Code for “Plethysm for characters of relative operads and PROPs” – demonstration notebook

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In[1]:=
```
Get["Lambda`",
 Path → If[$MachineName == "epsilon", $HomeDirectory, NotebookDirectory[]]]
```

Partitions

In[2]:=
```
tr[{5, 4, 4, 1}]
```
Out[2]= {4, 3, 3, 3, 1}

In[3]:=
```
hook[{5, 4, 4, 1}]
```
Out[3]= 21 021

In[4]:=
```
z[{5, 4, 4, 1}]
```
Out[4]= 160

In[5]:=
```
χ[{3, 2, 1}, {3, 3}]
```
Out[5]= −2

Projections

In[6]:=
```
TableForm[Comap[{Identity, view[3]}] /@
   {m[{5, 4, 4, 2}, x], m[{1}, x], m[{2}, x], m[{2, 1}, x], m[{1, 1, 1}, x]},
  TableHeadings → {None, {"f", "f in Λ3"}}]
```
Out[6]//TableForm=

f	f in Λ3
$m_{\{5,4,4,2\}}(x)$	0
$m_{\{1\}}(x)$	$x_1 + x_2 + x_3$
$m_{\{2\}}(x)$	$x_1^2 + x_2^2 + x_3^2$
$m_{\{2,1\}}(x)$	$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$
$m_{\{1,1,1\}}(x)$	$x_1 x_2 x_3$

In[7]:=
```
TableForm[Comap[{Identity, Expand @* view[3]}] /@
  {e[1, x], e[2, x], e[3, x], e[1, x] * e[2, y]},
 TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[7]//TableForm=

f	f in Λ3
$e_1(x)$	$x_1 + x_2 + x_3$
$e_2(x)$	$x_1 x_2 + x_1 x_3 + x_2 x_3$
$e_3(x)$	$x_1 x_2 x_3$
$e_1(x)\, e_2(y)$	$x_1 y_1 y_2 + x_2 y_1 y_2 + x_3 y_1 y_2 + x_1 y_1 y_3 + x_2 y_1 y_3 + x_3 y_1 y_3 + x_1 y_2 y_3 + x_2 y_2 y_3 + x_3 y_2 y_3$

In[8]:=
```
TableForm[Comap[{Identity, Expand @* view[3]}] /@
  {h[1, x], h[2, x], h[3, x], h[1, x] * h[2, y]},
 TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[8]//TableForm=

f	f in Λ3
$h_1(x)$	$x_1 + x_2 + x_3$
$h_2(x)$	$x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2$
$h_3(x)$	$x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 + x_3^3$
$h_1(x)\, h_2(y)$	$x_1 y_1^2 + x_2 y_1^2 + x_3 y_1^2 + x_1 y_1 y_2 + x_2 y_1 y_2 + x_3 y_1 y_2 + x_1 y_2^2 + x_2 y_2^2 + x_3 y_2^2 + x_1 y_1 y_3 + x_2 y_1 y_3$

In[9]:=
```
TableForm[Comap[{Identity, Expand @* view[3]}] /@
  {p[0, x], p[1, x], p[2, x], p[3, x], p[1, x] * p[2, y]},
 TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[9]//TableForm=

f	f in Λ3
1	1
$p_1(x)$	$x_1 + x_2 + x_3$
$p_2(x)$	$x_1^2 + x_2^2 + x_3^2$
$p_3(x)$	$x_1^3 + x_2^3 + x_3^3$
$p_1(x)\, p_2(y)$	$x_1 y_1^2 + x_2 y_1^2 + x_3 y_1^2 + x_1 y_2^2 + x_2 y_2^2 + x_3 y_2^2 + x_1 y_3^2 + x_2 y_3^2 + x_3 y_3^2$

In[10]:=
```
Reduce[view[5][2 * h[2, x]] == view[5][p[1, x]^2 + p[2, x]]]
```

Out[10]=

True

In[11]:=
```
alt[{2}, x]
```

Out[11]=

$$x_1^2$$

In[12]:=
```
alt[{2, 1}, x]
```

Out[12]=

$$x_1^2 x_2 - x_1 x_2^2$$

In[13]:=
```
alt[{3, 2, 1}, x]
```

Out[13]=

$$x_1^3 x_2^2 x_3 - x_1^2 x_2^3 x_3 - x_1^3 x_2 x_3^2 + x_1 x_2^3 x_3^2 + x_1^2 x_2 x_3^3 - x_1 x_2^2 x_3^3$$

In[14]:=
```
TableForm[Comap[{Identity, Expand @* view[3]}] /@
  {s[{1}, x], s[{2}, x], s[{2, 1}, x]}, TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[14]//TableForm=

f	f in Λ3
$s_{\{1\}}(x)$	$x_1 + x_2 + x_3$
$s_{\{2\}}(x)$	$x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2$
$s_{\{2,1\}}(x)$	$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2 x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$

Sanity check

Check that $s_\lambda = \det\left(e_{\lambda'_i - i + j}\right)_{1 \le i, j \le l(\lambda')}$, as expected

In[15]:=
```
chk[λ_List, n_Integer] := With[{λtr = tr[λ]}, Reduce[
   view[n][Det[Table[e[λtr[[i]] - i + j, x], {i, Length[λtr]}, {j, Length[λtr]}]]] ==
  view[n][s[λ, x]]]]
```

In[16]:=
```
chk[{2, 1}, 4]
```

Out[16]=
```
True
```

In[17]:=
```
chk[{3, 2, 1}, 4]
```

Out[17]=
```
True
```

In[18]:=
```
chk[{4, 2, 1}, 5]
```

Out[18]=
```
True
```

Conversions

In[19]:=
```
TableForm[Comap[{Identity, conv[m]}] /@
  {e[5, x], h[2, x], h[3, x], h[4, x], p[5, x], e[2, x] * h[3, y]}]
```

