```
In[1]:= Get["Lambda`",
          Path → If[$MachineName == "epsilon", $HomeDirectory, NotebookDirectory[]]]
```

Partitions

```
In[2]:= tr[{5, 4, 4, 1}]
Out[2]= {4, 3, 3, 3, 1}

In[3]:= hook[{5, 4, 4, 1}]
Out[3]= 21021

In[4]:= z[{5, 4, 4, 1}]
Out[4]= 160

In[5]:= χ[{3, 2, 1}, {3, 3}]
Out[5]= -2
```

Projections

```
In[6]:= TableForm[Comap[{Identity, view[3]}] /@
           {m[{5, 4, 4, 2}, x], m[{1}, x], m[{2}, x], m[{2, 1}, x], m[{1, 1, 1}, x]},
          TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[6]//TableForm=

f	f in Λ3
$m_{\{5,4,4,2\}}(x)$	0
$m_{\{1\}}(x)$	$x_1 + x_2 + x_3$
$m_{\{2\}}(x)$	$x_1^2 + x_2^2 + x_3^2$
$m_{\{2,1\}}(x)$	$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$
$m_{\{1,1,1\}}(x)$	$x_1 x_2 x_3$

```
In[7]:= TableForm[Comap[{Identity, Expand @* view[3]}] /@
           {e[1, x], e[2, x], e[3, x], e[1, x] ** e[2, y]},
          TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[7]//TableForm=

f	f in Λ3
$e_1(x)$	$x_1 + x_2 + x_3$
$e_2(x)$	$x_1 x_2 + x_1 x_3 + x_2 x_3$
$e_3(x)$	$x_1 x_2 x_3$
$e_1(x)\ e_2(y)$	$x_1 y_1 y_2 + x_2 y_1 y_2 + x_3 y_1 y_2 + x_1 y_1 y_3 + x_2 y_1 y_3 + x_3 y_1 y_3 + x_1 y_2 y_3 + x_2 y_2 y_3 + x_3 y_2 y_3$

```
In[8]:= TableForm[Comap[{Identity, Expand @* view[3]}] /@
           {h[1, x], h[2, x], h[3, x], h[1, x] ** h[2, y]},
          TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[8]//TableForm=

f	f in Λ3
$h_1(x)$	$x_1 + x_2 + x_3$
$h_2(x)$	$x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2$
$h_3(x)$	$x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 + x_3^3$
$h_1(x)\ h_2(y)$	$x_1 y_1^2 + x_2 y_1^2 + x_3 y_1^2 + x_1 y_1 y_2 + x_2 y_1 y_2 + x_3 y_1 y_2 + x_1 y_2^2 + x_2 y_2^2 + x_3 y_2^2 + x_1 y_1 y_3 + x_2 y_1 y_3$

In[9]:=
```
TableForm[Comap[{Identity, Expand @* view[3]}] /@
  {p[0, x], p[1, x], p[2, x], p[3, x], p[1, x] * p[2, y]},
 TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[9]//TableForm=

f	f in Λ3
1	1
$p_1(x)$	$x_1 + x_2 + x_3$
$p_2(x)$	$x_1^2 + x_2^2 + x_3^2$
$p_3(x)$	$x_1^3 + x_2^3 + x_3^3$
$p_1(x)\, p_2(y)$	$x_1 y_1^2 + x_2 y_1^2 + x_3 y_1^2 + x_1 y_2^2 + x_2 y_2^2 + x_3 y_2^2 + x_1 y_3^2 + x_2 y_3^2 + x_3 y_3^2$

In[10]:=
```
Reduce[view[5][2 * h[2, x]] == view[5][p[1, x]^2 + p[2, x]]]
```

Out[10]=

True

In[11]:=
```
alt[{2}, x]
```

Out[11]=

x_1^2

In[12]:=
```
alt[{2, 1}, x]
```

Out[12]=

$x_1^2 x_2 - x_1 x_2^2$

In[13]:=
```
alt[{3, 2, 1}, x]
```

Out[13]=

$x_1^3 x_2^2 x_3 - x_1^2 x_2^3 x_3 - x_1^3 x_2 x_3^2 + x_1 x_2^3 x_3^2 + x_1^2 x_2 x_3^3 - x_1 x_2^2 x_3^3$

In[14]:=
```
TableForm[Comap[{Identity, Expand @* view[3]}] /@
  {s[{1}, x], s[{2}, x], s[{2, 1}, x]}, TableHeadings → {None, {"f", "f in Λ3"}}]
```

Out[14]//TableForm=

f	f in Λ3
$s_{\{1\}}(x)$	$x_1 + x_2 + x_3$
$s_{\{2\}}(x)$	$x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2$
$s_{\{2,1\}}(x)$	$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2 x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$

Sanity check

Check that $s_\lambda = \det\left(e_{\lambda'_i - i + j}\right)_{1 \le i, j \le l(\lambda')}$, as expected

In[15]:=
```
chk[λ_List, n_Integer] := With[{λtr = tr[λ]}, Reduce[
   view[n][Det[Table[e[λtr[[i]] - i + j, x], {i, Length[λtr]}, {j, Length[λtr]}]]] ==
  view[n][s[λ, x]]]]
```

