Formality of a higher-codimensional Swiss-Cheese operad

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July 17, 2019

We study configurations of points in the complement of a linear subspace inside a Euclidean space, $\mathbb{R}^n \setminus \mathbb{R}^m$ with $n - m \geq 2$. We define a higher-codimensional Swiss-Cheese operad VSC_{mn} associated to such configurations, a variant of the classical Swiss-Cheese operad. The operad VSC_{mn} is weakly equivalent to the operad of locally constant factorization algebras on the stratified space $\{\mathbb{R}^m \subset \mathbb{R}^n\}$. We prove that this operad is formal over $\mathbb{R}$.

Contents

Introduction

The theory of operads deals with types of algebras, e.g. associative algebras, commutative algebras, Lie algebras, etc. The little disks operads D_n (for $n \geq 1$) are central to this theory. Their applications include deformation quantization of Poisson manifolds, the Deligne

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conjecture, embedding spaces of manifolds, factorization homology, and configuration spaces. Briefly, an element of $\mathsf{D}_n(k)$ is an embedding of k little n-disks with disjoint interiors in the unit n-disk D^n. The operadic structure on $\mathsf{D}_n = \{\mathsf{D}_n(k)\}_{k\geq 0}$ is obtained by considering the composition of such embeddings. Elements of D_n represent operations acting on n-fold loop spaces, and the operadic structure reflects the composition of such operations. Up to homotopy, grouplike D_n-algebras are exactly n-fold loop spaces [BV68; May72]. A fundamental property of D_n is its "formality", i.e. its cohomology $H^*(\mathsf{D}_n;\mathbb{Q})$ encodes its rational homotopy type [Kon99; Tam03; LV14; Pet14; FW15].

Voronov's Swiss-Cheese operads SC_n [Vor99] (for $n \geq 2$) are two-colored operads which encode, in some sense, central actions of D_n-algebras on D_{n-1}-algebras. Elements of $\mathsf{SC}_n(k,l)$ are roughly speaking given by configurations of k half-disks and l full disks in the unit upper half-disk. Under good assumptions, SC_n-algebras are exactly relative iterated loop spaces [Duc16; Que15; HLS16; Vie19] (see Section 2.2). Unlike the little disks operads, the Swiss-Cheese operads are not formal [Liv15; Wil17]. This non-formality is equivalent to the nonformality of the natural inclusion $\mathsf{D}_{n-1} \subset \mathsf{D}_n$ [TW18] by the arguments of [Wil17, Section 5.1].

In this paper, we introduce variants of the Swiss-Cheese operads VSC_{mn} for $n-2 \geq m \geq 1$. These operads encode the action of a D_n-algebra on a D_m-algebra by a central derivation (see Section 3.2). Elements of $\mathsf{VSC}_{mn}(k,l)$ are given by configurations of two kinds of little n-disks: l aerial disks which are forbidden from touching $D^m \subset D^n$, and k terrestrial disks which are centered on D^m.

Theorem A (See Theorem 5.31)**.** *For $n-2 \geq m \geq 1$, the higher-codimensional Swiss-Cheese operad VSC_{mn} is formal over $\mathbb{R}$, i.e. $H^*(\mathsf{VSC}_{mn};\mathbb{R})$ (viewed as a cooperad in the category of commutative dg-algebras) encodes the real homotopy type of VSC_{mn}.*

It follows from results of Livernet [Liv15] that $\mathsf{VSC}_{(n-1)n}$ is not formal (Remark 3.28).

Motivation The motivation for this paper comes from the study of configuration spaces. Indeed, the little disks operads are intimately linked to configuration spaces of Euclidean spaces. If we take an element of $\mathsf{D}_n(k)$ and we keep only the centers of the disks, then we obtain an element of the ordered configuration space $\mathrm{Conf}_{\mathbb{R}^n}(k)$. We thus obtain homotopy equivalences $\mathsf{D}_n(k) \simeq \mathrm{Conf}_{\mathbb{R}^n}(k)$ for all k, $n \geq 0$, although they do not preserve the operadic structures.

This observation was the starting point of a series of papers on configuration spaces of manifolds. Campos–Willwacher [CW16] and the author [Idr19] provided combinatorial models for the real homotopy types of configuration spaces of simply connected closed smooth manifolds. Campos, Lambrechts, Willwacher, and the author [Cam+18b] provided similar models for configuration spaces of compact, simply connected smooth manifolds with simply connected boundary of dimension ≥ 4. Campos, Ducoulombier, Willwacher, and the author [Cam+18a] studied framed configuration spaces of orientable closed smooth manifolds, i.e. configurations of points equipped with trivializations of the tangent spaces.

In each case, we used models for the little disks operads or their variants. Indeed, a closed manifold is locally homeomorphic to $\mathbb{R}^n$. As we saw above, configuration spaces of $\mathbb{R}^n$ are linked to the little disks operad D_n. The formality of D_n, and more precisely its proof by Kontsevich and Lambrechts–Volić, was essential in [Idr19; CW16]. A manifold with boundary is locally homeomorphic to the upper half-space $\mathbb{H}^n$. By analogy, configuration spaces of $\mathbb{H}^n$ are linked to the Swiss-Cheese operad SC_n. While SC_n is not formal, Willwacher [Wil15] defined a model for the real homotopy type of SC_n. We used this model extensively in [Cam+18b]. For framed configuration spaces [Cam+18a], we used the model for the framed little disks operad due to Khoroshkin–Willwacher [KW17].

The overarching goal of the present paper is to provide a stepping stone to study configuration spaces of more general manifolds. More precisely, let N be a closed manifold and $M \subset N$ be an embedded submanifold of codimension ≥ 2. Our goal is to study the configuration spaces of the complement $N \setminus M$, for example the complement of a knot $S^3 \setminus K$. Such a pair (N, M) is locally homeomorphic to the stratified space $(\mathbb{R}^n, \mathbb{R}^m)$. Using the analogy above, configuration spaces of $(\mathbb{R}^n, \mathbb{R}^m)$ are linked to the operad VSC_{mn}. Hence we hope that the proof of the formality of VSC_{mn} can be adapted in order to define models for the real homotopy type of $\operatorname{Conf}_{N\setminus M}(k)$.

Let us note that Willwacher studied another generalization of the Swiss-Cheese operad, called the “extended Swiss-Cheese operad” ESC_{mn} [Wil17]. He proved that this operad is formal for $n - m \geq 2$. The difference between ESC_{mn} and VSC_{mn} can be seen at the level of configuration spaces. In VSC_{mn}, the aerial disks are forbidden from touching the “ground” D^m, whereas this is allowed in ESC_{mn}. We refer to Remark 2.11 for a precise statement. The formality of ESC_{mn} is equivalent to the formality of $\mathsf{D}_m \subset \mathsf{D}_n$ [Wil17, Section 5.1]. It does not seem easy to adapt the argument for VSC_{mn}, as it is obtained by removing a subspace from ESC_{mn}, an operation which is usually difficult to deal with in homotopy theory.

