1 Introduction

The stochastic volitility model (SV)...

Markov Chain Monte Carlo...

The outline of this essay is following. We will introduce this model and derive the Bayesian posterior distribution and propose an MCMC to fit the model. The main difficulty of this model will be sampling a particular distribution with no analytic expression. We will list three possible methods appeared in literature to overcome this difficulty. Then we will fit the model on the same set of data with these methods and check their results and performance. At the end, we devote a section to compare the result of GARCH and SV on more data sets.

2 Basic Model

y is the financial serie we are interested in with zero mean. The model is

$$y(t) = \sqrt{h(t)}u(t)$$

$$\ln(h(t)) = \alpha + \delta \ln(h(t-1)) + \sigma_\nu \nu(t)$$

$$u(t), \nu(t) \sim N(0, 1)$$

h_t is a latent variable measuring the volitility of y, δ is the volatility persistence.

3 Bayesian Analysis

3.1 MCMC

Assume the prior distribution:

$$\alpha \sim N(\alpha_0, \sigma_\alpha^2)$$

$$\delta \sim N(\delta_0, \sigma_\delta^2)$$

$$\sigma_\nu^2 \sim IG(\frac{\nu_0}{2}, \frac{s_0}{2})$$

We have the marginal distribution

$$p(y, h, \delta, \alpha, \sigma_\nu^2) \propto \frac{1}{\sigma_\nu^{2+\nu_0}} \exp\left(-\frac{(\delta-\delta_0)^2}{2\sigma_\delta^2} - \frac{(\alpha-\alpha_0)^2}{2\sigma_\alpha^2} - \frac{s_0^2}{2\sigma_\nu^2}\right)$$
$$\times \prod_{t=2}^{N} \frac{1}{h_t^{\frac{3}{2}}\sigma_\nu} \exp\left(-\frac{y_t^2}{2h_t} - \frac{(\ln h_t - \delta \ln h_{t-1} - \alpha)^2}{2\sigma_\nu^2}\right) \tag{1}$$

From which we can derive the posterior distributions are

$$(\sigma_\nu^2 | h, \alpha, \delta) \sim IG\left(\frac{\nu_0 + N - 1}{2}, \frac{s'}{2}\right)$$

$$s' = s_0+(N-1)\alpha^2+(1+\delta^2)S_2-\delta^2(\ln h_N)^2-(\ln h_1)^2-2\alpha((1-\delta)S_1-\ln h_1+\delta\ln h_N)-2\delta S_3$$

$$(\delta|h,\alpha,\sigma_\nu^2) \sim N\left(\frac{\sigma_\nu^2\delta_0+\sigma_\delta^2(S_3-\alpha(S_1-\ln h_N))}{\sigma_\nu^2+\sigma_\delta^2(S_2-(\ln h_N)^2)}, \frac{\sigma_\nu^2\sigma_\delta^2}{\sigma_\nu^2+\sigma_\delta^2(S_2-(\ln h_N)^2)}\right)$$

$$(\alpha|h,\sigma_\nu^2,\delta) \sim N\left(\frac{\sigma_\alpha^2((1-\delta)S_1-\ln h_1+\delta\ln h_N)+\sigma_\nu^2\alpha_0}{\sigma_\nu^2+(N-1)\sigma_\alpha^2}, \frac{\sigma_\nu^2\sigma_\alpha^2}{\sigma_\nu^2+(N-1)\sigma_\alpha^2}\right)$$

$$S_1 = \sum_{t=1}^{N}\ln h_t \quad S_2 = \sum_{t=1}^{N}(\ln h_t)^2 \quad S_3 = \sum_{t=2}^{N}\ln h_t \ln h_{t-1}$$

$$p(h_t|h_{t+1},h_{t-1},\delta,\alpha,\sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}}\exp\left(-\frac{y_t^2}{2h_t}\right)\frac{1}{h_t}\exp\left(-\frac{(\ln h_t-\mu_t)^2}{2\sigma_t^2}\right) \tag{2}$$

$$\mu_t = \frac{\delta\ln h_{t+1}+\delta\ln h_{t-1}+(1-\delta)\alpha}{1+\delta^2}$$

$$\sigma^2 = \frac{\sigma_\nu^2}{1+\delta^2}$$

In addition, following Jacuqier (1994), we will not update h_1 and h_N with (2), we will update them by directly drawing from autoregressive model of $\ln h$.