Out[19]//TableForm=

$e_5(x)$	$m_{\{1,1,1,1,1\}}(x)$
$h_2(x)$	$m_{\{2\}}(x) + m_{\{1,1\}}(x)$
$h_3(x)$	$m_{\{3\}}(x) + m_{\{2,1\}}(x) + m_{\{1,1,1\}}(x)$
$h_4(x)$	$m_{\{4\}}(x) + m_{\{2,2\}}(x) + m_{\{3,1\}}(x) + m_{\{2,1,1\}}(x) + m_{\{1,1,1,1\}}(x)$
$p_5(x)$	$m_{\{5\}}(x)$
$e_2(x)\, h_3(y)$	$m_{\{1,1\}}(x)\, (m_{\{3\}}(y) + m_{\{2,1\}}(y) + m_{\{1,1,1\}}(y))$

In[20]:=
```
TableForm[Comap[{Identity, conv[e]}] /@
  {h[5, x], p[5, x], s[{2, 1, 1}, x], s[{1, 1, 1, 1, 1}, x]}]
```

Out[20]//TableForm=

$h_5(x)$	$e_1(x)^5 - 4\, e_1(x)^3\, e_2(x) + 3\, e_1(x)\, e_2(x)^2 + 3\, e_1(x)^2\, e_3(x) - 2\, e_2(x)\, e_3(x) - 2\, e_1(x)$ (
$p_5(x)$	$e_1(x)^5 - 5\, e_1(x)^3\, e_2(x) + 5\, e_1(x)\, e_2(x)^2 + 5\, e_1(x)^2\, e_3(x) - 5\, e_2(x)\, e_3(x) - 5\, e_1(x)$ (
$s_{\{2,1,1\}}(x)$	$e_1(x)\, e_3(x) - e_4(x)$
$s_{\{1,1,1,1,1\}}(x)$	$e_5(x)$

In[21]:=
```
TableForm[Comap[{Identity, conv[h]}] /@
  {e[2, x], e[3, x], e[4, x], p[2, x], p[3, x], p[4, x], s[{2, 1, 1}, x]}]
```

Out[21]//TableForm=

$e_2(x)$	$h_1(x)^2 - h_2(x)$
$e_3(x)$	$h_1(x)^3 - 2\, h_1(x)\, h_2(x) + h_3(x)$
$e_4(x)$	$h_1(x)^4 - 3\, h_1(x)^2\, h_2(x) + h_2(x)^2 + 2\, h_1(x)\, h_3(x) - h_4(x)$
$p_2(x)$	$-h_1(x)^2 + 2\, h_2(x)$
$p_3(x)$	$h_1(x)^3 - 3\, h_1(x)\, h_2(x) + 3\, h_3(x)$
$p_4(x)$	$-h_1(x)^4 + 4\, h_1(x)^2\, h_2(x) - 2\, h_2(x)^2 - 4\, h_1(x)\, h_3(x) + 4\, h_4(x)$
$s_{\{2,1,1\}}(x)$	$h_1(x)^2\, h_2(x) - h_2(x)^2 - h_1(x)\, h_3(x) + h_4(x)$

In[22]:=
```
TableForm[Comap[{Identity, conv[p]}] /@ {e[2, x], e[3, x],
   e[4, x], e[5, x], h[2, x], h[3, x], h[4, x], h[5, x], s[{2, 1}, x]}]
```

Out[22]//TableForm=

$e_2(x)$	$\frac{1}{2}\left(p_1(x)^2 - p_2(x)\right)$
$e_3(x)$	$\frac{1}{6}\left(p_1(x)^3 - 3\,p_1(x)\,p_2(x) + 2\,p_3(x)\right)$
$e_4(x)$	$\frac{1}{24}\left(p_1(x)^4 - 6\,p_1(x)^2\,p_2(x) + 3\,p_2(x)^2 + 8\,p_1(x)\,p_3(x) - 6\,p_4(x)\right)$
$e_5(x)$	$\frac{1}{120}\left(p_1(x)^5 - 10\,p_1(x)^3\,p_2(x) + 15\,p_1(x)\,p_2(x)^2 + 20\,p_1(x)^2\,p_3(x) - 20\,p_2(x)\,p_3(x) - 30\,p\right.$
$h_2(x)$	$\frac{1}{2}\left(p_1(x)^2 + p_2(x)\right)$
$h_3(x)$	$\frac{1}{6}\left(p_1(x)^3 + 3\,p_1(x)\,p_2(x) + 2\,p_3(x)\right)$
$h_4(x)$	$\frac{1}{24}\left(p_1(x)^4 + 6\,p_1(x)^2\,p_2(x) + 3\,p_2(x)^2 + 8\,p_1(x)\,p_3(x) + 6\,p_4(x)\right)$
$h_5(x)$	$\frac{1}{120}\left(p_1(x)^5 + 10\,p_1(x)^3\,p_2(x) + 15\,p_1(x)\,p_2(x)^2 + 20\,p_1(x)^2\,p_3(x) + 20\,p_2(x)\,p_3(x) + 30\,p\right.$
$s_{\{2,1\}}(x)$	$\frac{p_1(x)^3}{3} - \frac{p_3(x)}{3}$

Sanity checks:

In[23]:=
```
And @@ Flatten@Table[
   Simplify[conv[t][conv[t2][t[10, x]]] == t[10, x]], {t, {e, h, p}}, {t2, {e, h, p}}]
```

Out[23]=

True

In[24]:=
```
Simplify[conv[p][Sum[hook[λ] * s[λ, x], {λ, IntegerPartitions[10]}]] == p[1, x]^10]
```