Proof The proof of our theorem is inspired by Kontsevich's proof of the formality of the little disks operad and its improvement by Lambrechts–Volić [Kon99; LV14]. For technical reasons, it is necessary to consider the Fulton–MacPherson compactification $\mathsf{FM}_n(k)$ of $\operatorname{Conf}_k(\mathbb{R}^n)$ [AS94; FM94; Sin04]. The collection FM_n assembles to form a topological operad which has the same homotopy type as D_n. The goal is to find a zigzag, in the category of cooperads in commutative differential-graded algebras, between the cohomology of FM_n and the forms on FM_n. Kontsevich builds a cooperad graphs_n, spanned by graphs with two kinds of vertices (external and internal), with a differential designed to kill the relations in $H^*(\mathsf{FM}_n)$. There is a quotient quasi-isomorphism $\mathsf{graphs}_n \to H^*(\mathsf{FM}_n)$, and a quasi-isomorphism $\omega : \mathsf{graphs}_n \to \Omega(\mathsf{FM}_n)$ given by integrals.

Using the same pattern, we first define VFM_{mn}, a variant of VSC_{mn} inspired by the Fulton–MacPherson compactification. We build an intermediate cooperad of graphs, $\mathsf{vgraphs}_{mn}$, which serves as a bridge between the cohomology $H^*(\mathsf{VFM}_{mn})$ and the forms $\Omega^*(\mathsf{VFM}_{mn})$. The definition of $\mathsf{vgraphs}_{mn}$ is inspired by Willwacher's model for the Swiss-Cheese operad [Wil15] and the graph complex used in [KW17, Section 8]. The map $\mathsf{vgraphs}_{mn} \to \Omega^*(\mathsf{VFM}_{mn})$ is defined by integrals. Unfortunately, we cannot find a direct map $\mathsf{vgraphs}_{mn} \to H^*(\mathsf{VFM}_{mn})$, as the differential of $\mathsf{vgraphs}_{mn}$ depends on non-explicit integrals. However, using vanishing results on the cohomology of some graph complex, we are able to simplify $\mathsf{vgraphs}_{mn}$ up to homotopy, and then map it to $H^*(\mathsf{VFM}_{mn})$.

Outline In Section 1, we write down some necessary background on operads, the little disks operads, and the theory of piecewise algebraic forms. In Section 2, we define the Fulton–MacPherson compactification VFM_{mn} and we compare it with VSC_{mn} and with the operad $\mathsf{Disk}^{\mathrm{fr}}_{m\subset n}$ of locally constant factorization algebras on $\{\mathbb{R}^m \subset \mathbb{R}^n\}$ [AFT17]. We give examples of VSC_{mn}-algebras based on relative iterated loop spaces. In Section 3, we compute the cohomology of the operad VSC_{mn}. We also give a presentation by generators and relations of its homology $\mathsf{vsc}_{mn} = H_*(\mathsf{VSC}_{mn})$. In Section 4, we start by reviewing Kontsevich's proof of the formality of the little disks operad, and we define the cooperad $\mathsf{vgraphs}_{mn}$ that will be used to adapt that proof to VFM_{mn}. We moreover construct the map from $\mathsf{vgraphs}_{mn}$ to forms on VFM_{mn} using integrals. Finally, in Section 5, we complete the proof of the main theorem. We first show that certain integrals used in the definition of $\mathsf{vgraphs}_{mn}$ can be simplified. Then we construct the map into $H^*(\mathsf{VFM}_{mn})$ and we show that all our maps are quasi-isomorphisms. In Appendix A, we define twisting of relative cooperads. In Appendix B, we sketch a proof that $\mathsf{VFM}_{mn}(U, V)$ is an SA manifold and that the canonical projections are SA bundles.

Acknowledgments The author would like to thank Benoit Fresse, Matteo Felder, Muriel Livernet, and Thomas Willwacher for fruitful discussions and helpful comments. The author would also like to thank the anonymous referee for several helpful remarks and for pointing out an error. The author acknowledges support from ERC StG 678156–GRAPHCPX.

1. Prerequisites

Throughout this paper, we work with cohomologically graded dg-modules over the base field $\mathbb{R}$ (except in Section 3.1 where we work over $\mathbb{Z}$). For us, the cohomology (resp. homology) of a space is concentrated in nonnegative (resp. nonpositive) degrees.

1.1. Operads

We work extensively with operads and cooperads. General references include [LV12] and [Fre17, Part I(a)]. We assume basic proficiency with the theory.

For some applications, especially when working with cooperads, it is easier to label the inputs of an operation by elements of an arbitrary finite set rather than $\{1, \dots, k\}$. Briefly, we fix a base symmetric monoidal category $\mathcal{C}$ (e.g. dg-modules). Let Bij be the category of finite sets and bijections A symmetric collection is a functor $\mathsf{Bij} \to \mathcal{C}$. For $k \geq 0$, we let $\underline{k} = \{1, \dots, k\}$ (in particular $\underline{0} = \varnothing$). A symmetric collection M can equivalently be seen as a sequence $\{\mathsf{M}(n) \coloneqq \mathsf{M}(\underline{n})\}_{n \geq 0}$ with each $\mathsf{M}(n)$ equipped with an action of the symmetric group $\Sigma_n = \mathrm{Aut}_{\mathsf{Bij}}(\underline{n})$.

For a pair of finite sets $W \subset U$, we define the quotient $U/W = (U \setminus W) \sqcup \{*\}$. For example $U/\varnothing = U \sqcup \{*\}$. An operad P is a symmetric collection equipped with composition maps $\circ_W : \mathsf{P}(U/W) \otimes \mathsf{P}(W) \to \mathsf{P}(U)$, for each pair $W \subset U$, satisfying the usual axioms. Dually, a cooperad C is a symmetric collection equipped with cocomposition maps $\circ^\vee_W : \mathsf{C}(U) \to \mathsf{C}(U/W) \otimes \mathsf{C}(W)$. Following [Fre17], we call "Hopf cooperad" a cooperad in the category of commutative differential-graded algebras (CDGAs).

