Configuration spaces and Operads

Najib Idrissi

December 11th, 2018 @ Stockholm Topology Seminar

M: n-manifold

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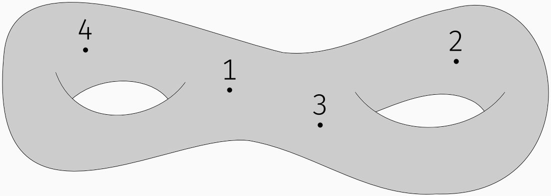
$$\mathrm{Conf}_r(M) := \{(x_1, \dots, x_r) \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}$$



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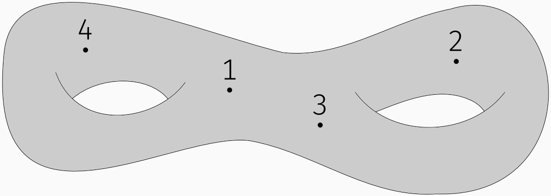

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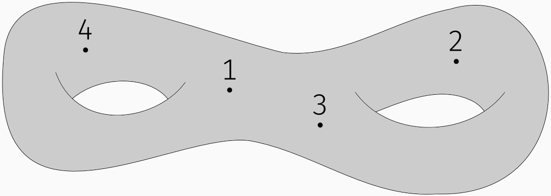

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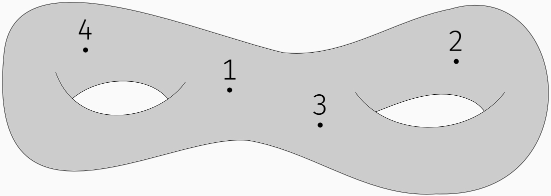

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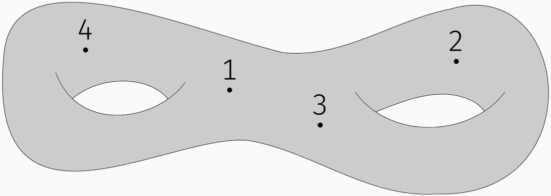

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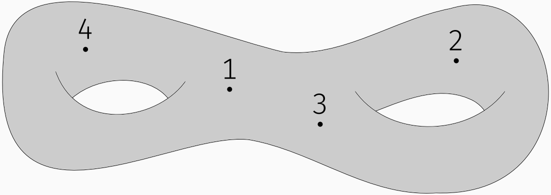

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- Motion planning [robotics]

Open question

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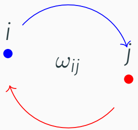
Simply connected closed manifolds

Homotopy invariance is still open.

We can also localize: $M \simeq_{\mathbb{Q}} N \implies \mathrm{Conf}_r(M) \simeq_{\mathbb{Q}} \mathrm{Conf}_r(N)$?

Configurations in a Euclidean spaces

Presentation of $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ [Arnold, Cohen]



- Generators: ω_{ij} of degree $n-1$ (for $1 \leq i \neq j \leq r$)
- Relations:

$$\omega_{ij}^2 = \omega_{ji} - (-1)^n \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$$

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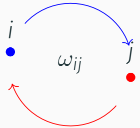

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Theorem (Arnold 1969)

Formality: $H^*(\mathrm{Conf}_k(\mathbb{C})) \sim_{\mathbb{C}} \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_k(\mathbb{C}))$, $\omega_{ij} \mapsto \mathrm{d}\log(z_i - z_j)$.

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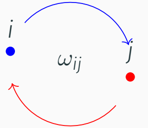

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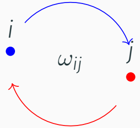

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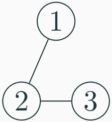
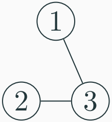
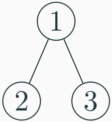
Corollary

The cohomology of $\mathrm{Conf}_k(\mathbb{R}^n)$ determines its rational homotopy type.

Arnold relations: $R_{123} =$  $+$  $+$ 

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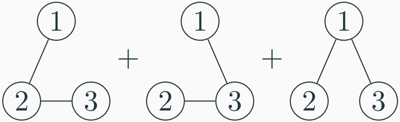

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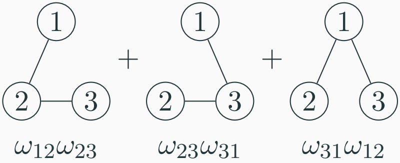
Arnold relations: $R_{123} =$ 

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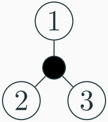
$\rightsquigarrow$ add "internal" vertices and a differential which contracts edges incident to these new vertices: 

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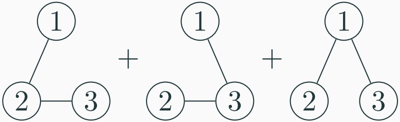

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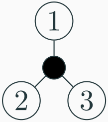

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Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 1)

We get a quasi-free CDGA $\mathtt{Graphs}_n(r)$ and a quasi-isomorphism $\mathtt{Graphs}_n(r) \xrightarrow{\sim} H^*(\mathrm{Conf}_r(\mathbb{R}^n))$.

