1 Model Description

$$y(t) = \sqrt{h(t)}u(t)$$

$$\ln(h(t)) = \alpha + \delta \ln(h(t-1)) + \sigma_\nu \nu(t)$$

$t = 1, ..., N$

$$u(t), \nu(t) \sim N(0, 1)$$

2 Model Fitting

2.1 MCMC

From the model we have:

$$(y(t)|h(t)) \sim N(0, h(t))$$

$$(h(t)|h(t-1), \omega) \sim N(\alpha + \delta \ln(h(t-1)), \sigma_\nu^2)$$

For parameters ω, assume the prior distribution:

$$\alpha \sim N(\alpha_0, \sigma_\alpha^2)$$

$$\delta \sim N(\delta_0, \sigma_\delta^2)$$

$$\sigma_\nu^2 \sim IG(\frac{\nu_0}{2}, \frac{s_0}{2})$$

We have the marginal distribution

$$\begin{aligned} p(y, h, \omega) \propto & \frac{1}{\sigma_\nu^{N+3+\nu_0}\sigma_\delta^3\sigma_\alpha^3} \exp\left(-\frac{(\delta-\delta_0)^2}{2\sigma_\delta^2} - \frac{(\alpha-\alpha_0)^2}{2\sigma_\alpha^2} - \frac{s_0^2}{2\sigma_\nu^2}\right) \\ \times \prod_t & \frac{1}{h(t)^{\frac{3}{2}}} \exp\left(-\frac{y(t)^2}{2h(t)} - \frac{(\ln h(t) - \delta \ln h(t-1) - \alpha)^2}{2\sigma_\nu^2}\right) \end{aligned} \quad (1)$$

From which we can derive the posterior distributions are

$$(\sigma_\nu^2|h, \alpha, \delta) \sim IG\left(\frac{\nu_0 + N + 1}{2}, \frac{s'}{2}\right)$$

$$s' = s_0 + (N+1)\alpha^2 + (1+\delta^2)S_2 + \delta^2 H_0^2 + H_{N+1}^2 - 2\alpha((1-\delta)S_1 - \delta H_0 + H_{N+1}) - 2\delta S_3$$

$$(\delta|h, \alpha, \sigma_\nu^2) \sim N\left(\frac{\sigma_\nu^2\delta_0 + \sigma_\delta^2(S_3 - \alpha(S_1 + H_0))}{\sigma_\nu^2 + \sigma_\delta^2(S_2 + H_0^2)}, \frac{\sigma_\nu^2\sigma_\delta^2}{\sigma_\nu^2 + \sigma_\delta^2(S_2 + H_0^2)}\right)$$

$$(\alpha|h, \sigma_\nu^2, \delta) \sim N\left(\frac{\sigma_\alpha^2((1-\delta)S_1 + H_{N+1} - \delta H_0) + \sigma_\nu^2\alpha_0}{\sigma_\nu^2 + (N+1)\sigma_\alpha^2}, \frac{\sigma_\nu^2\sigma_\alpha^2}{\sigma_\nu^2 + (N+1)\sigma_\alpha^2}\right)$$

$$p(h(t)|h(t+1),h(t-1),\omega) \propto \frac{1}{\sqrt{h(t)}} \exp\left(-\frac{y(t)^2}{2h(t)}\right) \tag{2}$$
$$\times \frac{1}{h(t)} \exp\left(-\frac{(\ln h(t)-\mu(t))^2}{2\sigma^2}\right)$$

where $\sigma^2 = \frac{\sigma_\nu^2}{1+\delta^2}$ and $\mu(t) = \frac{\delta \ln h(t+1) + \delta \ln h(t-1) + (1-\delta)\alpha}{1+\delta^2}$

Here $S_1 = \sum_{t=1}^N \ln h(t)$, $S_2 = \sum_{t=1}^N (\ln h(t))^2$, $S_3 = \sum_{t=1}^{N+1} \ln h(t) \ln h(t-1)$. In addition, we will augement the data by adding $h(0)$ and $h(N+1)$. We will draw them from unconditional distribution

$$\ln h \sim N\left(\frac{\alpha}{1-\delta}, \frac{\sigma_\nu^2}{1-\delta^2}\right)$$

in every iteration. We denote $H_0 = \ln h(0)$ and $H_{N+1} = \ln h(N+1)$

Possible samplings methods are

1. Gibbs sampling: Sufficient for updating ω but too tedious for h.

2. Rejection sampling: Approximate $p(h)$ by a blancket density $q(h)$ and choose a constant c such that $p(h) \leq cq(h)$ for any h. Consider a draw from q and accept it with probability $p(h)/cq(h)$. If rejected, draw again. Still not efficient enough for h as it has to compute the overall constant of $p(h)$ for every time point. Also, the choice of c can be tricky as it might leads to too high or low rejection rate.