We also deal with some special particular two-colored operads called "relative operads" [Vor99] or "Swiss-Cheese type operads" [Wil16]. A compact definition is the following: given an operad P, a relative P-operad is an operad in the category of right P-modules. If we unpack this definition, a relative P-operad is a "bisymmetric collection", i.e. a functor $\mathsf{Q} : \mathsf{Bij} \times \mathsf{Bij} \to \mathcal{C}$, equipped with two kinds of composition maps:

$$\begin{aligned} \circ_T &: \mathsf{Q}(U, V/T) \otimes \mathsf{P}(T) \to \mathsf{Q}(U, V) && \text{for } V \subset T, \\ \circ_{W,T} &: \mathsf{Q}(U/W, V) \otimes \mathsf{Q}(W, T) \to \mathsf{Q}(U, V \sqcup T) && \text{for } W \subset U, \end{aligned}$$

as well as a unit, satisfying associativity, equivariance, and unitality axioms. We will often write "the operad Q", and P will be implicit from the context. A relative C-cooperad is defined dually. We also apply the adjective "Hopf" to denote such cooperads in the category of CDGAs.

1.2. Little disks and variants

Fix some $n \geq 0$. Let us define the little n-disks operad D_n. Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed disk. The space $\mathsf{D}_n(r)$ is the space of maps $c : (D^n)^{\sqcup r} \to D^n$ such that: (i) each $c_i : D^n \to D^n$ is an embedding given by a composition of a translation and a positive rescaling; (ii) the interiors of two different little disks are disjoint, i.e. $c_i(\mathring{D}^n) \cap c_j(\mathring{D}^n) = \varnothing$ for $i \neq j$. Using composition of embeddings, the collection $\mathsf{D}_n = \{\mathsf{D}_n(r)\}_{r \geq 0}$ forms a topological operad.

Let us also describe two operads with the same homotopy type as D_n. The first comes from the link between the little disks operads and configuration spaces. We will need it for technical reasons. Given some space X, its rth (ordered) configuration space is:

$$\mathrm{Conf}_X(r) \coloneqq \{(x_1, \dots, x_k) \in X^k \mid \forall i \neq j,\, x_i \neq x_j\}. \tag{1.1}$$

More generally, for a finite set U, $\mathrm{Conf}_X(U)$ is the set of injections $U \hookrightarrow X$.

Given an element of $\mathsf{D}_n(r)$, we can forget the radii and keep the center of each disk. This map defines a homotopy equivalence $\mathsf{D}_n(r) \to \mathrm{Conf}_{\mathbb{R}^n}(r)$. However, it is not possible to see the operad structure of D_n on the configuration spaces directly. To remedy this deficiency, we can use the Fulton–MacPherson compactification $\mathsf{FM}_n(r)$ of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ [FM94; AS94] (see also [Sin04] and [LV14, Chapter 5]). First, note that there is an action of the group $\mathbb{R}^n \rtimes \mathbb{R}_{>0}$ of translations and positive rescaling on $\mathrm{Conf}_{R^n}(r)$. We can thus consider the quotient $\underline{\mathrm{Conf}}_n(r) \coloneqq \mathrm{Conf}_{\mathbb{R}^n}(r)/\mathbb{R}^n \rtimes \mathbb{R}_{>0}$, and the quotient map is a homotopy equivalence. There is an embedding:

$$(\theta_{ij}, \delta_{ijk}) : \underline{\mathrm{Conf}}_n(r) \to (S^{n-1})^{\binom{r}{2}} \times [0, +\infty]^{\binom{r}{3}}. \tag{1.2}$$

Given some $x \in \mathrm{Conf}_{\mathbb{R}^n}(r)$, we have $\theta_{ij}(x) \coloneqq (x_i - x_j)/\|x_i - x_j\|$ which records the direction from i to j, while $\delta_{ijk}(x) = \|x_i - x_j\|/\|x_i - x_k\|$ records the relative distance.

Then the Fulton–MacPherson compactification $\mathsf{FM}_n(r)$ is the closure of the image of the embedding above. It is a smooth manifold with corners and a semi-algebraic set (see Section 1.3), of dimension $nr - n - 1$ if $r \geq 2$ and reduced to a point otherwise. Its interior is $\underline{\mathrm{Conf}}_n(r)$, and the inclusion is a deformation retract [Sal01, Proposition 2.4]. For example, we have $\mathsf{FM}_n(0) = \mathsf{FM}_n(1) = \{*\}$ and $\mathsf{FM}_n(2) = S^{n-1}$. There is an operad structure on the collection FM_n, defined by explicit formulas in $(S^{n-1})^{\binom{r}{2}} \times [0, +\infty]^{\binom{r}{3}}$, see [LV14, Section 5.2]. The operads FM_n and D_n have the same homotopy type, see [Mar99] and [Sal01, Proposition 4.3].

The second variant of the little disks operads serves as motivation for the paper. The little disks operad D_n has the same homotopy type as the operad $\mathsf{Disk}_n^{\mathrm{fr}}$ of locally constant framed factorization algebras on $\mathbb{R}^n$. This operad is defined in the ∞-categorical setting in [Lur17, Definition 5.4.2.10] (where it is denoted by $\mathbb{E}_B^{\otimes}$ with B a singleton) or [AFT17, Notation 2.8]. Consider $\mathbb{R}^n$ with its canonical parallelization $\tau : \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\cong} T\mathbb{R}^n$. The space $\mathsf{Disk}_n^{\mathrm{fr}}(k)$ is given by collections of the form $(c_1, h_1, \dots, c_k, h_k)$ where the c_i are disjoint embeddings of $\mathbb{R}^n$ into $\mathbb{R}^n$ and h_i is a homotopy between $dc_i \circ \tau$ and τ. The collection $\mathsf{Disk}_n^{\mathrm{fr}}$ forms a topological operad by considering composition of embeddings. The proof that $\mathsf{Disk}_n^{\mathrm{fr}} \simeq \mathsf{D}_n$ is almost identical to the proof of [Sal01, Proposition 3.9]. The advantage of this definition is that we can take any stratified (parallelized) space instead of $\mathbb{R}^n$. This paper is devoted to the operad associated to $\{\mathbb{R}^m \subset \mathbb{R}^n\}$.

Remark 1.3. There is an issue with units (i.e. the component $\mathsf{Disk}_n^{\mathrm{fr}}(0)$) in the ∞-categorical setting, namely the connection with the Ran space [Lur17, Section 5.5]. What we wrote above is true in our strict setting, but the name "operad governing locally constant framed factorization algebras on $\mathbb{R}^n$" might be a bit abusive if we allow $k = 0$. This is of no consequence in this paper.

