1. A

$$\begin{aligned} f(\beta) &= (y - X\beta)^T W (y - X\beta) \\ &= \beta^T X^T W X \beta - 2y^T W X \beta + \cdots \\ &= \beta^T X^T W X \beta - 2y^T W^T X^T (X^T W X)^{-1} X^T W X \beta + \cdots \\ &= (\beta - (X^T W X)^{-1} X^T W y)^T X^T W X (\beta - (X^T W X)^{-1} X^T W y) + \cdots \end{aligned} \quad (1)$$

B See code

C See code

D See code

2. A

$$\begin{aligned} l(\beta) &= -\log(\prod_i \omega_i^{y_i} (1-\omega_i)^{1-y_i}) \\ &= -(\sum_i y_i \log \omega_i + (1-y_i)\log(1-\omega_i)) \end{aligned} \quad (2)$$

$$\frac{\partial \omega_i}{\partial \beta_j} = \frac{x_{ij}\exp(-x_i\beta)}{(1+\exp(-x_i\beta))^2} = x_{ij}\omega_i(1-\omega_i)$$

$$\frac{\partial l}{\partial \beta_j} = \sum_i (\frac{1-y_i}{1-\omega_i} - \frac{y_i}{\omega_i}) x_{ij}\omega_i(1-\omega_i) = \sum_i (\omega_i - y_i) x_{ij}$$

B See code.

C Write down the taylor expansion of l:

$$l(\beta) = l(\beta_0) + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j}\beta_j + \frac{1}{2}\sum_{jk}\frac{\partial^2 l(\beta_0)}{\partial \beta_k \partial \beta_j}\beta_j\beta_k + \cdots$$

Compute second order derivative:

$$\begin{aligned} &\frac{\partial^2 l}{\partial \beta_k \partial \beta_j} \\ =& \sum_i \omega_i(1-\omega_i) x_{ij} x_{ik} \end{aligned} \quad (3)$$

l can be written as a quadratic form of diagonal matrix W whose diagonal entries are $\omega_i(1-\omega_i)$

$$\begin{aligned} l(\beta) &= l(\beta_0) + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j}\Delta\beta_j + \frac{1}{2}\sum_{jk} x_{ij}x_{ik} f_i(\beta_0)\Delta\beta_j\Delta\beta_k + \cdots \\ &= \frac{1}{2}\Delta\beta^T X^T W X \Delta\beta + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j}\Delta\beta_j + \cdots \\ &= \frac{1}{2}(z - X\Delta\beta)^T W (z - X\Delta\beta) + \cdots \end{aligned} \quad (4)$$

Determine z:

$$
\begin{aligned}
z^T W X &= -\frac{\partial l(\beta_0)}{\partial \beta_i} \\
\sum_i z_i \omega_i (1-\omega_i) x_{ij} &= -\sum_i (\omega_i - y_i) x_{ij} \\
z_i &= \frac{y_i - \omega_i}{\omega_i (1-\omega_i)} \qquad (5)
\end{aligned}
$$

D See code

E See code