Out[24]=

True

Sanity checks

In[25]:=
```
With[{μ = {3, 2, 1}}, Simplify[view[6][p[μ, x]] ==
   view[6][Sum[χ[λ, μ] * s[λ, x], {λ, IntegerPartitions[Total[μ]]}]]]]
```

Out[25]=

True

Plethysm

In[26]:=
```
pl[p[3, x], p[5, x], x]
```

Out[26]=

$p_{15}(x)$

In[27]:=
```
Expand[pl[e[2, x], h[2, x], x]]
```

Out[27]=

$$\frac{p_1(x)^4}{8} + \frac{1}{4}\,p_1(x)^2\,p_2(x) - \frac{p_2(x)^2}{8} - \frac{p_4(x)}{4}$$

In[28]:=
```
view[6][e[2, x]] /. Thread[Table[Subscript[x, i], {i, 6}] → List @@ view[3][h[2, x]]]
```

Out[28]=

$$x_1^3\,x_2 + x_1^2\,x_2^2 + x_1\,x_2^3 + x_1^3\,x_3 + 2\,x_1^2\,x_2\,x_3 + 2\,x_1\,x_2^2\,x_3 + x_2^3\,x_3 + x_1^2\,x_3^2 + 2\,x_1\,x_2\,x_3^2 + x_2^2\,x_3^2 + x_1\,x_3^3 + x_2\,x_3^3$$

In[29]:=
```
Expand[view[3][pl[e[2, x], h[2, x], x]]]
```

Out[29]=

$$x_1^3\,x_2 + x_1^2\,x_2^2 + x_1\,x_2^3 + x_1^3\,x_3 + 2\,x_1^2\,x_2\,x_3 + 2\,x_1\,x_2^2\,x_3 + x_2^3\,x_3 + x_1^2\,x_3^2 + 2\,x_1\,x_2\,x_3^2 + x_2^2\,x_3^2 + x_1\,x_3^3 + x_2\,x_3^3$$

In[30]:=
```
pl[p[3, x], {p[4, y], f}, x, y]
```
Out[30]=

$$p_{12}(y)$$

In[31]:=
```
pl[p[3, x], {p[4, x], h[2, y]}, x, y]
```
Out[31]=

$$p_{12}(x)$$

In[32]:=
```
pl[p[2, x] × p[3, y], {p[4, x], h[2, y]}, x, y]
```
Out[32]=

$$\frac{1}{2}\left(p_3(y)^2 + p_6(y)\right) p_8(x)$$

In[33]:=
```
FullSimplify[pl[h[2, x] * e[3, y], {ℏ^2 * e[2, x] * e[1, y], ℏ * h[2, y]}, x, y]]
```
Out[33]=

$$\frac{1}{384}\hbar^7 \left(p_1(y)^2 \left(p_1(x)^2 - p_2(x)\right)^2 + 2\, p_2(y) \left(p_2(x)^2 - p_4(x)\right)\right) \left(\left(p_1(y)^2 + p_2(y)\right)^3 - 6\left(p_1(y)^2 + p_2(y)\right)\left(p_2(y)^2 + p_4(y)\right) + 8\left(p_3(y)^2 + p_6(y)\right)\right)$$

Orthogonality

Character of the regular representation, in two variables

In[34]:=
```
Table[reg2[n, x, y], {n, 4}] // Column
```
Out[34]=

$$p_1(x)\, p_1(y)$$
$$\frac{1}{2} p_1(x)^2 p_1(y)^2 + \frac{p_2(x)\, p_2(y)}{2}$$
$$\frac{1}{6} p_1(x)^3 p_1(y)^3 + \frac{1}{2} p_1(x)\, p_1(y)\, p_2(x)\, p_2(y) + \frac{p_3(x)\, p_3(y)}{3}$$
$$\frac{1}{24} p_1(x)^4 p_1(y)^4 + \frac{1}{4} p_1(x)^2 p_1(y)^2 p_2(x)\, p_2(y) + \frac{1}{8} p_2(x)^2 p_2(y)^2 + \frac{1}{3} p_1(x)\, p_1(y)\, p_3(x)\, p_3(y) + \frac{p_4(x)\, p_4(y)}{4}$$

In[35]:=
```
Table[conv[p][reg[n, x, y]] == reg2[n, x, y], {n, 5}] // Reduce
```
Out[35]=

True

In[36]:=
```
Table[conv[s][reg2[n, x, y]] == reg[n, x, y], {n, 5}] // Reduce
```
Out[36]=

True

Computation from 11/03/2025

In[37]:=
```
pl[h[2, x], {p[1, x]^3 × p[1, y], 0}, x, y] × h[2, z]
```
Out[37]=

$$\frac{1}{2} h_2(z) \left(p_1(x)^6 p_1(y)^2 + p_2(x)^3 p_2(y)\right)$$

In[38]:=
```
reg[2, y, z]
```
Out[38]=

$$s_{\{2\}}(y)\, s_{\{2\}}(z) + s_{\{1,1\}}(y)\, s_{\{1,1\}}(z)$$

```
adj[reg[2, y, z], %37]
```

Out[39]=

$$\frac{p_2(x)^3}{2}+\frac{1}{2}\left(\frac{p_1(x)^6}{2}-\frac{p_2(x)^3}{2}\right)+\frac{1}{2}\left(\frac{p_1(x)^6}{2}+\frac{p_2(x)^3}{2}\right)$$

```
Simplify[%39]
```

Out[40]=

$$\frac{1}{2}\left(p_1(x)^6+p_2(x)^3\right)$$

```
pl[h[2, x], p[1, x]^3, x] == %40
```

Out[41]=

True

Sanity check

Check that $h_n(x)\circ h_1(x)\,h_1(y)=\sum_{\lambda\dashv n}s_\lambda(x)\,s_\lambda(y)$ in two different ways.