In summary, the outline of the algorithm is

1. Initialize $h, \alpha, \delta, \sigma_\nu^2$
2. For $t = 2, 3, \cdots, N-1$, draw h_t from $p(h_t|h_{t+1}, h_{t-1}, \delta, \alpha, \sigma_\nu^2)$
3. Draw $\ln h_1$ from $N(\alpha+\delta\ln h_2, \sigma_\nu^2)$, $\ln h_N$ from $N(\alpha+\delta\ln h_{N-1}, \sigma_\nu^2)$
4. Draw σ_ν^2 from $(\sigma_\nu^2|h, \alpha, \delta)$
5. Draw δ from $(\delta|h, \alpha, \sigma_\nu^2)$
6. Draw α from $(\alpha|h, \delta, \sigma_\nu^2)$
7. Go to step 2

It is easy to simulate the posterior distribution of σ_ν^2. α and δ, so the only nontrivial part of the MCMC is step 2. Below we will give three sampling methods. The comparison of them will be the focus of this project.

3.2 Sampling Method 1: Metroplis-Hastings with Random Walk

Write (2) as a distribution of $\ln h$ rather than h

$$p(\ln h_t|h_{t+1},h_{t-1},\delta,\alpha,\sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}}\exp\left(-\frac{y_t^2}{2h_t}\right)\exp\left(-\frac{(\ln h_t-\mu_t)^2}{2\sigma_t^2}\right)$$

Given $\ln h_t^{i-1}$. Each time we simply propose a $\ln h_t^*$ by drawing

$$N(\ln h_t^{i-1}, e^2)$$

where e^2 is a preset parameter independent of other variables. And accept it with probability

$$\text{Min}(1, \frac{p(\ln h_t^*)}{p(\ln h_t^{i-1})})$$

The algorithm is:

1. Draw $\ln h_t^*$ from $N(\ln h_t^{i-1}, e^2)$
2. Accept this value with probability $\text{Min}(1, \frac{p(\ln h_t^{i*})}{p(\ln h_t^{i-1})})$
3. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

3.3 Sampling Method 2: Metroplis-Hastings with Accept-Reject Sampling

This is the method proposed in Jacquier(1994). The idea is to refine the process of the proposing update in MH. We can "approximate" (2) by an inverse gamma distribution:

$$q(h_t) = \frac{\lambda^\phi}{\Gamma(\phi)} h^{-(\phi+1)} e^{-\frac{\lambda}{h_t}}$$

where

$$\lambda = \frac{1 - 2e^{\sigma^2}}{1 - e^{\sigma^2}} + \frac{1}{2}$$

$$\phi = (\lambda - 1)e^{\mu_t + \frac{\sigma^2}{2}} + \frac{y_t^2}{2}$$

and σ^2 and μ_t are defined under (2). The reason of this choice is that we can choose a inverse gamma distribution which have same first and second moment with the lognormal part of (2), this then combines with the inverse gamma part of (2) to give the above inverse gamma distribution.

Define

$$c = 1.1 \left(\frac{p(h)}{q(h)} \right)_{h = \text{mode of } q}$$

We will propose candidate h_t^{i*} from $IG(\lambda, \phi)$ and accept it with $\text{Min}(1, \frac{p(h^*)}{cq(h^*)})$, if rejected, repropose until accepted. The winner of accept-reject process will be the candidate of MH process with transition kernel $f(h_t^*) = \text{Min}(p(h_t^*), cq(h_t^*))$. The actual algorithm will be

1. Draw h_t^* from $IG(\lambda, \phi)$, note that both λ and ϕ are functions of h_{t+1}, h_{t-1} and other parameters
2. Accept h_t^* with probability $\text{Min}(1, \frac{p(h_t^*)}{cq(h_t^*)})$
3. If rejected, go to step 1.
4. If $p(h_t^*) < cq(h_t^*)$, $h_t^i = h_t^*$. The algorithm ends.
5. Accept h_t^* with probability $\text{Min}(1, \frac{p(h_t^*)/q(h_t^*)}{p(h_t^{i-1})/q(h_t^{i-1})})$
6. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

