1 Introduction

The variance of returns of assets tends to change over time. The stochastic volatility model (SV) attempts to model this variance by assuming that they follow some latent stochastic process. It appeared first in the literature of theoretical finance on option pricing, though empirical models are usually described in discrete way. There is evidence the SV models offer increased flexibility over the GARCH family (Geweke(1994)).

However, even the simplest SV model is difficult to fit due to various reasons. Naive strategy of maximum likelihood estimation fails due to the fact that the marginal distribution is evaluated in high dimensional space. Monte Carlo Markov Chain (MCMC), on the other hand, faces difficulty in simulating some non-analytical distributions and people needs to resort to indirect approach such as Metroplis-Hastings algorithm or reject-accept sampling. The topic of this essay will be to analyse the MCMC algorithm for SV model. Instead of general theory, we will focus on a simple SV model where the underling stochastic process is a log-normal autoregressive process. As for the MCMC, we will implement some specific algorithms and test their performance on both artificial and real world data.

The outline of this essay is following. We will introduce the model in section 2. In section 3, we derive the Bayesian posterior distribution and review three MCMC algorithms to fit the model. In order to compare them quantitatively, in section 4 we held a "competition": We will generate a fake data with underlying model and test which method can recover the hidden parameters. In section 5, we will apply the model on the real world data with these methods and check their results and performance.

2 Basic Model

y_t is the financial series we are interested in with zero mean. The model is

$$y_t = \sqrt{h_t} u_t$$

$$\ln(h_t) = \alpha + \delta \ln(h_{t-1}) + \sigma_\nu \nu_t$$

$$u_t, \nu_t \sim N(0, 1)$$

h_t is a latent variable measuring the volatility of y, δ is the volatility persistence.

3 Bayesian Analysis

3.1 MCMC

Assume the prior distribution:

$$\alpha \sim N(\alpha_0, \sigma_\alpha^2)$$

$$\delta \sim N(\delta_0, \sigma_\delta^2)$$
$$\sigma_\nu^2 \sim IG(\frac{\nu_0}{2}, \frac{s_0}{2})$$

We have the marginal distribution

$$p(y, h, \delta, \alpha, \sigma_\nu^2) \propto \frac{1}{\sigma_\nu^{2+\nu_0}} \exp\left(-\frac{(\delta-\delta_0)^2}{2\sigma_\delta^2} - \frac{(\alpha-\alpha_0)^2}{2\sigma_\alpha^2} - \frac{s_0^2}{2\sigma_\nu^2}\right)$$
$$\times \prod_{t=2}^{N} \frac{1}{h_t^{\frac{3}{2}}\sigma_\nu} \exp\left(-\frac{y_t^2}{2h_t} - \frac{(\ln h_t - \delta \ln h_{t-1} - \alpha)^2}{2\sigma_\nu^2}\right) \quad (1)$$

From which we can derive the posterior distributions are

$$(\sigma_\nu^2 | h, \alpha, \delta) \sim IG\left(\frac{\nu_0 + N - 1}{2}, \frac{s'}{2}\right)$$

$$s' = s_0 + (N-1)\alpha^2 + (1+\delta^2)S_2 - \delta^2(\ln h_N)^2 - (\ln h_1)^2 - 2\alpha((1-\delta)S_1 - \ln h_1 + \delta \ln h_N) - 2\delta S_3$$

$$(\delta | h, \alpha, \sigma_\nu^2) \sim N\left(\frac{\sigma_\nu^2\delta_0 + \sigma_\delta^2(S_3 - \alpha(S_1 - \ln h_N))}{\sigma_\nu^2 + \sigma_\delta^2(S_2 - (\ln h_N)^2)}, \frac{\sigma_\nu^2\sigma_\delta^2}{\sigma_\nu^2 + \sigma_\delta^2(S_2 - (\ln h_N)^2)}\right)$$

