Code for "Plethysm for characters of relative operads and PROPs" – demonstration notebook

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```
In[1]:= Get["Lambda`",
    Path → If[$MachineName == "epsilon", $HomeDirectory, NotebookDirectory[]]]
```

Partitions

```
In[2]:= tr[{5, 4, 4, 1}]
```
```
Out[2]= {4, 3, 3, 3, 1}
```

```
In[3]:= hook[{5, 4, 4, 1}]
```
```
Out[3]= 21 021
```

```
In[4]:= z[{5, 4, 4, 1}]
```
```
Out[4]= 160
```

```
In[5]:= χ[{3, 2, 1}, {3, 3}]
```
```
Out[5]= -2
```

Projections

```
In[6]:= TableForm[Comap[{Identity, view[3]}] /@
    {m[{5, 4, 4, 2}, x], m[{1}, x], m[{2}, x], m[{2, 1}, x], m[{1, 1, 1}, x]},
    TableHeadings → {None, {"f", "f in ∧3"}}]
```

Out[6]//TableForm=

f	f in ∧3
$m_{\{5,4,4,2\}}(x)$	0
$m_{\{1\}}(x)$	$x_1 + x_2 + x_3$
$m_{\{2\}}(x)$	$x_1^2 + x_2^2 + x_3^2$
$m_{\{2,1\}}(x)$	$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$
$m_{\{1,1,1\}}(x)$	$x_1 x_2 x_3$

In[7]:= `TableForm[Comap[{Identity, Expand @* view[3]}] /@`
`{e[1, x], e[2, x], e[3, x], e[1, x] * e[2, y]},`
`TableHeadings → {None, {"f", "f in ⋀3"}}]`

Out[7]//TableForm=

f	f in ⋀3
$e_1(x)$	$x_1 + x_2 + x_3$
$e_2(x)$	$x_1 x_2 + x_1 x_3 + x_2 x_3$
$e_3(x)$	$x_1 x_2 x_3$
$e_1(x)\, e_2(y)$	$x_1 y_1 y_2 + x_2 y_1 y_2 + x_3 y_1 y_2 + x_1 y_1 y_3 + x_2 y_1 y_3 + x_3 y_1 y_3 + x_1 y_2 y_3 + x_2 y_2 y_3 + x_3 y_2 y_3$

In[8]:= `TableForm[Comap[{Identity, Expand @* view[3]}] /@`
`{h[1, x], h[2, x], h[3, x], h[1, x] * h[2, y]},`
`TableHeadings → {None, {"f", "f in ⋀3"}}]`

Out[8]//TableForm=

f	f in ⋀3
$h_1(x)$	$x_1 + x_2 + x_3$
$h_2(x)$	$x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2$
$h_3(x)$	$x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 + x_3^3$
$h_1(x)\, h_2(y)$	$x_1 y_1^2 + x_2 y_1^2 + x_3 y_1^2 + x_1 y_1 y_2 + x_2 y_1 y_2 + x_3 y_1 y_2 + x_1 y_2^2 + x_2 y_2^2 + x_3 y_2^2 + x_1 y_1 y_3 + x_2 y_1 y_3$

In[9]:= `TableForm[Comap[{Identity, Expand @* view[3]}] /@`
`{p[0, x], p[1, x], p[2, x], p[3, x], p[1, x] * p[2, y]},`
`TableHeadings → {None, {"f", "f in ⋀3"}}]`

Out[9]//TableForm=

f	f in ⋀3
1	1
$p_1(x)$	$x_1 + x_2 + x_3$
$p_2(x)$	$x_1^2 + x_2^2 + x_3^2$
$p_3(x)$	$x_1^3 + x_2^3 + x_3^3$
$p_1(x)\, p_2(y)$	$x_1 y_1^2 + x_2 y_1^2 + x_3 y_1^2 + x_1 y_2^2 + x_2 y_2^2 + x_3 y_2^2 + x_1 y_3^2 + x_2 y_3^2 + x_3 y_3^2$

In[10]:= `Reduce[view[5][2 * h[2, x]] == view[5][p[1, x]^2 + p[2, x]]]`

Out[10]=

`True`

In[11]:= `alt[{2}, x]`

Out[11]=

x_1^2

In[12]:= `alt[{2, 1}, x]`

Out[12]=

$x_1^2 x_2 - x_1 x_2^2$

In[13]:= `alt[{3, 2, 1}, x]`

Out[13]=

$x_1^3 x_2^2 x_3 - x_1^2 x_2^3 x_3 - x_1^3 x_2 x_3^2 + x_1 x_2^3 x_3^2 + x_1^2 x_2 x_3^3 - x_1 x_2^2 x_3^3$

In[14]:= `TableForm[Comap[{Identity, Expand @* view[3]}] /@`
`{s[{1}, x], s[{2}, x], s[{2, 1}, x]}, TableHeadings → {None, {"f", "f in ⋀3"}}]`

Out[14]//TableForm=

f	f in ⋀3
$s_{\{1\}}(x)$	$x_1 + x_2 + x_3$
$s_{\{2\}}(x)$	$x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2$
$s_{\{2,1\}}(x)$	$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2 x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$

Sanity check

Check that $s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \le i, j \le l(\lambda')}$, as expected

```
In[15]:= chk[λ_List, n_Integer] := With[{λtr = tr[λ]}, Reduce[
           view[n][Det[Table[e[λtr〚i〛 - i + j, x], {i, Length[λtr]}, {j, Length[λtr]}]]] ==
         view[n][s[λ, x]]]]
```