In[16]:=
```
chk[{2, 1}, 4]
```

Out[16]=

True

In[17]:=
```
chk[{3, 2, 1}, 4]
```

Out[17]=

True

In[18]:=
```
chk[{4, 2, 1}, 5]
```

Out[18]=

True

Conversions

In[19]:=
```
TableForm[Comap[{Identity, conv[m]}] /@
  {e[5, x], h[2, x], h[3, x], h[4, x], p[5, x], e[2, x] * h[3, y]}]
```

Out[19]//TableForm=

$e_5(x)$	$m_{\{1,1,1,1,1\}}(x)$
$h_2(x)$	$m_{\{2\}}(x) + m_{\{1,1\}}(x)$
$h_3(x)$	$m_{\{3\}}(x) + m_{\{2,1\}}(x) + m_{\{1,1,1\}}(x)$
$h_4(x)$	$m_{\{4\}}(x) + m_{\{2,2\}}(x) + m_{\{3,1\}}(x) + m_{\{2,1,1\}}(x) + m_{\{1,1,1,1\}}(x)$
$p_5(x)$	$m_{\{5\}}(x)$
$e_2(x)\, h_3(y)$	$m_{\{1,1\}}(x)\, (m_{\{3\}}(y) + m_{\{2,1\}}(y) + m_{\{1,1,1\}}(y))$

In[20]:=
```
TableForm[Comap[{Identity, conv[e]}] /@
  {h[5, x], p[5, x], s[{2, 1, 1}, x], s[{1, 1, 1, 1, 1}, x]}]
```

Out[20]//TableForm=

$h_5(x)$	$e_1(x)^5 - 4 e_1(x)^3 e_2(x) + 3 e_1(x)\, e_2(x)^2 + 3 e_1(x)^2 e_3(x) - 2 e_2(x)\, e_3(x) - 2 e_1(x)$ (
$p_5(x)$	$e_1(x)^5 - 5 e_1(x)^3 e_2(x) + 5 e_1(x)\, e_2(x)^2 + 5 e_1(x)^2 e_3(x) - 5 e_2(x)\, e_3(x) - 5 e_1(x)$ (
$s_{\{2,1,1\}}(x)$	$e_1(x)\, e_3(x) - e_4(x)$
$s_{\{1,1,1,1,1\}}(x)$	$e_5(x)$

In[21]:=
```
TableForm[Comap[{Identity, conv[h]}] /@
  {e[2, x], e[3, x], e[4, x], p[2, x], p[3, x], p[4, x], s[{2, 1, 1}, x]}]
```

Out[21]//TableForm=

$e_2(x)$	$h_1(x)^2 - h_2(x)$
$e_3(x)$	$h_1(x)^3 - 2 h_1(x)\, h_2(x) + h_3(x)$
$e_4(x)$	$h_1(x)^4 - 3 h_1(x)^2 h_2(x) + h_2(x)^2 + 2 h_1(x)\, h_3(x) - h_4(x)$
$p_2(x)$	$-h_1(x)^2 + 2 h_2(x)$
$p_3(x)$	$h_1(x)^3 - 3 h_1(x)\, h_2(x) + 3 h_3(x)$
$p_4(x)$	$-h_1(x)^4 + 4 h_1(x)^2 h_2(x) - 2 h_2(x)^2 - 4 h_1(x)\, h_3(x) + 4 h_4(x)$
$s_{\{2,1,1\}}(x)$	$h_1(x)^2 h_2(x) - h_2(x)^2 - h_1(x)\, h_3(x) + h_4(x)$

In[22]:=
```
TableForm[Comap[{Identity, conv[p]}] /@ {e[2, x], e[3, x],
   e[4, x], e[5, x], h[2, x], h[3, x], h[4, x], h[5, x], s[{2, 1}, x]}]
```

Out[22]//TableForm=

$e_2(x)$	$\frac{1}{2}\left(p_1(x)^2 - p_2(x)\right)$
$e_3(x)$	$\frac{1}{6}\left(p_1(x)^3 - 3 p_1(x)\, p_2(x) + 2 p_3(x)\right)$
$e_4(x)$	$\frac{1}{24}\left(p_1(x)^4 - 6 p_1(x)^2 p_2(x) + 3 p_2(x)^2 + 8 p_1(x)\, p_3(x) - 6 p_4(x)\right)$
$e_5(x)$	$\frac{1}{120}\left(p_1(x)^5 - 10 p_1(x)^3 p_2(x) + 15 p_1(x)\, p_2(x)^2 + 20 p_1(x)^2 p_3(x) - 20 p_2(x)\, p_3(x) - 30\right.$ p
$h_2(x)$	$\frac{1}{2}\left(p_1(x)^2 + p_2(x)\right)$
$h_3(x)$	$\frac{1}{6}\left(p_1(x)^3 + 3 p_1(x)\, p_2(x) + 2 p_3(x)\right)$
$h_4(x)$	$\frac{1}{24}\left(p_1(x)^4 + 6 p_1(x)^2 p_2(x) + 3 p_2(x)^2 + 8 p_1(x)\, p_3(x) + 6 p_4(x)\right)$
$h_5(x)$	$\frac{1}{120}\left(p_1(x)^5 + 10 p_1(x)^3 p_2(x) + 15 p_1(x)\, p_2(x)^2 + 20 p_1(x)^2 p_3(x) + 20 p_2(x)\, p_3(x) + 30\right.$ p
$s_{\{2,1\}}(x)$	$\frac{p_1(x)^3}{3} - \frac{p_3(x)}{3}$

Sanity checks:

In[23]:=
```
And @@ Flatten@Table[
   Simplify[conv[t][conv[t2][t[10, x]]] == t[10, x]], {t, {e, h, p}}, {t2, {e, h, p}}]
```
Out[23]=

True

In[24]:=
```
Simplify[conv[p][Sum[hook[λ] * s[λ, x], {λ, IntegerPartitions[10]}]] == p[1, x]^10]
```
Out[24]=

True

Sanity checks

In[25]:=
```
With[{μ = {3, 2, 1}}, Simplify[view[6][p[μ, x]] ==
   view[6][Sum[χ[λ, μ] * s[λ, x], {λ, IntegerPartitions[Total[μ]]}]]]]
```
Out[25]=

True

Plethysm

In[26]:=
```
pl[p[3, x], p[5, x], x]
```
Out[26]=

$$p_{15}(x)$$

In[27]:=
```
Expand[pl[e[2, x], h[2, x], x]]
```
Out[27]=

$$\frac{p_1(x)^4}{8} + \frac{1}{4} p_1(x)^2 p_2(x) - \frac{p_2(x)^2}{8} - \frac{p_4(x)}{4}$$

In[28]:=
```
view[6][e[2, x]] /. Thread[Table[Subscript[x, i], {i, 6}] → List @@ view[3][h[2, x]]]
```
Out[28]=

$$x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^3 x_3 + 2 x_1^2 x_2 x_3 + 2 x_1 x_2^2 x_3 + x_2^3 x_3 + x_1^2 x_3^2 + 2 x_1 x_2 x_3^2 + x_2^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3$$

In[29]:=
```
Expand[view[3][pl[e[2, x], h[2, x], x]]]
```
Out[29]=

$$x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^3 x_3 + 2 x_1^2 x_2 x_3 + 2 x_1 x_2^2 x_3 + x_2^3 x_3 + x_1^2 x_3^2 + 2 x_1 x_2 x_3^2 + x_2^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3$$

In[30]:=
```
pl[p[3, x], {p[4, y], f}, x, y]
```
Out[30]=

$$p_{12}(y)$$

In[31]:=
```
pl[p[3, x], {p[4, x], h[2, y]}, x, y]
```
Out[31]=

$$p_{12}(x)$$

In[32]:=
```
pl[p[2, x] × p[3, y], {p[4, x], h[2, y]}, x, y]
```
Out[32]=

$$\frac{1}{2} \left(p_3(y)^2 + p_6(y)\right) p_8(x)$$

In[33]:=
```
FullSimplify[pl[h[2, x] * e[3, y], {ℏ^2 * e[2, x] * e[1, y], ℏ * h[2, y]}, x, y]]
```
Out[33]=

$$\frac{1}{384} \hbar^7 \left(p_1(y)^2 \left(p_1(x)^2 - p_2(x)\right)^2 + 2 p_2(y) \left(p_2(x)^2 - p_4(x)\right)\right) \left(\left(p_1(y)^2 + p_2(y)\right)^3 - 6 \left(p_1(y)^2 + p_2(y)\right) \left(p_2(y)^2 + p_4(y)\right) + 8 \left(p_3(y)^2 + p_6(y)\right)\right)$$

Orthogonality

Character of the regular representation, in two variables

In[34]:=
```
Table[reg2[n, x, y], {n, 4}] // Column
```

Out[34]=

$$p_1(x)\,p_1(y)$$
$$\frac{1}{2}p_1(x)^2 p_1(y)^2 + \frac{p_2(x)\,p_2(y)}{2}$$
$$\frac{1}{6}p_1(x)^3 p_1(y)^3 + \frac{1}{2}p_1(x)\,p_1(y)\,p_2(x)\,p_2(y) + \frac{p_3(x)\,p_3(y)}{3}$$
$$\frac{1}{24}p_1(x)^4 p_1(y)^4 + \frac{1}{4}p_1(x)^2 p_1(y)^2 p_2(x)\,p_2(y) + \frac{1}{8}p_2(x)^2 p_2(y)^2 + \frac{1}{3}p_1(x)\,p_1(y)\,p_3(x)\,p_3(y) + \frac{p_4(x)\,p_4(y)}{4}$$

In[35]:=
```
Table[conv[p][reg[n, x, y]] == reg2[n, x, y], {n, 5}] // Reduce
```

Out[35]=

True

In[36]:=
```
Table[conv[s][reg2[n, x, y]] == reg[n, x, y], {n, 5}] // Reduce
```

Out[36]=

True

Computation from 11/03/2025

In[37]:=
```
pl[h[2, x], {p[1, x]^3 × p[1, y], 0}, x, y] × h[2, z]
```

Out[37]=

$$\frac{1}{2}h_2(z)\left(p_1(x)^6 p_1(y)^2 + p_2(x)^3 p_2(y)\right)$$

In[38]:=
```
reg[2, y, z]
```

Out[38]=

$$s_{\{2\}}(y)\,s_{\{2\}}(z) + s_{\{1,1\}}(y)\,s_{\{1,1\}}(z)$$

In[39]:=
```
adj[reg[2, y, z], 37]
```

Out[39]=

$$\frac{p_2(x)^3}{2} + \frac{1}{2}\left(\frac{p_1(x)^6}{2} - \frac{p_2(x)^3}{2}\right) + \frac{1}{2}\left(\frac{p_1(x)^6}{2} + \frac{p_2(x)^3}{2}\right)$$

In[40]:=
```
Simplify[39]
```

Out[40]=

$$\frac{1}{2}\left(p_1(x)^6 + p_2(x)^3\right)$$

In[41]:=
```
pl[h[2, x], p[1, x]^3, x] == 40
```

Out[41]=

True

Sanity check

Check that $h_n(x) \circ h_1(x)\,h_1(y) = \sum_{\lambda \vdash n} s_\lambda(x)\,s_\lambda(y)$ in two different ways.