1.3. Semi-algebraic sets and PA forms

For technical reasons, we will need to use the technology of semi-algebraic (SA) sets and piecewise algebraic (PA) forms [Har+11]. An SA set is a subset of $\mathbb{R}^N$ (for some N) which is obtained as finite unions of finite intersections of sets defined by polynomial equalities and inequalities. One can also define SA manifolds [Har+11, Definition 2.1] and SA bundles [Har+11, Definition 8.1].

Example 1.4. The Fulton–MacPherson compactification $\mathsf{FM}_n(k)$ is an SA manifold [LV14, Proposition 5.1.2]. The projection $p_{ij} : \mathsf{FM}_n(k) \to \mathsf{FM}_n(2)$ which forgets all points but two is an SA bundle [LV14, Theorem 5.3.2].

Given an SA set X, a minimal form on X is a form of the type $f_0 df_1 \wedge \dots \wedge df_k$, where the $f_i : X \to \mathbb{R}$ are SA maps [Har+11, Section 5.2]. A PA form is obtained as obtained by integrating a minimal form along the fibers of an SA bundle (or more generally, a strongly continuous family of chains) [Har+11, Section 5]. The complex $\Omega^*_{\mathrm{PA}}(X)$ of all PA forms on X is a CDGA. If X is compact, then $\Omega^*_{\mathrm{PA}}(X)$ is a model for the real homotopy type of X [Har+11, Theorem 6.1].

The reason we need to consider such PA forms is the existence of the map which integrates along the fibers of a PA bundle, which satisfies a good Stokes' formula [Har+11, Section 8]. This is necessary for our purposes, because the projection map p_{ij} above is a PA bundle but not a submersion [LV14, Example 5.9.1].

Let P be an operad in compact SA sets. Then the symmetric sequence $\Omega^*_{\mathrm{PA}}(\mathsf{P})$ is not actually a Hopf cooperad, because the Künneth quasi-isomorphisms go in the wrong direction. However, if C is a Hopf cooperad, then it is possible to define a morphism $\mathsf{C} \to \Omega^*_{\mathrm{PA}}(\mathsf{P})$ as a collection of maps $\mathsf{C}(U) \to \Omega^*_{\mathrm{PA}}(\mathsf{P}(U))$ making the obvious diagrams commute [LV14, Chapter 3]. Such a morphism is called a quasi-isomorphism if it is a quasi-isomorphism in each arity. Results of Fresse [Fre18] can be adapted to Ω^*_{PA} to show that if P is cofibrant in the category of Hopf cooperads satisfying $\mathsf{P}(0) = \{*\}$ (which is true for FM_n), then a Hopf cooperad which is quasi-isomorphic to $\Omega^*_{\mathrm{PA}}(\mathsf{P})$ encodes the real homotopy type of P. Briefly, Fresse shows that there is an operadic upgrade of the functor Ω^*_{PA} which turns operads into Hopf cooperads, and that this operadic upgrade is part of a Quillen equivalence. The constructions can be extended to colored operads. For simplicity, we will treat $\Omega^*_{\mathrm{PA}}(\mathsf{P})$ as if it were an actual Hopf cooperad, recalling that each time we have a morphism into $\Omega^*_{\mathrm{PA}}(\mathsf{P})$, we actually have a morphism in the sense defined above.

Definition 1.5. An operad P in compact SA sets is formal (over $\mathbb{R}$) if there exists a zigzag of quasi-isomorphisms of Hopf cooperads $H^*(\mathsf{P}; \mathbb{R}) \leftarrow \cdot \to \Omega^*_{\mathrm{PA}}(\mathsf{P})$.

Another common definition of "formality" merely requires the dg-operads $C_*(\mathsf{P})$ and $H_*(\mathsf{P})$ to be quasi-isomorphic. The notion defined above is stronger: if P is formal, we can forget cup products and dualize to recover formality of chains.

2. Definition of VFM_{mn} and comparison

From now on and until the end of the paper, we fix integers $n - 2 \geq m \geq 1$. (In some tangential remarks, we will consider $n = m + 1$.) We identify $\mathbb{R}^m$ as the subspace of $\mathbb{R}^n$ given by $\mathbb{R}^m \times \{0\}^{n-m}$.

2.1. The compactification and its boundary

We define the colored configuration spaces by:

$$\mathrm{Conf}_{mn}(U, V) \coloneqq \mathrm{Conf}_{\mathbb{R}^m}(U) \times \mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(V) \subset \mathrm{Conf}_{\mathbb{R}^n}(U \sqcup V). \tag{2.1}$$

Roughly speaking, $\mathrm{Conf}_{mn}(U, V)$ is the set of configurations of two kinds of points in $\mathbb{R}^n$: U "terrestrial" points in $\mathbb{R}^m$, and V "aerial" points in $\mathbb{R}^n \setminus \mathbb{R}^m$. We will reuse the terminology "aerial/terrestrial" throughout the paper.

There is a natural action of the group $\mathbb{R}^m \rtimes \mathbb{R}_{>0}$ of translations and positive rescalings on $\mathrm{Conf}_{mn}(U, V)$. We define the quotient:

$$\underline{\mathrm{Conf}}_{mn}(U, V) \coloneqq \mathrm{Conf}_{mn}(U, V) / (\mathbb{R}^m \rtimes \mathbb{R}_{>0}). \tag{2.2}$$

We can identify elements of $\underline{\mathrm{Conf}}_{mn}(U, V)$ with configurations of radius 1 such that the barycenter belongs to $\{0\}^m \times \mathbb{R}^{n-m} \subset \mathbb{R}^n$. Since $\mathbb{R}^m \rtimes \mathbb{R}_{>0}$ is contractible, the quotient map is a weak homotopy equivalence. If $\#U + 1\#V \geq 2$, then the action is free, smooth, and proper, thus $\underline{\mathrm{Conf}}_{mn}(U, V)$ is a manifold of dimension $m\#U + n\#V - m - 1$ [Lee03, Theorem 7.10]. However, if $\#U \leq 1$ and $\#V = 0$, then $\underline{\mathrm{Conf}}_{mn}(U, V)$ is merely a point. We define an embedding

$$(\theta_{ij}, \delta_{ijk}, \alpha_v) : \underline{\mathrm{Conf}}_{mn}(U, V) \hookrightarrow (S^{n-1})^{\binom{U \sqcup V}{2}} \times [0, +\infty]^{\binom{U \sqcup V}{3}} \times (S^{n-m-1})^V, \tag{2.3}$$

$$\theta_{ij}(x) \coloneqq (x_i - x_j)/\|x_i - x_j\|, \tag{2.4}$$

$$\delta_{ijk}(x) \coloneqq \|x_i - x_j\|/\|x_i - x_k\|, \tag{2.5}$$

$$\alpha_v(x) \coloneqq p_{(\mathbb{R}^m)^\perp}(x_v)/\|p_{(\mathbb{R}^m)^\perp}(v)\|, \tag{2.6}$$

where $p_{(\mathbb{R}^m)^\perp}$ is the orthogonal projection on $(\mathbb{R}^m)^\perp = \{0\}^m \times \mathbb{R}^{n-m} \subset \mathbb{R}^n$.