3.4 Sampling Method 3: Pure Accept-Reject Sampling

This method was used in Kim(1998).

$$\begin{aligned}
\ln p(\ln h_t|\cdots) &= -\frac{1}{2}\ln h_t - \frac{y_t^2}{2h_t} - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&\leq -\frac{1}{2}\ln h_t - \frac{y_t^2}{2}\left(\exp(-\mu_t)(1 + \mu_t - \ln h_t)\right) - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&= -\frac{(\ln h_t - \mu_t')^2}{2\sigma^2} + \text{constants} \qquad (3)
\end{aligned}$$

Where

$$\mu_t' = \mu_t + \frac{\sigma^2}{2}(y_t^2 \exp(-\mu_t) - 1)$$

This observation leads to a standard reject-accept sampling:

1. Draw $\ln h_t^*$ from $N(\mu_t', \sigma^2)$
2. Accept h_t^* with probability $\text{Min}(1, g(h_t^*)$, where

$$g(h_t) = \exp(\frac{y_t^2}{2}(\exp(-\mu_t))(1 + \mu_t - \ln h_t) - \frac{1}{h_t})$$

3. If rejected, go to step 1.
4. $h_t^i = h_t^*$

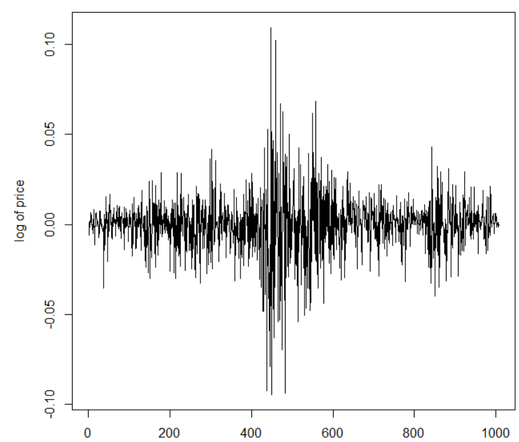

Figure 1: Daily change of S&P500 closing price from 2007-2010

4 Test of Three Sampling Method

4.1 Data

For this section, we use the S&P500 from 1/1/2007 to 12/31/2010. As in Jacquier(1994) and Gallant(1992), we study the change of log of closing price every trading day:

$$y_t = \log(price_{t+1}/price_t)$$

y are plotted in Figure 1. There are 1008 data points in the time series.

4.2 Result

We fit model for the data with five methods: MH random walk with three different "speed" e, and the other two methods. We will address them as "MH + RW with $e = \ldots$", "MH + RA", "RA", respectively. For prior we take $\delta \sim N(0, 10)$, $\alpha \sim N(0, 10)$ and $\sigma_\nu^2 \sim IG(\frac{1}{2}, \frac{1}{2})$. We simulate the MCMC for 12000 iterations, and all the posterior results are based on the sample collected after 4000th iteration. We list the result of parameters in Table 1. The numbers outside and inside bracket are posterior mean and standard deviations respectively. From Figure 2 to 6 we give all histograms of these variables. For each h_t, we compute its sample mean and plot the log of the mean in Figure 7

4.3 Time

In Table 2 we listed the time and rejection/repeat rate of each method for 12000 iterations. Repeat rate applies for methods involving MH algorithm,

Method	δ	α	σ_ν^2	$Cov(\delta,\alpha)$ $(\times 10^{-4})$	$Cov(\delta,\sigma_\nu^2)$ $(\times 10^{-4})$	$Cov(\alpha,\sigma_\nu^2)$ $(\times 10^{-4})$
MH +RW, $e = 0.05$	0.976(0.008)	-0.214(0.073)	0.06(0.012)	5.9	-0.12	-1.18
MH +RW,$e = 0.1$	0.977(0.008)	-0.199(0.067)	0.061(0.009)	5.07	-0.27	-2.3
MH + RW, $e = 0.3$	0.973(0.009)	-0.24(0.083)	0.068(0.015)	7.8	-0.83	-7.3
MH + RA	0.986(0.006)	-0.113(0.051)	0.029(0.004)	3.1	-0.05	-0.44
RA	0.974(0.01)	-0.223(0.089)	0.071(0.017)	8.97	-1	-8.9

