Real homotopy of configuration spaces

Najib Idrissi

Toric Topology Research Seminar @ Fields Institute (online)

Introduction: configuration spaces

M: n-manifold

Configuration spaces

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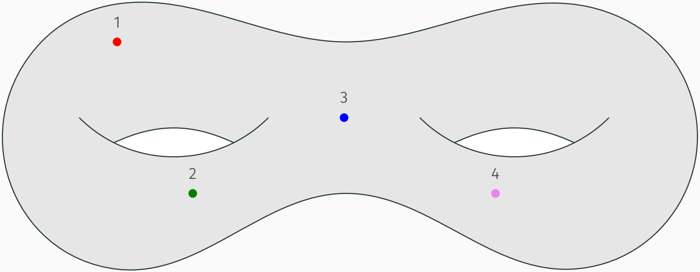
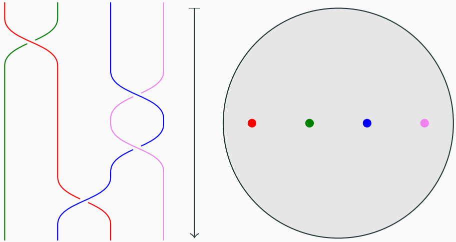
$$\mathrm{Conf}_M(r) \coloneqq \{(x_1, \ldots, x_r) \in M^r \mid \forall i \neq j,\ x_i \neq x_j\}$$



Applications

- braid groups;

Braid $\tau \in B_r$ = path in $\mathrm{Conf}_{D^2}(r)$



More generally $\mathrm{Conf}_{\Sigma}(r) \Rightarrow$ surface braid groups

Applications

- braid groups;
- iterated loop spaces;

$$\Omega^n X = \{\gamma : D^n \to X \mid \gamma(\partial D^n) = *\}$$

$\to$ has algebraic (operadic) structure encoded by Conf_{D^n} [May, Boardman–Vogt]

Applications

- braid groups;
- iterated loop spaces;
- Goodwillie–Weiss manifold calculus;

Goal: compute

$$\mathrm{Emb}(M, N) = \{f : M \hookrightarrow N\} \subset \mathrm{Map}(M, N)$$

$\rightarrow$ “approximated” by a subspace of

$$\prod_{r \geq 0} \mathrm{Map}(\mathrm{Conf}_M(r), \mathrm{Conf}_N(r))$$

under good conditions

Applications

- braid groups;
- iterated loop spaces;
- Goodwillie–Weiss manifold calculus;
- Gelfand–Fuks cohomology;

Characteristic classes of foliations live in

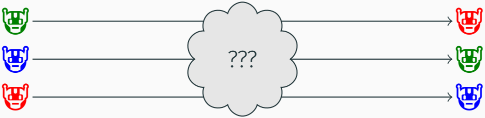
$$H^*_{\mathrm{cont}}(\Gamma_c(M, TM))$$

$\to$ computed by a spectral sequence involving configuration spaces [Cohen–Taylor]

Applications

- braid groups;
- iterated loop spaces;
- Goodwillie–Weiss manifold calculus;
- Gelfand–Fuks cohomology;
- motion planning.

Want to move several robots at the same time



$\iff$ find a section of:

$$\mathrm{Map}([0,1], \mathrm{Conf}_M(r)) \to \mathrm{Conf}_M(r) \times \mathrm{Conf}_M(r)$$
$$\gamma \mapsto (\gamma(0), \gamma(1))$$

Minimum number of domains of continuity ("topological complexity") depends on homotopy type of $\mathrm{Conf}_M(r)$ [Farber]

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Some evidence:

- $H_*(\mathrm{Conf}_M(r))$ ✓ (Bödigheimer–Cohen–Taylor, Benderky–Gitler)

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- $\Sigma^\infty\mathrm{Conf}_M(r)$ ✓ (Aouina–Klein)

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Goal

Find a model of $\mathrm{Conf}_M(r)$ from a model of M.

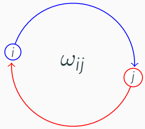
Closed manifolds

Building block: $\mathbb{R}^n$

Presentation of $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ [Arnold, Cohen]

- Generators: ω_{ij} of degree $n-1$ (for $1 \leq i \neq j \leq r$)
- Relations:

$$\omega_{ij}^2 = \omega_{ji} - (-1)^n \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$$



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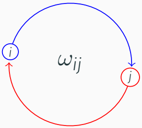

Theorem (Arnold 1969)

Formality: $H^*(\mathrm{Conf}_{\mathbb{C}}(r)) \sim_{\mathbb{C}} \Omega^*(\mathrm{Conf}_{\mathbb{C}}(r))$, $\omega_{ij} \mapsto \mathrm{d}\log(z_i - z_j)$.

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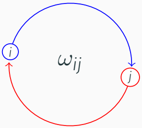

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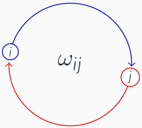

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Corollary

The cohomology of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ determines its rational homotopy type.

