The Lambrechts–Stanley Model of Configuration Spaces

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Young Topologists Meeting 2017, Stockholm

What is this all about?

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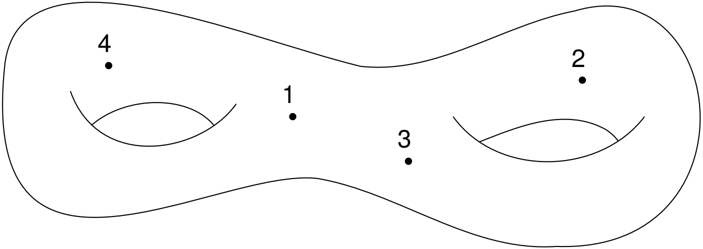
$$\mathrm{Conf}_k(M) := \{(x_1, \dots, x_k) \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}$$



The Great Question I

Fact: homeo/diffeomorphism invariance

$M \cong N \implies \mathrm{Conf}_k(M) \cong \mathrm{Conf}_k(N).$

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Answer

☹ No!

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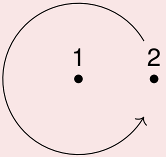
Question$_1$: homotopy invariance?

$$M \simeq N \overset{?}{\Rightarrow} \mathrm{Conf}_k(M) \simeq \mathrm{Conf}_k(N)?$$

Answer

☹ No! For example:

$\mathbb{R}^n \simeq \mathbb{R}^m$ but $\mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1} \not\simeq S^{m-1}$.



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In the counterexample: open manifolds of different dimensions

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😵 Still no!

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Answer

😵 Still no!

Theorem (Longoni & Salvatore, 2005)

$L_{7,1} \simeq L_{7,2}$ but $\mathrm{Conf}_2(L_{7,1}) \not\simeq \mathrm{Conf}_2(L_{7,2})$ (where $L_{p,q} = S^3/(\mathbb{Z}/p\mathbb{Z})$ are lens spaces)

The Great Question III

Still hope: the counterexample isn't simply connected $\pi_1(L_{p,q}) = \mathbb{Z}/p\mathbb{Z}$

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For M, N closed and simply connected, $M \simeq N \overset{?}{\Rightarrow} \mathrm{Conf}_k(M) \simeq \mathrm{Conf}_k(N)$

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For M, N closed and simply connected, $M \simeq N \overset{?}{\Rightarrow} \mathrm{Conf}_k(M) \simeq \mathrm{Conf}_k(N)$

Answer

Nobody knows...

Crash course: rational homotopy theory

Whole homotopy type $\rightarrow$ too much; work over $\mathbb{Q}$ / $\mathbb{R}$ instead

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Whole homotopy type $\to$ too much; work over $\mathbb{Q}$ / $\mathbb{R}$ instead

$M \simeq_{\mathbb{Q}} N$ if $M \xleftarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} N$ where $H_*(f_i; \mathbb{Q})$ is an iso ($\iff$ $\pi_*(f) \otimes \mathbb{Q}$ iso)

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$\implies$ for simply connected spaces, completely encoded in "rational model":

$A \simeq \Omega^*(M)$ "forms on M" (e.g. de Rham, piecewise polynomial...)

where A is an "explicit" CDGA (= Commutative Differential Graded Algebra)

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Question $\otimes$ $\mathbb{Q}$

[Same hypotheses] Does $M \simeq_{\mathbb{Q}} N \overset{?}{\Rightarrow} \mathrm{Conf}_k(M) \simeq_{\mathbb{Q}} \mathrm{Conf}_k(N)$?

Some previous results

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Remark

Similar results for unordered configuration spaces $B_k(M) = \mathrm{Conf}_k(M)/\Sigma_k$

A first version of the theorem

Theorem (I. 2016)

For M and N smooth, simply connected, closed manifolds of dimension at least 4:

$$M \simeq_{\mathbb{R}} N \implies \mathrm{Conf}_k(M) \simeq_{\mathbb{R}} \mathrm{Conf}_k(N).$$

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In fact: explicit (real) model $\mathtt{G}_A(k)$, conjectured by Lambrechts & Stanley.

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Remark

In $\dim \leq 3$, this is clear ($n = 1$: no manifolds; $n = 2$: only S^2; $n = 3$: only S^3 by Poincaré conjecture.)

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- 2015 [Cordova Bulens] $\mathtt{G}_A(2)$ model of $\mathrm{Conf}_2(M)$ for $\dim M = 2m$

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Idea

Study all of $\{\mathrm{Conf}_k(M)\}_{k\geq 0}$ at once: more structure!