The relations R_{ijk} are only satisfied up to homotopy in $\Omega^*(\mathrm{Conf}_r(\mathbb{R}^n))$.
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Let $\varphi \in \Omega^{n-1}(\mathrm{Conf}_2(\mathbb{R}^n))$ be the volume form.
For $\Gamma \in \mathtt{Graphs}_n(r)$ with i internal vertices:

$$\omega(\Gamma) := \int_{\mathrm{Conf}_{k+i}(\mathbb{R}^n) \to \mathrm{Conf}_k(\mathbb{R}^n)} \bigwedge_{(ij) \in E_\Gamma} \varphi_{ij}.$$

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⚠ I'm cheating! We have to compactify $\mathrm{Conf}_k(\mathbb{R}^n)$ to make sure $\int$ converges and to apply the Stokes formula correctly.

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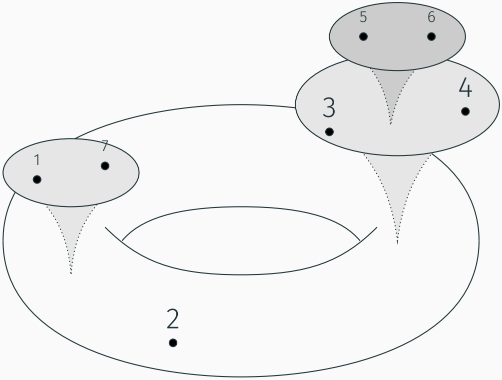
Fulton–MacPherson compactification $\mathrm{Conf}_k(M) \overset{\sim}{\hookrightarrow} \mathsf{FM}_M(k)$



Compactification

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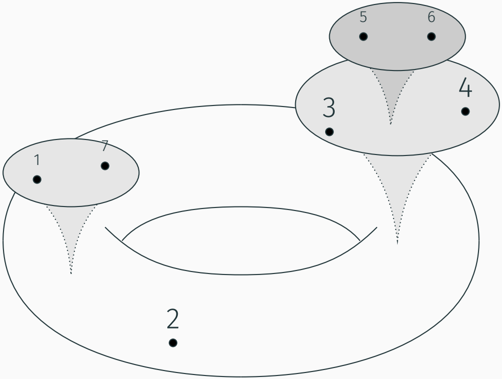
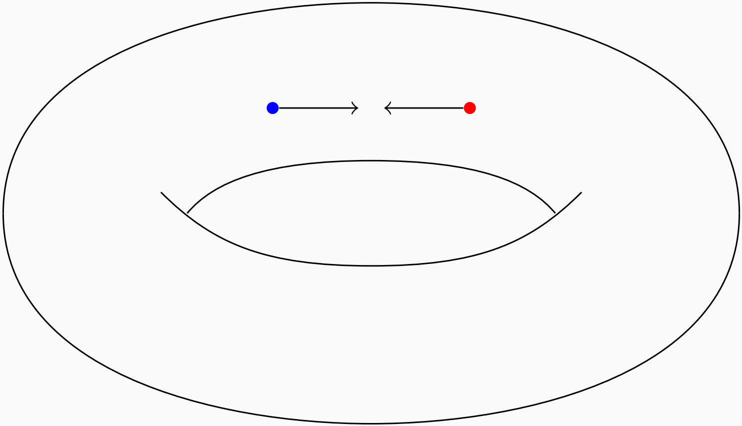
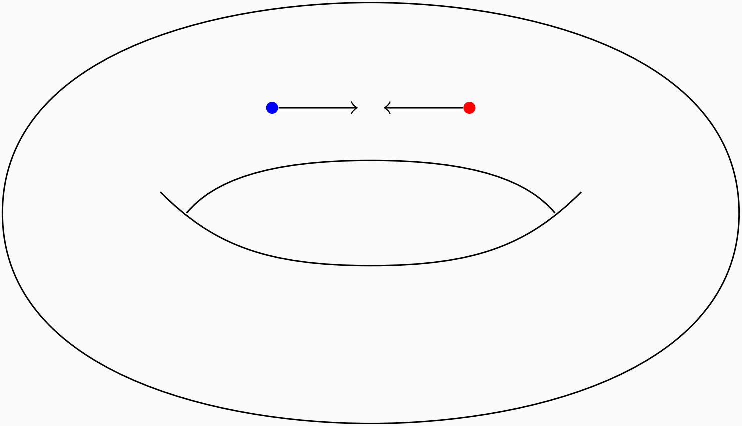