$$(\alpha | h, \sigma_\nu^2, \delta) \sim N\left(\frac{\sigma_\alpha^2((1-\delta)S_1 - \ln h_1 + \delta \ln h_N) + \sigma_\nu^2\alpha_0}{\sigma_\nu^2 + (N-1)\sigma_\alpha^2}, \frac{\sigma_\nu^2\sigma_\alpha^2}{\sigma_\nu^2 + (N-1)\sigma_\alpha^2}\right)$$

$$S_1 = \sum_{t=1}^{N} \ln h_t \quad S_2 = \sum_{t=1}^{N} (\ln h_t)^2 \quad S_3 = \sum_{t=2}^{N} \ln h_t \ln h_{t-1}$$

$$p(h_t | h_{t+1}, h_{t-1}, \delta, \alpha, \sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}} \exp\left(-\frac{y_t^2}{2h_t}\right) \frac{1}{h_t} \exp\left(-\frac{(\ln h_t - \mu_t)^2}{2\sigma_t^2}\right) \quad (2)$$

$$\mu_t = \frac{\delta \ln h_{t+1} + \delta \ln h_{t-1} + (1-\delta)\alpha}{1+\delta^2}$$

$$\sigma^2 = \frac{\sigma_\nu^2}{1+\delta^2}$$

In addition, following Jacuqier (1994), we will not update h_1 and h_N with (2), we will update them by directly drawing from autoregressive model of $\ln h$.

In summary, the outline of the algorithm is

1. Initialize $h, \alpha, \delta, \sigma_\nu^2$
2. For $t = 2, 3, \cdots, N-1$, draw h_t from $p(h_t|h_{t+1}, h_{t-1}, \delta, \alpha, \sigma_\nu^2)$
3. Draw $\ln h_1$ from $N(\alpha + \delta \ln h_2, \sigma_\nu^2)$, $\ln h_N$ from $N(\alpha + \delta \ln h_{N-1}, \sigma_\nu^2)$
4. Draw σ_ν^2 from $(\sigma_\nu^2|h, \alpha, \delta)$
5. Draw δ from $(\delta|h, \alpha, \sigma_\nu^2)$
6. Draw α from $(\alpha|h, \delta, \sigma_\nu^2)$
7. Go to step 2

It is easy to simulate the posterior distribution of σ_ν^2. α and δ, so the only nontrivial part of the MCMC is step 2. Below we will give three sampling methods. The comparison of them will be the focus of this project.

3.2 Sampling Method 1: Metroplis-Hastings with Random Walk

Write (2) as a distribution of $\ln h$ rather than h

$$p(\ln h_t|h_{t+1}, h_{t-1}, \delta, \alpha, \sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}} \exp\left(-\frac{y_t^2}{2h_t}\right) \exp\left(-\frac{(\ln h_t - \mu_t)^2}{2\sigma_t^2}\right)$$

Given $\ln h_t^{i-1}$. Each time we simply propose a $\ln h_t^*$ by drawing

$$N(\ln h_t^{i-1}, e^2)$$

where e^2 is a preset parameter independent of other variables. And accept it with probability

$$\text{Min}(1, \frac{p(\ln h_t^*)}{p(\ln h_t^{i-1})})$$

The algorithm is:

1. Draw $\ln h_t^*$ from $N(\ln h_t^{i-1}, e^2)$
2. Accept this value with probability $\text{Min}(1, \frac{p(\ln h_t^{i*})}{p(\ln h_t^{i-1})})$
3. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

3.3 Sampling Method 2: Metroplis-Hastings with Accept-Reject Sampling

This is the method proposed in Jacquier(1994). The idea is to refine the process of the proposing update in MH. We can "approximate" (2) by an inverse gamma distribution:

$$q(h_t) = \frac{\lambda^\phi}{\Gamma(\phi)} h^{-(\phi+1)} e^{-\frac{\lambda}{h_t}}$$

where

$$\lambda = \frac{1 - 2e^{\sigma^2}}{1 - e^{\sigma^2}} + \frac{1}{2}$$

$$\phi = (\lambda - 1) e^{\mu_t + \frac{\sigma^2}{2}} + \frac{y_t^2}{2}$$

and σ^2 and μ_t are defined under (2). The reason of this choice is that we can choose a inverse gamma distribution which have same first and second moment with the lognormal part of (2), this then combines with the inverse gamma part of (2) to give the above inverse gamma distribution.

