

1 Basic concept

1.1 Bias-variance decomposition

Define $e(x) = f(x) - \hat{f}(x)$ and $p(e)$ as its pdf.

$$\text{MSE} = E(e^2) = \int e^2 p de = (\int e p de)^2 + (\int e^2 p de - (\int e p de)^2) = E(e)^2 + \text{Var}(e) \quad (1)$$

1.2 A simple example

(A) Y follows binomial distribution $B(n, \pi_h)$. $E(Y) = n\pi_h$, $\text{Var}(Y) = n\pi_h(1 - \pi_h)$. One can estimate $f(0)$ by

$$f(0)_{\text{estimate}} = \frac{Y}{nh} \quad (2)$$

(B)

$$\begin{aligned} \pi_h &\approx hf(0) + \frac{f''(0)}{2} \int x^2 dx \\ &= hf(0) + \frac{f''(0)h^3}{24} \end{aligned} \quad (3)$$

$$\begin{aligned} \text{MSE}(0) &= (E(\hat{f}(0)) - f(0))^2 + \text{Var}(\hat{f}(0)) \\ &\approx \left(\frac{\pi_h}{h} - f(0)\right)^2 + \frac{\pi_h}{nh^2}(1 - \pi_h) \\ &= \left(\frac{f''(0)}{24}\right)^2 h^4 + \frac{1}{nh} \left(f(0) + \frac{f''(0)h^2}{24}\right) \left(1 - hf(0) + \frac{f''(0)h^3}{24}\right) \\ &\approx \left(\frac{f''(0)}{24}\right)^2 h^4 + \frac{1}{nh} f(0)(1 - hf(0)) \end{aligned} \quad (4)$$

(C)

$$\frac{\partial \text{MSE}(0)}{\partial h} = 4Ah^3 - \frac{f(0)}{nh^2} \quad (5)$$

In order to minimize the MSE, $h = \left(\frac{f(0)}{4An}\right)^{\frac{1}{5}}$