```
In[16]:= chk[{2, 1}, 4]
```
Out[16]=
True

```
In[17]:= chk[{3, 2, 1}, 4]
```
Out[17]=
True

```
In[18]:= chk[{4, 2, 1}, 5]
```
Out[18]=
True

Conversions

```
In[19]:= TableForm[Comap[{Identity, conv[m]}] /@
         {e[5, x], h[2, x], h[3, x], h[4, x], p[5, x], e[2, x] * h[3, y]}]
```
Out[19]//TableForm=

$e_5(x)$	$m_{\{1,1,1,1,1\}}(x)$
$h_2(x)$	$m_{\{2\}}(x) + m_{\{1,1\}}(x)$
$h_3(x)$	$m_{\{3\}}(x) + m_{\{2,1\}}(x) + m_{\{1,1,1\}}(x)$
$h_4(x)$	$m_{\{4\}}(x) + m_{\{2,2\}}(x) + m_{\{3,1\}}(x) + m_{\{2,1,1\}}(x) + m_{\{1,1,1,1\}}(x)$
$p_5(x)$	$m_{\{5\}}(x)$
$e_2(x) \, h_3(y)$	$m_{\{1,1\}}(x) \, (m_{\{3\}}(y) + m_{\{2,1\}}(y) + m_{\{1,1,1\}}(y))$

```
In[20]:= TableForm[Comap[{Identity, conv[e]}] /@
         {h[5, x], p[5, x], s[{2, 1, 1}, x], s[{1, 1, 1, 1, 1}, x]}]
```
Out[20]//TableForm=

$h_5(x)$	$e_1(x)^5 - 4 e_1(x)^3 e_2(x) + 3 e_1(x) e_2(x)^2 + 3 e_1(x)^2 e_3(x) - 2 e_2(x) e_3(x) - 2 e_1(x) \,$
$p_5(x)$	$e_1(x)^5 - 5 e_1(x)^3 e_2(x) + 5 e_1(x) e_2(x)^2 + 5 e_1(x)^2 e_3(x) - 5 e_2(x) e_3(x) - 5 e_1(x) \,$
$s_{\{2,1,1\}}(x)$	$e_1(x) e_3(x) - e_4(x)$
$s_{\{1,1,1,1,1\}}(x)$	$e_5(x)$

