CS 203 Assignment 1

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Question 1.

Solution:

(a) It is given that a person will enjoy a free meal if the cards picked represent the alphabets of the word $TREAT$. We call this event A and let $P(A)$ denote the probability of this event.
Since picking a card in any deck is independent of the other, the probability of getting the letters of the required word will be $\frac{1}{20^5}$. Now, given we pick the letters above, the number of different possible configurations in which we can pick them is $\frac{5!}{2!}$.
Using the definition of probability of an event, the number of favorable outcomes is $\frac{5!}{2!}$, with the probability of each outcome being $\frac{1}{20^5}$. Thus, $P(A) = \frac{5!}{20^5 X 2!}$

(b) Assuming a meal can belong to any one of the following classes, cheap or expensive, we need to devise a strategy so that the number of favourable outcomes is significantly reduced in case of an expensive meal, thus bringing down its total probability.
Also, since it is mentioned in the question to just tweak the 'winning word', I assume that the number of letters in the new word would be same as that previously.
Another assumption I make is that in no case should the probability of a free meal be more than its value in part (a), even for a cheap meal, since it could incur a loss for the restaurant. Thus, our problem boils down to decreasing the number of possible arrangements of letters in case of an expensive meal, which can be done if we use the word $TREET$ instead. The number of favourable outcomes now would be $\frac{5!}{2!X2!}$, making $P(A)$ half as likely when the meal is expensive.
Our new strategy:

- If the meal is cheap, use the old word $TREAT$
- If the meal is expensive, use the new word $TREET$

Question 2.

Solution:
Let the event that i^{th} machine produces 1 be represented as $f(M_i, 1)$.
It is given in the question that $P(f(M_1, 1))$ is $\frac{3}{4}$ and $P(f(M_1, 0))$ is $\frac{1}{4}$.
With this notation, we can define probability of the required event using conditional probabilty:

$$P(\frac{f(M_1, 0)}{f(M_n, 1)}) = \frac{P(\frac{f(M_n,1)}{f(M_1,0)}) \cdot P(f(M_1, 0))}{P(f(M_n, 1))} \tag{1}$$

Now, we need to calculate $P(f(M_i, 1))$. It can be recursively defined in terms of $P(f(M_{i-1}, 1))$ and $P(f(M_{i-1}, 0))$ as:

$$P(f(M_i, 1)) = \frac{1}{4} \cdot P(f(M_{i-1}, 1)) + \frac{3}{4} \cdot P(f(M_{i-1}, 0)) \tag{2}$$

We know that $P(f(M_{i-1},1)) + P(f(M_{i-1},0)) = 1$, since the i^{th} machine either produces a 1 or a 0, we can re-arrange the recursion entirely in terms of $P(f(M_{i-1},1))$ as:

$$P(f(M_i,1)) = \frac{3}{4} - \frac{1}{2} \cdot P(f(M_{i-1},1)) \tag{3}$$

Solving the above recursion, we get:

$$P(f(M_i,1)) = \frac{(-1)^{n-1} \cdot (2(-1)^{n-1} + 2^{-(n-1)})}{4} \tag{4}$$

If we observe carefully , we can see that if we are given that M_1 produces 0, then the probability that M_2 produces 0 will be $\frac{1}{4}$ (same) and that it produces 1 will be $\frac{3}{4}$ (inverted). Hence the subsequent problems starting at M_2 can be considered equivalent to the original problem starting at M_1.
Therefore, we can say $P(\frac{f(M_n,1)}{f(M_1,0)})$ is equal to $P(f(M_{n-1},1))$.

Putting the values in equation 1, using equation 4 and considering the above equivalence gives us:

$$P(\frac{f(M_1,0)}{f(M_n,1)}) = \frac{-1}{2} \cdot \frac{(2(-2)^{n-2}+1)}{(2(-2)^{n-1}+1)} \tag{5}$$

Equation 5 gives us the required answer in terms of n for $n >= 2$. For the trivial case of $n = 1$, the answer is 0.

Question 3.