Definition 2.7. The space $\mathsf{VFM}_{mn}(U, V)$ is the closure of the image of the embedding $(\theta_{ij}, \delta_{ijk}, \vartheta_v)$ inside $(S^{n-1})^{\binom{U \sqcup V}{2}} \times [0, +\infty]^{\binom{U \sqcup V}{3}} \times (S^{n-m-1})^V$.

Example 2.8. We have $\mathsf{VFM}_{mn}(U, \varnothing) = \mathsf{FM}_m(U)$ and $\mathsf{VFM}_{mn}(0, 1) = S^{n-m-1}$.

Proposition 2.9 (Appendix B)**.** *The space $\mathsf{VFM}_{mn}(U, V)$ is a compact semi-algebraic manifold and a smooth manifold with corners. Its dimension is $m\#U + n\#V - m - 1$ if $2\#V + \#U \geq 2$, and zero otherwise. The projections $p_{U,V} : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U, V)$ are SA bundles.*

Proposition 2.10. *The collection VFM_{mn} assembles to form a relative FM_n-operad in compact SA sets.*

Proof. We can define operadic structure maps on VFM_{mn} by straightforward formulas in the coordinates $(S^{n-1})^{\cdots} \times [0, +\infty]^{\cdots} \times (S^{n-m-1})^{\cdots}$. For example, for $T \subset V$, $x \in \mathsf{VFM}_{mn}(U, V/T)$, and $y \in \mathsf{FM}_n(T)$, we have $\theta_{ij}(x \circ_T y) = \theta_{ij}(y)$ if $i, j \in T$, and $\theta_{ij}(x \circ_T y) = \theta_{[i][j]}(x)$ otherwise. The other formulas are defined similarly case by case. Compare with [LV14, Section 5.2] for the case of FM_n. ☐

Remark 2.11. The operad VFM_{mn} is not homotopy equivalent to the operad ESC_{mn} considered by Willwacher [Wil17]. Recall that $\mathsf{ESC}_{mn}(U, V) \coloneqq \mathsf{D}_n(U \sqcup V) \times_{\mathsf{D}_n(U)} \mathsf{D}_m(U)$, where $\mathsf{D}_n(U \sqcup V) \to \mathsf{D}_n(U)$ is the projection that forgets disks and $\mathsf{D}_m(U) \to \mathsf{D}_n(U)$ is the usual embedding. The crucial difference is that we do not allow "aerial" points to be on $\mathbb{R}^m$. For example, $\mathsf{VFM}_{mn}(0, 1) = S^{n-m-1} \not\simeq \mathsf{ESC}_{mn}(0, 1) \simeq *$, and $\mathsf{VFM}_{mn}(1, 1) \simeq S^{n-m-1} \not\simeq \mathsf{ESC}_{mn}(1, 1) \simeq S^{n-1}$.

Proposition 2.12. *We have a decomposition:*

$$\partial \mathsf{VFM}_{mn}(U, V) = \bigcup_{T \in \mathcal{BF}'(V)} \operatorname{im} \circ_T \cup \bigcup_{(W,T) \in \mathcal{BF}''(U;V)} \operatorname{im} \circ_{W,T},$$

where the boundary faces are indexed by:

$$\mathcal{BF}'(V) \coloneqq \{T \subset V \mid \#T \geq 2\},$$

$$\mathcal{BF}''(U; V) \coloneqq \{(W \subset U, T \subset V) \mid \#(W \cup T) < \#(U \cup V) \text{ and } 2 \cdot \#T + \#W \geq 2\}.$$

Each of these boundary faces is a compact SA subset of the boundary, and the intersection of two distinct faces is of positive codimension.

Proof. Adapting the proofs of [LV14, Proposition 5.4.1] is straightforward. $\square$

If $p : E \to B$ is an SA bundle, then its fiberwise boundary is $E^\partial = \bigcup_{x \in B} \partial p^{-1}(x)$, see [Har+11, Definition 8.1]. If p is an SA bundle of rank k, then $p^\partial : E^\partial \to B$ is an SA bundle of rank $k-1$. It is instructive to compute the fiberwise boundary of $\mathrm{pr}_1 : [0,1]^2 \to [0,1]$. We see that E^∂ is not $\partial E = \bigcup_{x \in B} p^{-1}(x) \cap \partial E$. The fiberwise Stokes formula for PA forms reads $d(p_* \alpha) = p_*(d\alpha) \pm p_*^\partial \alpha$ [Har+11, Proposition 8.12].

Proposition 2.13. *The fiberwise boundary of the canonical projection $p_{U,V} : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U,V)$ is given by:*

$$\mathsf{VFM}_{mn}^\partial(U,V) = \bigcup_{T \in \mathcal{BF}'(V,J)} \operatorname{im} \circ_T \cup \bigcup_{(W,T) \in \mathcal{BF}''(U,I;V,J)} \operatorname{im} \circ_{W,T} \subset \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J),$$

where the subsets $\mathcal{BF}'(V,J) \subset \mathcal{BF}'(V \sqcup J)$ and $\mathcal{BF}''(U,I;V,J) \subset \mathcal{BF}''(U \sqcup I, V \sqcup J)$ are respectively defined by the conditions $\#(T \cap J) \leq 1$ and by $((U \subset W$ and $V \subset T)$ or $(V \cap T = \varnothing$ and $\#(U \cap W) \leq 1))$.

Proof. We can adapt the proof of [LV14, Proposition 5.7.1] immediately. We use that $\mathsf{VFM}_{mn}(U,V)$ is a compact manifold with interior $\underline{\mathrm{Conf}}_{mn}(U,V)$. Thus $\mathsf{VFM}_{mn}^\partial(U,V) \cap p_{U,V}^{-1}(\underline{\mathrm{Conf}}_{mn}(U,V)) = (\partial \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J)) \cap p_{U,V}^{-1}(\underline{\mathrm{Conf}}_{mn}(U,V))$. Since $p_{U,V}$ is a bundle map, we just have to compute the closure of $(\partial \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J)) \cap p_{U,V}^{-1}(\underline{\mathrm{Conf}}_{mn}(U,V))$ to get $\mathsf{VFM}_{mn}^\partial(U,V)$. We then use the decomposition of Proposition 2.12 and we check that $\mathcal{BF}'(V,J)$ and $\mathcal{BF}''(U,I;V,J)$ index the faces which are sent to the boundary under the map $p_{U,V}$. $\square$

2.2. Comparison with other operads

In this section, we compare VFM_{mn} with two operads: the operad $\mathsf{Disk}_{m \subset n}^{\mathrm{fr}}$ of locally constant factorization algebras on $\{\mathbb{R}^m \subset \mathbb{R}^n\}$, and an operad VSC_{mn} that is analogous to the Swiss-Cheese operad SC_n.