```
In[21]:= TableForm[Comap[{Identity, conv[h]}] /@
         {e[2, x], e[3, x], e[4, x], p[2, x], p[3, x], p[4, x], s[{2, 1, 1}, x]}]
```
Out[21]//TableForm=

$e_2(x)$	$h_1(x)^2 - h_2(x)$
$e_3(x)$	$h_1(x)^3 - 2 h_1(x) h_2(x) + h_3(x)$
$e_4(x)$	$h_1(x)^4 - 3 h_1(x)^2 h_2(x) + h_2(x)^2 + 2 h_1(x) h_3(x) - h_4(x)$
$p_2(x)$	$-h_1(x)^2 + 2 h_2(x)$
$p_3(x)$	$h_1(x)^3 - 3 h_1(x) h_2(x) + 3 h_3(x)$
$p_4(x)$	$-h_1(x)^4 + 4 h_1(x)^2 h_2(x) - 2 h_2(x)^2 - 4 h_1(x) h_3(x) + 4 h_4(x)$
$s_{\{2,1,1\}}(x)$	$h_1(x)^2 h_2(x) - h_2(x)^2 - h_1(x) h_3(x) + h_4(x)$

In[22]:= `TableForm[Comap[{Identity, conv[p]}] /@ {e[2, x], e[3, x],`
      `e[4, x], e[5, x], h[2, x], h[3, x], h[4, x], h[5, x], s[{2, 1}, x]}]`

Out[22]//TableForm=

$e_2(x)$	$\frac{1}{2}\left(p_1(x)^2 - p_2(x)\right)$
$e_3(x)$	$\frac{1}{6}\left(p_1(x)^3 - 3\,p_1(x)\,p_2(x) + 2\,p_3(x)\right)$
$e_4(x)$	$\frac{1}{24}\left(p_1(x)^4 - 6\,p_1(x)^2\,p_2(x) + 3\,p_2(x)^2 + 8\,p_1(x)\,p_3(x) - 6\,p_4(x)\right)$
$e_5(x)$	$\frac{1}{120}\left(p_1(x)^5 - 10\,p_1(x)^3\,p_2(x) + 15\,p_1(x)\,p_2(x)^2 + 20\,p_1(x)^2\,p_3(x) - 20\,p_2(x)\,p_3(x) - 30\,p\right.$
$h_2(x)$	$\frac{1}{2}\left(p_1(x)^2 + p_2(x)\right)$
$h_3(x)$	$\frac{1}{6}\left(p_1(x)^3 + 3\,p_1(x)\,p_2(x) + 2\,p_3(x)\right)$
$h_4(x)$	$\frac{1}{24}\left(p_1(x)^4 + 6\,p_1(x)^2\,p_2(x) + 3\,p_2(x)^2 + 8\,p_1(x)\,p_3(x) + 6\,p_4(x)\right)$
$h_5(x)$	$\frac{1}{120}\left(p_1(x)^5 + 10\,p_1(x)^3\,p_2(x) + 15\,p_1(x)\,p_2(x)^2 + 20\,p_1(x)^2\,p_3(x) + 20\,p_2(x)\,p_3(x) + 30\,p\right.$
$s_{\{2,1\}}(x)$	$\frac{p_1(x)^3}{3} - \frac{p_3(x)}{3}$

Sanity checks:

In[23]:= `And @@ Flatten@Table[`
      `Simplify[conv[t][conv[t2][t[10, x]]] == t[10, x]], {t, {e, h, p}}, {t2, {e, h, p}}]`

Out[23]=

True

In[24]:= `Simplify[conv[p][Sum[hook[λ] * s[λ, x], {λ, IntegerPartitions[10]}]] == p[1, x]^10`

Out[24]=

True

Sanity checks

In[25]:= `With[{μ = {3, 2, 1}}, Simplify[view[6][p[μ, x]] ==`
      `view[6][Sum[χ[λ, μ] * s[λ, x], {λ, IntegerPartitions[Total[μ]]}]]]]`

Out[25]=

True

Plethysm

In[26]:= `pl[p[3, x], p[5, x], x]`

Out[26]=

$p_{15}(x)$

In[27]:= `Expand[pl[e[2, x], h[2, x], x]]`

Out[27]=

$$\frac{p_1(x)^4}{8} + \frac{1}{4}\,p_1(x)^2\,p_2(x) - \frac{p_2(x)^2}{8} - \frac{p_4(x)}{4}$$

In[28]:= `view[6][e[2, x]] /. Thread[Table[Subscript[x, i], {i, 6}] → List @@ view[3][h[2, x]]]`

Out[28]=

$x_1^3\,x_2 + x_1^2\,x_2^2 + x_1\,x_2^3 + x_1^3\,x_3 + 2\,x_1^2\,x_2\,x_3 + 2\,x_1\,x_2^2\,x_3 + x_2^3\,x_3 + x_1^2\,x_3^2 + 2\,x_1\,x_2\,x_3^2 + x_2^2\,x_3^2 + x_1\,x_3^3 + x_2\,x_3^3$

In[29]:= `Expand[view[3][pl[e[2, x], h[2, x], x]]]`

Out[29]=

$x_1^3\,x_2 + x_1^2\,x_2^2 + x_1\,x_2^3 + x_1^3\,x_3 + 2\,x_1^2\,x_2\,x_3 + 2\,x_1\,x_2^2\,x_3 + x_2^3\,x_3 + x_1^2\,x_3^2 + 2\,x_1\,x_2\,x_3^2 + x_2^2\,x_3^2 + x_1\,x_3^3 + x_2\,x_3^3$

In[30]:= `pl[p[3, x], {p[4, y], f}, x, y]`

Out[30]=

$$p_{12}(y)$$

In[31]:= `pl[p[3, x], {p[4, x], h[2, y]}, x, y]`

Out[31]=

$$p_{12}(x)$$

In[32]:= `pl[p[2, x] × p[3, y], {p[4, x], h[2, y]}, x, y]`

Out[32]=

$$\frac{1}{2}\left(p_3(y)^2 + p_6(y)\right)p_8(x)$$

In[33]:= `FullSimplify[pl[h[2, x] * e[3, y], {ℏ^2 * e[2, x] * e[1, y], ℏ * h[2, y]}, x, y]]`

Out[33]=

$$\frac{1}{384}\hbar^7\left(p_1(y)^2\left(p_1(x)^2 - p_2(x)\right)^2 + 2\,p_2(y)\left(p_2(x)^2 - p_4(x)\right)\right)$$
$$\left(\left(p_1(y)^2 + p_2(y)\right)^3 - 6\left(p_1(y)^2 + p_2(y)\right)\left(p_2(y)^2 + p_4(y)\right) + 8\left(p_3(y)^2 + p_6(y)\right)\right)$$

Orthogonality

Character of the regular representation, in two variables

In[34]:= `Table[reg2[n, x, y], {n, 4}] // Column`

Out[34]=

$$p_1(x)\,p_1(y)$$
$$\frac{1}{2}p_1(x)^2 p_1(y)^2 + \frac{p_2(x)\,p_2(y)}{2}$$
$$\frac{1}{6}p_1(x)^3 p_1(y)^3 + \frac{1}{2}p_1(x)\,p_1(y)\,p_2(x)\,p_2(y) + \frac{p_3(x)\,p_3(y)}{3}$$
$$\frac{1}{24}p_1(x)^4 p_1(y)^4 + \frac{1}{4}p_1(x)^2 p_1(y)^2 p_2(x)\,p_2(y) +$$
$$\frac{1}{8}p_2(x)^2 p_2(y)^2 + \frac{1}{3}p_1(x)\,p_1(y)\,p_3(x)\,p_3(y) + \frac{p_4(x)\,p_4(y)}{4}$$

In[35]:= `Table[conv[p][reg[n, x, y]] == reg2[n, x, y], {n, 5}] // Reduce`

Out[35]=

True

In[36]:= `Table[conv[s][reg2[n, x, y]] == reg[n, x, y], {n, 5}] // Reduce`

Out[36]=

True

Computation from 11/03/2025

In[37]:= `pl[h[2, x], {p[1, x]^3 × p[1, y], 0}, x, y] × h[2, z]`

Out[37]=

$$\frac{1}{2}h_2(z)\left(p_1(x)^6 p_1(y)^2 + p_2(x)^3 p_2(y)\right)$$

In[38]:= `reg[2, y, z]`

Out[38]=

$$s_{\{2\}}(y)\,s_{\{2\}}(z) + s_{\{1,1\}}(y)\,s_{\{1,1\}}(z)$$