Solution:
For the trivial case, when $I = [1...n]$, the formula given for $P(B)$ reduces as below, since J has only 1 possible value I:

$$P(B) = P(\cap_{i \in I} A_i)$$

Hence, our formula holds for the trivial case when $I = [1...n]$.
Now, let us prove it holds for any subset I of $[1...n]$. Let the intersection of all the sets A_i *where* $i \in I$ be represented as a new event C.
Essentially, $B = C \cap (\cup_{i \notin I} A_i)^{\mathsf{c}}$ using De-morgan's law.
Our inclusion-exclusion formula is:

$$P(\cup_{i\in[n]} A_i) = \sum_{S \subseteq [n], S \neq \emptyset} (-1)^{|S|+1} \cdot P(\cap_{i \in S} A_i)$$

We need to prove:

$$P(B) = \sum_{J \supseteq I]} (-1)^{|J|-|I|} \cdot P(\cap_{i \in J} A_i) \tag{6}$$

So, Let us consider the domain of C now, where we define the new probabilities of all the events wrt to the domain C as $p(A_i) = P(A_i|C) = \frac{P(A_i \cap C)}{P(C)}$. Using the principle of inclusion-exclusion stated above,

$$p(\cup_{i \in I^c} A_i) = \sum_{I_{prime} \subseteq I^c, I_{prime} \neq \emptyset} (-1)^{|I_{prime}|+1} \cdot p(\cap_{i \in I^c_{prime}} A_i)$$

Hence,

$$p((\cup_{i \in I^c} A_i)^c) = 1 - \sum_{I_{prime} \subseteq I^c, I_{prime} \neq \emptyset} (-1)^{|I_{prime}|+1} \cdot p(\cap_{i \in I^c_{prime}} A_i)$$

Multipying both sides by $P(C)$, and using our definition of $p(A_i)$'s, we get:

$$P(C \cap (\cup_{i \in I^c} A_i)^c) = P(C) - \left[P(C) \cdot \sum_{I_{prime} \subseteq I^c, I_{prime} \neq \emptyset} (-1)^{|I_{prime}|+1} \cdot p(\cap_{i \in I^c_{prime}} A_i) \right]$$
$$\implies P(C) - \sum_{I_{prime} \subseteq I^c, I_{prime} \neq \emptyset} (-1)^{|I_{prime}|+1} \cdot P(\cap_{i \in I^c_{prime}} (A_i \cap C))$$

Our LHS is B, by our definition of B, and now we just need to show that our RHS is same as the RHS of equation (6).

Here, a thing to observe is that all the individual terms in equation (6) contain an intersection of all the elements of A_i, which has been substituted with C.
Also, $I_{prime} \subseteq I^c, I_{prime} \neq \emptyset$ is equivalent to $J \supset I$, since every value of I_{prime} can be mapped to a corresponding value of J, with the map defined as $I \cup I_{prime}$. Thus, the cardinality of I_{prime} will be $|J| - |I|$.
Hence, replacing the terms of RHS in a manner consistent with above gives:

$$RHS = P(C) - \sum_{J \supset I} (-1)^{|J|-|I|+1} \cdot P(\cap_{i \in J} A_i) \tag{7}$$

The last thing left is to merge $P(C)$ in the summation.
The case to consider here is when $J = I$, that is, we take the intersection of all events in I, which is nothing but C. The multiplicand associated with this term should be positive. Thus, equation (7) can be re-written as:

$$RHS = P(C) + \sum_{J \supset I} (-1)^{|J|-|I|} \cdot P(\cap_{i \in J} A_i)$$
$$\implies \sum_{J \supseteq I} (-1)^{|J|-|I|} \cdot P(\cap_{i \in J} A_i)$$

Here, the power associated with the term $P(C)$ is $|I| - |I|$, which is 0, hence multiplicand is positive.
RHS is congruent to equation (6). Thus, we prove equation (6) holds true.

Question 4.