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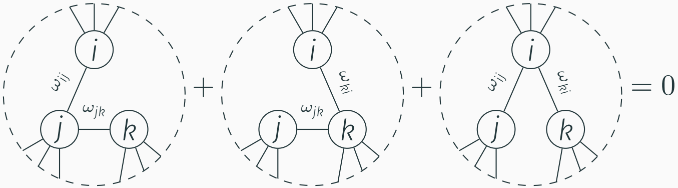
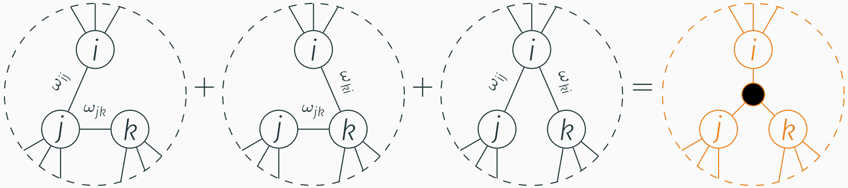
$H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$: graphs on r vertices mod local three-terms relations.



$$H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) \xleftarrow[\text{proj.}]{\sim} \mathsf{Graphs}_n(r) \xrightarrow[\int]{\sim} \Omega^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$$

Replace relations by differentials:



Idea of Kontsevich's proof

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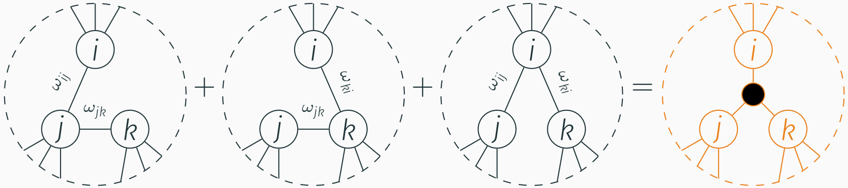

Key point: integrals of internal components vanish.

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- $r \geq 3$: more complicated.

Theorem (I)

M: simply connected closed smooth manifold, A: any Poincaré duality model of M, then:

$$\mathsf{G}_A(r) \simeq_{\mathbb{R}} \Omega^*(\mathrm{Conf}_M(r)), \quad \forall r \geq 0.$$

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Corollary (I, CW)

$M \simeq_{\mathbb{R}} N \implies \mathrm{Conf}_M(r) \simeq_{\mathbb{R}} \mathrm{Conf}_N(r).$

Proof

Inspired by the ideas of Kontsevich: graphs decorated by elements of A, replace relations by internal vertices, map into Ω^* by integrals

$$\mathsf{G}_A(r) \xleftarrow{\sim} \mathsf{Graphs}_R \xrightarrow{\sim} \Omega^*(\mathrm{Conf}_M(r))$$

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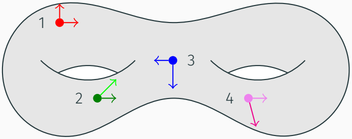
Remark

Get another bigger model: Graphs_R (cf. CW).
Benefit: quasi-free, good for homological algebra.

$$\mathrm{Conf}^{\mathrm{fr}}_M(r) = \left\{ \begin{aligned} &(x, B_1, \ldots, B_r) \mid \\ &x \in \mathrm{Conf}_M(r), \\ &B_i : \text{ basis of } T_{x_i}M \end{aligned} \right\}$$



Framed configurations

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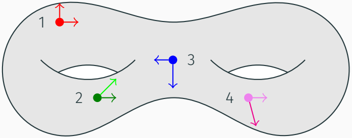

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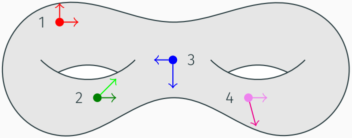

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Theorem (CDIW)

Graphical model for (oriented) $\mathrm{Conf}^{\mathrm{fr}}_M(r)$ based on graphs decorated by cohomology classes of M + cohomology of $\mathrm{BSO}(n)$.

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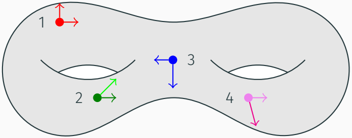

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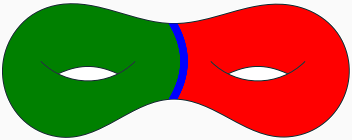
Problem: depends on non-explicit integrals; no homotopy invariance yet.

Manifolds with boundary

Manifold gluing



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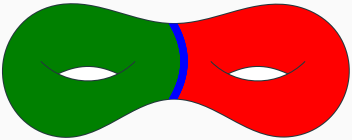

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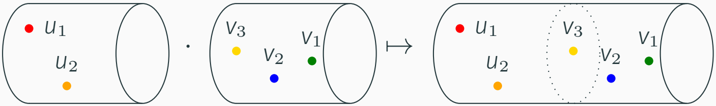

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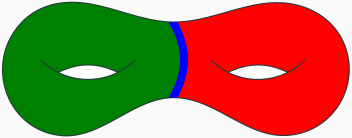

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Goal: compute configuration spaces "by induction"

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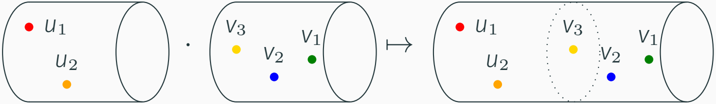

$\mathrm{Conf}_{M'}$ is a left module, $\mathrm{Conf}_{M''}$ is a right module, and:

$$\mathrm{Conf}_M \simeq \mathrm{Conf}_{M'} \otimes^{\mathbb{L}}_{\mathrm{Conf}_{N\times\mathbb{R}}} \mathrm{Conf}_{M''}.$$

Theorem (CILW)

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Quotient of $\mathsf{mGraphs}_{M'}$ = small "Lambrechts–Stanley-like" model, depends on Poincaré–Lefschetz duality model of $(M, \partial M)$.

