1 Hierarchical models: data-analysis problems

1.1 Math tests

1.

2. Rewrite the distributions in terms of precision

$$(y_{ij}|\theta_i, \omega) \sim N(\theta_i, (\omega)^{-1})$$

$$(\theta_i|\omega, \lambda) \sim N(\mu, (\omega\lambda)^{-1})$$

Choose prior for parameters

$$\omega \sim \Gamma(\frac{d}{2}, \frac{\eta}{2})$$

$$\lambda \sim \Gamma(\frac{h}{2}, \frac{h}{2})$$

The joint distribution of everything is, suppose n data are grouped into m groups:

$$\text{constant} \times \omega^{n+m}\lambda^m e^{-\frac{\omega\sum_{ij}(y_{ij}-\theta_i)^2}{2} - \frac{\omega\lambda\sum_i(\theta_i-\mu)^2}{2} - \frac{\omega\eta}{2} - \frac{h\lambda}{2}} \omega^{\frac{d}{2}-1}\lambda^{\frac{h}{2}-1}$$

From which we have, suppose each group have g_i elements

$$(\omega|y, \lambda, \theta, \mu) \sim \Gamma(\frac{d+n+m}{2}, \frac{\eta}{2} + \frac{\sum_{ij}(y_{ij}-\theta_i)^2}{2} + \frac{\lambda\sum_i(\theta_i-\mu)^2}{2})$$

$$(\lambda|y, \omega, \theta, \mu) \sim \Gamma(\frac{h+m}{2}, \frac{h}{2} + \frac{\omega\sum_i(\theta_i-\mu)^2}{2})$$

$$(\theta_i|y, \omega, \lambda, \mu) \sim N(\frac{\lambda\mu + \sum_j y_{ij}}{\lambda + g_i}, \frac{1}{\omega(\lambda+g_i)})$$

$$(\mu|y, \omega, \lambda, \theta) \sim N(\frac{\sum_i \theta_i}{m}, \frac{1}{m\omega\lambda})$$

We will update the parameter according to these distribution. For the code, see mathtest.r.

3. See "mathtest\mathtest.r"

1.2 Price elasticity of demand

Let $y = \log V$, $x = (1, \log P, d, d\log P)$. Where $d = 0, 1$ is an indicator whether a particular store display ads or not. Each data point are indexed by the store indices $i = 1, ..., n$ and time index $t = 1, ..., p_i$. Each x and β are four vectors whose vector index will be suppresed. We want to build the model:

$$y_{it} = x_{it}\beta_i + \epsilon_i$$

we have the prior distribution

$$\epsilon_i \sim N(0, \sigma^2)$$

$$\beta_i \sim N(\mu, \Sigma)$$
$$(\mu, \Sigma) \sim NIW(m, \lambda, \psi, \nu)$$
$$\sigma^2 \sim U(0, \infty)$$

from which we have

$$(y_{it}, \beta_i, \Sigma, \mu, \sigma^2) \sim N(X_i\beta_i, \sigma^2 I)N(\mu, \Sigma)NIW(m, \lambda, \Psi, \nu)$$

The conditional distributions are:

$$(\beta|y_{it}, \mu, \Sigma, \sigma^2) \sim N(\mu_i', \Sigma_i')$$

where

$$\mu_i' = \Sigma_i'(\frac{X_i^T Y_i}{\sigma^2} + \Sigma^{-1}\mu)$$
$$\Sigma_i'^{-1} = \frac{X_i^T X_i}{\sigma^2} + \Sigma^{-1}$$
$$(\mu, \Sigma|\beta_i) \sim NIW(m', \lambda', \Psi', \nu')$$

where

$$m' = \frac{\lambda m + \sum \beta_i}{\lambda + n}$$
$$\lambda' = \lambda + n$$
$$\Psi' = \Psi + \sum_i (\beta_i - \bar{\beta})(\beta_i - \bar{\beta})^T + \frac{\lambda n}{\lambda + n}\sum_i (\beta_i - \mu)(\beta_i - \mu)^T$$
$$\nu' = \nu + n$$
$$(\sigma^{-2}|y_{it}, \beta_i) \sim \Gamma(\frac{s}{2} + 1, \sum_i \frac{(Y_i - X_i\beta_i)^T(Y_i - X_i\beta_i)}{2})$$

s is the total number of observations.