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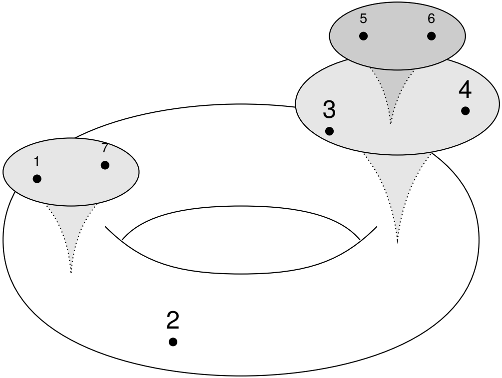
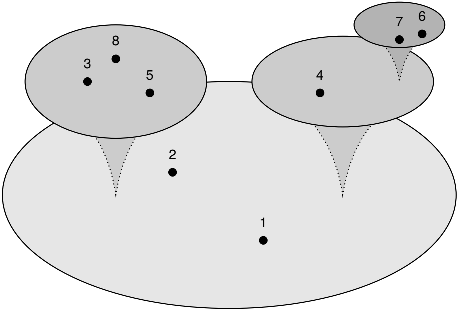
Fulton–MacPherson compactification $\mathrm{Conf}_k(M) \xrightarrow{\sim} \mathtt{FM}_M(k)$



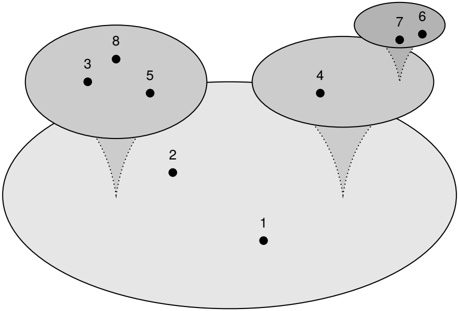
What about Euclidean spaces?

Can also compactify $\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\sim} \mathtt{FM}_{\mathbb{R}^n}(k)$



What about Euclidean spaces?

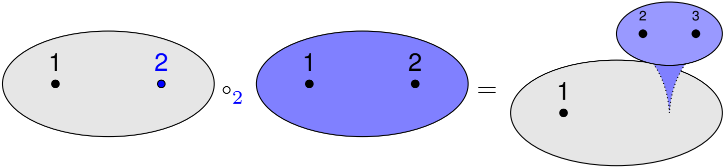
Can also compactify $\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\sim} \mathrm{Conf}_k(\mathbb{R}^n)/\mathrm{Aff}(\mathbb{R}^n) \overset{\sim}{\hookrightarrow} \mathtt{FM}_{\mathbb{R}^n}(k)$



(+ normalization to deal with $\mathbb{R}^n$ being noncompact)

Operads

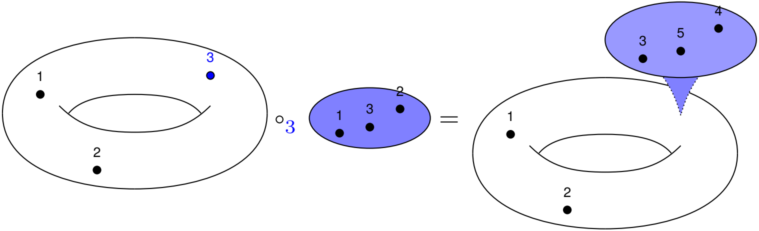
$\mathtt{FM}_{\mathbb{R}^n} = \{\mathtt{FM}_{\mathbb{R}^n}(k)\}_{k\geq 0}$ is an operad: we can insert an infinitesimal configuration into another



$$\mathtt{FM}_{\mathbb{R}^n}(k) \times \mathtt{FM}_{\mathbb{R}^n}(l) \xrightarrow{\circ_i} \mathtt{FM}_{\mathbb{R}^n}(k+l-1), \quad 1 \leq i \leq k$$

Modules over operads

M framed $\implies$ $\mathtt{FM}_M = \{\mathtt{FM}_M(k)\}_{k\geq 0}$ is a right $\mathtt{FM}_{\mathbb{R}^n}$-module: we can insert an infinitesimal configuration into a configuration on M



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The proof

Theorem (Kontsevich, Lambrechts and Volić; also different proofs by Tamarkin, and by Fresse and Willwacher)

"Explicit" formality morphisms of operads

$$H^*(\mathtt{FM}_{\mathbb{R}^n}) \xleftarrow{\sim} \mathtt{Graphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathtt{FM}_{\mathbb{R}^n})$$

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Idea: adapt the proof with *labeled* graph complexes

Theorem (I. 2016, complete version)

Same hypotheses:

$$\begin{tikzcd} \mathtt{G}_A & \mathtt{Graphs}_R \arrow[l, "\sim"'] \arrow[r, "\sim"] & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M) \\ \circlearrowleft^{\dagger} & \circlearrowleft^{\dagger} & \circlearrowleft^{\ddagger} \\ H^*(\mathtt{FM}_{\mathbb{R}^n}) & \mathtt{Graphs}_n \arrow[l, "\sim"'] \arrow[r, "\sim"] & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_{\mathbb{R}^n}) \end{tikzcd}$$

† When $\chi(M) = 0$ $\qquad$ ‡ When M is framed

Thank you for your attention!

arXiv:1608.08054

Slides online: https://operad.fr/talk/ytm2017/