Surfaces

Splitting

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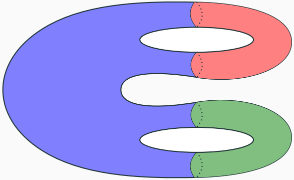

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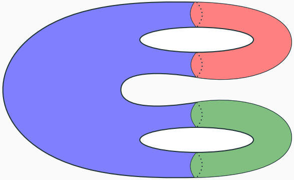

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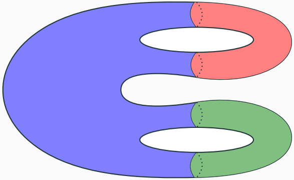

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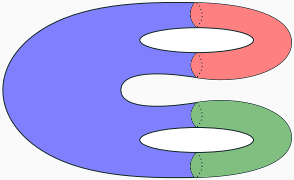

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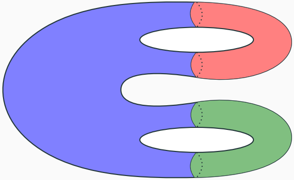

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+ cyclic formality of the little disks operad:

Theorem (CIW)

$\mathrm{Conf}^{\mathrm{fr}}_{S^2 \setminus \{1,\ldots,2g\}}$ and $\mathrm{Conf}^{\mathrm{fr}}_{S^1 \times \mathbb{R}}$ together with all their algebraic (monoid, orientation reversal, left/right actions) structures are formal.

Result

Description of $\Sigma_g \implies \mathrm{Conf}^{\mathrm{fr}}_{\Sigma_g}$ is an "iterated Hochschild complex"

$$\mathrm{Conf}^{\mathrm{fr}}_{\Sigma_g} \simeq \hat{\bigotimes}^{(1,1),\ldots,(g,g)}_{\mathrm{Conf}^{\mathrm{fr}}_{S^1\times\mathbb{R}}} \mathrm{Conf}^{\mathrm{fr}}_{S^2\setminus\{1,\ldots,2g\}}.$$

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Proof: cohomology of the $\bigotimes$ above, ...

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Proof: ... general rational homotopy theory, ...

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Proof: … formal version of Kontsevich's integrals, …

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Proof: … and combinatorics.

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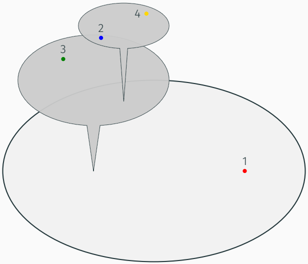
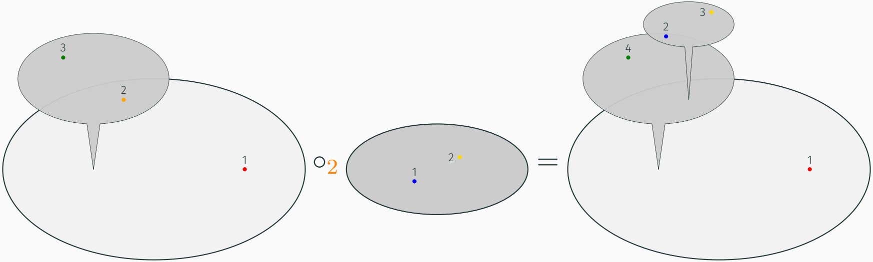
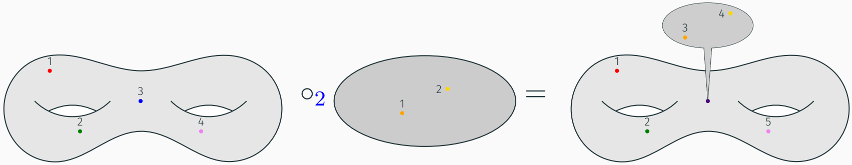
Need to compactify configuration spaces for integrals to converge: add virtual configurations with infinitesimally close points



Get a new algebraic structure: an operad



Right module structure on compactification of Conf_M



if M is parallelized; otherwise, need framed configurations.

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Some applications:

- Goodwillie–Weiss manifold calculus;
- factorization homology.

N. Idrissi. "The Lambrechts–Stanley Model of Configuration Spaces". In: *Invent. Math* 216.1 (2019), pp. 1–68. ISSN: 1432-1297. DOI: 10.1007/s00222-018-0842-9. arXiv: 1608.08054.

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Thank you for your attention!