In[42]:=
```
List @@ Expand[view[5][h[1, x] × h[1, y]]]
```
Out[42]=

$\{x_1 y_1, x_2 y_1, x_3 y_1, x_4 y_1, x_5 y_1, x_1 y_2, x_2 y_2, x_3 y_2, x_4 y_2, x_5 y_2, x_1 y_3, x_2 y_3,$
$x_3 y_3, x_4 y_3, x_5 y_3, x_1 y_4, x_2 y_4, x_3 y_4, x_4 y_4, x_5 y_4, x_1 y_5, x_2 y_5, x_3 y_5, x_4 y_5, x_5 y_5\}$

In[43]:=
```
rules = Table[Subscript[z, i] → %42[[i]], {i, Length[%42]}]
```
Out[43]=

$\{z_1 \to x_1 y_1, z_2 \to x_2 y_1, z_3 \to x_3 y_1, z_4 \to x_4 y_1, z_5 \to x_5 y_1, z_6 \to x_1 y_2,$
$z_7 \to x_2 y_2, z_8 \to x_3 y_2, z_9 \to x_4 y_2, z_{10} \to x_5 y_2, z_{11} \to x_1 y_3, z_{12} \to x_2 y_3, z_{13} \to x_3 y_3,$
$z_{14} \to x_4 y_3, z_{15} \to x_5 y_3, z_{16} \to x_1 y_4, z_{17} \to x_2 y_4, z_{18} \to x_3 y_4, z_{19} \to x_4 y_4,$
$z_{20} \to x_5 y_4, z_{21} \to x_1 y_5, z_{22} \to x_2 y_5, z_{23} \to x_3 y_5, z_{24} \to x_4 y_5, z_{25} \to x_5 y_5\}$

In[44]:=
```
With[{n = 1},
 Simplify[view[5][reg[n, x, y]] == (view[Length[%43]][h[n, z]] /. %43)]]
```
Out[44]=

True

With the actual bisymmetric plethysm:

In[45]:=
```
Reduce[Table[pl[h[n, x], {h[1, x] * h[1, y], g}, x, y] == reg2[n, x, y], {n, 6}]]
```
Out[45]=

True

Computations

Stable twisted cohomology of automorphism groups of free groups

We have: $Q(q, p) = \begin{cases} \mathrm{triv}_p, & q = 0 \mid 1, p \geq 1; \\ 0, & \text{otherwise.} \end{cases}$

Moreover $\mathcal{H} = \omega(\mathcal{P}) = \omega(\mathcal{F}(S(Q)))$ is the involution applied to the saturation of Q.
Unlike the Albanese cohomology, this is infinite dimensional in any fixed degree (because $Q(1, 0)$ contains an element of degree 0), so we can only compute the truncation up to some arity.

In[46]:=
```
TableForm[Table[{chQ[d, x, y]}, {d, 0, 6}],
 TableHeadings → {Range[0, 6], {"ch(Q) in degree d"}}]
```
Out[46]//TableForm=

	ch(Q) in degree d
0	$e_1(x)\, h_1(y)$
1	$h_1(y) + e_1(x)\, h_2(y)$
2	$h_2(y) + e_1(x)\, h_3(y)$
3	$h_3(y) + e_1(x)\, h_4(y)$
4	$h_4(y) + e_1(x)\, h_5(y)$
5	$h_5(y) + e_1(x)\, h_6(y)$
6	$h_6(y) + e_1(x)\, h_7(y)$

In[47]:=
```
chH_upto[2, 5, x, y] // printPoly[S, x, y]
```
Out[47]=

$S_{3,41}^{\oplus 3} \oplus S_{3,32} \oplus S_{3,31^2}^{\oplus 6} \oplus S_{3,2^21}^{\oplus 2} \oplus S_{3,21^3}^{\oplus 3} \oplus S_{21,41} \oplus S_{21,32}^{\oplus 5} \oplus S_{21,31^2}^{\oplus 7} \oplus S_{21,2^21}^{\oplus 8} \oplus S_{21,21^3}^{\oplus 9} \oplus S_{21,1^5}^{\oplus 3} \oplus S_{2,31}^{\oplus 3} \oplus S_{2,21^2}^{\oplus 6} \oplus S_{2,2^2} \oplus$
$S_{2,1^4}^{\oplus 2} \oplus S_{1^3,32}^{\oplus 2} \oplus S_{1^3,31^2} \oplus S_{1^3,2^21}^{\oplus 6} \oplus S_{1^3,21^3}^{\oplus 6} \oplus S_{1^3,1^5}^{\oplus 5} \oplus S_{1^2,31} \oplus S_{1^2,21^2}^{\oplus 5} \oplus S_{1^2,2^2}^{\oplus 4} \oplus S_{1^2,1^4}^{\oplus 5} \oplus S_{1,21}^{\oplus 3} \oplus S_{1,1^3}^{\oplus 4} \oplus S_{\emptyset,1^2}^{\oplus 2}$