Define

$$c = 1.1 \left(\frac{p(h)}{q(h)} \right)_{h = \text{mode of } q}$$

We will propose candidate h_t^{i*} from $IG(\lambda, \phi)$ and accept it with Min$(1, \frac{p(h^*)}{cq(h^*)})$, if rejected, repropose until accepted. The winner of accept-reject process will be the candidate of MH process with transition kernel $f(h_t^*) =$Min$(p(h_t^*), cq(h_t^*))$. The actual algorithm will be

1. Draw h_t^* from $IG(\lambda, \phi)$, note that both λ and ϕ are functions of h_{t+1}, h_{t-1} and other parameters
2. Accept h_t^* with probability Min$(1, \frac{p(h_t^*)}{cq(h_t^*)})$
3. If rejected, go to step 1.
4. If $p(h_t^*) < cq(h_t^*)$, $h_t^i = h_t^*$. The algorithm ends.
5. Accept h_t^* with probability Min$(1, \frac{p(h_t^*)/q(h_t^*)}{p(h_t^{i-1})/q(h_t^{i-1})})$
6. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

3.4 Sampling Method 3: Pure Accept-Reject Sampling

This method was used in Kim(1998).

$$\begin{aligned}
\ln p(\ln h_t | \cdots) &= -\frac{1}{2} \ln h_t - \frac{y_t^2}{2h_t} - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&\leq -\frac{1}{2} \ln h_t - \frac{y_t^2}{2} \left(\exp(-\mu_t)(1 + \mu_t - \ln h_t) \right) - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&= -\frac{(\ln h_t - \mu_t')^2}{2\sigma^2} + \text{constants} \qquad (3)
\end{aligned}$$

Where

$$\mu'_t = \mu_t + \frac{\sigma^2}{2}(y_t^2 \exp(-\mu_t) - 1)$$

This observation leads to a standard reject-accept sampling:

1. Draw $\ln h_t^*$ from $N(\mu'_t, \sigma^2)$
2. Accept h_t^* with probability $\mathrm{Min}(1, g(h_t^*)$, where

$$g(h_t) = \exp(\frac{y_t^2}{2}(\exp(-\mu_t))(1 + \mu_t - \ln h_t) - \frac{1}{h_t})$$

3. If rejected, go to step 1.
4. $h_t^i = h_t^*$

4 Test of Three Sampling Method: Artificial Data

4.1 Data

We will set $\alpha = 0$, $\delta = 0.95$, $\sigma_\nu^2 = 0.1$, $\ln h_1 = 0$ and generate a sequence of 1000 $\ln h_t$ and y_t following the model.

$$y_t = \sqrt{h_t} u_t$$
$$\ln(h_t) = \alpha + \delta \ln(h_{t-1}) + \sigma_\nu \nu_t$$
$$u_t, \nu_t \sim N(0, 1)$$

4.2 Result

We fit the artificial data with five methods: MH random walk with three different "speed" e, and the other two methods. We will address them as "MH + RW with $e = \ldots$", "MH + RA", "RA", respectively.

For h, we take the prediction of each model on h to be the mean of corresponding values in sample. We present the squared error $(h_t - \bar{h_t})^2$ for each t

5 Test of Three Sampling Method: S&P500

5.1 Data

For this section, we use the S&P500 from 1/1/2007 to 12/31/2010. As in Jacquier(1994) and Gallant(1992), we study the change of log of closing price:

$$y_t = \log(price_{t+1}/price_t)$$

Method	δ	α	σ_ν^2
MH +RW, $e = 0.05$			
MH +RW,$e = 0.1$			
MH + RW, $e = 0.3$			
MH + RA			
RA			