```
adj[reg[2, y, z], ✎37]
```

Out[39]=

$$\frac{p_2(x)^3}{2} + \frac{1}{2}\left(\frac{p_1(x)^6}{2} - \frac{p_2(x)^3}{2}\right) + \frac{1}{2}\left(\frac{p_1(x)^6}{2} + \frac{p_2(x)^3}{2}\right)$$

```
Simplify[✎39]
```

Out[40]=

$$\frac{1}{2}\left(p_1(x)^6 + p_2(x)^3\right)$$

```
pl[h[2, x], p[1, x]^3, x] == ✎40
```

Out[41]=

True

Sanity check

Check that $h_n(x) \circ h_1(x)\, h_1(y) = \sum_{\lambda \vdash n} s_\lambda(x)\, s_\lambda(y)$ in two different ways.

In[42]:= `List @@ Expand[view[5][h[1, x] × h[1, y]]]`

Out[42]=

$\{x_1 y_1, x_2 y_1, x_3 y_1, x_4 y_1, x_5 y_1, x_1 y_2, x_2 y_2, x_3 y_2, x_4 y_2, x_5 y_2, x_1 y_3, x_2 y_3,$
$x_3 y_3, x_4 y_3, x_5 y_3, x_1 y_4, x_2 y_4, x_3 y_4, x_4 y_4, x_5 y_4, x_1 y_5, x_2 y_5, x_3 y_5, x_4 y_5, x_5 y_5\}$

```
rules = Table[Subscript[z, i] → ✎42〚i〛, {i, Length[✎42]}]
```

Out[43]=

$\{z_1 \to x_1 y_1, z_2 \to x_2 y_1, z_3 \to x_3 y_1, z_4 \to x_4 y_1, z_5 \to x_5 y_1, z_6 \to x_1 y_2,$
$z_7 \to x_2 y_2, z_8 \to x_3 y_2, z_9 \to x_4 y_2, z_{10} \to x_5 y_2, z_{11} \to x_1 y_3, z_{12} \to x_2 y_3, z_{13} \to x_3 y_3,$
$z_{14} \to x_4 y_3, z_{15} \to x_5 y_3, z_{16} \to x_1 y_4, z_{17} \to x_2 y_4, z_{18} \to x_3 y_4, z_{19} \to x_4 y_4,$
$z_{20} \to x_5 y_4, z_{21} \to x_1 y_5, z_{22} \to x_2 y_5, z_{23} \to x_3 y_5, z_{24} \to x_4 y_5, z_{25} \to x_5 y_5\}$