The characters of $\mathcal{P}$ and $\mathcal{H}$ in degree d, output arity m, input arity n

In[48]:=
```
TableForm[
 Map[printPoly[S, x, y], Table[chH[2, m, n, x, y], {m, 0, 5}, {n, 0, 5}], {2}],
 TableHeadings → {Range[0, 5], Range[0, 5]}]
```

Out[48]//TableForm=

	0	1	2	3	4	5
0	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	0
1	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	$S^{\oplus 3}_{1,21} \oplus S^{\oplus 4}_{1,1^3}$	0	0
2	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	$S^{\oplus 3}_{2,31} \oplus S^{\oplus 6}_{2,21^2} \oplus S_{2,2^2} \oplus S^{\oplus 2}_{2,1^4} \oplus S_{1^2,31} \oplus S^{\oplus 5}_{1^2,21^2} \oplus S^{\oplus 4}_{1^2,2^2} \oplus S^{\oplus 5}_{1^2,1^4}$	0
3	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	$S^{\oplus 3}_{3,}$
4	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	0
5	0	0	$S^{\oplus 2}_{\emptyset,1^2}$	0	0	0

Albanese cohomology for IA_∞

We have: $\mathcal{Q}'(q,p) = \begin{cases} \mathrm{triv}_p[p] & q = 0,\ p \geq 1 \\ \mathrm{triv}_p[p-1] & q = 1,\ p \geq 2 \\ 0 & \text{otherwise.} \end{cases}$

Moreover $\mathcal{H}' = \omega(\mathcal{P}') = \omega(S(\mathcal{Q}'))$ is the involution applied to the saturation of $\mathcal{Q}'$.

The following function gives the character of $\mathcal{Q}'$ in homological degree d:

In[49]:=
```
TableForm[Table[{chQ_prime[d, x, y]}, {d, 0, 6}],
 TableHeadings → {Range[0, 6], {"ch(𝒬') in degree d"}}]
```

Out[49]//TableForm=

	ch($\mathcal{Q}'$) in degree d
0	0
1	$h_1(y) + h_1(x)\, h_2(y)$
2	$h_2(y) + h_1(x)\, h_3(y)$
3	$h_3(y) + h_1(x)\, h_4(y)$
4	$h_4(y) + h_1(x)\, h_5(y)$
5	$h_5(y) + h_1(x)\, h_6(y)$
6	$h_6(y) + h_1(x)\, h_7(y)$

Now, the saturation $\mathcal{P}' = S(\mathcal{Q}')$ has character $\sum_{n\geq 1} h_n(x) \circ (\mathrm{ch}(\mathcal{Q}'), 0)$. The first function gives $\mathrm{ch}(\mathcal{P}')$ up to degree d, the second in degree exactly d (without the $(-\hbar)^d$)

In[50]:=
```
TableForm[Table[{d, printPoly[S, x, y][chP_prime[d, x, y]]}, {d, 3}],
 TableHeadings → {None, {"d", "ch(𝒫') in degree d"}}]
```

Out[50]//TableForm=

d	ch($\mathcal{P}'$) in degree d
1	$S_{1,2} \oplus S_{\emptyset,1}$
2	$S_{2,4} \oplus S_{2,2^2} \oplus S_{1^2,31} \oplus S^{\oplus 2}_{1,3} \oplus S_{1,21} \oplus S^{\oplus 2}_{\emptyset,2}$
3	$S_{3,6} \oplus S_{3,42} \oplus S_{3,2^3} \oplus S_{21,51} \oplus S_{21,42} \oplus S_{21,321} \oplus S^{\oplus 2}_{2,5} \oplus S^{\oplus 2}_{2,41} \oplus S^{\oplus 2}_{2,32} \oplus S_{2,2^21} \oplus S_{1^3,41^2} \oplus S_{1^3,3^2} \oplus S_{1^2,5} \oplus S^{\oplus 2}_{1^2,41} \oplus S$

Calculations

Now we can find the decomposition of $\mathrm{ch}(\mathcal{H}')$ in some degree:

In[52]:=
```
Get[FileNameJoin[NotebookDirectory[], "chH_prime.mx"]]
```

In[53]:=
```
printPoly[V, x, y][chH_prime[1, x, y]]
```

Out[53]=

$V_{1,1^2} \oplus V_{\emptyset,1}$

In[54]:=
```
printPoly[V, x, y][chH_prime[2, x, y]]
```
Out[54]=

$$V_{2,21^2} \oplus V_{1^2,2^2} \oplus V_{1^2,1^4} \oplus V_{1,21} \oplus V^{\oplus 2}_{1,1^3} \oplus V^{\oplus 2}_{\emptyset,1^2}$$

In[55]:=
```
printPoly[V, x, y][chH_prime[3, x, y]]
```
Out[55]=

$$V_{3,31^3} \oplus V_{3,2^3} \oplus V_{21,321} \oplus V_{21,2^21^2} \oplus V_{21,21^4} \oplus V_{2,31^2} \oplus V^{\oplus 2}_{2,2^21} \oplus V^{\oplus 2}_{2,21^3} \oplus V_{2,1^5} \oplus V_{1^3,3^2} \oplus$$
$$V_{1^3,2^21^2} \oplus V_{1^3,1^6} \oplus V_{1^2,32} \oplus V^{\oplus 2}_{1^2,2^21} \oplus V^{\oplus 2}_{1^2,21^3} \oplus V^{\oplus 2}_{1^2,1^5} \oplus V^{\oplus 3}_{1,21^2} \oplus V^{\oplus 2}_{1,2^2} \oplus V^{\oplus 4}_{1,1^4} \oplus V_{\emptyset,21} \oplus V^{\oplus 3}_{\emptyset,1^3}$$