Table 1: Results of fitting SV model on artificial data generated from $\alpha = 0$, $\delta = 0.95$, $\sigma_\nu^2 = 0.1$

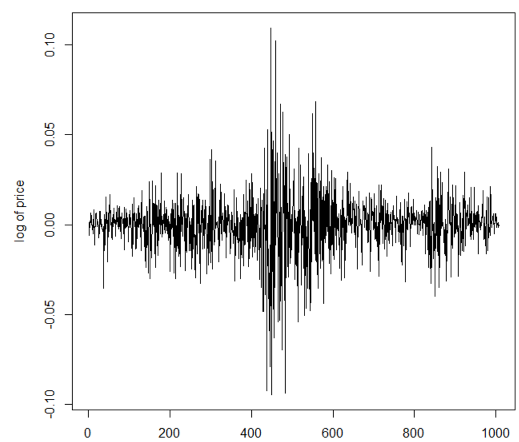

Figure 1: Daily change of S&P500 closing price from 2007-2010

y are plotted in Figure 1. There are 1008 data points in the time series.

5.2 Result

For prior we take $\delta \sim N(0, 10)$, $\alpha \sim N(0, 10)$ and $\sigma_\nu^2 \sim IG(\frac{1}{2}, \frac{1}{2})$. We simulate the MCMC for 12000 iterations, and all the posterior results are based on the sample collected after 4000th iteration. We list the result of parameters in Table 2. The numbers outside and inside bracket are posterior mean and standard deviations respectively. It is observed that MH + RA gives a highest persistence and smallest standard deviation on all parameters, and pure RA always gives the highest standard deviation and covariance. From Figure 2 to 6 we give all histograms of these variables.

For each h_t, we compute its sample mean and plot the log of the mean in Figure 7. It is a relief to observe that all five pictures reflects the abrupt change in the middle of Figure 1 (caused by the financial crisis) by showing a peak at

Method	δ	α	σ_ν^2	$Cov(\delta,\alpha)$ $(\times 10^{-4})$	$Cov(\delta,\sigma_\nu^2)$ $(\times 10^{-4})$	$Cov(\alpha,\sigma_\nu^2)$ $(\times 10^{-4})$
MH +RW, $e = 0.05$	0.976(0.008)	-0.214(0.073)	0.06(0.012)	5.9	-0.12	-1.18
MH +RW,$e = 0.1$	0.977(0.008)	-0.199(0.067)	0.061(0.009)	5.07	-0.27	-2.3
MH + RW, $e = 0.3$	0.973(0.009)	-0.24(0.083)	0.068(0.015)	7.8	-0.83	-7.3
MH + RA	0.986(0.006)	-0.113(0.051)	0.029(0.004)	3.1	-0.05	-0.44
RA	0.974(0.01)	-0.223(0.089)	0.071(0.017)	8.97	-1	-8.9

Table 2: Results of fitting SV model on S&P500 daily log change

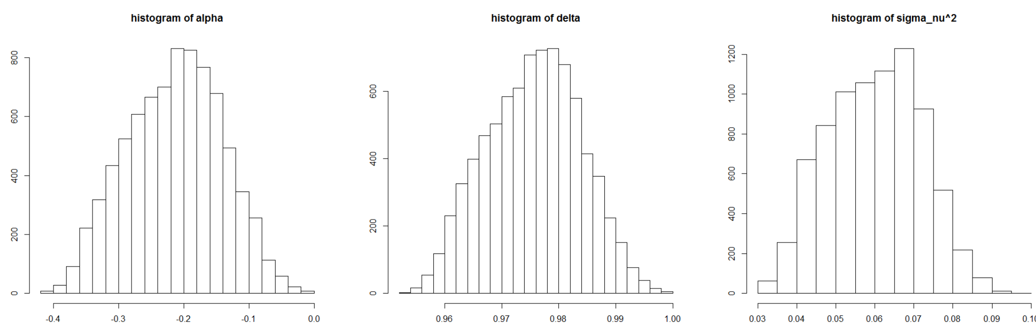

Figure 2: Histograms of MH + RW with $e = 0.05$

the same position. Their trend are also very similar, but curiously MH + RA gives a smoother plot.

