Curved Koszul duality for algebras over unital operads

Najib Idrissi

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Tool of choice: Koszul duality.

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$\implies$ Koszul complex $K_A := (A \otimes A^{¡}, d_\kappa(\Sigma e) = e)$; A is Koszul if K_A is acyclic

Example

$T(E)$ and $S(E)$ are both Koszul.

Bar/cobar adjunction:

$$\Omega : \{\text{coaug.coalgebras}\} \leftrightarrows \{\text{aug.algebras}\} : B$$

where $BA = (T^c(\Sigma\bar{A}), d_B)$ and $\Omega C = (T(\Sigma^{-1}\bar{C}), d_\Omega)$.

Quadratic algebras – Koszul resolutions

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Much smaller resolution!

Examples

$A = T(E) \implies \Omega A^{¡} = A = T(E)$ versus $\Omega BA = TT^cF(E)$

$A = S(E) \implies \Omega A^{¡} = T\Lambda^c(E)$ versus $\Omega BA = TT^cS(E)$.

QLC algebras – curved KD

Quadratic-linear-constant algebra: $A = T_+(E)/(R)$ with $R \subset E^{\otimes 2} \oplus E \oplus \mathbb{R}1$

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Goal: do this for more general types of unital algebras.

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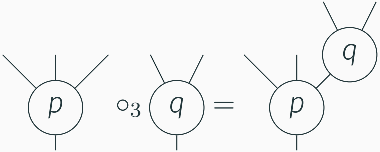
Operad $\mathsf{P} = \{\mathsf{P}(n)\}_{n \geq 0}$: combinatorial object that encodes a type of algebra.



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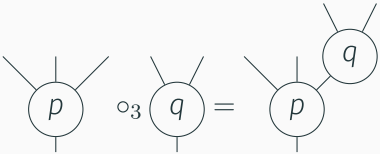

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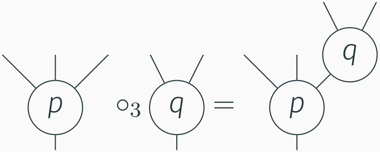

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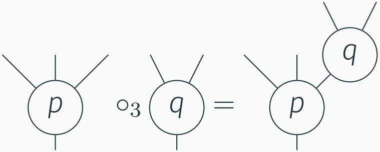

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$\mathsf{e}_n := H_*(\mathsf{E}_n) = \mathtt{Com} \circ \mathtt{Lie}_n$, $n \geq 2$ = Poisson n-algebras.

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Formally similar definitions: Koszul dual cooperad $\mathsf{P}^{¡} = \mathrm{FOp}^c(\Sigma E, \Sigma^2 R)$ and its linear dual $\mathsf{P}^! = \mathrm{FOp}(E^*)/(R^\perp)$.

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$\mathtt{Ass}^! = \mathtt{Ass}$; $\mathtt{Com}^! = \mathtt{Lie}$, $\mathtt{Lie}^! = \mathtt{Com}$; $\mathtt{e}_n^! = \mathtt{e}_n\{-n\}$.

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Theorem (Ginzburg–Kapranov '94, Getzler–Jones '94, Getzler '95...)

If P is quadratic and Koszul, then $\mathrm{P}_\infty := \Omega \mathrm{P}^{¡} \xrightarrow{\sim} \mathrm{P}$.

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Examples

$\mathtt{Ass}_\infty = A_\infty$-algebras, $\mathtt{Com}_\infty = C_\infty$-algebras, $\mathtt{Lie}_\infty = L_\infty$-algebras...

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Example

For a Lie algebra $\mathfrak{g}$, $\Omega_\kappa B_\kappa \mathfrak{g} = (L(C^{CE}_{*-1}(\mathfrak{g})), d)$.

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$\mathsf{P} = \mathsf{Ass}$: recovers the classical Koszul duality of associative algebras.
A: quadratic Com-algebra $\implies U(A^!) = (A_{\mathsf{Ass}})^!$ [Löfwall].

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Theorem (Hirsh–Millès '12)

If $qu\mathsf{P}$ is Koszul, then $u\mathsf{P}_\infty := \Omega(u\mathsf{P}^{\mathsf{i}}) \xrightarrow{\sim} u\mathsf{P}$: resolution of $u\mathsf{P}$

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The second condition is difficult to check!

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Theorem (I. '18)

If qA is Koszul then $\Omega_\kappa A^{¡} \xrightarrow{\sim} A$ is a resolution.

M: framed n-manifold, A: $u\mathsf{E}_n$-algebra ($\exists$ version for unframed manifolds.)

Application 1: factorization homology

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Compute $\int_M A = \operatorname{hocolim}_{(D^n)^{\sqcup k} \hookrightarrow M} A^{\otimes k}$.

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Upshot: data is separated in three + resolution

Chains of factorization homology over $\mathbb{R}$

If we work over $\mathbb{R}$ and we just want chains:

$$C_*(\textstyle\int_M A) \simeq C_*(\mathsf{E}_M) \circ^{\mathbb{L}}_{C_*(u\mathsf{E}_n)} C_*(A).$$

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$\implies$ we need to resolve A as a $u\mathtt{e}_n$-algebra.

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- $d = 0$
- curvature: $\theta(\bar{x}_i \wedge \bar{\xi}_j) = -\delta_{ij}$.

Weyl algebra $\mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$

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Action of $u\mathbf{e}_n$: free symmetric algebra and $\{x_i, \xi_j\} = \delta_{ij}1$

$\implies$ quadratic-(linear-)constant presentation

Quadratization $qA = S(x_i, \xi_j)$ free symmetric algebra + zero bracket

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(If we had applied curved KD at the level of operads instead:
$\Omega_\kappa B_\kappa A \supset (\underbrace{SL}_{\text{cobar}}\underbrace{S^cL^c}_{\text{bar}}\underbrace{S(x_i, \xi_j)}_{A}, d)$, + resolution of the unit...)

Computation of $\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$

We can also compute

$$\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) \simeq \mathsf{LS}_M \circ_{u\mathsf{e}_n} \left(SLS^c(\bar{x}_i, \bar{\xi}_j), d\right)$$

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Theorem (I. '18, see also Markarian '17, Döppenschmitt '18)

$\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) \simeq C_*^{CE}(\mathcal{M}^{n-*} \otimes \mathbb{R}\langle 1, x_i, \xi_j \rangle) \simeq \mathbb{R}$.

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Intuition: quantum observable with values in $A \rightsquigarrow$ “expectation” lives in $\int_M A$, should be a number.

Operad P + P-algebra $A \implies$ notion of *A*-modules

Operad $\mathtt{P}$ + $\mathtt{P}$-algebra $A \implies$ notion of A-modules

Examples

$\mathtt{P} = \mathtt{Ass} \to (A, A)$ bimodules; $\mathtt{P} = \mathtt{Com} \to A$-modules; $\mathtt{P} = \mathtt{Lie} \to$ representations of the Lie algebra.

Application 2: derived enveloping algebra

Operad $\mathtt{P}$ + $\mathtt{P}$-algebra $A \implies$ notion of A-modules

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Proposition

For $A = \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$, the derived enveloping algebra $U^{\mathbb{L}}_{u\mathsf{e}_n}(A)$ is q.iso to the underived one.

Thank you for your attention!

These slides: `https://idrissi.eu`