In[56]:=
```
printPoly[V, x, y][chH_prime[4, x, y]]
```
Out[56]=

$$V_{4,41^4} \oplus V_{4,32^21} \oplus V_{31,421^2} \oplus V_{31,3^22} \oplus V_{31,32^21} \oplus V_{31,321^3} \oplus V_{31,31^5} \oplus V_{31,2^31^2} \oplus V_{3,41^3} \oplus V^{\oplus 2}_{3,32^2} \oplus V^{\oplus 2}_{3,321^2} \oplus V^{\oplus 2}_{3,31^4} \oplus$$
$$V^{\oplus 2}_{3,2^31} \oplus V_{3,2^21^3} \oplus V_{3,21^5} \oplus V_{21^2,431} \oplus V_{21^2,3^21^2} \oplus V_{21^2,32^21} \oplus V_{21^2,321^3} \oplus V_{21^2,2^31^2} \oplus V_{21^2,2^21^4} \oplus V_{21^2,21^6} \oplus$$
$$V_{21,421} \oplus V^{\oplus 2}_{21,3^21} \oplus V^{\oplus 2}_{21,32^2} \oplus V^{\oplus 4}_{21,321^2} \oplus V^{\oplus 2}_{21,31^4} \oplus V^{\oplus 3}_{21,2^31} \oplus V^{\oplus 5}_{21,2^21^3} \oplus V^{\oplus 3}_{21,21^5} \oplus V_{21,1^7} \oplus V_{2^2,42^2} \oplus V_{2^2,3^21^2} \oplus$$
$$V_{2^2,321^3} \oplus V_{2^2,2^21^4} \oplus V_{2^2,2^4} \oplus V^{\oplus 3}_{2,321} \oplus V^{\oplus 3}_{2,31^3} \oplus V^{\oplus 5}_{2,2^21^2} \oplus V^{\oplus 6}_{2,21^4} \oplus V^{\oplus 4}_{2,2^3} \oplus V^{\oplus 2}_{2,1^6} \oplus V_{1^4,4^2} \oplus V_{1^4,3^21^2} \oplus V_{1^4,2^21^4} \oplus$$
$$V_{1^4,2^4} \oplus V_{1^4,1^8} \oplus V_{1^3,43} \oplus V^{\oplus 2}_{1^3,3^21} \oplus V^{\oplus 2}_{1^3,321^2} \oplus V^{\oplus 2}_{1^3,2^31} \oplus V^{\oplus 3}_{1^3,2^21^3} \oplus V^{\oplus 2}_{1^3,21^5} \oplus V^{\oplus 2}_{1^3,1^7} \oplus V^{\oplus 3}_{1^2,321} \oplus V_{1^2,31^3} \oplus$$
$$V^{\oplus 2}_{1^2,3^2} \oplus V^{\oplus 8}_{1^2,2^21^2} \oplus V^{\oplus 5}_{1^2,21^4} \oplus V_{1^2,2^3} \oplus V^{\oplus 5}_{1^2,1^6} \oplus V_{1,32} \oplus V_{1,31^2} \oplus V^{\oplus 6}_{1,2^21} \oplus V^{\oplus 7}_{1,21^3} \oplus V^{\oplus 7}_{1,1^5} \oplus V^{\oplus 2}_{\emptyset,21^2} \oplus V^{\oplus 2}_{\emptyset,2^2} \oplus V^{\oplus 5}_{\emptyset,1^4}$$