In[42]:=
```
List @@ Expand[view[5][h[1, x] × h[1, y]]]
```

Out[42]=

$\{x_1 y_1, x_2 y_1, x_3 y_1, x_4 y_1, x_5 y_1, x_1 y_2, x_2 y_2, x_3 y_2, x_4 y_2, x_5 y_2, x_1 y_3, x_2 y_3, x_3 y_3, x_4 y_3, x_5 y_3, x_1 y_4, x_2 y_4, x_3 y_4, x_4 y_4, x_5 y_4, x_1 y_5, x_2 y_5, x_3 y_5, x_4 y_5, x_5 y_5\}$

```
rules = Table[Subscript[z, i] → %42[[i]], {i, Length[%42]}]
```

Out[43]=

$\{z_1 \to x_1 y_1, z_2 \to x_2 y_1, z_3 \to x_3 y_1, z_4 \to x_4 y_1, z_5 \to x_5 y_1, z_6 \to x_1 y_2, z_7 \to x_2 y_2, z_8 \to x_3 y_2, z_9 \to x_4 y_2, z_{10} \to x_5 y_2, z_{11} \to x_1 y_3, z_{12} \to x_2 y_3, z_{13} \to x_3 y_3, z_{14} \to x_4 y_3, z_{15} \to x_5 y_3, z_{16} \to x_1 y_4, z_{17} \to x_2 y_4, z_{18} \to x_3 y_4, z_{19} \to x_4 y_4, z_{20} \to x_5 y_4, z_{21} \to x_1 y_5, z_{22} \to x_2 y_5, z_{23} \to x_3 y_5, z_{24} \to x_4 y_5, z_{25} \to x_5 y_5\}$

```
With[{n = 1},
 Simplify[view[5][reg[n, x, y]] == (view[Length[%43]][h[n, z]] /. %43)]]
```

Out[44]=

True

With the actual bisymmetric plethysm:

In[45]:=
```
Reduce[Table[pl[h[n, x], {h[1, x] * h[1, y], g}, x, y] == reg2[n, x, y], {n, 6}]]
```

Out[45]=

True

Computations

Stable twisted cohomology of automorphism groups of free groups

We have: $Q(q,p)=\begin{cases}\mathrm{triv}_p, & q=0\mid 1, p\geq 1;\\ 0, & \text{otherwise.}\end{cases}$

Moreover $\mathcal{H}=\omega(\mathcal{P})=\omega(\mathcal{F}(S(Q)))$ is the involution applied to the saturation of Q.

Unlike the Albanese cohomology, this is infinite dimensional in any fixed degree (because $Q(1,0)$ contains an element of degree 0), so we can only compute the truncation up to some arity.

In[46]:=
```
TableForm[Table[{chQ[d, x, y]}, {d, 0, 6}],
 TableHeadings → {Range[0, 6], {"ch(𝒬) in degree d"}}]
```

Out[46]//TableForm=

	ch($\mathcal{Q}$) in degree d
0	$e_1(x)\, h_1(y)$
1	$h_1(y) + e_1(x)\, h_2(y)$
2	$h_2(y) + e_1(x)\, h_3(y)$
3	$h_3(y) + e_1(x)\, h_4(y)$
4	$h_4(y) + e_1(x)\, h_5(y)$
5	$h_5(y) + e_1(x)\, h_6(y)$
6	$h_6(y) + e_1(x)\, h_7(y)$

In[47]:=
```
chH_upto[2, 5, x, y] // printPoly[S, x, y]
```

Out[47]=

$$S^{\oplus 3}_{3,41} \oplus S_{3,32} \oplus S^{\oplus 6}_{3,31^2} \oplus S^{\oplus 2}_{3,2^21} \oplus S^{\oplus 3}_{3,21^3} \oplus S_{21,41} \oplus S^{\oplus 5}_{21,32} \oplus S^{\oplus 7}_{21,31^2} \oplus S^{\oplus 8}_{21,2^21} \oplus S^{\oplus 9}_{21,21^3} \oplus S^{\oplus 3}_{21,1^5} \oplus S^{\oplus 3}_{2,31} \oplus S^{\oplus 6}_{2,21^2} \oplus S_{2,2^2} \oplus S^{\oplus 2}_{2,1^4} \oplus S^{\oplus 2}_{1^3,32} \oplus S_{1^3,31^2} \oplus S^{\oplus 6}_{1^3,2^21} \oplus S^{\oplus 6}_{1^3,21^3} \oplus S^{\oplus 5}_{1^3,1^5} \oplus S_{1^2,31} \oplus S^{\oplus 5}_{1^2,21^2} \oplus S^{\oplus 4}_{1^2,2^2} \oplus S^{\oplus 5}_{1^2,1^4} \oplus S^{\oplus 3}_{1,21} \oplus S^{\oplus 4}_{1,1^3} \oplus S^{\oplus 2}_{\emptyset,1^2}$$

The characters of $\mathcal{P}$ and $\mathcal{H}$ in degree d, output arity m, input arity n

In[48]:=
```
TableForm[
 Map[printPoly[S, x, y], Table[chH[2, m, n, x, y], {m, 0, 5}, {n, 0, 5}], {2}],
 TableHeadings → {Range[0, 5], Range[0, 5]}]
```