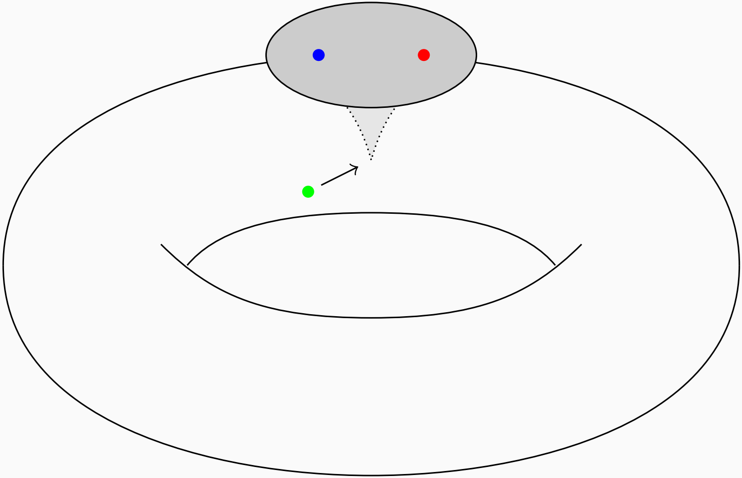
M closed manifold $\implies$ semi-algebraic stratified manifold $\dim = nk$

Animation n°1





Animation n°2



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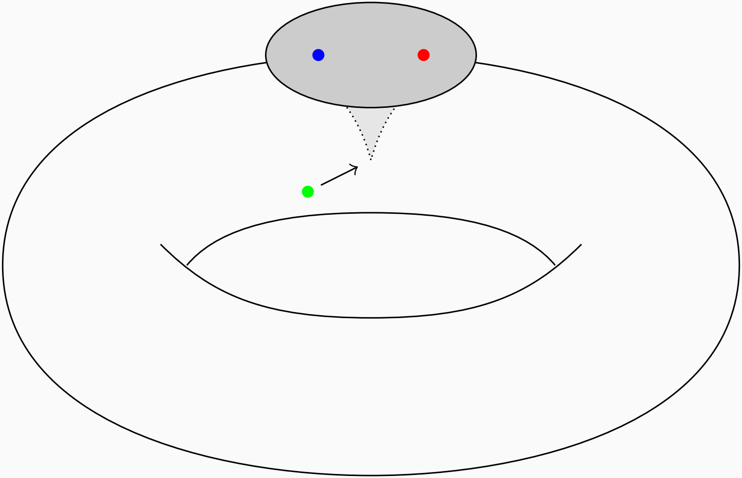
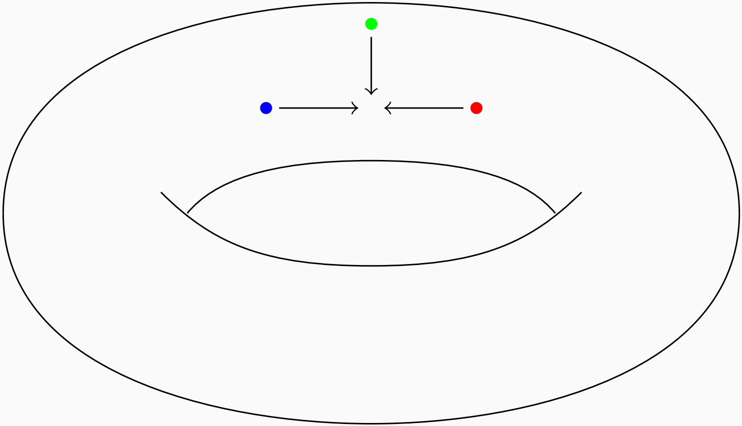