5.3 Time

In Table 3 we listed the time and rejection/repeat rate of each method for 12000 iterations. Repeat rate applies for methods involving MH algorithm, while rejection rate applies for methods involving RA sampling. Note that only the HM+RA method have both.

Method	Time for 12000 iteration (second)	reject rate	repeated rate
MH + RW, $e = 0.05$	213.42		0.1
MH + RW, $e = 0.1$	210.64		0.18
MH + RW, $e = 0.3$	215.87		0.44
MH +RA	451.58	0.09	0.0002
RA	171.39	0.008	

Table 3: Time and reject/repeat rate of each method

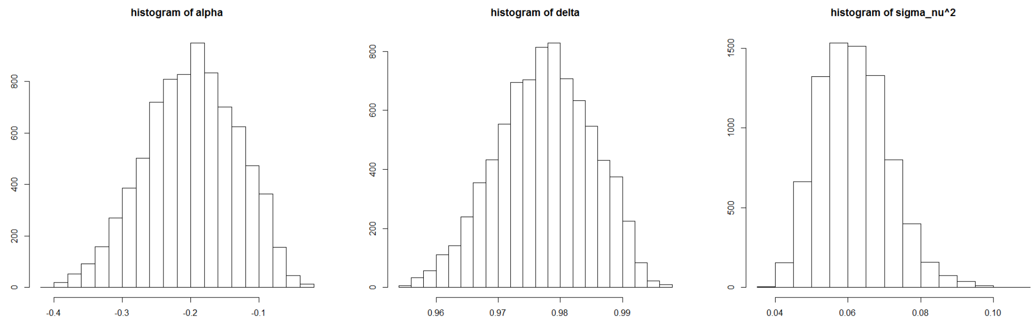

Figure 3: Histograms of MH + RW with $e = 0.1$

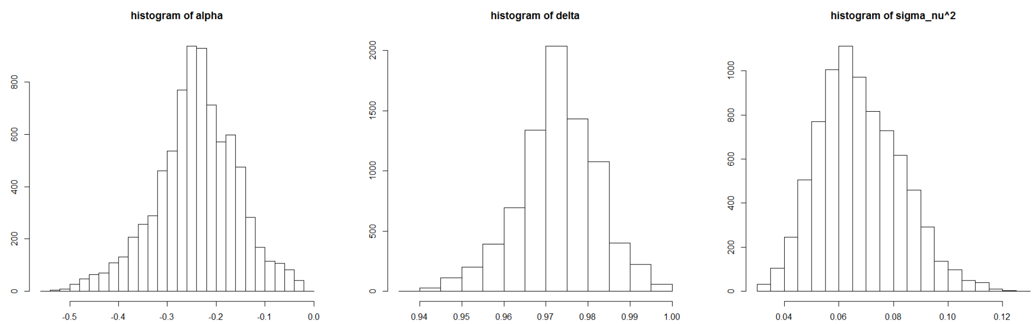

Figure 4: Histograms of MH + RW with $e = 0.3$

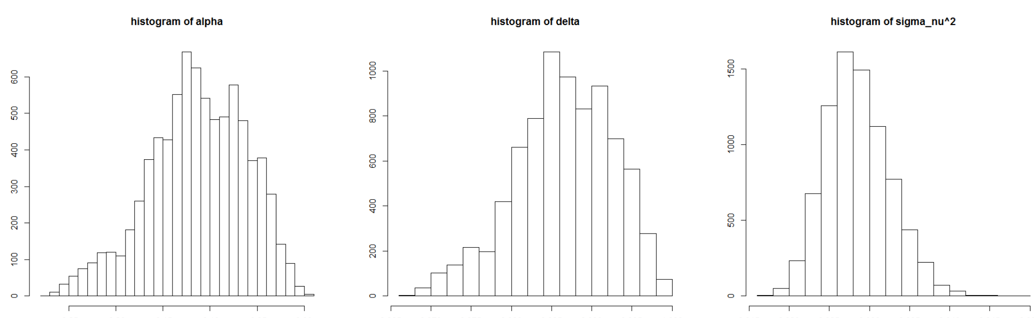

Figure 5: Histograms of MH + RA

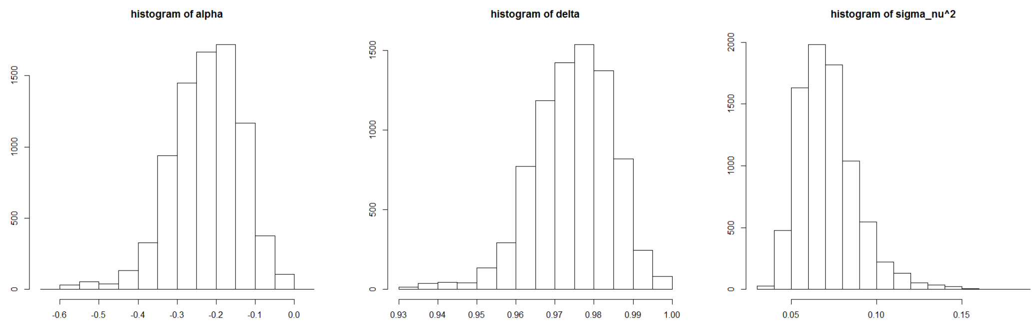

Figure 6: Histograms of RA

5.4 Autocorrelation

We give the autocorrelation function of parameters fit by all methods in Figure 8 to 12. The autocorrelation function are computed and plotted using acf() in stats library of R.

6 Reference

1. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian Analysis of Stochastic Volatility Models", Journal of Business & Economic Statistics, 1994, Vol. 12, No. 4

2. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian analysis of stochastic volatility models with fat-tails and correlated errors", Journal of Econometrics, 122(2004) 185-212

3. John Geweke, "Prior for Macroeconomic Time Series and Their Application", Econometric Theory, Vol. 10, 609-632

4. John Geweke, "Bayesian Comparison of Econometric Models", working paper, Federal Reserve Bank of Minneapolis Research Department

5. Siddhartha Chib and Edward Greenberg, "Understanding the Metropolis-Hastings Algorithm", The American Statistician, Vol. 49, No. 4. (Nov., 1995), pp.327-335

6. A. Ronald Gallant, Peter E. Rossi, George Tauchen, "Stock Prices and Volume", The Revew of Financial Studies, Vol. 5, No. 2. (1992), pp. 199-242

7. Sangjoon Kim, Neil Shephard, Siddhartha Chib, "Stochastic Volatility: Likelihood Inference and Comparison with ARCH Model", Review of E-conomic Studies (1998) 65, 361-393