Out[48]//TableForm=

	0	1	2	3	4	5
0	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	0
1	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	$S^{\oplus 3}_{1,21} \oplus S^{\oplus 4}_{1,1^3}$	0	0
2	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	$S^{\oplus 3}_{2,31} \oplus S^{\oplus 6}_{2,21^2} \oplus S_{2,2^2} \oplus S^{\oplus 2}_{2,1^4} \oplus S_{1^2,31} \oplus S^{\oplus 5}_{1^2,21^2} \oplus S^{\oplus 4}_{1^2,2^2} \oplus S^{\oplus 5}_{1^2,1^4}$	0
3	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	$S^{\oplus 3}_{3,}$
4	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	0
5	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	0

Albanese cohomology for IA_∞

We have: $\mathcal{Q}'(q, p) = \begin{cases} \mathrm{triv}_p[p] & q = 0, p \geq 1 \\ \mathrm{triv}_p[p-1] & q = 1, p \geq 2 \\ 0 & \text{otherwise.} \end{cases}$

Moreover $\mathcal{H}' = \omega(\mathcal{P}') = \omega(S(\mathcal{Q}'))$ is the involution applied to the saturation of $\mathcal{Q}'$.

The following function gives the character of $\mathcal{Q}'$ in homological degree d:

In[49]:=
```
TableForm[Table[{chQ_prime[d, x, y]}, {d, 0, 6}],
 TableHeadings → {Range[0, 6], {"ch(𝒬') in degree d"}}]
```

Out[49]//TableForm=

	ch($\mathcal{Q}'$) in degree d
0	0
1	$h_1(y) + h_1(x)\, h_2(y)$
2	$h_2(y) + h_1(x)\, h_3(y)$
3	$h_3(y) + h_1(x)\, h_4(y)$
4	$h_4(y) + h_1(x)\, h_5(y)$
5	$h_5(y) + h_1(x)\, h_6(y)$
6	$h_6(y) + h_1(x)\, h_7(y)$

Now, the saturation $\mathcal{P}' = S(\mathcal{Q}')$ has character $\sum_{n \geq 1} h_n(x) \circ (\mathrm{ch}(\mathcal{Q}'), 0)$. The first function gives $\mathrm{ch}(\mathcal{P}')$ up to degree d, the second in degree exactly d (without the $(-\hbar)^d$)

In[50]:=
```
TableForm[Table[{d, printPoly[S, x, y][chP_prime[d, x, y]]}, {d, 3}],
 TableHeadings → {None, {"d", "ch(𝒫') in degree d"}}]
```

Out[50]//TableForm=

d	ch(𝒫') in degree d
1	$S_{1,2} \oplus S_{\emptyset,1}$
2	$S_{2,4} \oplus S_{2,2^2} \oplus S_{1^2,31} \oplus S^{\oplus 2}_{1,3} \oplus S_{1,21} \oplus S^{\oplus 2}_{\emptyset,2}$
3	$S_{3,6} \oplus S_{3,42} \oplus S_{3,2^3} \oplus S_{21,51} \oplus S_{21,42} \oplus S_{21,321} \oplus S^{\oplus 2}_{2,5} \oplus S^{\oplus 2}_{2,41} \oplus S^{\oplus 2}_{2,32} \oplus S_{2,2^2 1} \oplus S_{1^3,41^2} \oplus S_{1^3,3^2} \oplus S_{1^2,5} \oplus S^{\oplus 2}_{1^2,41} \oplus S$

Calculations

Now we can find the decomposition of ch($\mathcal{H}$') in some degree:

In[51]:=
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Get[FileNameJoin[NotebookDirectory[], "chH_prime.mx"]]
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In[52]:=
```
printPoly[V, x, y][chH_prime[1, x, y]]
```

Out[52]=

$$V_{1,1^2} \oplus V_{\emptyset,1}$$

In[53]:=
```
printPoly[V, x, y][chH_prime[2, x, y]]
```

Out[53]=

$$V_{2,21^2} \oplus V_{1^2,2^2} \oplus V_{1^2,1^4} \oplus V_{1,21} \oplus V^{\oplus 2}_{1,1^3} \oplus V^{\oplus 2}_{\emptyset,1^2}$$

In[54]:=
```
printPoly[V, x, y][chH_prime[3, x, y]]
```

Out[54]=

$$V_{3,31^3} \oplus V_{3,2^3} \oplus V_{21,321} \oplus V_{21,2^2 1^2} \oplus V_{21,21^4} \oplus V_{2,31^2} \oplus V^{\oplus 2}_{2,2^2 1} \oplus V^{\oplus 2}_{2,21^3} \oplus V_{2,1^5} \oplus V_{1^3,3^2} \oplus$$
$$V_{1^3,2^2 1^2} \oplus V_{1^3,1^6} \oplus V_{1^2,32} \oplus V^{\oplus 2}_{1^2,2^2 1} \oplus V^{\oplus 2}_{1^2,21^3} \oplus V^{\oplus 2}_{1^2,1^5} \oplus V^{\oplus 3}_{1,21^2} \oplus V^{\oplus 2}_{1,2^2} \oplus V^{\oplus 4}_{1,1^4} \oplus V_{\emptyset,21} \oplus V^{\oplus 3}_{\emptyset,1^3}$$