2.2.1. Comparison with $\mathsf{Disk}_{m \subset n}^{\mathrm{fr}}$

Definition 2.14 (Cf. [AFT17, Section 4.3])**.** As a space, $\mathsf{Disk}_{m \subset n}^{\mathrm{fr}}(U,V)$ is given by framed embeddings $\gamma = (\gamma_i)_{i \in U \sqcup V} : (\mathbb{R}^n)^{\sqcup U} \sqcup (\mathbb{R}^n)^{\sqcup V} \to \mathbb{R}^n$ such that (i) for $u \in U$, $\gamma_u(\mathbb{R}^m) \subset \mathbb{R}^m$, and (ii) for $v \in V$, $\gamma_v(\mathbb{R}^n) \subset \mathbb{R}^n \setminus \mathbb{R}^m$. It is a relative operad (Section 1.1) over the operad $\mathsf{Disk}_n^{\mathrm{fr}}$ (Section 1.2) using composition of embeddings.

Proposition 2.15. *There exists a zigzag a weak homotopy equivalences of operads between $(\mathsf{Disk}_{m \subset n}^{\mathrm{fr}}, \mathsf{Disk}_n^{\mathrm{fr}})$ and $(\mathsf{VFM}_{mn}, \mathsf{FM}_n)$.*

Proof. Briefly, recall that for a topological operad P, the Boardman–Vogt resolution $W\mathsf{P} \xrightarrow{\sim} \mathsf{P}$ is a canonical resolution of P. The elements of $W\mathsf{P}(k)$ are rooted trees with k leaves, such that each internal vertex is labeled by an element of P (of arity equal to the number of incoming edges), and each internal edge is labeled by a time parameter

$t \in [0,1]$. If an edge is labeled by $t = 0$, then the tree is identified with the quotient obtained by contracting the edge and applying the operadic structure of P on the labels of the corresponding vertices. The operadic structure of $W\mathsf{P}$ is obtained by grafting trees, assigning the time parameter $t = 1$ to the new edge. The map $W\mathsf{P} \to \mathsf{P}$ simply contracts all the edges, and applies the operadic structure of P to the vertices.

A weak equivalence $W\mathsf{Disk}_n^{\mathrm{fr}} \xrightarrow{\sim} \mathsf{FM}_n$ (or its analogue for D_n) was constructed in [Mar99, Section 3] and [Sal01, Proposition 3.9]. It is obtained by taking an element of $W\mathsf{Disk}_n^{\mathrm{fr}}$, rescaling the disks by the time parameter associated to the edge, and keeping the centers of the disks (applying the operadic composition if the time parameter goes to zero). Using the same method as in [Mar99; Sal01], there is a weak equivalence $W\mathsf{Disk}_{m\subset n}^{\mathrm{fr}} \to \mathsf{VFM}_{mn}$ (compatible with the operad structures) that extends the map $\mathsf{Disk}_{m\subset n}^{\mathrm{fr}} \to \mathrm{Conf}_{mn}$ which keeps only the centers of the disks.

It remains to check that the induced map $W\mathsf{Disk}_{m\subset n}^{\mathrm{fr}} \to \mathsf{VFM}_{mn}$ is a homotopy equivalence in each arity. It is sufficient to check that $\mathrm{Conf}_{mn} \to \mathsf{VFM}_{mn}$ is a weak equivalence and to conclude by the 2-out-of-3 property. This is done by considering an explicit deformation retract, see [Sal01, Proposition 2.5]. □

2.2.2. Comparison with VSC_{mn} and a conjecture

We can also compare VFM_{mn} with the higher-codimensional Swiss-Cheese operad VSC_{mn}. Let $D^m = D^n \cap \mathbb{R}^m$ for convenience.

Definition 2.16. The space $\mathsf{VSC}_{mn}(U,V)$ is the space of maps $c : (D^n)^{U \sqcup V} \hookrightarrow D^n$ satisfying: (i) for all i, $c_i : D^n \hookrightarrow D^n$ is an embedding obtained by composing a translation and a positive rescaling; (ii) for $i \neq j$, we have $c_i(\mathring{D}^n) \cap c_j(\mathring{D}^n) = \varnothing$; (iii) for $u \in U$, we have $c_u(D^m) \subset D^m$; (iv) for $v \in V$, we have $c_v(D^n) \cap D^m = \varnothing$. Using composition of embeddings, VSC_{mn} is a relative D_n-operad.

Remark 2.17. When $n = m+1$, this operad is not the usual Swiss-Cheese operad SC_n. In fact, SC_n is a suboperad of $\mathsf{VSC}_{(n-1)n}$, by considering the connected components where all the aerial disks are in the upper half-disk.

Proposition 2.18. *There exists a zigzag of weak homotopy equivalences of operads* $(\mathsf{VSC}_{mn}, \mathsf{D}_n) \simeq (\mathsf{VFM}_{mn}, \mathsf{FM}_n)$.

Proof. There is an embedding of operads $\mathsf{D}_n \subset \mathsf{Disk}_n^{\mathrm{fr}}$. Indeed, an embedding of D^n into D^n given by translation and positive rescaling can be restricted to an embedding of $\mathbb{R}^n$ into $\mathbb{R}^n$, and such an embedding clearly preserves the framing up to homotopy. Similarly, we have an embedding $\mathsf{VSC}_{mn} \subset \mathsf{Disk}_{m\subset n}^{\mathrm{fr}}$. We also have obvious factorizations:

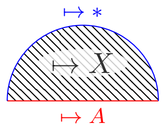
$$\begin{tikzcd} \mathsf{D}_n(U) \arrow[rr, hook] \arrow[dr, "\sim"'] & & \mathsf{Disk}_n^{\mathrm{fr}}(U), \\ & \mathrm{Conf}_{\mathbb{R}^n}(U) \arrow[ur, "\sim"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{VSC}_{mn}(U,V) \arrow[rr, hook] \arrow[dr, "\sim"'] & & \mathsf{Disk}_{m\subset n}^{\mathrm{fr}}(U,V). \\ & \mathrm{Conf}_{mn}(U,V) \arrow[ur, "\sim"'] & \end{tikzcd}$$

Hence we can conclude by the 2-out-of-3 property and Proposition 2.15. □

Let us also give examples of VSC_{mn}-algebras. For motivation, let us first recall algebras over D_n and SC_n. For a pointed space $* \in X$, the iterated loop space $\Omega^n X$ is the space of maps $\gamma : D^n \to X$ such that $\gamma(\partial D^n) = *$. The space $\Omega^n X$ is an algebra over D_n. Conversely, the recognition principle states that any "grouplike" D_n-algebra is weakly equivalent to an iterated loop space [BV68; May72].