Table 1: Results of fitting SV model on S&P500 daily log change

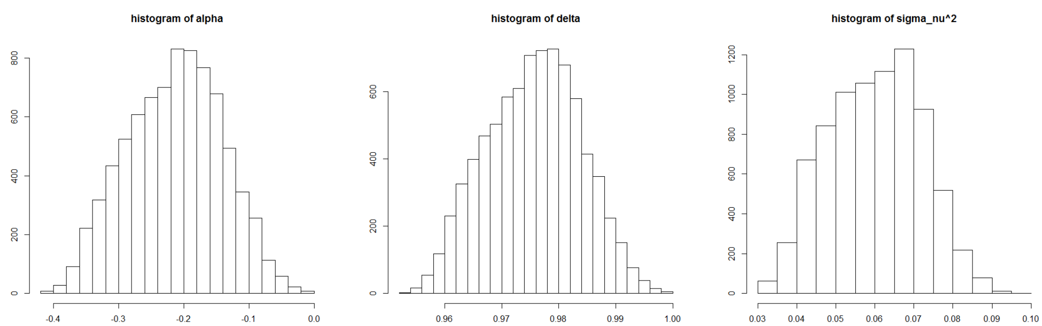

Figure 2: Histograms of MH + RW with $e = 0.05$

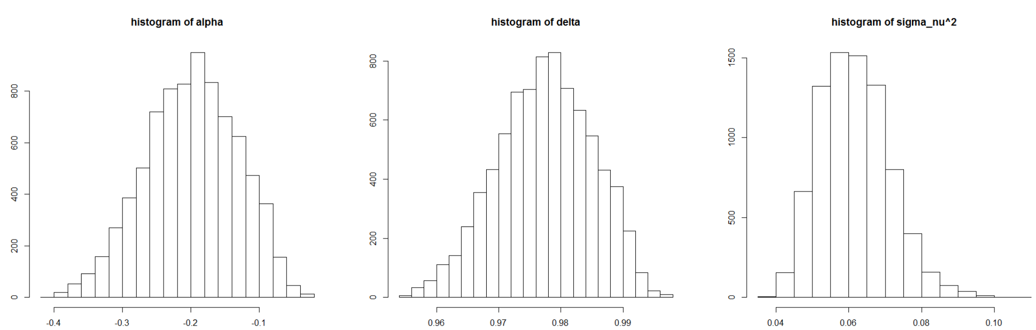

Figure 3: Histograms of MH + RW with $e = 0.1$

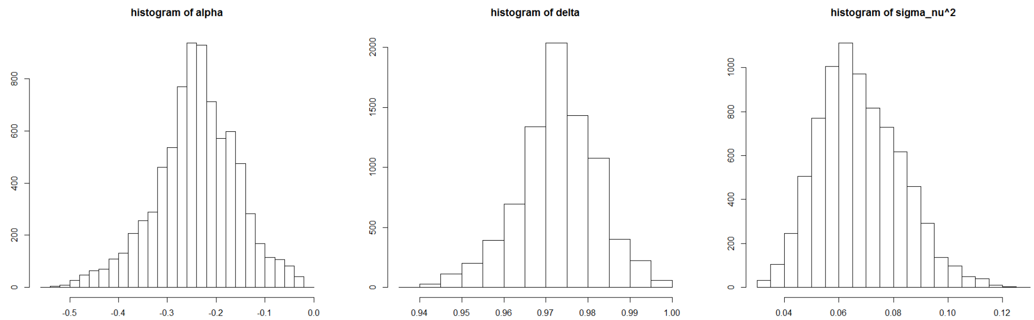

Figure 4: Histograms of MH + RW with $e = 0.3$

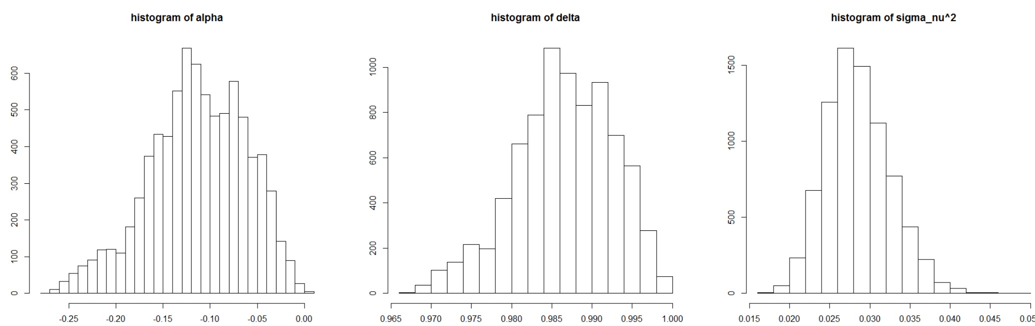

Figure 5: Histograms of MH + RA

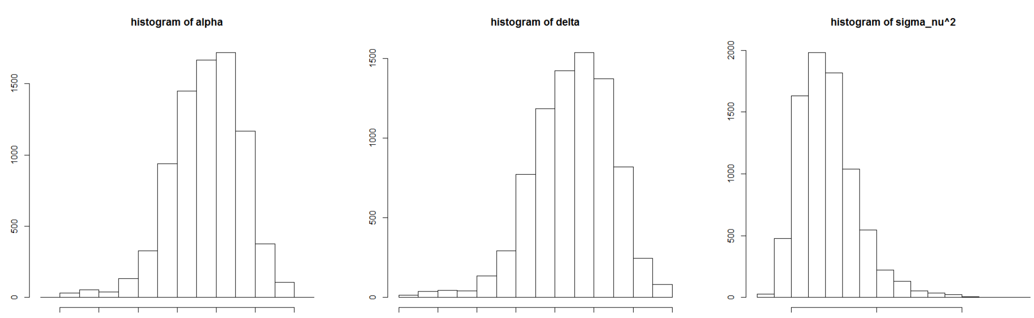

Figure 6: Histograms of RA

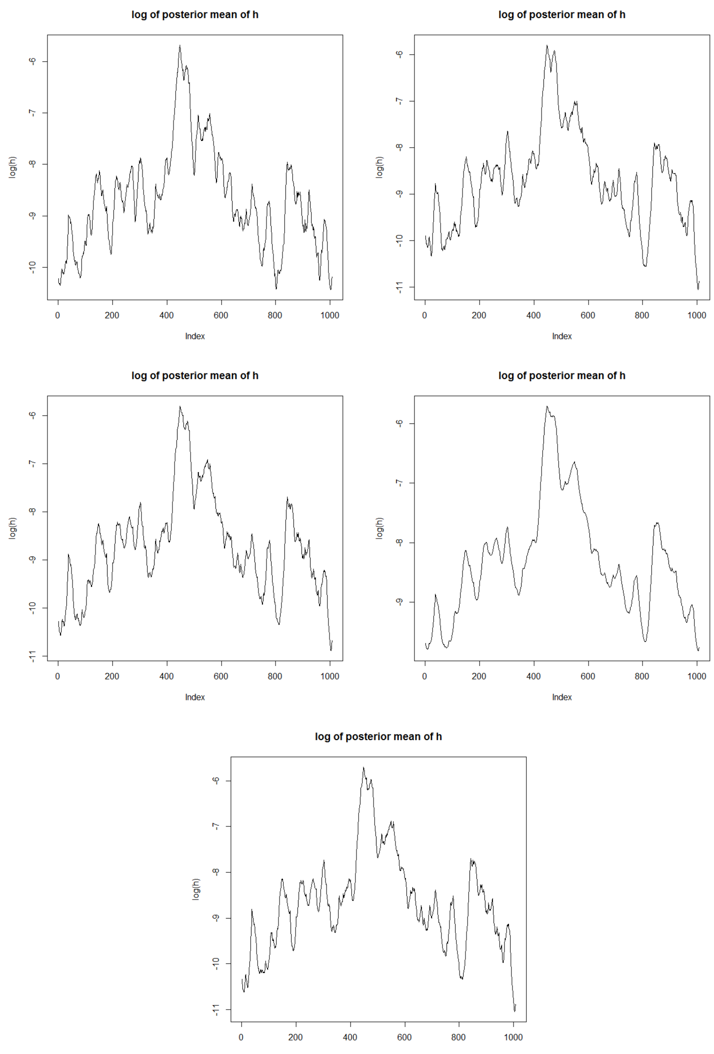