In[55]:=
```
printPoly[V, x, y][chH_prime[4, x, y]]
```

Out[55]=

$$V_{4,41^4} \oplus V_{4,32^2 1} \oplus V_{31,421^2} \oplus V_{31,3^2 2} \oplus V_{31,32^2 1} \oplus V_{31,321^3} \oplus V_{31,31^5} \oplus V_{31,2^3 1^2} \oplus V_{3,41^3} \oplus V^{\oplus 2}_{3,32^2} \oplus V^{\oplus 2}_{3,321^2} \oplus V^{\oplus 2}_{3,31^4} \oplus$$
$$V^{\oplus 2}_{3,2^3 1} \oplus V_{3,2^2 1^3} \oplus V_{3,21^5} \oplus V_{21^2,431} \oplus V_{21^2,3^2 1^2} \oplus V_{21^2,32^2 1} \oplus V_{21^2,321^3} \oplus V_{21^2,2^3 1^2} \oplus V_{21^2,2^2 1^4} \oplus V_{21^2,21^6} \oplus$$
$$V_{21,421} \oplus V^{\oplus 2}_{21,3^2 1} \oplus V^{\oplus 2}_{21,32^2} \oplus V^{\oplus 4}_{21,321^2} \oplus V^{\oplus 2}_{21,31^4} \oplus V^{\oplus 3}_{21,2^3 1} \oplus V^{\oplus 5}_{21,2^2 1^3} \oplus V^{\oplus 3}_{21,21^5} \oplus V_{21,1^7} \oplus V_{2^2,42^2} \oplus V_{2^2,3^2 1^2} \oplus$$
$$V_{2^2,321^3} \oplus V_{2^2,2^2 1^4} \oplus V_{2^2,2^4} \oplus V^{\oplus 3}_{2,321} \oplus V^{\oplus 3}_{2,31^3} \oplus V^{\oplus 5}_{2,2^2 1^2} \oplus V^{\oplus 6}_{2,21^4} \oplus V^{\oplus 4}_{2,2^3} \oplus V^{\oplus 2}_{2,1^6} \oplus V_{1^4,4^2} \oplus V_{1^4,3^2 1^2} \oplus V_{1^4,2^2 1^4} \oplus$$
$$V_{1^4,2^4} \oplus V_{1^4,1^8} \oplus V_{1^3,43} \oplus V^{\oplus 2}_{1^3,3^2 1} \oplus V^{\oplus 2}_{1^3,321^2} \oplus V^{\oplus 2}_{1^3,2^3 1} \oplus V^{\oplus 3}_{1^3,2^2 1^3} \oplus V^{\oplus 2}_{1^3,21^5} \oplus V^{\oplus 2}_{1^3,1^7} \oplus V^{\oplus 3}_{1^2,321} \oplus V_{1^2,31^3} \oplus$$
$$V^{\oplus 2}_{1^2,3^2} \oplus V^{\oplus 8}_{1^2,2^2 1^2} \oplus V^{\oplus 5}_{1^2,21^4} \oplus V_{1^2,2^3} \oplus V^{\oplus 5}_{1^2,1^6} \oplus V_{1,32} \oplus V_{1,31^2} \oplus V^{\oplus 6}_{1,2^2 1} \oplus V^{\oplus 7}_{1,21^3} \oplus V^{\oplus 7}_{1,1^5} \oplus V^{\oplus 2}_{\emptyset,21^2} \oplus V^{\oplus 2}_{\emptyset,2^2} \oplus V^{\oplus 5}_{\emptyset,1^4}$$

