1 A simple Gaussian location model

(A) This is equivalent with determine the distribution x under a gaussian distribution with precision ω and mean μ where $\omega \sim Gamma(\frac{d}{2}, \frac{\eta}{2\kappa})$, the result should be

$$p(\theta) \sim (1 + \frac{\kappa(\theta-\mu)^2}{\eta})^{-\frac{d+1}{2}} \tag{1}$$

Compare to the problem we have $\nu = d$, $s^2 = \frac{\eta}{d\kappa}$ and $m = \mu$

(B)

$$\begin{aligned} p(\theta,\omega|\vec{y}) &= \frac{p(\vec{y}|\theta,\omega)p(\theta,\omega)}{\int p(\vec{y}|\theta',\omega')p(\theta',\omega')d\theta' d\omega'} \\ &\propto \omega^{\frac{n}{2}} e^{-\frac{\omega \sum_i (y_i-\theta)^2}{2}} \omega^{\frac{d+1}{2}-1} e^{-\omega\frac{\kappa(\theta-\mu)^2}{2}} e^{-\omega\frac{\eta}{2}} \\ &\propto \omega^{\frac{d+n+1}{2}-1} e^{-\frac{\omega(n+\kappa)(\theta-\frac{\mu\kappa+\sum_i y_i}{n+\kappa})^2}{2}} e^{-\frac{\omega(\eta+\sum_i y_i^2+\kappa\mu^2-\frac{(\mu\kappa+\sum_i y_i)^2}{n+\kappa})}{2}} \end{aligned} \tag{2}$$

After rearanging parameters

$$\mu^* = \frac{\mu\kappa + \sum_i y_i}{n+\kappa} \tag{3}$$

$$\kappa^* = \kappa + n \tag{4}$$

$$d^* = d + n \tag{5}$$

$$\eta^* = \eta + \sum_i y_i^2 + \kappa\mu^2 - \frac{(\mu\kappa+\sum_i y_i)^2}{n+\kappa} \tag{6}$$

The distribution is of the same form as $p(\theta, \omega)$

$$p(\theta,\omega|\vec{y}) \propto \omega^{\frac{d^*+1}{2}-1} e^{-\frac{\omega\kappa^*(\theta-\mu^*)^2}{2}} e^{-\frac{\omega\eta^*}{2}} \tag{7}$$

(C) Keeping only the part dependent on θ

$$\begin{aligned} p(\theta|\vec{y},\omega) &= \frac{p(\vec{y},\theta,\omega)}{p(\vec{y},\omega)} \\ &\propto e^{-\frac{\omega\kappa^*(\theta-\mu^*)^2}{2}} \sim N(\mu^*, (\omega\kappa^*)^{-1}) \end{aligned} \tag{8}$$

(D)

$$\begin{aligned} p(\omega|\vec{y}) &= \frac{p(\omega,\vec{y})}{p(\vec{y})} \\ &\propto \int p(\omega,\vec{y},\theta')d\theta' \\ &\propto \omega^{\frac{d^*+1}{2}-1} e^{-\frac{\omega\eta^*}{2}} \int e^{-\frac{\omega\kappa^*(\theta'-\mu^*)^2}{2}} d\theta' \\ &\propto \omega^{\frac{d^*}{2}-1} e^{-\frac{\omega\eta^*}{2}} \sim \Gamma(\frac{d^*}{2}, \frac{\eta^*}{2}) \end{aligned} \tag{9}$$

(E)

$$p(\theta|\vec{y}) = \int p(\theta, \omega'|\vec{y}) d\omega' \tag{10}$$

Since $p(\theta, \omega|\vec{y})$ is same to $p(\theta, \omega)$ up to a parameter redefinition. So the result should be same to part A with parameter redefinition.

$$p(\theta|\vec{y}) \sim (1 + \frac{\kappa^*(\theta - \mu^*)^2}{\eta^*})^{-\frac{d^*+1}{2}} \tag{11}$$

(F) False. In the limit κ, d and η goes to zero

$$p(\theta) \propto \frac{1}{\sqrt{1+\theta^2}} \tag{12}$$

When $\theta \to \infty$, $p(\theta) \to \frac{1}{\theta}$ which cannot be normalized.

(G) True. The starred paramters are still in healthy condition under this limit.

(H)

2 The conjugate Gaussian linear model

2.1 Basics

(A)

$$\begin{aligned}
p(\vec{\beta}|\vec{y}, \omega) &= \frac{p(\vec{\beta}, \vec{y}, \omega)}{p(\vec{y}, \omega)} \\
&\propto p(\vec{y}|\vec{\beta}, \omega) p(\vec{\beta}|\omega) \\
&\propto e^{-\frac{\omega(\vec{y}-X\vec{\beta})^T \Lambda (\vec{y}-X\vec{\beta})}{2}} e^{-\frac{\omega(\vec{\beta}-\vec{m})^T K (\vec{\beta}-\vec{m})}{2}} \\
&\propto e^{-\frac{\omega(\vec{\beta}-\vec{m}^*)^T K^* (\vec{\beta}-\vec{m}^*)}{2}} &(13) \\
K^* &= X^T \Lambda X + K &(14) \\
\vec{m}^* &= K^{*-1}(X^T \Lambda \vec{y} + K\vec{m}) &(15)
\end{aligned}$$

