1 Basic concept

1.1 Bias-variance decomposition

Define $e(x) = f(x) - \hat{f}(x)$ and $p(e)$ as its pdf.

$$\text{MSE} = E(e^2) = \int e^2 pde = (\int epde)^2 + (\int e^2 pde - (\int epde)^2) = E(e)^2 + Var(e) \tag{1}$$

1.2 A simple example

(A) Y follows binomal distribution $B(n, \pi_h)$. $E(Y) = n\pi_h$, $Var(Y) = n\pi_h(1-\pi_h)$. One can estimate $f(0)$ by

$$f(0)_{\text{estimate}} = \frac{Y}{nh} \tag{2}$$

(B)

$$\begin{aligned} \pi_h &\approx hf(0) + \frac{f''(0)}{2}\int x^2 dx \\ &= hf(0) + \frac{f''(0)h^3}{24} \end{aligned} \tag{3}$$

Choosing h so that both $h \ll 1$ and $\pi_h \ll 1$ are true, we have

$$\begin{aligned} \text{MSE}(0) &= (E(\hat{f}(0)) - f(0))^2 + Var(\hat{f}(0)) \\ &\approx (\frac{\pi_h}{h} - f(0))^2 + \frac{\pi_h}{nh^2} \\ &= (\frac{f''(0)}{24})^2 h^4 + \frac{1}{nh}(f(0) + \frac{f''(0)h^2}{24}) \\ &\approx (\frac{f''(0)}{24})^2 h^4 + \frac{1}{nh} f(0) \end{aligned} \tag{4}$$

(C)

$$\frac{\partial \text{MSE}(0)}{\partial h} = 4Ah^3 - \frac{f(0)}{nh^2} \tag{5}$$

In order to minimize the MSE, $h = (\frac{f(0)}{4An})^{\frac{1}{5}}$

2 Curve Fitting by linear smoothing

(A)

$$y_{\text{estimate}} = \beta_{\text{estimate}} x$$

$$
\begin{aligned}
&= (\vec{x}^T\vec{x})^{-1}\vec{x}^T\vec{y}x \\
&= \frac{\sum_i x_i y_i}{\sum_i x_i^2}x \\
&= \sum_i \frac{x_i x}{\vec{x}^2} y_i
\end{aligned} \tag{6}
$$

So we have $\omega_i = \frac{x_i x}{\vec{x}^2}$

(B) See "linearsmoothing.R" and "weight.R"

3 Cross validation

(A) See "predictionerror.R"

(B) See "testmodel.R"

(C)

4 Local polynomial regression

(A) Define matrix R where $R_{ij}(\vec{x}) = (x_i - x)^{j-1}$. Then $g(x_i, a) = (R\vec{a})_i$. The cost function can be written as

$$
\sum_{i=1}^{n} \omega_i (y_i - (R\vec{a})_i)^2 \tag{7}
$$

Optimizing this with $\vec{a}$ is equivalent to linear regression on $f(\vec{x}) = \vec{x}^T\vec{a}$ given observed data $X = R$ and $\vec{y}$ and weighted cost function. The answer is

$$
\vec{a} = (R^T\Omega R)^{-1}R^T\Omega\vec{y} \tag{8}
$$

$$
\Omega = \mathrm{diag}(\omega_1, \omega_2, \cdots) \tag{9}
$$

The estimate of $f(x)$ is just $g(x, a) = a_0$.

(B) Call the old weight function ω and the new ones γ

$$
\begin{aligned}
(R^T\Omega R)^{-1} &= \begin{pmatrix} \sum_i \omega_i & \sum_i R_{i2}\omega_i \\ \sum_i R_{i2}\omega_i & \sum_i R_{i2}^2\omega_i \end{pmatrix}^{-1} \\
&= \frac{1}{(\sum_i \omega_i)(\sum_i R_{i2}^2\omega_i) - (\sum_i R_{i2}\omega_i)^2} \begin{pmatrix} \sum_i R_{i2}^2\omega_i & -\sum_i R_{i2}\omega_i \\ -\sum_i R_{i2}\omega_i & \sum_i \omega_i \end{pmatrix}
\end{aligned} \tag{10}
$$

$$
R^T\Omega\vec{y} = \begin{pmatrix} \sum_i \omega_i y_i \\ \sum_i R_{i2}\omega_i y_i \end{pmatrix} \tag{11}
$$

$$
\begin{aligned}
f(x) = a_0 &= \frac{(\sum_i R_{i2}^2\omega_i)(\sum_i \omega_i y_i) - (\sum_i R_{i2}\omega_i)(\sum_i R_{i2}\omega_i y_i)}{(\sum_i \omega_i)(\sum_i R_{i2}^2\omega_i) - (\sum_i R_{i2}\omega_i)^2} \\
&= \sum_i \frac{((\sum_j R_{j2}^2\omega_j) - (\sum_j R_{j2}\omega_j)R_{i2})\omega_i}{\sum_k((\sum_j R_{j2}^2\omega_j) - (\sum_j R_{j2}\omega_j)R_{k2})\omega_k} y_i \qquad (12)
\end{aligned}
$$

Define weight function $\gamma(x_i, x) = ((\sum_j R_{j2}^2\omega_j) - (\sum_j R_{j2}\omega_j)R_{i2})\omega_i$, the above can be written as a weighted sum of y_i. Further evaluate γ:

$$
\begin{aligned}
\gamma(x_i, x) &= ((\sum_j R_{j2}^2\omega_j) - (\sum_j R_{j2}\omega_j)R_{i2})\omega_i \\
&= \frac{1}{h^2}((\sum_j (x_j - x)^2 K_j) - (\sum_j (x_j - x)K_j)(x_i - x))K_i \\
&= \frac{1}{h^2}(s_2 - s_1(x_i - x))K_i \qquad (13)
\end{aligned}
$$

Here we used K_i to denote $K(\frac{x-x_i}{h})$. Since the function will be normalized so we can ignore $\frac{1}{h^2}$.

(C) From $y = f(x) + \epsilon$

$$
\begin{aligned}
\mathrm{Var}(\vec{y}) &= \sigma^2 \qquad (14) \\
\mathrm{Mean}(\vec{y}) &= \mathrm{Mean}(f(x)) \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Mean}(a_0) &= \sum_i \gamma_i \mathrm{Mean}(f(x)) = \mathrm{Mean}(f(x)) \\
\mathrm{Var}(a_0) &= \mathrm{Tr}\mathrm{Var}(H\vec{y}) \\
&= \sum_i |\gamma(x_i, x)|^2\sigma^2 \qquad (16)
\end{aligned}
$$

(D)

$$
\begin{aligned}
E(\sigma^2) &\propto E((\vec{y} - H\vec{y})^T(\vec{y} - H\vec{y})) \\
&\propto E(\vec{y}^T\vec{y}) - 2E(\vec{y}^T H\vec{y}) + E(\vec{y}^T H^2\vec{y}) \\
&\propto \sigma^2 + E(f(x))^2 - \qquad (17)
\end{aligned}
$$