Animation n°3



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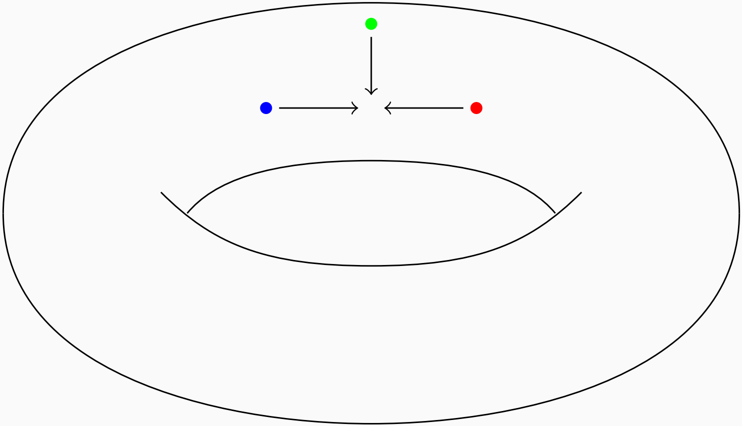
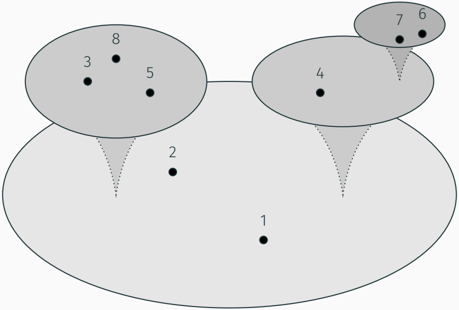


Compactification of $\mathrm{Conf}_k(\mathbb{R}^n)$

We have to "normalize" $\mathrm{Conf}_k(\mathbb{R}^n)$ to mitigate the non-compacity of $\mathbb{R}^n$:

$$\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\simeq} \mathrm{Conf}_k(\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_{>0}) \xhookrightarrow{\sim} \mathsf{FM}_n(k)$$



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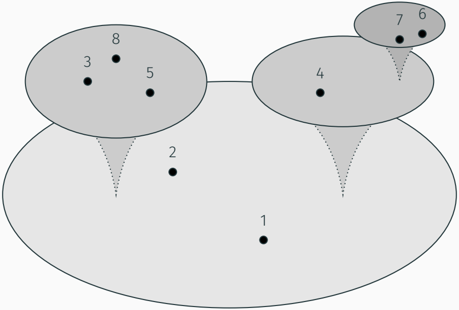
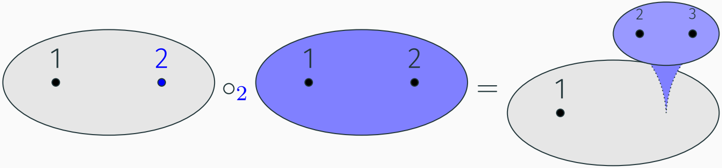


$\implies$ semi-algebraic stratified manifold $\dim = nk - n - 1$

We see a new structure on FM_n: an operad! We can "insert" an infinitesimal configuration in another one:



$$\mathsf{FM}_n(k) \times \mathsf{FM}_n(l) \xrightarrow{\circ_i} \mathsf{FM}_n(k+l-1), \quad 1 \leq i \leq k$$

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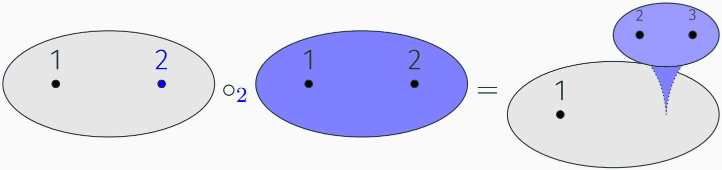

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Remark

Weakly equivalent to the "little disks operad".

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The operad $\mathtt{FM}_n$ is formal over $\mathbb{R}$:

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Formality has important applications, e.g. Deligne conjecture, deformation quantization of Poisson manifolds, etc.

Remark

$H_*(\mathtt{FM}_n)$ is the operad governing Poisson n-algebras for $n \geq 2$.

Poincaré duality

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Theorem (Lambrechts–Stanley 2008)

Any simply connected closed manifold admits a Poincaré duality model $A \sim \Omega^*(M)$.

The Lambrechts–Stanley model

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$\mathsf{G}_A(r)$: (conjectural) model of $\mathrm{Conf}_r(M) = M^{\times k} \setminus \bigcup_{i \neq j} \Delta_{ij}$

$:= \{x_i = x_j\}$

The Lambrechts–Stanley model

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- $r \geq 3$: more complicated.

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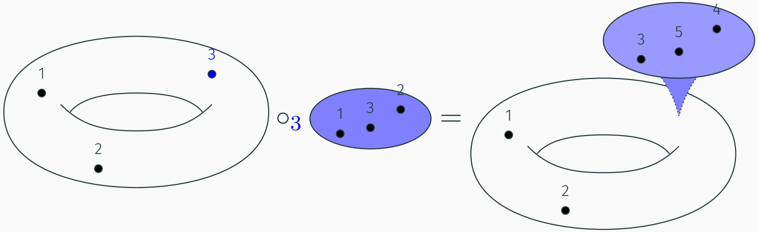
Remark

$\dim M \leq 3$: only spheres (Poincaré conjecture) and we know that G_A is a model, but adapting the proof is problematic!