(B)

$$\begin{aligned}
p(\omega|\vec{y}) &\propto p(\omega, \vec{y}) \\
&\propto \int p(\vec{y}|\vec{\beta}', \omega) p(\vec{\beta}'|\omega) p(\omega) d\vec{\beta}' \\
&\propto \omega^{\frac{d^*+1}{2}-1} e^{-\frac{\omega\eta^*}{2}} \int e^{-\frac{\omega(\vec{\beta}'-\vec{m}^*)^T K^* (\vec{\beta}'-\vec{m}^*)}{2}} d\vec{\beta}' \\
&\propto \omega^{\frac{d^*}{2}-1} e^{-\frac{\omega\eta^*}{2}} \sim \Gamma(\frac{d^*}{2}, \frac{\eta^*}{2}) &(16) \\
d^* &= d + n &(17) \\
\eta^* &= \eta + \vec{y}^T \Lambda \vec{y} + \vec{m}^T K \vec{m} - \vec{m}^{*T} K^* \vec{m}^* &(18)
\end{aligned}$$

(C) Similar to previous problems

$$p(\omega, \vec{\beta}|\vec{y}) \propto \omega^{\frac{d^*+1}{2}-1} e^{-\frac{\omega\eta^*}{2}} e^{-\frac{\omega(\vec{\beta}-\vec{m}^*)^T K^*(\vec{\beta}-\vec{m}^*)}{2}} \tag{19}$$

Again, reuse the derivation where we derive t-distribution from normal/gamma prior

$$\begin{aligned} p(\vec{\beta}|\vec{y}) &= \int p(\omega', \vec{\beta}|\vec{y}) d\omega' \\ &\propto (1 + \frac{(\vec{\beta}-\vec{m}^*)^T K^* (\vec{\beta}-\vec{m}^*)}{\eta^*})^{-\frac{d^*+1}{2}} \end{aligned} \tag{20}$$

(D) The script "gdpgrowth.R" read data from "gdpgrowth.csv" and use the above linear model to fit the relation between column GR6096 and DENS60, using

$$\vec{\beta}_{\text{estimate}} = \vec{m}^* = K^{*-1} X^T \Lambda \vec{y} \tag{21}$$

2.2 A heavy tailed error model

(A) Let $\vec{\alpha} = X\vec{\beta}$

$$\begin{aligned} p(\vec{y}|X, \vec{\beta}, \omega) &= \int p(\vec{y}|X, \vec{\beta}, \omega\Lambda) \prod_i p(\lambda_i) d\lambda_i \\ &\propto \prod_i \lambda_i^{\frac{1}{2}} e^{-\frac{\omega\lambda_i(y_i-\alpha_i)^2}{2}} p(\lambda_i) d\lambda_i \end{aligned} \tag{22}$$

This is the familiar normal/gamma prior, the result should be

$$p(\vec{y}|X, \vec{\beta}, \omega) \propto \prod_i (1 + \frac{\lambda_i (y_i - \alpha_i)^2}{h})^{-\frac{h+1}{2}} \tag{23}$$

(B)

$$\begin{aligned} p(\lambda_i|\vec{y}, \vec{\beta}, \omega) &\propto p(\vec{y}|\vec{\beta}, \omega, \Lambda) p(\lambda_i) \\ &\propto e^{-\lambda_i(\frac{h+\omega(y_i-\alpha_i)^2}{2})} \lambda_i^{\frac{h+1}{2}-1} \\ &\sim \Gamma(\frac{h+1}{2}, \frac{h+\omega(y_i-\alpha_i)^2}{2}) \end{aligned} \tag{24}$$

(C) See script "MCMC on gdpgrowth.R" and "gibbssampler.R". We generate the cycle

$$(\vec{\beta}_1|\vec{y}, \omega_0, \Lambda_0), (\omega_1|\vec{y}, \Lambda_0) \to (\Lambda_1|\vec{y}, \vec{\beta}_1, \omega_1) \to (\vec{\beta}_2|\vec{y}, \omega_1, \Lambda_1), (\omega_2|\vec{y}, \Lambda_1) \to \cdots \tag{25}$$

where

$$
\begin{aligned}
(\vec{\beta}|\vec{y}, \omega, \Lambda) &\sim N(\vec{m}^*(\Lambda, \vec{y}), \omega K^*(\Lambda)) \\
(\omega|\vec{y}, \Lambda) &\sim \Gamma(\frac{d^*}{2}, \frac{\eta^*(\Lambda, \vec{y})}{2}) \\
(\lambda_i|\vec{y}, \vec{\beta}, \omega) &\sim \Gamma(\frac{h}{2}, \frac{h + \omega(y_i - \alpha_i)^2}{2}) \qquad (26)
\end{aligned}
$$