For details, see "cheese\cheese.r"

1.3 A hierachical probit model with data augmentation

Let $i = 1, ..m$ denote quantities belongs to a certain state, $j = 1, ...g_i$ the number of samples within a given state. $n = \sum_i g_i$ the total data points. β is a f dimensional vector indicating f factors.

$$(z_{ij}|\mu_i, \beta) \sim \begin{cases} N_+(\mu_i + x_{ij}^T\beta, \sigma^2) & y_{ij} = 1, \\ N_-(\mu_i + x_{ij}^T\beta, \sigma^2) & y_{ij} = 0 \end{cases}$$

$$(\mu_i|\mu_0, \lambda) \sim N(\mu, \lambda^2)$$

$$(\lambda, \mu_0) \sim NIG(m_0, \frac{d_0}{2}, \frac{\eta_0}{2})$$

$$\beta \sim N(0, \tau^2 I)$$

$$\tau^2, \sigma^2 \sim U(0, \infty)$$

Besides z, we have these updates for gibbs sampler.

$$(\lambda^2, \mu_0 | z_{ij}, \mu_i) \sim NIG(\frac{1}{m}\sum \mu_i, \frac{d+m}{2}, \frac{\eta + \sum_i (\mu_i - \mu_0)^2}{2})$$

$$(\mu_i | \beta, z_{ij}, \lambda^2, \mu, \sigma^2) \sim N(\frac{\sigma^2 \mu + \lambda^2 \sum_j (z_{ij} - x_{ij}^T \beta)}{\sigma^2 + \lambda^2 g_i}, \frac{\lambda^2 \sigma^2}{\sigma^2 + \lambda^2 g_i})$$

$$(\beta | \mu_i, z_{ij}, \lambda^2, \sigma^2, \tau^2) \sim N(\frac{1}{\sigma^2} V X (z_{ij} - \mu_i), V)$$

where

$$V^{-1} = \frac{1}{\tau^2} I + \frac{1}{\sigma^2} X^T X$$

$$(\sigma^{-2} | z_{ij}, \beta, \mu_i) \sim \Gamma(\frac{n}{2}, \frac{\sum_{ij} (z_{ij} - \mu_i - X\beta)^2}{2})$$

$$(\tau^{-2} | \beta) \sim \Gamma(\frac{f}{2}, \frac{\beta^2}{2})$$

2 Gene expression over time

$$y_{ijrt} = f_{it} + h_{ijt} + \epsilon + \tau_{ij}$$

Here $i = 1 \cdots g$ stand for groups, $j = 1 \cdots n_i$ stands for genes within each group, sometimes will use $k = 1 \cdots n$ to label all genes without specifying gropus. $r = 1, 2, 3$ stands for replicat of a gene.

$$\epsilon \sim N(0, \sigma_\epsilon^2)$$

$$\tau_{ij} \sim N(0, \sigma_{\tau k}^2)$$

$$f_{it} \sim N(0, \sigma_{fi}^2 C)$$

$$h_{kt} \sim N(0, \sigma_{hk}^2 C)$$

$$\sigma_{\tau k}^2, \sigma_\epsilon^2, \sigma_{fi}^2, \sigma_{hk}^2 \sim \text{Inverse}$$

C is a 12×12 matrix computed from Mater(5/2) covariance function with $\tau_2 = 0$ and $b = 1$

$$C_{ij} = e^{-\frac{(i-j)^2}{2}}$$

The marginal distribution of all paramters are

$$\prod_k (\sigma_{\tau k}^{-2-3} \sigma_{hk}^{-2-3}) \prod_i \sigma_{fi}^{-2-n_i} \sigma_\epsilon^{-2-n}$$
$$\times e^{-\frac{1}{2}\sum_{kr}(y-f-h)^2(\frac{1}{\sigma_\epsilon}+\frac{1}{\sigma_{\tau k}^2})-\frac{1}{2}\sum_i \frac{1}{\sigma_{fi}^2} f_i^T C^{-1} f_i - \frac{1}{2}\sum_k \frac{1}{\sigma_{hk}^2} h_k^T C^{-1} h_k} \tag{1}$$

From which we have posterior distribution

$$(h_k|\cdots) \sim N(V_{hk}\frac{\sigma_\epsilon^2+\sigma_{\tau k}^2}{\sigma_\epsilon^2 \sigma_{\tau k}^2}\sum_r (y_{kr}-f_k), V_{hk})$$

$$V_{hk}^{-1} = 3\frac{\sigma_\epsilon^2+\sigma_{\tau k}^2}{\sigma_\epsilon^2 \sigma_{\tau k}^2} + \frac{1}{\sigma_{hk}} C^{-1}$$

$$(f_i|\cdots) \sim N(V_{fi}\sum_{jr}(\frac{1}{\sigma_\epsilon^2}+\frac{1}{\sigma_{\tau j}})(y_{ijr}-h_{ij}), V_{fi})$$

$$V_{fi}^{-1} = 3(\frac{n_i}{\sigma_\epsilon^2}+\sum_j \frac{1}{\sigma_{\tau j}}) + \frac{1}{\sigma_{ni}} C^{-1}$$

$$(\sigma_\epsilon^{-2}|\cdots) \sim \Gamma(\frac{2+n}{2}, \frac{\text{RMS of all data}}{2})$$

$$(\sigma_{\tau k}^{-2}|\cdots) \sim \Gamma(\frac{5}{2}, \frac{\text{RMS of gene k}}{2})$$

$$(\sigma_{hk}^{-2}|\cdots) \sim \Gamma(\frac{5}{2}, \frac{h_k^T C^{-1} h_k}{2})$$

$$(\sigma_{fi}^{-2}|\cdots) \sim \Gamma(\frac{3+n_i}{2}, \frac{f_i^T C^{-1} f_i}{2})$$