CIS 623 Activity

Name:

A. Let **sum**, `double` be defined as:

```
sum         :: [Integer] -> Integer
sum []       =  0   -- sum.1
sum (x:xs) =  x + sum xs   -- sum.2

double      :: [Integer] -> [Integer]
double []      = []   -- double.1
double (z:zs) = (2 * z) : double zs   -- double.2
```

Prove, by using mathematical induction, that

```
sum (double xs) = 2 * sum xs
```

for any finite list `xs`. Present your proof in Feijen's convention.

Solution We prove the following proposition by mathmatical induction.

Proposition for any finite list `xs`,

```
sum (double xs) = 2 * sum xs
```

Proof

Case `[]` (Base case)

$$
\begin{aligned}
& \texttt{sum}(\texttt{double}\ [\,]) \quad (\textit{i.e. L.H.S.}) \\
= \quad & \{\ \text{double.1}\ \} \\
& \texttt{sum}\ [\,] \\
= \quad & \{\ \text{sum.1}\ \} \\
& 0
\end{aligned}
$$

$$
\begin{aligned}
& 2 * \texttt{sum}\ [\,] \quad (\textit{i.e. R.H.S.}) \\
= \quad & \{\ \text{sum.1}\ \} \\
& 2 * 0 \\
= \quad & \{\ \text{arithmetic}\ \} \\
& 0
\end{aligned}
$$

Case `x:xs` (Induction Step)

$$
\begin{aligned}
& \texttt{sum}\ (\texttt{double}(\texttt{x} : \texttt{xs})) \quad (\textit{i.e. L.H.S.}) \\
= & \quad \{\ \text{double.2}\ \} \\
& \texttt{sum}\ (2 * \texttt{x} : \texttt{double xs}) \\
= & \quad \{\ \text{sum.2}\ \} \\
& 2 * \texttt{x} + \texttt{sum}\ (\texttt{double xs}) \\
= & \quad \{\ \text{Induction assumption}\ \} \\
& 2 * \texttt{x} + 2 * \texttt{sum xs}
\end{aligned}
$$

$$
\begin{aligned}
& 2 * \texttt{sum}\ (\texttt{x} : \texttt{xs}) \quad (\textit{i.e. R.H.S.}) \\
= & \quad \{\ \text{sum.2}\ \} \\
& 2 * (\texttt{x} + \texttt{sum xs}) \\
= & \quad \{\ \text{arithemetic}\ \} \\
& 2 * \texttt{x} + 2 * \texttt{sum xs}
\end{aligned}
$$

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B. Let **length**, ++ be defined as:

```
length []      = 0
length (x:xs) = 1 + length xs

[] ++ zs      = zs
(w:ws) ++ zs = w:(ws++zs)
```

Prove, by using mathematical induction, that

```
length (xs ++ ys) = length xs  + length ys
```

for any finite list `xs`. Present your proof in Feijen's convention.

Solution We prove the following proposition by mathmatical induction.

Proposition for any finite list `xs`,

```
length (xs ++ ys) = length xs  + length ys
```

Proof

Case `[]` (Base case)

$$
\begin{array}{ll}
 & \texttt{length}\ ([]\ ++\ \texttt{ys}) \quad (\textit{i.e. L.H.S.}) \\
= & \quad \{\ ++.1\ \} \\
 & \texttt{length ys}
\end{array}
$$

$$
\begin{array}{ll}
 & \texttt{length}\ [] + \texttt{length ys} \quad (\textit{i.e. R.H.S.}) \\
= & \quad \{\ \text{length.1}\ \} \\
 & \texttt{0} + \texttt{length ys} \\
= & \quad \{\ \text{arithmetic}\ \} \\
 & \texttt{length ys}
\end{array}
$$

Case `x:xs` (Induction Step)

$$
\begin{array}{ll}
 & \texttt{length}((\texttt{x} : \texttt{xs})\ ++\ \texttt{ys}) \quad (\textit{i.e. L.H.S.}) \\
= & \quad \{\ ++.2\ \} \\
 & \texttt{length}(\texttt{x} : (\texttt{xs}\ ++\ \texttt{ys}) \\
= & \quad \{\ \text{length.2}\ \} \\
 & \texttt{1} + \texttt{length}(\texttt{xs}\ ++\ \texttt{ys}) \\
= & \quad \{\ \text{induction assumption}\ \} \\
 & \texttt{1} + \texttt{length xs} + \texttt{length ys}
\end{array}
$$

$$
\begin{aligned}
& \texttt{length}\,(\texttt{x} : \texttt{xs}) \;+\; \texttt{length}\,\texttt{ys} \quad (\textit{i.e. R.H.S.}) \\
= \quad & \{ \text{ length.2 } \} \\
& \texttt{1} \;+\; \texttt{length}\,\texttt{xs} \;+\; \texttt{length}\,\texttt{ys}
\end{aligned}
$$

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C. Let `shunt` be defined as:

```
shunt          :: [a] -> [a] -> [a]
shunt [] ys       = ys
shunt (x:xs) ys = shunt xs (x:ys)
```

Prove, by using mathematical induction, that the following proposition is true:

```
Propostion:  The statement

''for all finite list zs,
shunt (shunt xs zs) [] = shunt zs xs''

holds for any finite list xs.
```

Present your proof in Feijen's convention.

Solution We prove the following proposition by mathmatical induction.

Proposition The statement

$$(\forall \text{ finite } \texttt{zs})[\texttt{shunt}\ (\texttt{shunt xs zs})\ [\,] \ = \ \texttt{shunt zs xs}].$$

is true for all finite list `xs`.

Proof

Case `[]` (Base case)

$$\begin{array}{cl} & \texttt{shunt}\ (\texttt{shunt xs}\ [\,])\ [\,] \quad (\textit{i.e. L.H.S.}) \\ = & \quad \{\ \text{shunt.1}\ \} \\ & \texttt{shunt zs}\ [] \quad (\textit{i.e. R.H.S.}) \end{array}$$

Case `x:xs` (Induction Step)

$$\begin{array}{cl} & \texttt{shunt}\ (\texttt{shunt}\ (\texttt{x} : \texttt{xs})\ \texttt{zs})\ [\,] \quad (\textit{i.e. L.H.S.}) \\ = & \quad \{\ \text{shunt.2}\ \} \\ & \texttt{shunt}\ (\texttt{shunt xs}(\texttt{x} : \texttt{zs}))\ [\,] \\ = & \quad \{\ \text{induction hypothesis: the quantified variable } \texttt{zs} \text{ replaced by } \texttt{x:zs}\ \} \\ & \texttt{shunt}\ (\texttt{x} : \texttt{zs})\ \texttt{xs} \\ = & \quad \{\ \text{shunt.2}\ \} \\ & \texttt{shunt zs}\ (\texttt{x} : \texttt{xs}) \quad (\textit{i.e. R.H.S.}) \end{array}$$

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