In[57]:=
```
printPoly[V, x, y][chH_prime[5, x, y]]
```
Out[57]=

$$V_{5,51^5} \oplus V_{5,42^21^2} \oplus V_{5,3^22^2} \oplus V_{41,521^3} \oplus V_{41,4321} \oplus V_{41,42^3} \oplus V_{41,42^21^2} \oplus V_{41,421^4} \oplus V_{41,41^6} \oplus V_{41,3^31} \oplus V_{41,3^221^2} \oplus$$
$$V_{41,32^31} \oplus V_{41,32^21^3} \oplus V_{4,51^4} \oplus V^{\oplus 2}_{4,42^21} \oplus V^{\oplus 2}_{4,421^3} \oplus V^{\oplus 2}_{4,41^5} \oplus V^{\oplus 2}_{4,3^221} \oplus V^{\oplus 2}_{4,32^3} \oplus V^{\oplus 3}_{4,32^21^2} \oplus V_{4,321^4} \oplus V_{4,31^6} \oplus$$
$$V_{4,3^3} \oplus V_{4,2^31^3} \oplus V_{32,52^21} \oplus V_{32,43^2} \oplus V_{32,4321} \oplus V_{32,431^3} \oplus V_{32,42^21^2} \oplus V_{32,421^4} \oplus V_{32,3^22^2} \oplus V_{32,3^221^2} \oplus$$
$$V_{32,3^21^4} \oplus V_{32,32^31} \oplus V_{32,32^21^3} \oplus V_{32,321^5} \oplus V_{32,2^41^2} \oplus V_{31^2,531^2} \oplus V_{31^2,4^22} \oplus V_{31^2,4321} \oplus V_{31^2,431^3} \oplus V_{31^2,42^3} \oplus$$
$$V_{31^2,42^21^2} \oplus V_{31^2,421^4} \oplus V_{31^2,3^31} \oplus V^{\oplus 2}_{31^2,3^221^2} \oplus V_{31^2,32^31} \oplus V^{\oplus 2}_{31^2,32^21^3} \oplus V_{31^2,321^5} \oplus V_{31^2,31^7} \oplus V_{31^2,2^31^4} \oplus$$
$$V_{31^2,2^5} \oplus V_{31,521^2} \oplus V^{\oplus 2}_{31,432} \oplus V^{\oplus 2}_{31,431^2} \oplus V^{\oplus 4}_{31,42^21} \oplus V^{\oplus 4}_{31,421^3} \oplus V^{\oplus 2}_{31,41^5} \oplus V^{\oplus 5}_{31,3^221} \oplus V^{\oplus 3}_{31,3^21^3} \oplus V^{\oplus 3}_{31,32^3} \oplus$$
$$V^{\oplus 8}_{31,32^21^2} \oplus V^{\oplus 6}_{31,321^4} \oplus V^{\oplus 3}_{31,31^6} \oplus V^{\oplus 2}_{31,3^3} \oplus V^{\oplus 3}_{31,2^41} \oplus V^{\oplus 4}_{31,2^31^3} \oplus V^{\oplus 2}_{31,2^21^5} \oplus V_{31,21^7} \oplus V_{3,42^2} \oplus V^{\oplus 3}_{3,421^2} \oplus V^{\oplus 3}_{3,41^4} \oplus$$
$$V^{\oplus 4}_{3,3^22} \oplus V_{3,3^21^2} \oplus V^{\oplus 9}_{3,32^21} \oplus V^{\oplus 7}_{3,321^3} \oplus V^{\oplus 6}_{3,31^5} \oplus V^{\oplus 7}_{3,2^31^2} \oplus V^{\oplus 4}_{3,2^21^4} \oplus V^{\oplus 3}_{3,21^6} \oplus V_{3,2^4} \oplus V_{2^21,532} \oplus V_{2^21,4^21^2} \oplus$$
$$V_{2^21,4321} \oplus V_{2^21,431^3} \oplus V_{2^21,42^21^2} \oplus V_{2^21,3^22^2} \oplus V_{2^21,3^221^2} \oplus V^{\oplus 2}_{2^21,3^21^4} \oplus V_{2^21,32^31} \oplus V_{2^21,32^21^3} \oplus V_{2^21,321^5} \oplus$$
$$V_{2^21,2^41^2} \oplus V_{2^21,2^31^4} \oplus V_{2^21,2^21^6} \oplus V_{21^3,541} \oplus V_{21^3,4^21^2} \oplus V_{21^3,4321} \oplus V_{21^3,431^3} \oplus V_{21^3,3^22^2} \oplus V_{21^3,3^221^2} \oplus$$
$$V_{21^3,3^21^4} \oplus V_{21^3,32^31} \oplus V_{21^3,32^21^3} \oplus V_{21^3,321^5} \oplus V_{21^3,2^41^2} \oplus V_{21^3,2^31^4} \oplus V_{21^3,2^21^6} \oplus V_{21^3,21^8} \oplus V_{21^2,531} \oplus V^{\oplus 2}_{21^2,4^21} \oplus$$
$$V^{\oplus 2}_{21^2,432} \oplus V^{\oplus 4}_{21^2,431^2} \oplus V^{\oplus 2}_{21^2,42^21} \oplus V^{\oplus 2}_{21^2,421^3} \oplus V^{\oplus 5}_{21^2,3^221} \oplus V^{\oplus 6}_{21^2,3^21^3} \oplus V^{\oplus 3}_{21^2,32^3} \oplus V^{\oplus 7}_{21^2,32^21^2} \oplus V^{\oplus 6}_{21^2,321^4} \oplus$$
$$V^{\oplus 2}_{21^2,31^6} \oplus V^{\oplus 4}_{21^2,2^41} \oplus V^{\oplus 6}_{21^2,2^31^3} \oplus V^{\oplus 6}_{21^2,2^21^5} \oplus V^{\oplus 3}_{21^2,21^7} \oplus V_{21^2,1^9} \oplus V^{\oplus 3}_{21,431} \oplus V^{\oplus 3}_{21,42^2} \oplus V^{\oplus 4}_{21,421^2} \oplus V_{21,41^4} \oplus$$
