Curved Koszul duality for algebras over unital operads

Najib Idrissi

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Why?

- Compute derived invariants : derived tensor product, derived mapping space...
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Tool of choice: Koszul duality.

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Quadratic algebras – Koszul duals

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$\implies$ Koszul complex $K_A := (A \otimes A^{¡}, d_\kappa(\Sigma e) = e)$; A is Koszul if K_A is acyclic

Example

$F(E)$ and $S(E)$ are both Koszul.

Bar/cobar adjunction:

$$\Omega : \{\text{coaug.coalgebras}\} \leftrightarrows \{\text{aug.algebras}\} : B$$

where $BA = (F^c(\Sigma\bar{A}), d_B)$ and $\Omega C = (F(\Sigma^{-1}\bar{C}), d_\Omega)$.

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Much smaller resolution!

Examples

$A = F(E) \implies \Omega A^{\text{i}} = A = F(E)$ versus $\Omega BA = FF^cF(E)$

$A = S(E) \implies \Omega A^{\text{i}} = F\Lambda^c(E)$ versus $\Omega BA = FF^cS(E)$.

QLC algebras – curved KD

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Goal: do this for more general types of unital algebras.

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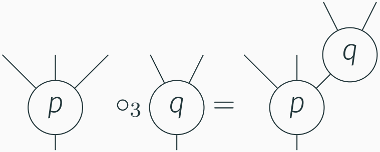
Operad $\mathsf{P} = \{\mathsf{P}(n)\}_{n \geq 0}$: combinatorial object that encodes a type of algebra.



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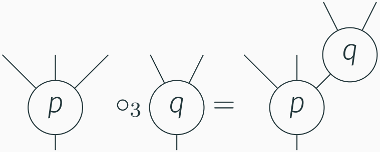

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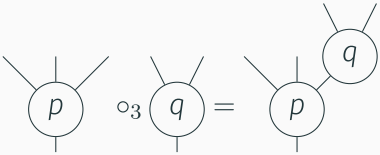

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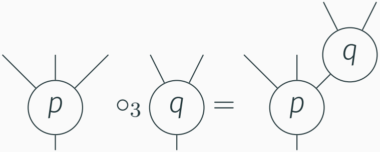

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$\mathsf{e}_n := H_*(\mathsf{E}_n)$, $n \geq 2$ = Poisson n-algebras.

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$\mathtt{Ass}^! = \mathtt{Ass}$; $\mathtt{Com}^! = \mathtt{Lie}$, $\mathtt{Lie}^! = \mathtt{Com}$; $\mathtt{e}_n^! = \mathtt{e}_n\{-n\}$.

Formally similar definitions: bar/cobar adjunction

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$\mathtt{Ass}_\infty = A_\infty$-algebras, $\mathtt{Com}_\infty = C_\infty$-algebras, $\mathtt{Lie}_\infty = L_\infty$-algebras…

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Big resolution of operadic algebras

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Example

For a Lie algebra $\mathfrak{g}$, $\Omega_\kappa B_\kappa \mathfrak{g} = (L(C^{CE}_{*-1}(\mathfrak{g})), d)$.

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Example

$\mathsf{P} = \mathsf{Ass}$: recovers the classical Koszul duality of associative algebras.

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Bar/cobar extends to the curved setting

Theorem (Hirsh–Millès ’12)

If $qu\mathrm{P}$ is Koszul, then $u\mathrm{P}_\infty := \Omega(u\mathrm{P}^{\text{i}}) \xrightarrow{\sim} u\mathrm{P}$: resolution of $u\mathrm{P}$

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Generalization of bar/cobar adjunction:

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Theorem (I. ’18)

If qA is Koszul then $\Omega_\kappa A^{¡} \xrightarrow{\sim} A$ is a resolution.

M: framed n-manifold, A: $u\mathsf{E}_n$-algebra

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Goal

Compute $\int_M A = \mathrm{hocolim}_{(D^n)^{\sqcup k} \hookrightarrow M} A^{\otimes k}$.

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Theorem (Francis 2015)

$\int_M A \simeq \mathsf{E}_M \circ^{\mathbb{L}}_{u\mathsf{E}_n} A = \operatorname{hocoeq}\big(\mathsf{E}_M \circ u\mathsf{E}_n \circ A \rightrightarrows \mathsf{E}_M \circ A\big)$, where:

$$u\mathsf{E}_n(k) = \operatorname{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k\times}, \mathbb{R}^n); \quad \mathsf{E}_M(k) = \operatorname{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k\times}, M).$$

($\exists$ version for unframed manifolds.)

If we work over $\mathbb{R}$ and we just want chains:

$$C_*(\textstyle\int_M A) \simeq C_*(\mathsf{E}_M) \circ^{\mathbb{L}}_{C_*(u\mathsf{E}_n)} C_*(A).$$

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The operad $C_*(u\mathsf{E}_n)$ is formal: $C_*(u\mathsf{E}_n) \simeq u\mathsf{e}_n \coloneqq H_*(u\mathsf{E}_n)$.

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M closed, simply connected, smooth, $\dim M \geq 4 \implies$
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$\implies$ we need to resolve A as a $u\mathsf{e}_n$-algebra.

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(If we had applied curved KD at the level of operads instead:
$\Omega_\kappa B_\kappa A \supset (\underbrace{SL}_{\text{cobar}}\underbrace{S^cL^c}_{\text{bar}}\underbrace{S(x_i, \xi_j)}_{A}, d)$, + resolution of the unit...)

We can also compute

$$\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) \simeq \mathsf{LS}_M \circ_{u\mathsf{e}_n} (SLS^c(\bar{x}_i, \bar{\xi}_j), d)$$

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Computation of $\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$

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Intuition: quantum observable with values in $A \rightsquigarrow$ “expectation” lives in $\int_M A$, should be a number.

Operad P + P-algebra $A \implies$ notion of A-modules

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Proposition

For $A = \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$, the derived enveloping algebra $U^{\mathbb{L}}_{u\mathsf{e}_n}(A)$ is q.iso to the underived one.

Thank you for your attention!

These slides: `https://idrissi.eu`