Figure 7: Log of mean of h_t of five methods, in the order of appearance in previous tables

Method	Time for 12000 iteration (second)	reject rate	repeated rate
MH + RW, $e = 0.05$	213.42		0.1
MH + RW, $e = 0.1$	210.64		0.18
MH + RW, $e = 0.3$	215.87		0.44
MH +RA	451.58	0.09	0.0002
RA	171.39	0.008	

Table 2: Time and reject/repeat rate of each method

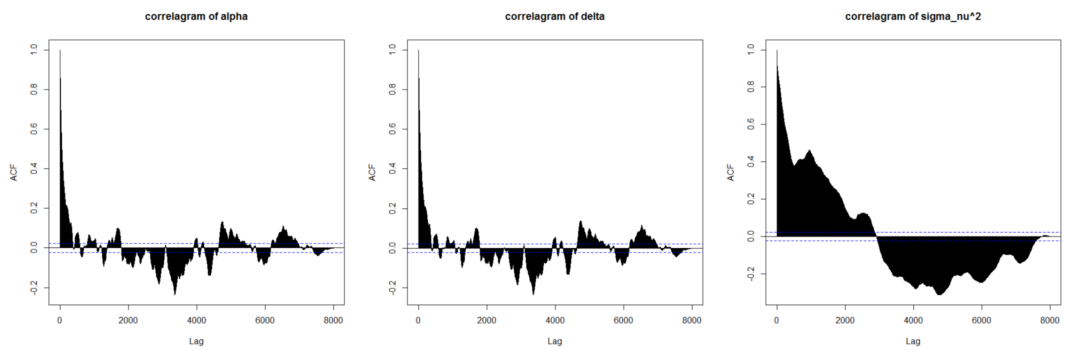

Figure 8: Correlagram of MH + RW with $e = 0.05$

while rejection rate applies for methods involving RA sampling. Note that only the HM+RA method have both.

4.4 Autocorrelation

We give the autocorrelation function of parameters fit by all methods in Figure 8 to 12. The autocorrelation function are computed and plotted using acf() in stats library of R.

5 Comparison with GARCH

5.1 Data

We will use multiple data in this section: S&P500, Nasdaq composite and Nikkei 225.

6 Reference

1. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian Analysis of Stochastic Volatility Models", Journal of Business & Economic Statistics, 1994, Vol. 12, No. 4