$$V^{\oplus 5}_{21,3^22} \oplus V^{\oplus 9}_{21,3^21^2} \oplus V^{\oplus 12}_{21,32^21} \oplus V^{\oplus 16}_{21,321^3} \oplus V^{\oplus 7}_{21,31^5} \oplus V^{\oplus 14}_{21,2^31^2} \oplus V^{\oplus 15}_{21,2^21^4} \oplus V^{\oplus 9}_{21,21^6} \oplus V^{\oplus 6}_{21,2^4} \oplus V^{\oplus 3}_{21,1^8} \oplus V_{2^2,52^2} \oplus$$
$$V^{\oplus 2}_{2^2,432} \oplus V^{\oplus 2}_{2^2,431^2} \oplus V^{\oplus 2}_{2^2,42^21} \oplus V^{\oplus 2}_{2^2,421^3} \oplus V^{\oplus 3}_{2^2,3^221} \oplus V^{\oplus 4}_{2^2,3^21^3} \oplus V^{\oplus 2}_{2^2,32^3} \oplus V^{\oplus 4}_{2^2,32^21^2} \oplus V^{\oplus 5}_{2^2,321^4} \oplus V_{2^2,31^6} \oplus$$
$$V^{\oplus 3}_{2^2,2^41} \oplus V^{\oplus 3}_{2^2,2^31^3} \oplus V^{\oplus 3}_{2^2,2^21^5} \oplus V_{2^2,21^7} \oplus V_{2,421} \oplus V_{2,41^3} \oplus V^{\oplus 3}_{2,3^21} \oplus V^{\oplus 7}_{2,32^2} \oplus V^{\oplus 10}_{2,321^2} \oplus V^{\oplus 8}_{2,31^4} \oplus V^{\oplus 11}_{2,2^31} \oplus V^{\oplus 15}_{2,2^21^3} \oplus$$
$$V^{\oplus 12}_{2,21^5} \oplus V^{\oplus 5}_{2,1^7} \oplus V_{1^5,5^2} \oplus V_{1^5,4^21^2} \oplus V_{1^5,3^22^2} \oplus V_{1^5,3^21^4} \oplus V_{1^5,2^41^2} \oplus V_{1^5,2^21^6} \oplus V_{1^5,1^{10}} \oplus V_{1^4,54} \oplus V^{\oplus 2}_{1^4,4^21} \oplus$$
$$V^{\oplus 2}_{1^4,431^2} \oplus V^{\oplus 2}_{1^4,3^221} \oplus V^{\oplus 3}_{1^4,3^21^3} \oplus V^{\oplus 2}_{1^4,32^3} \oplus V_{1^4,32^21^2} \oplus V^{\oplus 2}_{1^4,321^4} \oplus V^{\oplus 2}_{1^4,2^41} \oplus V^{\oplus 3}_{1^4,2^31^3} \oplus V^{\oplus 3}_{1^4,2^21^5} \oplus V^{\oplus 2}_{1^4,21^7} \oplus$$
$$V^{\oplus 2}_{1^4,1^9} \oplus V^{\oplus 3}_{1^3,431} \oplus V_{1^3,421^2} \oplus V^{\oplus 2}_{1^3,4^2} \oplus V_{1^3,3^22} \oplus V^{\oplus 8}_{1^3,3^21^2} \oplus V^{\oplus 5}_{1^3,32^21} \oplus V^{\oplus 7}_{1^3,321^3} \oplus V_{1^3,31^5} \oplus V^{\oplus 7}_{1^3,2^31^2} \oplus V^{\oplus 11}_{1^3,2^21^4} \oplus$$
$$V^{\oplus 6}_{1^3,21^6} \oplus V^{\oplus 5}_{1^3,2^4} \oplus V^{\oplus 5}_{1^3,1^8} \oplus V_{1^2,43} \oplus V_{1^2,421} \oplus V^{\oplus 6}_{1^2,3^21} \oplus V^{\oplus 3}_{1^2,32^2} \oplus V^{\oplus 11}_{1^2,321^2} \oplus V^{\oplus 4}_{1^2,31^4} \oplus V^{\oplus 11}_{1^2,2^31} \oplus V^{\oplus 18}_{1^2,2^21^3} \oplus$$
$$V^{\oplus 13}_{1^2,21^5} \oplus V^{\oplus 9}_{1^2,1^7} \oplus V^{\oplus 5}_{1,321} \oplus V^{\oplus 3}_{1,31^3} \oplus V^{\oplus 2}_{1,3^2} \oplus V^{\oplus 15}_{1,2^21^2} \oplus V^{\oplus 14}_{1,21^4} \oplus V^{\oplus 5}_{1,2^3} \oplus V^{\oplus 12}_{1,1^6} \oplus V_{\emptyset,32} \oplus V^{\oplus 4}_{\emptyset,2^21} \oplus V^{\oplus 5}_{\emptyset,21^3} \oplus V^{\oplus 7}_{\emptyset,1^5}$$

In[58]:=
```
printPoly[V, x, y][chH_prime[2, x, y]] // TeXForm // CopyToClipboard
```

In[59]:=
```
Table[chH_prime[d, x, y] /. s[λ_, _] :→ hook[λ], {d, 1, 7}]
```
Out[59]=

```
{2, 12, 162, 4221, 182106, 11705807, 1046227328}
```

In[60]:=
```
TeXForm[
 Transpose[Table[{d, NumberForm[Length[Expand[chH_prime[d, x, y]]], DigitBlock → 3],
    NumberForm[chH_prime[d, x, y] /. _s → 1, DigitBlock → 3]}, {d, 1, 9}]]]
```

Out[60]//TeXForm=
```
\left(
\begin{array}{ccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 2 & 6 & 21 & 69 & 219 & 663 & \text{1,915} & \text{5,182} & \text{13,330} \\
 2 & 8 & 34 & 152 & 720 & \text{3,634} & \text{19,266} & \text{107,018} & \text{619,606
\end{array}
\right)
```

In[61]:=
```
Table[Length[Expand[chH_prime[d, x, y]]], {d, 1, 10}]
```

Out[61]=
```
{2, 6, 21, 69, 219, 663, 1915, 5182, 13 330, 32 876}
```