```
With[{n = 1},
 Simplify[view[5][reg[n, x, y]] == (view[Length[✎43]][h[n, z]] /. ✎43)]]
```

Out[44]=

True

With the actual bisymmetric plethysm:

In[45]:= `Reduce[Table[pl[h[n, x], {h[1, x] * h[1, y], g}, x, y] == reg2[n, x, y], {n, 6}]]`

Out[45]=

True

Computations

Stable twisted cohomology of automorphism groups of free groups

We have: $Q(q, p) = \begin{cases} \mathrm{triv}_p, & q = 0 \mid 1, p \geq 1; \\ 0, & \text{otherwise.} \end{cases}$

Moreover $\mathcal{H} = \omega(\mathcal{P}) = \omega(\mathcal{F}(S(Q)))$ is the involution applied to the saturation of Q.

Unlike the Albanese cohomology, this is infinite dimensional in any fixed degree (because $Q(1, 0)$ contains an element of degree 0), so we can only compute the truncation up to some arity.

In[46]:= `TableForm[Table[{chQ[d, x, y]}, {d, 0, 6}],`
    `TableHeadings → {Range[0, 6], {"ch(Q) in degree d"}}]`

Out[46]//TableForm=

	ch(Q) in degree d
0	$e_1(x) \, h_1(y)$
1	$h_1(y) + e_1(x) \, h_2(y)$
2	$h_2(y) + e_1(x) \, h_3(y)$
3	$h_3(y) + e_1(x) \, h_4(y)$
4	$h_4(y) + e_1(x) \, h_5(y)$
5	$h_5(y) + e_1(x) \, h_6(y)$
6	$h_6(y) + e_1(x) \, h_7(y)$

In[47]:= `chH_upto[2, 5, x, y] // printPoly[S, x, y]`

Out[47]=

$S_{3,41}^{\oplus 3} \oplus S_{3,32} \oplus S_{3,31^2}^{\oplus 6} \oplus S_{3,2^21}^{\oplus 2} \oplus S_{3,21^3}^{\oplus 3} \oplus S_{21,41} \oplus S_{21,32}^{\oplus 5} \oplus S_{21,31^2}^{\oplus 7} \oplus S_{21,2^21}^{\oplus 8} \oplus S_{21,21^3}^{\oplus 9} \oplus S_{21,1^5}^{\oplus 3} \oplus S_{2,31}^{\oplus 3} \oplus S_{2,21^2}^{\oplus 6} \oplus S_{2,2^2} \oplus$
$S_{2,1^4}^{\oplus 2} \oplus S_{1^3,32}^{\oplus 2} \oplus S_{1^3,31^2} \oplus S_{1^3,2^21}^{\oplus 6} \oplus S_{1^3,21^3}^{\oplus 6} \oplus S_{1^3,1^5}^{\oplus 5} \oplus S_{1^2,31} \oplus S_{1^2,21^2}^{\oplus 5} \oplus S_{1^2,2^2}^{\oplus 4} \oplus S_{1^2,1^4}^{\oplus 5} \oplus S_{1,21}^{\oplus 3} \oplus S_{1,1^3}^{\oplus 4} \oplus S_{\emptyset,1^2}^{\oplus 2}$

The characters of $\mathcal{P}$ and $\mathcal{H}$ in degree d, output arity m, input arity n

In[48]:= `TableForm[`
    `Map[printPoly[S, x, y], Table[chH[2, m, n, x, y], {m, 0, 5}, {n, 0, 5}], {2}],`
    `TableHeadings → {Range[0, 5], Range[0, 5]}]`

Out[48]//TableForm=

	0	1	2	3	4	5
0	0	0	$S_{\emptyset,1^2}^{\oplus 2}$	0	0	0
1	0	0	$S_{\emptyset,1^2}^{\oplus 2}$	$S_{1,21}^{\oplus 3} \oplus S_{1,1^3}^{\oplus 4}$	0	0
2	0	0	$S_{\emptyset,1^2}^{\oplus 2}$	0	$S_{2,31}^{\oplus 3} \oplus S_{2,21^2}^{\oplus 6} \oplus S_{2,2^2} \oplus S_{2,1^4}^{\oplus 2} \oplus S_{1^2,31} \oplus S_{1^2,21^2}^{\oplus 5} \oplus S_{1^2,2^2}^{\oplus 4} \oplus S_{1^2,1^4}^{\oplus 5}$	0
3	0	0	$S_{\emptyset,1^2}^{\oplus 2}$	0	0	$S_{3,4}^{\oplus 3}$
4	0	0	$S_{\emptyset,1^2}^{\oplus 2}$	0	0	0
5	0	0	$S_{\emptyset,1^2}^{\oplus 2}$	0	0	0

Albanese cohomology for IA$_\infty$

We have: $Q'(q, p) = \begin{cases} \mathrm{triv}_p[p] & q = 0,\, p \geq 1 \\ \mathrm{triv}_p[p-1] & q = 1,\, p \geq 2 \\ 0 & \text{otherwise.} \end{cases}$

Moreover $\mathcal{H}' = \omega(\mathcal{P}') = \omega(S(Q'))$ is the involution applied to the saturation of Q'.

The following function gives the character of Q' in homological degree d:

In[49]:= `TableForm[Table[{chQ_prime[d, x, y]}, {d, 0, 6}],`
    `TableHeadings → {Range[0, 6], {"ch(Q') in degree d"}}]`

Out[49]//TableForm=

	ch(Q') in degree d
0	0
1	$h_1(y) + h_1(x) \, h_2(y)$
2	$h_2(y) + h_1(x) \, h_3(y)$
3	$h_3(y) + h_1(x) \, h_4(y)$
4	$h_4(y) + h_1(x) \, h_5(y)$
5	$h_5(y) + h_1(x) \, h_6(y)$