Solution:
Let P_i be the probability if the player wins in the i^{th} turn, given he "takes" the card in that turn.
It is implicit in the meaning of taking the i^{th} card, that he consistently "skips" every card in each of the preceding $i-1$ turns.
We get the probability that a player wins if he "takes" the 1^{st} card as:

$$P_1 = \frac{x}{x+y} \tag{8}$$

For proving that no strategy is better than taking a card in the 1^{st} turn, it is sufficient to prove that:

$$(P_1 \geq P_i) \quad \forall i > 1 \tag{9}$$

For a test example, let us consider $i = 2$. Given the player skips the 1^{st} card, it can either be black or red. The respective probabilities for "skipping" a black or a red card in the 1^{st} turn are $\frac{x}{x+y}$ and $\frac{y}{x+y}$. Using partition theorem for probability:

$$P_2 = \left(\frac{x}{x+y} \cdot \frac{x-1}{x+y-1} + \frac{y}{x+y} \cdot \frac{x}{x+y-1}\right) \tag{10}$$

On simplifying, we get:

$$P_2 = (\frac{x}{x+y} \cdot \frac{x-1}{x+y-1} + \frac{x}{x+y} \cdot \frac{y}{x+y-1}) \implies \frac{x}{x+y} \cdot (\frac{x-1}{x+y-1} + \frac{y}{x+y-1}) \implies \frac{x}{x+y} \tag{11}$$

Thus, (9) holds for i = 2.
Let the probability of getting j *black* cards and $i-1-j$ *red* cards in the 1^{st} $i-1$ "skipped" turns be $b_{i-1}(j)$
Thus,

$$b_{i-1}(j) = \binom{i-1}{j} \cdot \frac{(x)...(x-(j-1) \cdot (y)...(y-(i-j-2))}{(x+y)...(x+y-(i-2))} \tag{12}$$

On simpifying the above eqn, we get:

$$b_{i-1}(j) = \frac{\binom{y}{i-1-j} \cdot \binom{x}{j}}{\binom{x+y}{i-1}} \tag{13}$$

After placing i-1 cards, the probability of getting a black i^{th} card will be dependent on j, the number of black cards in the starting $i-1$ turns. Hence, the expression for P_i becomes:

$$P_i = \sum_{j=0}^{\min\{x,i-1\}} \frac{\binom{y}{i-1-j} \cdot \binom{x}{j}}{\binom{x+y}{i-1}} \cdot \frac{x-j}{x+y-(i-1)} \tag{14}$$

On taking the terms devoid of j outside the expression, and using the combination formula $\binom{n}{x} = \frac{n}{x} \cdot \binom{n-1}{x-1}$ and $\sum_{j=0}^{x1} \binom{n1}{x1-j} \cdot \binom{n2}{j} = \binom{n1+n2}{x1}$, we get:

$$\implies P_i = \frac{1}{\binom{x+y}{i-1} \cdot (x+y-(i-1))} \sum_{j=0}^{\min\{x,i-1\}} \binom{y}{i-1-j} \cdot \binom{x}{j} \cdot (x-j)$$

$$\implies \frac{1}{\binom{x+y}{i-1} \cdot (x+y-(i-1))} \left[\sum_{j=0}^{\min\{x,i-1\}} \binom{y}{i-1-j} \cdot \binom{x}{j} \cdot x \quad - \sum_{j=0}^{\min\{x,i-1\}} \binom{y}{i-1-j} \cdot \binom{x}{j} \cdot j \right]$$

$$\implies \frac{1}{\binom{x+y}{i-1} \cdot (x+y-(i-1))} \left[\sum_{j=0}^{\min\{x,i-1\}} \binom{y}{i-1-j} \cdot \binom{x}{j} \cdot x \quad - \sum_{j=0}^{\min\{x,i-1\}} \binom{y}{i-1-j} \cdot \binom{x-1}{j-1} \cdot x \right]$$

$$\implies \frac{x}{\binom{x+y}{i-1} \cdot (x+y-(i-1))} \left[\binom{x+y}{i-1} \quad - \binom{x+y-1}{i-2} \right]$$

$$\implies \frac{x}{(x+y-(i-1))} \left[1 \quad - \frac{i-1}{x+y} \right]$$

$$\implies \frac{x}{(x+y-(i-1))} \left[\frac{(x+y-(i-1))}{x+y} \right]$$

$$\implies P_i \cong \frac{x}{x+y}$$

Equation (9) holds true. Hence, we proved that any strategy P_i can at most be as good as P_1 but not better.