1 Basic Model

$$y(t) = \sqrt{h(t)}u(t)$$

$$\ln(h(t)) = \alpha + \delta \ln(h(t-1)) + \sigma_\nu \nu(t)$$

$t = 1, ..., N$

$$u(t), \nu(t) \sim N(0,1)$$

2 Model Fitting

2.1 MCMC

From the model we have:

$$(y_t|h_t) \sim N(0, h_t)$$

$$(h_t|h_{t-1}, \omega) \sim N(\alpha + \delta \ln h_{t-1}, \sigma_\nu^2)$$

For other parameters, assume the prior distribution:

$$\alpha \sim N(\alpha_0, \sigma_\alpha^2)$$

$$\delta \sim N(\delta_0, \sigma_\delta^2)$$

$$\sigma_\nu^2 \sim IG(\frac{\nu_0}{2}, \frac{s_0}{2})$$

We have the marginal distribution

$$\begin{aligned} p(y, h, \delta, \alpha, \sigma_\nu^2) &\propto \frac{1}{\sigma_\nu^{2+\nu_0}} \exp\left(-\frac{(\delta-\delta_0)^2}{2\sigma_\delta^2} - \frac{(\alpha-\alpha_0)^2}{2\sigma_\alpha^2} - \frac{s_0^2}{2\sigma_\nu^2}\right) \\ &\times \prod_{t=2}^{N} \frac{1}{h_t^{\frac{3}{2}}\sigma_\nu} \exp\left(-\frac{y_t^2}{2h_t} - \frac{(\ln h_t - \delta \ln h_{t-1} - \alpha)^2}{2\sigma_\nu^2}\right) \end{aligned} \tag{1}$$

From which we can derive the posterior distributions are

$$(\sigma_\nu^2|h, \alpha, \delta) \sim IG\left(\frac{\nu_0 + N - 1}{2}, \frac{s'}{2}\right)$$

$$s' = s_0 + (N-1)\alpha^2 + (1+\delta^2)S_2 - \delta^2(\ln h_N)^2 - (\ln h_1)^2 - 2\alpha((1-\delta)S_1 - \ln h_1 + \delta \ln h_N) - 2\delta S_3$$

$$(\delta|h, \alpha, \sigma_\nu^2) \sim N\left(\frac{\sigma_\nu^2\delta_0 + \sigma_\delta^2(S_3 - \alpha(S_1 - \ln h_N))}{\sigma_\nu^2 + \sigma_\delta^2(S_2 - (\ln h_N)^2)}, \frac{\sigma_\nu^2\sigma_\delta^2}{\sigma_\nu^2 + \sigma_\delta^2(S_2 - (\ln h_N)^2)}\right)$$

$$(\alpha|h, \sigma_\nu^2, \delta) \sim N\left(\frac{\sigma_\alpha^2((1-\delta)S_1 - \ln h_1 + \delta \ln h_N) + \sigma_\nu^2\alpha_0}{\sigma_\nu^2 + (N-1)\sigma_\alpha^2}, \frac{\sigma_\nu^2\sigma_\alpha^2}{\sigma_\nu^2 + (N-1)\sigma_\alpha^2}\right)$$

$$S_1 = \sum_{t=1}^{N} \ln h_t \quad S_2 = \sum_{t=1}^{N} (\ln h_t)^2 \quad S_3 = \sum_{t=2}^{N} \ln h_t \ln h_{t-1}$$

$$p(h_t|h_{t+1},h_{t-1},\delta,\alpha,\sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}}\exp\left(-\frac{y_t^2}{2h_t}\right)\frac{1}{h_t}\exp\left(-\frac{(\ln h_t-\mu_t)^2}{2\sigma_t^2}\right) \tag{2}$$

$$\mu_t = \frac{\delta\ln h_{t+1}+\delta\ln h_{t-1}+(1-\delta)\alpha}{1+\delta^2}$$

$$\sigma^2 = \frac{\sigma_\nu^2}{1+\delta^2}$$

In addition, h_1 an h_N cannot be updated with 2, we will update them by directly drawing from autoregressive model of $\ln h$.

In summary, the outline of the algorithm is

1. Initialize $h,\alpha,\delta,\sigma_\nu^2$
2. For $t=2,3,\cdots,N-1$, draw h_t from $p(h_t|h_{t+1},h_{t-1},\delta,\alpha,\sigma_\nu^2)$
3. Draw $\ln h_1$ from $N(\alpha+\delta\ln h_2,\sigma_n u^2)$, $\ln h_N$ from $N(\alpha+\delta\ln h_{N-1},\sigma_n u^2)$
4. Draw σ_ν^2 from $(\sigma_\nu^2|h,\alpha,\delta)$
5. Draw δ from $(\delta|h,\alpha,\sigma_\nu^2)$
6. Draw α from $(\alpha|h,\delta,\sigma_\nu^2)$
7. Go to step 2

It is easy to simulate the posterior distribution of σ_ν^2. α and δ, so the only nontrivial part of the MCMC is how to simulate $p(h_t|h_{t+1},h_{t-1},\delta,\alpha,\sigma_\nu^2)$ which is a product of an inverse gamma and a lognormal distribution. Below we will give three sampling methods. The comparison of them will be the focus of this project.