6	$h_6(y) + h_1(x) \, h_7(y)$

Now, the saturation $\mathcal{P}' = S(Q')$ has character $\sum_{n \geq 1} h_n(x) \circ (\mathrm{ch}(Q'), 0)$. The first function gives $\mathrm{ch}(\mathcal{P}')$ up to degree d, the second in degree exactly d (without the $(-\hbar)^d$)

In[50]:= `TableForm[Table[{d, printPoly[S, x, y][chP_prime[d, x, y]]}, {d, 3}],`
   `TableHeadings → {None, {"d", "ch(𝒫') in degree d"}}]`

Out[50]//TableForm=

d	ch(𝒫') in degree d
1	$S_{1,2} \oplus S_{\emptyset,1}$
2	$S_{2,4} \oplus S_{2,2^2} \oplus S_{1^2,31} \oplus S_{1,3}^{\oplus 2} \oplus S_{1,21} \oplus S_{\emptyset,2}^{\oplus 2}$
3	$S_{3,6} \oplus S_{3,42} \oplus S_{3,2^3} \oplus S_{21,51} \oplus S_{21,42} \oplus S_{21,321} \oplus S_{2,5}^{\oplus 2} \oplus S_{2,41}^{\oplus 2} \oplus S_{2,32}^{\oplus 2} \oplus S_{2,2^21} \oplus S_{1^3,41^2} \oplus S_{1^3,3^2} \oplus S_{1^2,5} \oplus S_{1^2,41}^{\oplus 2} \oplus$ S

Calculations

Now we can find the decomposition of ch(𝓗') in some degree:

In[51]:= `Get[FileNameJoin[NotebookDirectory[], "chH_prime.mx"]]`

In[52]:= `printPoly[V, x, y][chH_prime[1, x, y]]`

Out[52]=

$V_{1,1^2} \oplus V_{\emptyset,1}$

In[53]:= `printPoly[V, x, y][chH_prime[2, x, y]]`

Out[53]=

$V_{2,21^2} \oplus V_{1^2,2^2} \oplus V_{1^2,1^4} \oplus V_{1,21} \oplus V_{1,1^3}^{\oplus 2} \oplus V_{\emptyset,1^2}^{\oplus 2}$

In[54]:= `printPoly[V, x, y][chH_prime[3, x, y]]`

Out[54]=

$V_{3,31^3} \oplus V_{3,2^3} \oplus V_{21,321} \oplus V_{21,2^21^2} \oplus V_{21,21^4} \oplus V_{2,31^2} \oplus V_{2,2^21}^{\oplus 2} \oplus V_{2,21^3}^{\oplus 2} \oplus V_{2,1^5} \oplus V_{1^3,3^2} \oplus$
$V_{1^3,2^21^2} \oplus V_{1^3,1^6} \oplus V_{1^2,32} \oplus V_{1^2,2^21}^{\oplus 2} \oplus V_{1^2,21^3}^{\oplus 2} \oplus V_{1^2,1^5}^{\oplus 2} \oplus V_{1,21^2}^{\oplus 3} \oplus V_{1,2^2}^{\oplus 2} \oplus V_{1,1^4}^{\oplus 4} \oplus V_{\emptyset,21} \oplus V_{\emptyset,1^3}^{\oplus 3}$

In[55]:= `printPoly[V, x, y][chH_prime[4, x, y]]`

Out[55]=

$V_{4,41^4} \oplus V_{4,32^21} \oplus V_{31,421^2} \oplus V_{31,3^22} \oplus V_{31,32^21} \oplus V_{31,321^3} \oplus V_{31,31^5} \oplus V_{31,2^31^2} \oplus V_{3,41^3} \oplus V_{3,32^2}^{\oplus 2} \oplus V_{3,321^2}^{\oplus 2} \oplus V_{3,31^4}^{\oplus 2} \oplus$
$V_{3,2^31}^{\oplus 2} \oplus V_{3,2^21^3} \oplus V_{3,21^5} \oplus V_{21^2,431} \oplus V_{21^2,3^21^2} \oplus V_{21^2,32^21} \oplus V_{21^2,321^3} \oplus V_{21^2,2^31^2} \oplus V_{21^2,2^21^4} \oplus V_{21^2,21^6} \oplus$
$V_{21,421} \oplus V_{21,3^21}^{\oplus 2} \oplus V_{21,32^2}^{\oplus 2} \oplus V_{21,321^2}^{\oplus 4} \oplus V_{21,31^4}^{\oplus 2} \oplus V_{21,2^31}^{\oplus 3} \oplus V_{21,2^21^3}^{\oplus 5} \oplus V_{21,21^5}^{\oplus 3} \oplus V_{21,1^7} \oplus V_{2^2,42^2} \oplus V_{2^2,3^21^2} \oplus$
$V_{2^2,321^3} \oplus V_{2^2,2^21^4} \oplus V_{2^2,2^4} \oplus V_{2,321}^{\oplus 3} \oplus V_{2,31^3}^{\oplus 3} \oplus V_{2,2^21^2}^{\oplus 5} \oplus V_{2,21^4}^{\oplus 6} \oplus V_{2,2^3}^{\oplus 4} \oplus V_{2,1^6}^{\oplus 2} \oplus V_{1^4,4^2} \oplus V_{1^4,3^21^2} \oplus V_{1^4,2^21^4} \oplus$
$V_{1^4,2^4} \oplus V_{1^4,1^8} \oplus V_{1^3,43} \oplus V_{1^3,3^21}^{\oplus 2} \oplus V_{1^3,321^2}^{\oplus 2} \oplus V_{1^3,2^31}^{\oplus 2} \oplus V_{1^3,2^21^3}^{\oplus 3} \oplus V_{1^3,21^5}^{\oplus 2} \oplus V_{1^3,1^7}^{\oplus 2} \oplus V_{1^2,321}^{\oplus 3} \oplus V_{1^2,31^3} \oplus$