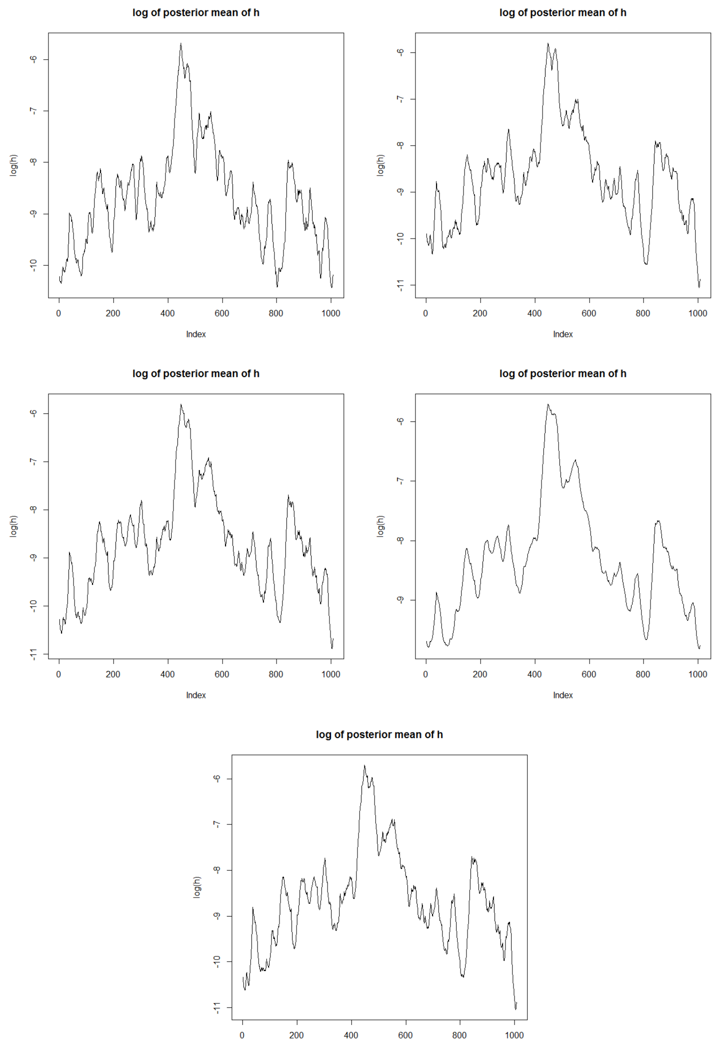

Figure 7: Log of mean of h_t of five methods, in the order of appearance in previous tables

