \kappa + n \quad (4)$$

$$d^* = d + n \quad (5)$$

$$\eta^* = \eta + \sum_i y_i^2 + \kappa\mu^2 - \frac{(\mu\kappa + \sum_i y_i)^2}{n + \kappa} \quad (6)$$

The distribution is of the same form as $p(\theta, \omega)$

$$p(\theta, \omega | \vec{y}) \propto \omega^{\frac{d^*+1}{2} - 1} e^{-\frac{\omega\kappa^*(\theta - \mu^*)^2}{2}} e^{-\frac{\omega\eta^*}{2}} \quad (7)$$

- (C) Keeping only the part dependent on θ

$$\begin{aligned} p(\theta | \vec{y}, \omega) &= \frac{p(\vec{y}, \theta, \omega)}{p(\vec{y}, \omega)} \\ &\propto e^{-\frac{\omega\kappa^*(\theta - \mu^*)^2}{2}} \sim N(\mu^*, (\omega\kappa^*)^{-1}) \end{aligned} \quad (8)$$

- (D)

$$\begin{aligned} p(\omega | \vec{y}) &= \frac{p(\omega, \vec{y})}{p(\vec{y})} \\ &\propto \int p(\omega, \vec{y}, \theta') d\theta' \\ &\propto \omega^{\frac{d^*+1}{2} - 1} e^{-\frac{\omega\eta^*}{2}} \int e^{-\frac{\omega\kappa^*(\theta' - \mu^*)^2}{2}} d\theta' \\ &\propto \omega^{\frac{d^*}{2} - 1} e^{-\frac{\omega\eta^*}{2}} \sim \Gamma(\frac{d^*}{2}, \frac{\eta^*}{2}) \end{aligned} \quad (9)$$

(E)

$$p(\theta|\vec{y}) = \int p(\theta, \omega'|\vec{y}) d\omega' \quad (10)$$

Since $p(\theta, \omega|\vec{y})$ is same to $p(\theta, \omega)$ up to a parameter redefinition. So the result should be same to part A with parameter redefinition.

$$p(\theta|\vec{y}) \sim (1 + \frac{\kappa^*(\theta - \mu^*)^2}{\eta^*})^{-\frac{d^*+1}{2}} \quad (11)$$

(F) False. In the limit κ , d and η goes to zero

$$p(\theta) \propto \frac{1}{\sqrt{1 + \theta^2}} \quad (12)$$

When $\theta \rightarrow \infty$, $p(\theta) \rightarrow \frac{1}{\theta}$ which cannot be normalized.

(G) True. The starred paramters are still in healthy condition under this limit.

(H)

2 The conjugate Gaussian linear model

2.1 Basics

(A)

$$\begin{aligned} p(\vec{\beta}|\vec{y}, \omega) &= \frac{p(\vec{\beta}, \vec{y}, \omega)}{p(\vec{y}, \omega)} \\ &\propto p(\vec{y}|\vec{\beta}, \omega) p(\vec{\beta}|\omega) \\ &\propto e^{-\frac{\omega(\vec{y} - X\vec{\beta})^T \Lambda (\vec{y} - X\vec{\beta})}{2}} e^{-\frac{\omega(\vec{\beta} - \vec{m})^T K (\vec{\beta} - \vec{m})}{2}} \\ &\propto e^{-\frac{\omega(\vec{\beta} - \vec{m}^*)^T K^* (\vec{\beta} - \vec{m}^*)}{2}} \end{aligned} \quad (13)$$

$$K^* = X^T \Lambda X + K \quad (14)$$

$$\vec{m}^* = K^{*-1}(X^T \Lambda \vec{y} + K\vec{m}) \quad (15)$$

(B)

$$\begin{aligned} p(\omega|\vec{y}) &\propto p(\omega, \vec{y}) \\ &\propto \int p(\vec{y}|\vec{\beta}', \omega) p(\vec{\beta}'|\omega) p(\omega) d\vec{\beta}' \\ &\propto \omega^{\frac{d^*+1}{2}-1} e^{-\frac{\omega\eta^*}{2}} \int e^{-\frac{\omega(\vec{\beta}' - \vec{m}^*)^T K^* (\vec{\beta}' - \vec{m}^*)}{2}} d\vec{\beta}' \\ &\propto \omega^{\frac{d^*}{2}-1} e^{-\frac{\omega\eta^*}{2}} \sim \Gamma(\frac{d^*}{2}, \frac{\eta^*}{2}) \end{aligned} \quad (16)$$

$$d^* = d + n \quad (17)$$

$$\eta^* = \eta + \vec{y}^T \Lambda \vec{y} + \vec{m}^T K \vec{m} - \vec{m}^{*T} K^* \vec{m}^* \quad (18)$$