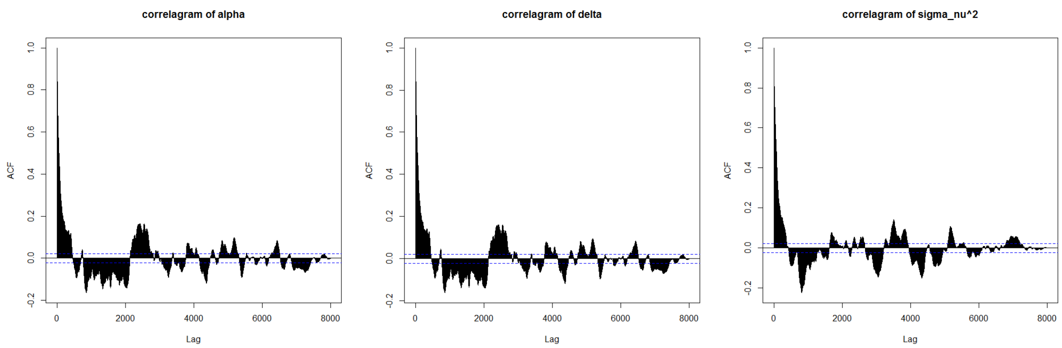

Figure 9: Correlagram of MH + RW with $e = 0.1$

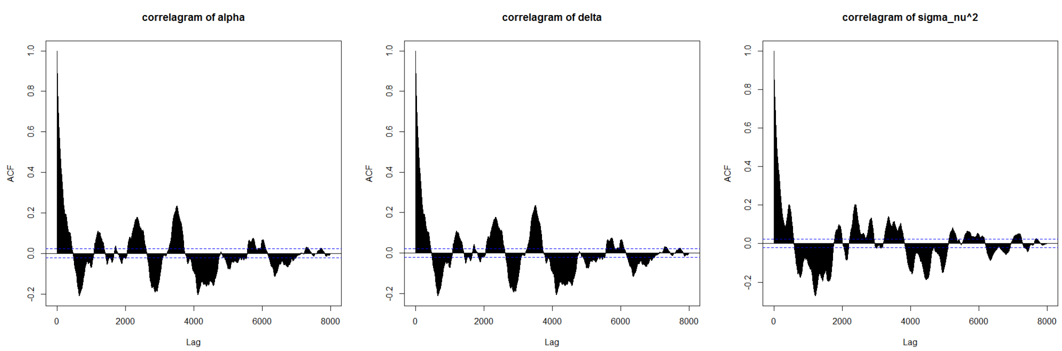

Figure 10: Correlagram of MH + RW with $e = 0.3$

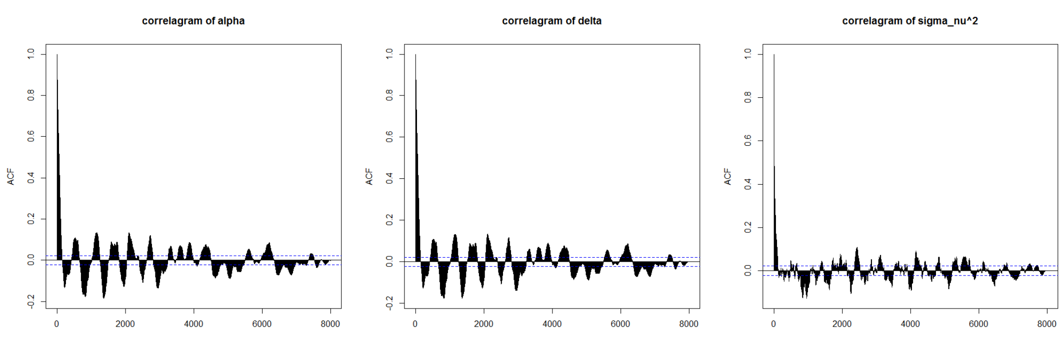

Figure 11: Correlagram of MH + RA

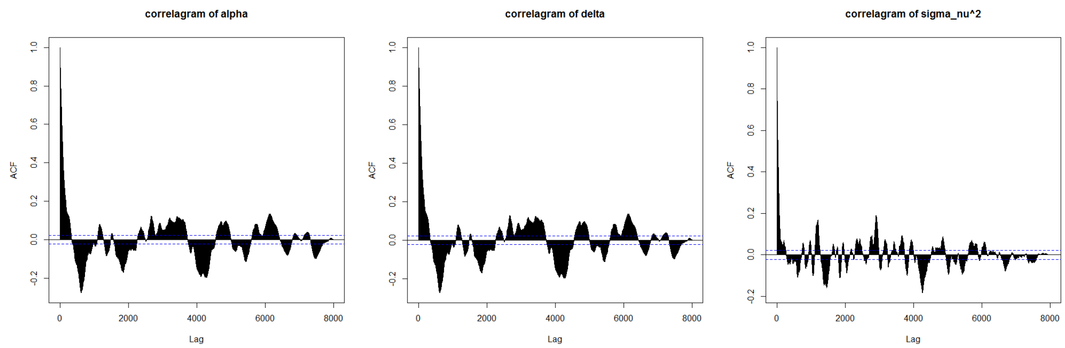

Figure 12: Correlagram of RA

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