M parallelized $\implies$ $\mathsf{FM}_M = \{\mathsf{FM}_M(k)\}_{k\geq 0}$ is a right FM_n-module :



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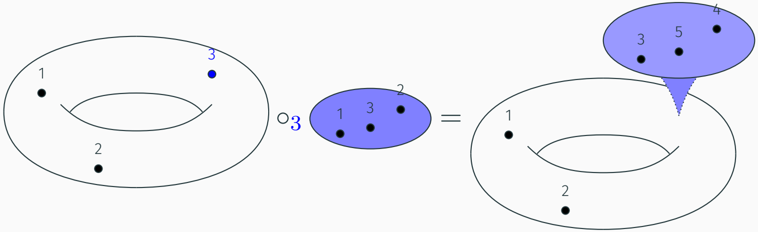

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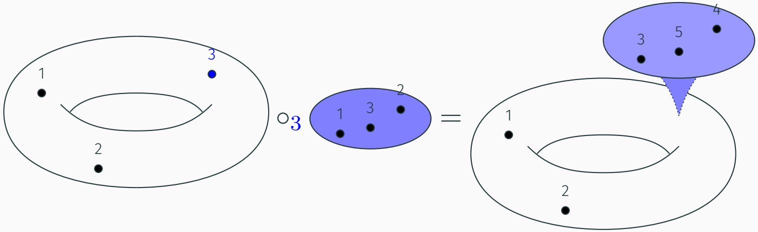

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A bit of abstract nonsense:

Proposition

$\chi(M) = 0 \implies \mathsf{G}_A = \{\mathsf{G}_A(k)\}_{k\geq 0}$ is a Hopf right $H^*(\mathsf{FM}_n)$-comodule.

Theorem (I. 2016)

M: closed simply connected smooth manifold, $\dim M \geq 4$

$$\begin{array}{ccccc}
\mathsf{G}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R & \dashrightarrow^{\sim} & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M) \\
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Conclusion

Not only do we have a model of each $\mathrm{Conf}_r(M)$, but for their richer structure if we look at them all at once.

Consider the space of embeddings: $\mathrm{Emb}(M, N) = \{f : M \hookrightarrow N\}$.

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Goodwillie–Weiss manifold calculus [Boavida–Weiss, Turchin]: for parallelized manifolds of codimension ≥ 3,

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Remark

Requires something like $\mathrm{Mor}^h_{\mathrm{Conf}_\bullet(\mathbb{R}^n)}(\mathrm{Conf}_\bullet(M), \mathrm{Conf}_\bullet(N)) \simeq_{\mathbb{R}} \mathrm{Mor}^h_{\mathrm{Conf}_\bullet(\mathbb{R}^n)^{\mathbb{R}}}(\mathrm{Conf}_\bullet(M)^{\mathbb{R}}, \mathrm{Conf}_\bullet(N)^{\mathbb{R}})$

Application 2: factorization homology

Schematically, factorization homology = homology where $\otimes$ replaces $\oplus$.
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Theorem (Campos–I.–Lambrechts–Willwacher 2018)

For manifolds with boundary: homotopy invariance of $\mathrm{Conf}_r(-)$, generalization of the Lambrechts–Stanley model (and more); under good conditions, including $\dim M \geq \dots$

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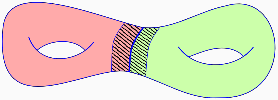

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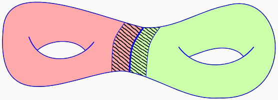

Allows to compute Conf_r by "induction":

Roughly: we use 2-colored labeled graphs.

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Real model of this module based on graph complexes (little hope of analogue of Lambrechts–Stanley model…)

Should allow us to compute e.g. embedding spaces of non-parallelized manifolds. (Not enough, though: need partially framed configurations for the larger manifold N.)

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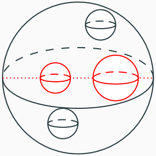
There exists an operad VSC_{mn} which models the local situation $\mathbb{R}^n \setminus \mathbb{R}^m$:

 $\in \mathsf{VSC}_{13}(2, 2)$

Theorem (I. 2018)

The operad VSC_{mn} is formal over $\mathbb{R}$.

Thank you for your attention!

These slides: `https://idrissi.eu`