(C) Similar to previous problems

$$p(\omega, \vec{\beta} | \vec{y}) \propto \omega^{\frac{d^*+1}{2}-1} e^{-\frac{\omega \eta^*}{2}} e^{-\frac{\omega(\vec{\beta}-\vec{m}^*)^T K^* (\vec{\beta}-\vec{m}^*)}{2}} \quad (19)$$

Again, reuse the derivation where we derive t-distribution from normal/gamma prior

$$\begin{aligned} p(\vec{\beta} | \vec{y}) &= \int p(\omega', \vec{\beta} | \vec{y}) d\omega' \\ &\propto (1 + \frac{(\vec{\beta} - \vec{m}^*)^T K^* (\vec{\beta} - \vec{m}^*)}{\eta^*})^{-\frac{d^*+1}{2}} \end{aligned} \quad (20)$$

(D) The script "gdpgrowth.R" read data from "gdpgrowth.csv" and use the above linear model to fit the relation between column GR6096 and DEF60, using

$$\vec{\beta}_{\text{estimate}} = \vec{m}^* = K^{*-1} X^T \Lambda \vec{y} \quad (21)$$

2.2 A heavy tailed error model

(A) Let $\vec{\alpha} = X \vec{\beta}$

$$\begin{aligned} p(\vec{y} | X, \vec{\beta}, \omega) &= \int p(\vec{y} | X, \vec{\beta}, \omega, \Lambda) \prod_i p(\lambda_i) d\lambda_i \\ &\propto \prod_i \lambda_i^{\frac{1}{2}} e^{-\frac{\omega \lambda_i (y_i - \alpha_i)^2}{2}} p(\lambda_i) d\lambda_i \end{aligned} \quad (22)$$

This is the familiar normal/gamma prior, the result should be

$$p(\vec{y} | X, \vec{\beta}, \omega) \propto \prod_i (1 + \frac{\lambda_i (y_i - \alpha_i)^2}{h})^{-\frac{h+1}{2}} \quad (23)$$

(B)

$$\begin{aligned} p(\lambda_i | \vec{y}, \vec{\beta}, \omega) &\propto p(\vec{y} | \vec{\beta}, \omega, \Lambda) p(\lambda_i) \\ &\propto e^{-\lambda_i (\frac{h+\omega(y_i-\alpha_i)^2}{2})} \lambda_i^{\frac{h+1}{2}-1} \\ &\sim \Gamma(\frac{h+1}{2}, \frac{h+\omega(y_i-\alpha_i)^2}{2}) \end{aligned} \quad (24)$$

(C) See script "MCMC on gdpgrowth.R" and "gibbsampler.R". We generate the cycle

$$(\vec{\beta}_1 | \vec{y}, \omega_0, \Lambda_0), (\omega_1 | \vec{y}, \Lambda_0) \rightarrow (\Lambda_1 | \vec{y}, \vec{\beta}_1, \omega_1) \rightarrow (\vec{\beta}_2 | \vec{y}, \omega_1, \Lambda_1), (\omega_2 | \vec{y}, \Lambda_1) \rightarrow \dots \quad (25)$$

where

$$\begin{aligned}
(\vec{\beta}|\vec{y}, \omega, \Lambda) &\sim N(\vec{m}^*(\Lambda, \vec{y}), \omega K^*(\Lambda)) \\
(\omega|\vec{y}, \Lambda) &\sim \Gamma(\frac{d^*}{2}, \frac{\eta^*(\Lambda, \vec{y})}{2}) \\
(\lambda_i|\vec{y}, \vec{\beta}, \omega) &\sim \Gamma(\frac{h}{2}, \frac{h + \omega(y_i - \alpha_i)^2}{2})
\end{aligned} \tag{26}$$