In[56]:=
```
printPoly[V, x, y][chH_prime[5, x, y]]
```

Out[56]=

$$V_{5,51^{5}} \oplus V_{5,42^{2}1^{2}} \oplus V_{5,3^{2}2^{2}} \oplus V_{41,521^{3}} \oplus V_{41,4321} \oplus V_{41,42^{3}} \oplus V_{41,42^{2}1^{2}} \oplus V_{41,421^{4}} \oplus V_{41,41^{6}} \oplus V_{41,3^{3}1} \oplus V_{41,3^{2}21^{2}} \oplus$$
$$V_{41,32^{3}1} \oplus V_{41,32^{2}1^{3}} \oplus V_{4,51^{4}} \oplus V^{\oplus 2}_{4,42^{2}1} \oplus V^{\oplus 2}_{4,421^{3}} \oplus V^{\oplus 2}_{4,41^{5}} \oplus V^{\oplus 2}_{4,3^{2}21} \oplus V^{\oplus 2}_{4,32^{3}} \oplus V^{\oplus 3}_{4,32^{2}1^{2}} \oplus V_{4,321^{4}} \oplus V_{4,31^{6}} \oplus$$
$$V_{4,3^{3}} \oplus V_{4,2^{3}1^{3}} \oplus V_{32,52^{2}1} \oplus V_{32,43^{2}} \oplus V_{32,4321} \oplus V_{32,431^{3}} \oplus V_{32,42^{2}1^{2}} \oplus V_{32,421^{4}} \oplus V_{32,3^{2}2^{2}} \oplus V_{32,3^{2}21^{2}} \oplus$$
$$V_{32,3^{2}1^{4}} \oplus V_{32,32^{3}1} \oplus V_{32,32^{2}1^{3}} \oplus V_{32,321^{5}} \oplus V_{32,2^{4}1^{2}} \oplus V_{31^{2},531^{2}} \oplus V_{31^{2},4^{2}2} \oplus V_{31^{2},4321} \oplus V_{31^{2},431^{3}} \oplus V_{31^{2},42^{3}} \oplus$$
$$V_{31^{2},42^{2}1^{2}} \oplus V_{31^{2},421^{4}} \oplus V_{31^{2},3^{3}1} \oplus V^{\oplus 2}_{31^{2},3^{2}21^{2}} \oplus V_{31^{2},32^{3}1} \oplus V^{\oplus 2}_{31^{2},32^{2}1^{3}} \oplus V_{31^{2},321^{5}} \oplus V_{31^{2},31^{7}} \oplus V_{31^{2},2^{3}1^{4}} \oplus$$
$$V_{31^{2},2^{5}} \oplus V_{31,521^{2}} \oplus V^{\oplus 2}_{31,432} \oplus V^{\oplus 2}_{31,431^{2}} \oplus V^{\oplus 4}_{31,42^{2}1} \oplus V^{\oplus 4}_{31,421^{3}} \oplus V^{\oplus 2}_{31,41^{5}} \oplus V^{\oplus 5}_{31,3^{2}21} \oplus V^{\oplus 3}_{31,3^{2}1^{3}} \oplus V^{\oplus 3}_{31,32^{3}} \oplus$$
$$V^{\oplus 8}_{31,32^{2}1^{2}} \oplus V^{\oplus 6}_{31,321^{4}} \oplus V^{\oplus 3}_{31,31^{6}} \oplus V^{\oplus 2}_{31,3^{3}} \oplus V^{\oplus 3}_{31,2^{4}1} \oplus V^{\oplus 4}_{31,2^{3}1^{3}} \oplus V^{\oplus 2}_{31,2^{2}1^{5}} \oplus V_{31,21^{7}} \oplus V_{3,42^{2}} \oplus V^{\oplus 3}_{3,421^{2}} \oplus V^{\oplus 3}_{3,41^{4}} \oplus$$
$$V^{\oplus 4}_{3,3^{2}2} \oplus V_{3,3^{2}1^{2}} \oplus V^{\oplus 9}_{3,32^{2}1} \oplus V^{\oplus 7}_{3,321^{3}} \oplus V^{\oplus 6}_{3,31^{5}} \oplus V^{\oplus 7}_{3,2^{3}1^{2}} \oplus V^{\oplus 4}_{3,2^{2}1^{4}} \oplus V^{\oplus 3}_{3,21^{6}} \oplus V_{3,2^{4}} \oplus V_{2^{2}1,532} \oplus V_{2^{2}1,4^{2}1^{2}} \oplus$$
$$V_{2^{2}1,4321} \oplus V_{2^{2}1,431^{3}} \oplus V_{2^{2}1,42^{2}1^{2}} \oplus V_{2^{2}1,3^{2}2^{2}} \oplus V_{2^{2}1,3^{2}21^{2}} \oplus V^{\oplus 2}_{2^{2}1,3^{2}1^{4}} \oplus V_{2^{2}1,32^{3}1} \oplus V_{2^{2}1,32^{2}1^{3}} \oplus V_{2^{2}1,321^{5}} \oplus$$
$$V_{2^{2}1,2^{4}1^{2}} \oplus V_{2^{2}1,2^{3}1^{4}} \oplus V_{2^{2}1,2^{2}1^{6}} \oplus V_{21^{3},541} \oplus V_{21^{3},4^{2}1^{2}} \oplus V_{21^{3},4321} \oplus V_{21^{3},431^{3}} \oplus V_{21^{3},3^{2}2^{2}} \oplus V_{21^{3},3^{2}21^{2}} \oplus$$
$$V_{21^{3},3^{2}1^{4}} \oplus V_{21^{3},32^{3}1} \oplus V_{21^{3},32^{2}1^{3}} \oplus V_{21^{3},321^{5}} \oplus V_{21^{3},2^{4}1^{2}} \oplus V_{21^{3},2^{3}1^{4}} \oplus V_{21^{3},2^{2}1^{6}} \oplus V_{21^{3},21^{8}} \oplus V_{21^{2},531} \oplus V^{\oplus 2}_{21^{2},4^{2}1} \oplus$$
$$V^{\oplus 2}_{21^{2},432} \oplus V^{\oplus 4}_{21^{2},431^{2}} \oplus V^{\oplus 2}_{21^{2},42^{2}1} \oplus V^{\oplus 2}_{21^{2},421^{3}} \oplus V^{\oplus 5}_{21^{2},3^{2}21} \oplus V^{\oplus 6}_{21^{2},3^{2}1^{3}} \oplus V^{\oplus 3}_{21^{2},32^{3}} \oplus V^{\oplus 7}_{21^{2},32^{2}1^{2}} \oplus V^{\oplus 6}_{21^{2},321^{4}} \oplus$$
$$V^{\oplus 2}_{21^{2},31^{6}} \oplus V^{\oplus 4}_{21^{2},2^{4}1} \oplus V^{\oplus 6}_{21^{2},2^{3}1^{3}} \oplus V^{\oplus 6}_{21^{2},2^{2}1^{5}} \oplus V^{\oplus 3}_{21^{2},21^{7}} \oplus V_{21^{2},1^{9}} \oplus V^{\oplus 3}_{21,431} \oplus V^{\oplus 3}_{21,42^{2}} \oplus V^{\oplus 4}_{21,421^{2}} \oplus V_{21,41^{4}} \oplus$$