3. Metropolis-Hastings algorithm: Using a transition kernel $f(h)$, make the $n+1$th draw from $h(t)$, accept it with probability

$$\min\left(\frac{p(h(t)^{n+1})/f(h(t)^{n+1})}{p(h(t)^n)/f(h(t)^n)}, 1\right)$$

Finally, the method we are going to use is a combination of all above:

1. Given h, use Gibbs sampling to sample posterior distribution of ω.

2. Given ω and $h(t \neq t^*)$. Draw from $q(h(t^*))$ and accept it with probability $\min(\frac{p(h(t^*))}{cq(h(t^*))}, 1)$. Repeat until the first acceptance. We choose q to be the marginal probability density of an inverse gamma matrix:

$$q(h) = \frac{\lambda^\phi}{\Gamma(\phi)} h^{-(\phi+1)} e^{-\frac{\lambda}{h}}$$

where

$$\phi = \frac{1-2e^{\sigma^2}}{1-e^{\sigma^2}} + \frac{1}{2}$$

$$\lambda = (\phi-1)e^{\mu(t)+\frac{\sigma^2}{2}} + \frac{y(t)^2}{2}$$

The reason of this choice is that we chose a gamma distribution that has the same first and second moment as the second line of (2). This gamma distribution is further combined with the first line of (2), which is itself a gamma distribution, and give rise q.

Instead of the criteria given in rejection sampling, we will choose c by

$$c = 1.1 \left(\frac{p(h)}{q(h)} \right)_{h=\text{mode of } q}$$

We do not need to compute the normalizing constant for p as it will be compensated by c.

3. After the acceptance in 2, we began the procedure in Metropolis-Hastings algorithm with the same acceptance probability. When a of h^{n+1} is rejected, we will let it inherit the value of h^n instead of redraw.

2.2 AR(1)

We need to subtract from financial series an "average" value to get the variance $y(t)$ and we will use an AR(1) model to fit what we should subtract. More specifically, for a financial series $r(t)$, we will use use the ar.ols function in stats package of R to fit the model

$$r(t) = \alpha + \beta r(t-1) + \epsilon$$

Where ϵ is a normally distributed error and α and β are parameters. Then $y(t)$ is the residue of this model. It is possible to include this model as part of STOV and fit the whole parameter space using MCMC but we will not pursue that here.

2.3 Estimation of Prior

Using the fact that the unconditional distribution of $\ln h(t)$ is $N(\frac{\alpha}{1-\delta}, \frac{\sigma_\nu^2}{1-\delta^2})$, we can assume:

$$\frac{\alpha}{1-\delta} \sim N(\bar{h}, \sigma_h^2)$$

Where $\bar{h}$ and σ_h^2 are the sample mean and variation of $\ln h(t)$. If we assume a reasonable distribution of δ, we can choose the prior of σ_ν^2 such that it's peaked at $(1-\delta^2)\sigma_h^2$. As for α, with the above assumption, we can have:

$$\begin{pmatrix} \alpha \\ \delta \end{pmatrix} = N\left(\begin{pmatrix} \bar{h}(1-\bar{\delta}) \\ \bar{\delta} \end{pmatrix}, \begin{pmatrix} \sigma_h^2(1-\bar{\delta})^2 + \bar{h}^2\sigma_\delta^2 & -\bar{h}\sigma_\delta^2 \\ -\bar{h}\sigma_\delta^2 & \sigma_\delta^2 \end{pmatrix} \right) \tag{3}$$

there fore we will be able to compute the prior mean and variance of α.

3 Data Source

4 Reference

1. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian Analysis of Stochastic Volatility Models", Journal of Business & Economic Statistics, 20:1, 69-87

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3. John Geweke, "Prior for Macroeconomic Time Series and Their Application", Econometric Theory, Vol. 10, 609-632

4. John Geweke, "Bayesian Comparison of Econometric Models", working paper, Federal Reserve Bank of Minneapolis Research Department