For a pair of pointed topological spaces $* \in A \subset X$, there is an associated relative iterated loop space $\Omega^n(X, A) = \mathrm{hofib}(\Omega^{n-1}A \to \Omega^{n-1}X)$. Concretely, consider the upper half-disk $D^n_h = D^n \cap \mathbb{H}^n$. Its boundary ∂D^n_h is obtained by gluing the lower disk $\partial_- D^n_h = D^n \cap \partial\mathbb{H}^n \cong D^{n-1}$ to the upper hemisphere $\partial_+ D^n_h = \partial D^n \cap \mathbb{H}^n \cong D^{n-1}$ along the circle $\partial D^n \cap \partial\mathbb{H}^n \cong S^{n-2}$.

The relative iterated loop space $\Omega^n(X, A)$ is then the space of maps $\gamma : D^n_h \to X$ such that $\gamma(\partial_- D^n_h) \subset A$ and $\gamma(\partial_+ D^n_h) = *$. For example, we have that $\Omega^1(X, A) = \{\gamma : [0, 1] \to X \mid \gamma(0) \in A, \gamma(1) = *\}$. A sketch for $n = 2$ is on the side. The pair $(\Omega^n(X, A), \Omega^n X)$ is an algebra over the operad SC_n. Conversely, a relative recognition principle states that any algebra over that operad satisfying good properties is weakly equivalent to such a pair [Duc16; Que15; HLS16; Vie19].



By analogy, we define the (n, m)-relative iterated loop space:

$$\Omega^{n,m}(X, A) \coloneqq \{\gamma : D^n \to X \mid \gamma(D^m) \subset A \text{ and } \gamma(\partial D^n) = *\}. \tag{2.19}$$

The pair $(\Omega^{n,m}(X, A), \Omega^n X)$ is an algebra over the operad VSC_{mn}. We conjecture that an analogous relative recognition principle holds: any VSC_{mn}-algebra satisfying appropriate conditions should be weakly equivalent to such a pair.

3. (Co)homology of VSC_{mn}

In this section, we compute the integral cohomology of $\mathsf{VSC}_{mn}(U, V)$ (Definition 2.16). We then give a presentation of the operad $H_*(\mathsf{VSC}_{mn})$ by generators and relations. Unless specified, the ring of coefficients is $\mathbb{Z}$ in this section.

3.1. The cohomology as a Hopf cooperad

We will first compute the cohomology of $\mathrm{Conf}_W(l)$ with $W \coloneqq \mathbb{R}^n \setminus \mathbb{R}^m$ for $n - m \geq 2$. The computation is inspired by the methods of [Sin13]. We prove that it free as an abelian group, thus we will be able to apply Künneth's formula to get the cohomology of $\mathsf{VSC}_{mn}(k, l) \simeq \mathrm{Conf}_{\mathbb{R}^m}(k) \times \mathrm{Conf}_W(l)$ as a tensor product. Then we study what maps are induced on cohomology by the operad structure of VSC_{mn}.

Let us consider the Fadell–Neuwirth fibration, which forgets (for example) the last point of a configuration. Note that W with l points removed is homotopy equivalent to $S^{n-m-1} \vee \bigvee^l S^{n-1}$. Therefore we get fibrations, for $n - m \geq 2$:

$$S^{n-m-1} \vee \bigvee^l S^{n-1} \hookrightarrow \mathrm{Conf}_W(l+1) \xrightarrow{\pi} \mathrm{Conf}_W(l) \tag{3.1}$$

Definition 3.2. The Poincaré polynomial of X is $\mathscr{P}(X) \coloneqq \sum_{i\geq 0}(\operatorname{rk} H^i(X)) \cdot t^i$. For P, $Q \in \mathbb{N}[[t]]$, we say that $P \preceq Q$ if the coefficients of $Q - P$ are nonnegative.

Proposition 3.3. *For $n - m \geq 2$, the Poincaré polynomial of $\mathrm{Conf}_W(l)$ satisfies:*

$$\mathscr{P}(\mathrm{Conf}_W(l)) \preceq \prod_{i=0}^{l-1}(1 + t^{n-m-1} + it^{n-1}). \tag{3.4}$$

Moreover, if the equality is reached and the homology of $\mathrm{Conf}_W(l-1)$ is free as a $\mathbb{Z}$-module, then the homology of the total space $\mathrm{Conf}_W(l)$ is free too.

Proof. We use the fibrations of Equation (3.1), the Serre spectral sequence, and induction to deduce the proposition.

Note that for $n - m = 2$ then the base $\mathrm{Conf}_W(l-1)$ is not simply connected. However, we can adapt the arguments of [Coh76, Lemma 6.3] to show that the coefficient system is trivial. Let $c_1, c_2 \in \mathrm{Conf}_W(l-1)$ be two configurations and $F_1, F_2 = \pi^{-1}(c_1), \pi^{-1}(c_2) \subset \mathrm{Conf}_W(l)$ the fibers ($\simeq S^{n-m-1} \vee (S^{n-1})^{\vee l}$). Any path $\gamma \in \Omega_{c_1,c_2}\mathrm{Conf}_W(l-1)$ lifts to a path in $\mathrm{Conf}_W(l)$ by putting the lth point far from all the others (e.g. outside a ball B enclosing the compact subset $\mathrm{im}(\gamma)$). We want the induced isomorphism $h_\gamma : H_*(F_1) \to H_*(F_2)$ to be the identity.

