1 Model Description

$$y(t) = \sqrt{h(t)}u(t)$$

$$\ln(h(t)) = \alpha + \delta \ln(h(t-1)) + \sigma_\nu \nu(t)$$

$t = 1, ..., N$

$$u(t), \nu(t) \sim N(0, 1)$$

2 Model Fitting

2.1 MCMC

From the model

$$(y(t)|h(t)) \sim N(0, h(t))$$

$$(h(t)|h(t-1), \omega) \sim N(\alpha + \delta \ln(h(t-1)), \sigma_\nu^2)$$

For parameters ω, assume the prior distribution:

$$\alpha \sim N(\bar{\alpha}, \sigma_\alpha^2)$$

$$\delta \sim N(\bar{\delta}, \sigma_\delta^2)$$

$$\sigma_\alpha, \sigma_\delta \sim \frac{1}{\sigma_\alpha^2}, \frac{1}{\sigma_\delta^2}$$

$$\sigma_\nu^2 \sim IG(\frac{\nu_0}{2}, \frac{s_0^2}{2})$$

We have the marginal distribution

$$p(y, h, \omega) \propto \frac{1}{\sigma_\nu^{N+2+\nu_0} \sigma_\delta^3 \sigma_\alpha^3} \exp\left(-\frac{(\delta - \bar{\delta})^2}{2\sigma_\delta^2} - \frac{(\alpha - \bar{\alpha})}{2\sigma_\alpha^2} - \frac{s_0^2}{2\sigma_\nu^2}\right) \tag{1}$$
$$\times \prod_t \frac{1}{h(t)^{\frac{3}{2}}} \exp\left(-\frac{y(t)^2}{2h(t)} - \frac{(\ln h(t) - \delta \ln h(t-1) - \alpha)^2}{2\sigma_\nu^2}\right)$$

From which we can write the posterior distribution or marginal distribution

$$\sigma_\nu^2 \sim IG\left(\frac{\nu_0 + N}{2}, \frac{s_0^2 + \sum_t (\ln h(t) - \delta \ln h(t-1) - \alpha)^2}{2}\right)$$

$$\delta \sim N\left(\frac{\sigma_\nu^2 \bar{\delta} + \sigma_\delta^2 \sum \ln h(t-1)(\ln h(t) - \alpha)}{\sigma_\nu^2 + \sigma_\delta^2 \sum (\ln h(t))^2}, \frac{\sigma_\nu^2 \sigma_\delta^2}{\sigma_\nu^2 + \sigma_\delta^2 \sum (\ln h(t))^2}\right)$$

$$\alpha \sim N\left(\frac{\sigma_\alpha^2 (1 - \delta) \sum \ln h(t) + \sigma_\nu^2 \bar{\alpha}}{\sigma_\nu^2 + N\sigma_\alpha^2}, \frac{\sigma_\nu^2 \sigma_\alpha^2}{\sigma_\nu^2 + N\sigma_\alpha^2}\right)$$

$$\sigma_\alpha^2 \sim IG\left(\frac{1}{2}, \frac{(\alpha - \bar{\alpha})^2}{2}\right)$$

$$\sigma_\delta^2 \sim IG\left(\frac{1}{2}, \frac{(\delta-\bar{\delta})^2}{2}\right)$$

$$p(h(t)|h(t+1), h(t-1), \omega) \propto \frac{1}{\sqrt{h(t)}} \exp\left(-\frac{y(t)^2}{2h(t)}\right) \tag{2}$$
$$\times \frac{1}{h(t)} \exp\left(-\frac{(\ln h(t) - \mu(t))^2}{2\sigma^2}\right)$$

where $\sigma^2 = \frac{\sigma_\nu^2}{1+\delta^2}$ and $\mu(t) = \frac{\delta \ln h(t+1) + \delta \ln h(t-1) + (1-\delta)\alpha}{1+\delta^2}$

Possible samplings methods are

1. Gibbs sampling: Sufficient for updating ω but too tedious for h.
2. Rejection sampling: Approximate $p(h)$ by a blancket density $q(h)$ and choose a constant c such that $p(h) \le cq(h)$ for any h. Consider a draw from q and accept it with probability $p(h)/cq(h)$. If rejected, draw again. Still not efficient enough for h as it has to compute the overall constant of $p(h)$ for every time point. Also, the choice of c can be tricky as it might leads to too high or low rejection rate.
3. Metropolis-Hastings algorithm: Using a transition kernel $f(h)$, make the $n+1$th draw from $h(t)$, accept it with probability
$$\min\left(\frac{p(h(t)^{n+1})/f(h(t)^{n+1})}{p(h(t)^n)/f(h(t)^n)}, 1\right)$$

Finally, the method we are going to use is a combination of all above:

1. Given h, use Gibbs sampling to sample posterior distribution of ω.
2. Given ω and $h(t \neq t^*)$. Draw from $q(h(t^*))$ and accept it with probability $\min(\frac{p(h(t^*))}{cq(h(t^*))}, 1)$. Repeat until the first acceptance. We choose q to be the marginal probability density of an inverse gamma matrix:
$$q(h) = \frac{\lambda^\phi}{\Gamma(\phi)} h^{-(\phi+1)} e^{-\frac{\lambda}{h}}$$
where
$$\phi = \frac{1-2e^{\sigma^2}}{1-e^{\sigma^2}} + \frac{1}{2}$$
$$\lambda = (\phi-1)e^{\mu(t)+\frac{\sigma^2}{2}} + \frac{y(t)^2}{2}$$
The reason of this choice is that we chose a gamma distribution that has the same first and second moment as the second line of (2). This gamma distribution is further combined with the first line of (2), which is itself a gamma distribution, and give rise q.

Instead of the criteria given in rejection sampling, we will choose c by

$$c = 1.1 \left(\frac{p(h)}{q(h)} \right)_{h = \text{mode of } q}$$

We do not need to compute the normalizing constant for p as it will be compensated by c.

3. After the acceptance in 2, we began the procedure in Metropolis-Hastings algorithm with the same acceptance probability. When a of h^{n+1} is rejected, we will let it inherit the value of h^n instead of redraw.

2.2 AR(1)

3 Data Source

4 Reference

1. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian Analysis of Stochastic Volatility Models", Journal of Business & Economic Statistics, 20:1, 69-87

2. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian analysis of stochastic volatility models with fat-tails and correlated errors", Journal of Econometrics, 122(2004) 185-212

3. John Geweke, "Prior for Macroeconomic Time Series and Their Application", Econometric Theory, Vol. 10, 609-632