$V_{1^2,3^2}^{\oplus 2} \oplus V_{1^2,2^21^2}^{\oplus 8} \oplus V_{1^2,21^4}^{\oplus 5} \oplus V_{1^2,2^3} \oplus V_{1^2,1^6}^{\oplus 5} \oplus V_{1,32} \oplus V_{1,31^2} \oplus V_{1,2^21}^{\oplus 6} \oplus V_{1,21^3}^{\oplus 7} \oplus V_{1,1^5}^{\oplus 7} \oplus V_{\emptyset,21^2}^{\oplus 2} \oplus V_{\emptyset,2^2}^{\oplus 2} \oplus V_{\emptyset,1^4}^{\oplus 5}$

In[56]:= **printPoly[V, x, y][chH_prime[5, x, y]]**

Out[56]=

$V_{5,51^5} \oplus V_{5,42^21^2} \oplus V_{5,3^22^2} \oplus V_{41,521^3} \oplus V_{41,4321} \oplus V_{41,42^3} \oplus V_{41,42^21^2} \oplus V_{41,421^4} \oplus V_{41,41^6} \oplus V_{41,3^31} \oplus V_{41,3^221^2} \oplus$
$V_{41,32^31} \oplus V_{41,32^21^3} \oplus V_{4,51^4} \oplus V^{\oplus2}_{4,42^21} \oplus V^{\oplus2}_{4,421^3} \oplus V^{\oplus2}_{4,41^5} \oplus V^{\oplus2}_{4,3^221} \oplus V^{\oplus2}_{4,32^3} \oplus V^{\oplus3}_{4,32^21^2} \oplus V_{4,321^4} \oplus V_{4,31^6} \oplus$
$V_{4,3^3} \oplus V_{4,2^31^3} \oplus V_{32,52^21} \oplus V_{32,43^2} \oplus V_{32,4321} \oplus V_{32,431^3} \oplus V_{32,42^21^2} \oplus V_{32,421^4} \oplus V_{32,3^22^2} \oplus V_{32,3^221^2} \oplus$
$V_{32,3^21^4} \oplus V_{32,32^31} \oplus V_{32,32^21^3} \oplus V_{32,321^5} \oplus V_{32,2^41^2} \oplus V_{31^2,531^2} \oplus V_{31^2,4^22} \oplus V_{31^2,4321} \oplus V_{31^2,431^3} \oplus V_{31^2,42^3} \oplus$
$V_{31^2,42^21^2} \oplus V_{31^2,421^4} \oplus V_{31^2,3^31} \oplus V^{\oplus2}_{31^2,3^221^2} \oplus V_{31^2,32^31} \oplus V^{\oplus2}_{31^2,32^21^3} \oplus V_{31^2,321^5} \oplus V_{31^2,31^7} \oplus V_{31^2,2^31^4} \oplus$
$V_{31^2,2^5} \oplus V_{31,521^2} \oplus V^{\oplus2}_{31,432} \oplus V^{\oplus2}_{31,431^2} \oplus V^{\oplus4}_{31,42^21} \oplus V^{\oplus4}_{31,421^3} \oplus V^{\oplus2}_{31,41^5} \oplus V^{\oplus5}_{31,3^221} \oplus V^{\oplus3}_{31,32^13} \oplus V_{31,32^3} \oplus$
$V^{\oplus8}_{31,32^21^2} \oplus V^{\oplus6}_{31,321^4} \oplus V^{\oplus3}_{31,31^6} \oplus V^{\oplus2}_{31,3^3} \oplus V^{\oplus3}_{31,2^41} \oplus V^{\oplus4}_{31,2^31^3} \oplus V^{\oplus2}_{31,2^21^5} \oplus V_{31,21^7} \oplus V_{3,42^2} \oplus V^{\oplus3}_{3,421^2} \oplus V^{\oplus3}_{3,41^4} \oplus$
$V^{\oplus4}_{3,3^22} \oplus V_{3,3^21^2} \oplus V^{\oplus9}_{3,32^21} \oplus V^{\oplus7}_{3,321^3} \oplus V^{\oplus6}_{3,31^5} \oplus V^{\oplus7}_{3,2^31^2} \oplus V^{\oplus4}_{3,2^21^4} \oplus V^{\oplus3}_{3,21^6} \oplus V_{3,2^4} \oplus V_{2^21,532} \oplus V_{2^21,4^21^2} \oplus$
$V_{2^21,4321} \oplus V_{2^21,431^3} \oplus V_{2^21,42^21^2} \oplus V_{2^21,3^22^2} \oplus V_{2^21,3^221^2} \oplus V^{\oplus2}_{2^21,3^21^4} \oplus V_{2^21,32^31} \oplus V_{2^21,32^21^3} \oplus V_{2^21,321^5} \oplus$
$V_{2^21,2^41^2} \oplus V_{2^21,2^31^4} \oplus V_{2^21,2^21^6} \oplus V_{21^3,541} \oplus V_{21^3,4^21^2} \oplus V_{21^3,4321} \oplus V_{21^3,431^3} \oplus V_{21^3,3^22^2} \oplus V_{21^3,3^221^2} \oplus$
$V_{21^3,3^21^4} \oplus V_{21^3,32^31} \oplus V_{21^3,32^21^3} \oplus V_{21^3,321^5} \oplus V_{21^3,2^41^2} \oplus V_{21^3,2^31^4} \oplus V_{21^3,2^21^6} \oplus V_{21^3,21^8} \oplus V_{21^2,531} \oplus V^{\oplus2}_{21^2,4^21} \oplus$
$V^{\oplus2}_{21^2,432} \oplus V^{\oplus4}_{21^2,431^2} \oplus V^{\oplus2}_{21^2,42^21} \oplus V^{\oplus2}_{21^2,421^3} \oplus V^{\oplus5}_{21^2,3^221} \oplus V^{\oplus6}_{21^2,3^21^3} \oplus V^{\oplus3}_{21^2,32^3} \oplus V^{\oplus7}_{21^2,32^21^2} \oplus V^{\oplus6}_{21^2,321^4} \oplus$
$V^{\oplus2}_{21^2,31^6} \oplus V^{\oplus4}_{21^2,2^41} \oplus V^{\oplus6}_{21^2,2^31^3} \oplus V^{\oplus6}_{21^2,2^21^5} \oplus V^{\oplus3}_{21^2,21^7} \oplus V_{21^2,1^9} \oplus V^{\oplus3}_{21,431} \oplus V^{\oplus3}_{21,42^2} \oplus V^{\oplus4}_{21,421^2} \oplus V_{21,41^4} \oplus$
