CS 345A Assignment 7

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Easy

Algorithm:

Thus, we construct the network G' as follows:

- First, we construct a bipartite graph (G) from the set of balloons(B) to the set of conditions(C). An edge (B_i, C_j) exists in this graph if the balloon B_i contains condition C_j in the set S_i. All such edges are assigned a capacity of 1.
- We add a source vertex S to G, and draw edges from S to all vertices in B, with a capacity 2.
- Finally, we add a sink vertex T to $G \cup S$, and draw edges from each vertex in the set C to T, with a capacity k, for keeping a provision that any condition can be measured by k different balloons.

Thus, we get our final network $G' = G \cup S \cup T$.
If the max flow in the network G' is found to be less than $k|C|$, we can say that there is no solution for the problem, satisfying the required constraints. Else the solution is reported.

Correctness:

Let's first of all analyze how the constraints of the original problem are catered to in G'. The edges from balloons to the disjointed set of conditions is based on whether they can be checked or not using the balloon in question. This ensures that no edge exists from a balloon to a condition that cannot be measured by it. The capacity of each of these edges is kept at 1, which keeps check of the fact that a balloon must not be used to recheck the same condition again.

The source vertex S is connected to every balloon with an edge of capacity 2 this ascertains that the path through a balloon cannot carry a flow of more than two, i.e. a balloon cannot make more than two measurements.

Lastly, the sink vertex T with edges from all condition vertices of maximum capacity k which captures the possibility of each condition getting checked by at least k distinct balloons as a flow of k units is allowed for each edge to the sink which must originate from k different balloons as the maximum capacity of edges is capped at 1 for balloons to conditions.

A point to note here is that we can assume an Integral flow through all edges without loss of generality using the Integrality theorem proved in the lecture.

Since the problem statement specifies that any condition is to be measured by least k different balloons, there needs to be a flow of at least k through each of the edges running from vertices in set C to T, in order to find a valid solution to this problem.
Since the maximum capacity of these edges is k, we can say that the flow through these edges should be exactly k in the flow instance corresponding to a valid solution for the problem (since flow cannot be greater than edge capacity).
Hence, we need to find a flow such that all edges from set C to t are fully saturated (i.e. each edge (C_j, T) carries a flow of $k \quad \forall j \in (1, n)$). Such a flow, if exists should correspond to the maximum flow through this network (using Maxflow-MinCut Theorem), since it is equal to the capacity of the cut A defined as: $A = S \cup B \cup C$ and $\bar{A} = T$.

Theorem: The existence of a solution of the Balloon measurement problem is same as the existence of a Maximum flow of $k|C|$ possible in the final construction(G').

Implication 1: If there exists a maximum flow in the construction G' with a maximum flow of $k|C|$. Shown above, every edge from C to T carries a flow of k which in turn implies that there exist at least k distinct balloons to measure each condition. As, the construction of G' ensures all other constraints (Illustrated above) we have a solution for our balloon measurement problem.

Implication 2: Let us assume that a solution for the problem exists with a maximum flow $< k|C|$ in G'. Using the arguments above we can clearly see that at least one such edge exits that does not carry a flow of k which implies that the condition corresponding to that edge cannot be verified by at least k distinct balloons. Thus, violating our original assumption of existence of such solution.

Thus, we establish an equivalence between existence of a solution of the Balloon measurement problem and the existence of a Maximum flow of $k|C|$ in the final construction(G').

Time Complexity:

The bipartite graph G has $m+n$ vertices, and $O(nm)$ edges. We add m and n extra edges to this graph when we add vertex S and T, respectively. The graph G' has $V = m + n + 2$ and $E = O(mn + m + n)$.Thus, the overall construction time for this graph will be of the order $O(|E| + |V|)$, approximated as $O(mn + m + n)$. Once it is constructed, we find the max-flow through the network using Ford-Fulkerson's algorithm in $O(|E|c_{max})$ time. Here, we can replace c_{max} by m, since the maximum edge capacity in the constructed network can be at most m in the worst case when $k = m$.
Thus, the overall running time of this algorithm would be $O((mn + m + n)(1 + m))$ worst case, which is polynomial in both m and n, and can be approximated as $O(m^2n)$.