$$V^{\oplus 5}_{21,3^{2}2} \oplus V^{\oplus 9}_{21,3^{2}1^{2}} \oplus V^{\oplus 12}_{21,32^{2}1} \oplus V^{\oplus 16}_{21,321^{3}} \oplus V^{\oplus 7}_{21,31^{5}} \oplus V^{\oplus 14}_{21,2^{3}1^{2}} \oplus V^{\oplus 15}_{21,2^{2}1^{4}} \oplus V^{\oplus 9}_{21,21^{6}} \oplus V^{\oplus 6}_{21,2^{4}} \oplus V^{\oplus 3}_{21,1^{8}} \oplus V_{2^{2},52^{2}} \oplus$$
$$V^{\oplus 2}_{2^{2},432} \oplus V^{\oplus 2}_{2^{2},431^{2}} \oplus V^{\oplus 2}_{2^{2},42^{2}1} \oplus V^{\oplus 2}_{2^{2},421^{3}} \oplus V^{\oplus 3}_{2^{2},3^{2}21} \oplus V^{\oplus 4}_{2^{2},3^{2}1^{3}} \oplus V^{\oplus 2}_{2^{2},32^{3}} \oplus V^{\oplus 4}_{2^{2},32^{2}1^{2}} \oplus V^{\oplus 5}_{2^{2},321^{4}} \oplus V_{2^{2},31^{6}} \oplus$$
$$V^{\oplus 3}_{2^{2},2^{4}1} \oplus V^{\oplus 3}_{2^{2},2^{3}1^{3}} \oplus V^{\oplus 3}_{2^{2},2^{2}1^{5}} \oplus V_{2^{2},21^{7}} \oplus V_{2,421} \oplus V_{2,41^{3}} \oplus V^{\oplus 3}_{2,3^{2}1} \oplus V^{\oplus 7}_{2,32^{2}} \oplus V^{\oplus 10}_{2,321^{2}} \oplus V^{\oplus 8}_{2,31^{4}} \oplus V^{\oplus 11}_{2,2^{3}1} \oplus V^{\oplus 15}_{2,2^{2}1^{3}} \oplus$$
$$V^{\oplus 12}_{2,21^{5}} \oplus V^{\oplus 5}_{2,1^{7}} \oplus V_{1^{5},5^{2}} \oplus V_{1^{5},4^{2}1^{2}} \oplus V_{1^{5},3^{2}2^{2}} \oplus V_{1^{5},3^{2}1^{4}} \oplus V_{1^{5},2^{4}1^{2}} \oplus V_{1^{5},2^{2}1^{6}} \oplus V_{1^{5},1^{10}} \oplus V_{1^{4},54} \oplus V^{\oplus 2}_{1^{4},4^{2}1} \oplus$$
$$V^{\oplus 2}_{1^{4},431^{2}} \oplus V^{\oplus 2}_{1^{4},3^{2}21} \oplus V^{\oplus 3}_{1^{4},3^{2}1^{3}} \oplus V^{\oplus 2}_{1^{4},32^{3}} \oplus V_{1^{4},32^{2}1^{2}} \oplus V^{\oplus 2}_{1^{4},321^{4}} \oplus V^{\oplus 2}_{1^{4},2^{4}1} \oplus V^{\oplus 3}_{1^{4},2^{3}1^{3}} \oplus V^{\oplus 3}_{1^{4},2^{2}1^{5}} \oplus V^{\oplus 2}_{1^{4},21^{7}} \oplus$$
$$V^{\oplus 2}_{1^{4},1^{9}} \oplus V^{\oplus 3}_{1^{3},431} \oplus V_{1^{3},421^{2}} \oplus V^{\oplus 2}_{1^{3},4^{2}} \oplus V_{1^{3},3^{2}2} \oplus V^{\oplus 8}_{1^{3},3^{2}1^{2}} \oplus V^{\oplus 5}_{1^{3},32^{2}1} \oplus V^{\oplus 7}_{1^{3},321^{3}} \oplus V_{1^{3},31^{5}} \oplus V^{\oplus 7}_{1^{3},2^{3}1^{2}} \oplus V^{\oplus 11}_{1^{3},2^{2}1^{4}} \oplus$$
$$V^{\oplus 6}_{1^{3},21^{6}} \oplus V^{\oplus 5}_{1^{3},2^{4}} \oplus V^{\oplus 5}_{1^{3},1^{8}} \oplus V_{1^{2},43} \oplus V_{1^{2},421} \oplus V^{\oplus 6}_{1^{2},3^{2}1} \oplus V^{\oplus 3}_{1^{2},32^{2}} \oplus V^{\oplus 11}_{1^{2},321^{2}} \oplus V^{\oplus 4}_{1^{2},31^{4}} \oplus V^{\oplus 11}_{1^{2},2^{3}1} \oplus V^{\oplus 18}_{1^{2},2^{2}1^{3}} \oplus$$
$$V^{\oplus 13}_{1^{2},21^{5}} \oplus V^{\oplus 9}_{1^{2},1^{7}} \oplus V^{\oplus 5}_{1,321} \oplus V^{\oplus 3}_{1,31^{3}} \oplus V^{\oplus 2}_{1,3^{2}} \oplus V^{\oplus 15}_{1,2^{2}1^{2}} \oplus V^{\oplus 14}_{1,21^{4}} \oplus V^{\oplus 5}_{1,2^{3}} \oplus V^{\oplus 12}_{1,1^{6}} \oplus V_{\emptyset,32} \oplus V^{\oplus 4}_{\emptyset,2^{2}1} \oplus V^{\oplus 5}_{\emptyset,21^{3}} \oplus V^{\oplus 7}_{\emptyset,1^{5}}$$

In[57]:=
```
printPoly[V, x, y][chH_prime[2, x, y]] // TeXForm // CopyToClipboard
```

In[58]:=
```
Table[chH_prime[d, x, y] /. s[λ_, _] ⧴ hook[λ], {d, 1, 7}]
```

Out[58]=

{2, 12, 162, 4221, 182 106, 11 705 807, 1 046 227 328}

In[59]:=
```
TeXForm[
  Transpose[Table[{d, NumberForm[Length[Expand[chH_prime[d, x, y]]], DigitBlock → 3],
     NumberForm[chH_prime[d, x, y] /. _s → 1, DigitBlock → 3]}, {d, 1, 9}]]]
```

Out[59]//TeXForm=
```
\left(
\begin{array}{ccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 2 & 6 & 21 & 69 & 219 & 663 & \text{1,915} & \text{5,182} & \text{13,330} \\
 2 & 8 & 34 & 152 & 720 & \text{3,634} & \text{19,266} & \text{107,018} & \text{619,606
\end{array}
\right)
```

In[60]:=
```
Table[Length[Expand[chH_prime[d, x, y]]], {d, 1, 10}]
```

Out[60]=

{2, 6, 21, 69, 219, 663, 1915, 5182, 13 330, 32 876}