2.2 Sampling Method 1: Metroplis-Hastings with Random Walk

Write (2) as a distribution of $\ln h$ rather than h

$$p(\ln h_t|h_{t+1},h_{t-1},\delta,\alpha,\sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}}\exp\left(-\frac{y_t^2}{2h_t}\right)\exp\left(-\frac{(\ln h_t-\mu_t)^2}{2\sigma_t^2}\right)$$

Given $\ln h_t^{i-1}$. Each time we simply propose a $\ln h_t^{i*}$ by drawing

$$N(\ln h_t^{i-1},e^2)$$

where e^2 is a preset parameter independent of other variables. And accept it with probability

$$\mathrm{Min}(1,\frac{p(\ln h_t^{i*})}{p(\ln h_t^{i-1})})$$

The algorithm is:

1. Draw $\ln h_t^*$ from $N(\ln h_t^{i-1}, e^2)$
2. Accept this value with probability Min$(1, \frac{p(\ln h_t^{i*})}{p(\ln h_t^{i-1})})$
3. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

2.3 Sampling Method 2: Metroplis-Hastings with Accept-Reject Sampling

This is the method proposed in . The idea is to refine the process of the proposing update in MH. We can "approximate" (2) by an inverse gamma distribution:

$$q(h_t) = \frac{\lambda^\phi}{\Gamma(\phi)} h^{-(\phi+1)} e^{-\frac{\lambda}{h_t}}$$

where

$$\lambda = \frac{1 - 2e^{\sigma^2}}{1 - e^{\sigma^2}} + \frac{1}{2}$$

$$\phi = (\lambda - 1)e^{\mu_t + \frac{\sigma^2}{2}} + \frac{y_t^2}{2}$$

and σ^2 and μ_t are defined under (2).

Define

$$c = 1.2 \left(\frac{p(h)}{q(h)} \right)_{h=\text{mode of } q}$$

We will propose candidate h_t^{i*} from $IG(\lambda, \phi)$ and accept it with Min$(1, \frac{p(h^*)}{cq(h^*)})$, if rejected, repropose until accepted. The winner of accept-reject process will be the candidate of MH process withtransition kernel $f(h_t^*)$ =Min$(p(h_t^*), cq(h_t^*))$. The actual algorithm will be

1. Draw h_t^* from $IG(\lambda, \phi)$, note that both λ and ϕ are functions of h_{t+1}, h_{t-1} and other parameters
2. Accept h_t^* with probability Min$(1, \frac{p(h_t^*)}{cq(h_t^*)})$
3. If rejected, go to step 1.
4. If $p(h_t^*) < cq(h_t^*)$, $h_t^i = h_t^*$. The algorithm ends.
5. Accept h_t^* with probability Min$(1, \frac{p(h_t^*)/q(h_t^*)}{p(h_t^{i-1})/q(h_t^{i-1})})$
6. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

2.4 Sampling Method 3: Pure Accept-Reject Sampling

$$\begin{aligned}
\ln p(\ln h_t|\cdots) &= -\frac{1}{2}\ln h_t - \frac{y_t^2}{2h_t} - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&\leq -\frac{1}{2}\ln h_t - \frac{y_t^2}{2}\left(\exp(-\mu_t)(1+\mu_t - \ln h_t)\right) - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&= -\frac{(\ln h_t - \mu_t')^2}{2\sigma^2} + \text{constants} \qquad (3)
\end{aligned}$$

Where

$$\mu_t' = \mu_t + \frac{\sigma^2}{2}(y_t^2 \exp(-\mu_t) - 1)$$

This observation leads to a standard reject-accept sampling:

1. Draw $\ln h_t^*$ from $N(\mu_t', \sigma^2)$
2. Accept h_t^* with probability $\mathrm{Min}(1, g(h_t^*)$, where

$$g(h_t) = \exp(\frac{y_t^2}{2}(\exp(-\mu_t))(1+\mu_t - \ln h_t) - \frac{1}{h_t})$$

3. If rejected, go to step 1.
4. $h_t^i = h_t^*$

3 Test of Three Sampling Method

3.1 Data

3.2 Result

4 Comparison with GARCH

4.1 Data

5 Reference

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