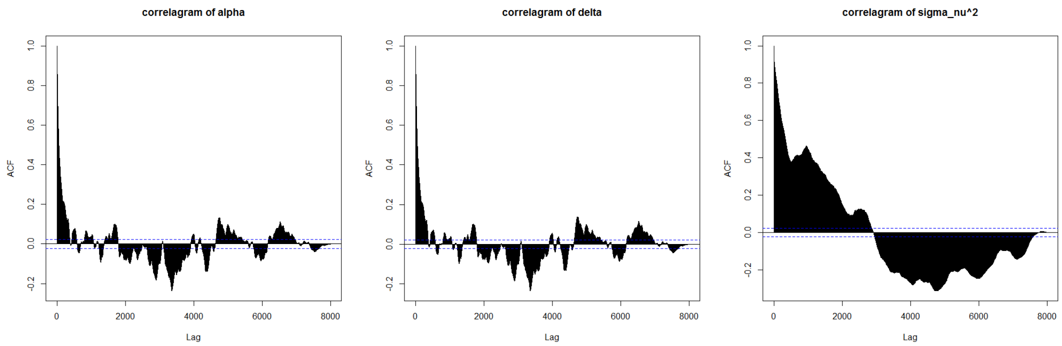

Figure 8: Correlagram of MH + RW with $e = 0.05$

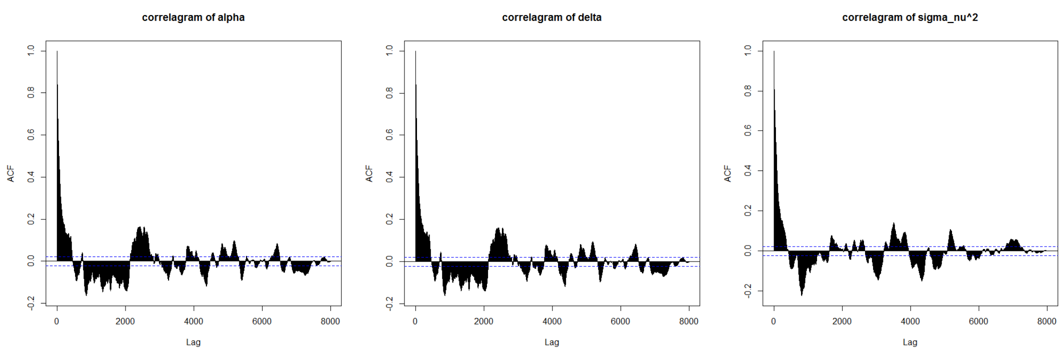

Figure 9: Correlagram of MH + RW with $e = 0.1$

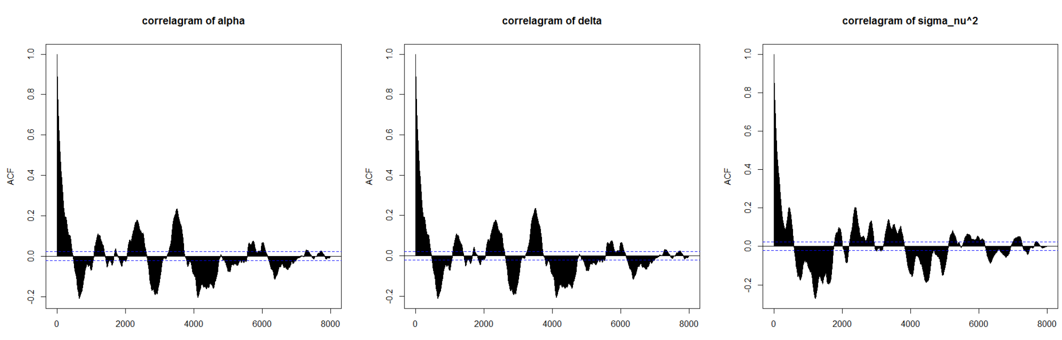

Figure 10: Correlagram of MH + RW with $e = 0.3$

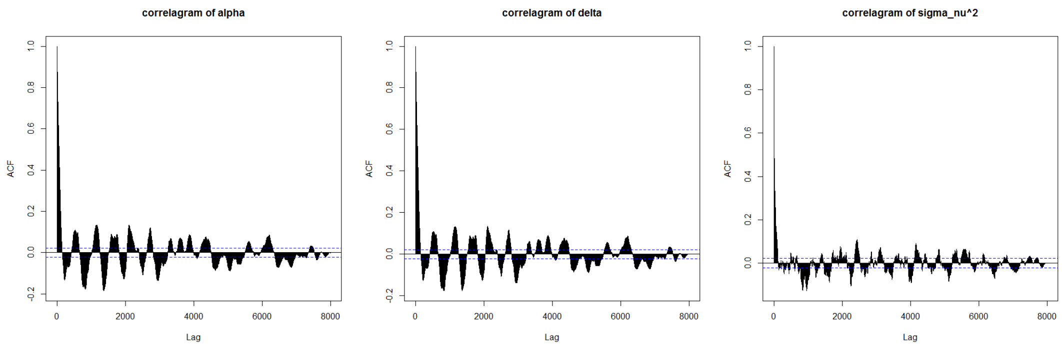

Figure 11: Correlagram of MH + RA

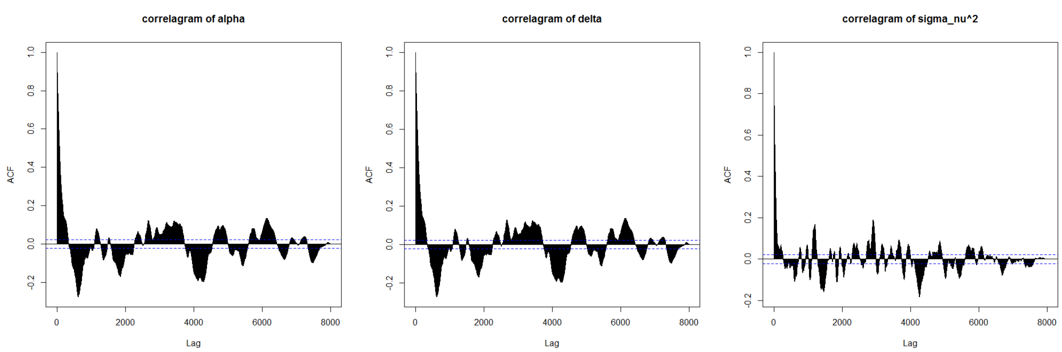

Figure 12: Correlagram of RA