$V^{\oplus5}_{21,3^22} \oplus V^{\oplus9}_{21,3^21^2} \oplus V^{\oplus12}_{21,32^21} \oplus V^{\oplus16}_{21,321^3} \oplus V^{\oplus7}_{21,31^5} \oplus V^{\oplus14}_{21,2^31^2} \oplus V^{\oplus15}_{21,2^21^4} \oplus V^{\oplus9}_{21,21^6} \oplus V^{\oplus6}_{21,2^4} \oplus V^{\oplus3}_{21,1^8} \oplus V_{2^2,52^2} \oplus$
$V^{\oplus2}_{2^2,432} \oplus V^{\oplus2}_{2^2,431^2} \oplus V^{\oplus2}_{2^2,42^21} \oplus V^{\oplus2}_{2^2,421^3} \oplus V^{\oplus3}_{2^2,3^221} \oplus V^{\oplus4}_{2^2,3^21^3} \oplus V^{\oplus2}_{2^2,32^3} \oplus V^{\oplus4}_{2^2,32^21^2} \oplus V^{\oplus5}_{2^2,321^4} \oplus V_{2^2,31^6} \oplus$
$V^{\oplus3}_{2^2,2^41} \oplus V^{\oplus3}_{2^2,2^31^3} \oplus V^{\oplus3}_{2^2,2^21^5} \oplus V_{2^2,21^7} \oplus V_{2,421} \oplus V_{2,41^3} \oplus V^{\oplus3}_{2,3^21} \oplus V^{\oplus7}_{2,32^2} \oplus V^{\oplus10}_{2,321^2} \oplus V^{\oplus8}_{2,31^4} \oplus V^{\oplus11}_{2,2^31} \oplus V^{\oplus15}_{2,2^21^3} \oplus$
$V^{\oplus12}_{2,21^5} \oplus V^{\oplus5}_{2,1^7} \oplus V_{1^5,5^2} \oplus V_{1^5,4^21^2} \oplus V_{1^5,3^22^2} \oplus V_{1^5,3^21^4} \oplus V_{1^5,2^41^2} \oplus V_{1^5,2^21^6} \oplus V_{1^5,1^{10}} \oplus V_{1^4,54} \oplus V^{\oplus2}_{1^4,4^21} \oplus$
$V^{\oplus2}_{1^4,431^2} \oplus V^{\oplus2}_{1^4,3^221} \oplus V^{\oplus3}_{1^4,3^21^3} \oplus V^{\oplus2}_{1^4,32^3} \oplus V_{1^4,32^21^2} \oplus V^{\oplus2}_{1^4,321^4} \oplus V^{\oplus2}_{1^4,2^41} \oplus V^{\oplus3}_{1^4,2^31^3} \oplus V^{\oplus3}_{1^4,2^21^5} \oplus V_{1^4,21^7} \oplus$
$V^{\oplus2}_{1^4,1^9} \oplus V^{\oplus3}_{1^3,431} \oplus V_{1^3,421^2} \oplus V^{\oplus2}_{1^3,4^2} \oplus V_{1^3,3^22} \oplus V^{\oplus8}_{1^3,3^21^2} \oplus V^{\oplus5}_{1^3,32^21} \oplus V^{\oplus7}_{1^3,321^3} \oplus V_{1^3,31^5} \oplus V^{\oplus7}_{1^3,2^31^2} \oplus V^{\oplus11}_{1^3,2^21^4} \oplus$
$V^{\oplus6}_{1^3,21^6} \oplus V^{\oplus5}_{1^3,2^4} \oplus V^{\oplus5}_{1^3,1^8} \oplus V_{1^2,43} \oplus V_{1^2,421} \oplus V^{\oplus6}_{1^2,3^21} \oplus V^{\oplus3}_{1^2,32^2} \oplus V^{\oplus11}_{1^2,321^2} \oplus V^{\oplus4}_{1^2,31^4} \oplus V^{\oplus11}_{1^2,2^31} \oplus V^{\oplus18}_{1^2,2^21^3} \oplus$
$V^{\oplus13}_{1^2,21^5} \oplus V^{\oplus9}_{1^2,1^7} \oplus V^{\oplus5}_{1,321} \oplus V^{\oplus3}_{1,31^3} \oplus V^{\oplus2}_{1,3^2} \oplus V^{\oplus15}_{1,2^21^2} \oplus V^{\oplus14}_{1,21^4} \oplus V^{\oplus5}_{1,2^3} \oplus V^{\oplus12}_{1,1^6} \oplus V_{\emptyset,32} \oplus V^{\oplus4}_{\emptyset,2^21} \oplus V^{\oplus5}_{\emptyset,21^3} \oplus V^{\oplus7}_{\emptyset,1^5}$

In[57]:= **printPoly[V, x, y][chH_prime[2, x, y]] // TeXForm // CopyToClipboard**

In[58]:= **Table[chH_prime[d, x, y] /. s[λ_, _] :> hook[λ], {d, 1, 7}]**

Out[58]=

$\{2, 12, 162, 4221, 182\,106, 11\,705\,807, 1\,046\,227\,328\}$

In[59]:= **TeXForm[**
  Transpose[Table[{d, NumberForm[Length[Expand[chH_prime[d, x, y]]], DigitBlock → 3],
    NumberForm[chH_prime[d, x, y] /. _s → 1, DigitBlock → 3]}, {d, 1, 9}]]]

Out[59]//TeXForm=

```
\left(
\begin{array}{ccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 2 & 6 & 21 & 69 & 219 & 663 & \text{1,915} & \text{5,182} & \text{13,330} \\
 2 & 8 & 34 & 152 & 720 & \text{3,634} & \text{19,266} & \text{107,018} & \text{619,606}
\end{array}
\right)
```

In[60]:= **Table[Length[Expand[chH_prime[d, x, y]]], {d, 1, 10}]**

Out[60]=

$\{2, 6, 21, 69, 219, 663, 1915, 5182, 13\,330, 32\,876\}$