It is clear that h_γ does not affect the fundamental class of $S^{n-m-1} = S^1$, as we can choose a representative with the lth point rotating around the axis R^m outside the ball B. The class of the ith S^{n-1} in the fiber corresponds to the lth point rotating around the ith point. This can be represented by concatenating a path η_{ij} from l to i with a small sphere σ_i around i. Consider the path γ_i given by the ith coordinate of the path γ. Then the image of the class h_γ under h_γ can be represented by $\eta_{ij} \cdot \gamma_i \cdot \sigma_i$ (see Figure 3.1 for an example). But this $(n-1)$st homology class is homologous to $\eta_{ij} \cdot \sigma_i$ for any path γ. □

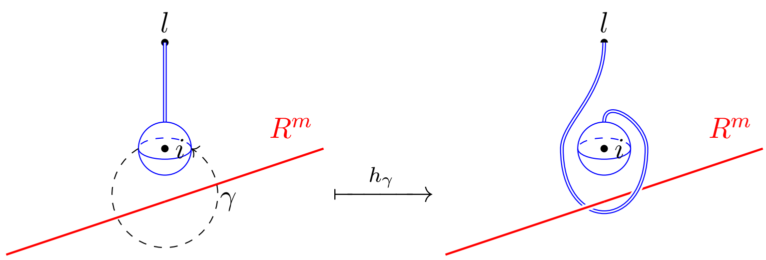


Figure 3.1: The effect of h_γ on the fundamental class (in blue) of S^{n-1}

Remark 3.5. If $n - m = 1$, then $\mathrm{Conf}_W(l)$ is not connected, so applying the spectral sequence is more difficult. However, in this case we simply have $W \cong \mathbb{R}^n \sqcup \mathbb{R}^n$ (identifying the upper and the lower open half-spaces of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself) so $\mathrm{Conf}_W(l) = \bigsqcup_{l=l'+l''} \mathrm{Conf}_{\mathbb{R}^n}(l') \times \mathrm{Conf}_{\mathbb{R}^n}(l'') \times_{\Sigma_{l'} \times \Sigma_{l''}} \Sigma_l$. Its Poincaré polynomial is

$\sum_{l=l'+l''} \frac{l!}{(l')!(l'')!} \prod_{i=0}^{l'-1}(1+it^{n-1}) \prod_{j=0}^{l''-1}(1+jt^{n-1})$, which is equal to the RHS of (3.4) by a simple induction argument.

Let us recall the usual presentation of $\mathsf{e}_n^\vee(l) \coloneqq H^*(\mathrm{Conf}_{\mathbb{R}^n}(l))$ [Arn69; Coh76]. It is a quotient of a free graded symmetric algebra, on generators ω_{ij} of degree $n-1$:

$$\mathsf{e}_n^\vee(l) = S(\omega_{ij})_{1\le i\ne j\le l}/(\omega_{ji} - (-1)^n\omega_{ij},\, \omega_{ij}^2,\, \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij}). \tag{3.6}$$

Definition 3.7. We define an algebra, with generators η_i of degree $n-m-1$:

$$\mathsf{vsc}_{mn}^\vee(0,l) \coloneqq \mathsf{e}_n^\vee(l) \otimes S(\eta_i)_{1\le i\le l}/(\eta_i^2, \eta_i\omega_{ij} - \eta_j\omega_{ij}). \tag{3.8}$$

Remark 3.9. This is very similar to the Lambrechts–Stanley model G_A [Idr19] applied to $A = H^*(D^n \setminus D^m)$ with vanishing diagonal class.

Lemma 3.10. *The Poincaré polynomial of $\mathsf{vsc}_{mn}^\vee(0,l)$ is the RHS of (3.4).* □

Proposition 3.11. *For $n-2 \ge m \ge 1$, we have a well-defined algebra map $\mathsf{vsc}_{mn}^\vee(0,l) \to H^*(\mathrm{Conf}_W(l))$, given on generators by $\omega_{ij} \mapsto \theta_{ij}^*(\mathrm{vol}_{n-1})$ and $\eta_i \mapsto \alpha_i^*(\mathrm{vol}_{n-m-1})$ (with θ_{ij}, α_i as in Section 2).*

Proof. It is clear that $\alpha_i^*(\mathrm{vol}_{n-m-1})$ squares to zero. Thus the only thing we need to check is that $\eta_i\omega_{ij} = \eta_j\omega_{ij}$ holds in $H^*(\mathrm{Conf}_W(l))$. It is sufficient to check this on $\mathrm{Conf}_W(2)$, as the three classes involved are pulled back from there. The product $S^{n-1} \times S^{n-m-1}$ maps into $\mathrm{Conf}_W(2)$: the first factor describes the rotation of point 2 around point 1, while the second factor describes the rotation of the pair around the axis $\mathbb{R}^m$. Let σ be the pushforward of the fundamental class of $S^{n-1} \times S^{n-m-1}$ along this embedding. Thanks to Equation (3.4), $H^{n-m-1+n-1}(\mathrm{Conf}_2(W))$ is at most one-dimensional. Since both cohomology classes $\eta_1\omega_{12}$ and $\eta_2\omega_{12}$ evaluates to 1 on σ, it follows that $\eta_1\omega_{12} = \eta_2\omega_{12}$. □

Proposition 3.12. *For $n-2 \ge m \ge 1$, the map $\mathsf{vsc}_{mn}^\vee(0,l) \to H^*(\mathrm{Conf}_W(l))$ is an isomorphism.*

Proof. Our proof is inspired by the proof of [Sin13, Theorem 4.9]. Let us check that it is an isomorphism over each field $\Bbbk \in \{\mathbb{Q}, \mathbb{F}_p\}$. Then by induction and using the Serre spectral sequence of Proposition 3.11, we show that the homology of $\mathrm{Conf}_W(l)$ is free as a $\mathbb{Z}$-module. This will then imply that the map of the proposition is an isomorphism over $\mathbb{Z}$ using the universal coefficients theorem.

Given Inequality (3.4), the universal coefficients theorem, and the fact that we are working over a field, it is sufficient to show that the map tensored with $\Bbbk$ is injective. We may create classes in $H_*(\mathrm{Conf}_W(l);\Bbbk)$ using trees where vertices are possibly decorated by a loop (to represent the homology class corresponding to η_i). The Jacobi relation is satisfied, as these homology classes come from the subspace $\mathrm{Conf}_{\mathbb{R}^n}(l)$ (with $\mathbb{R}^n$ being e.g. the upper half-space). Similarly, the classes from $\mathsf{vsc}_{mn}^\vee(l)$ correspond to graphs modded out by the Arnold relation, with each connected component possibly decorated by η. The duality pairing between homology and cohomology classes correspond to the pairing between graphs and trees of [Sin13]. As this pairing is nondegenerate [Sin13, Theorem 4.7], we therefore get that $\mathsf{vsc}_{mn}^\vee(l) \to H^*(\mathrm{Conf}_W(l))$ is injective. □

Definition 3.13. More generally, we define, for integers $k, l \geq 0$:

$$\mathsf{vsc}_{mn}^\vee(k,l) \coloneqq \mathsf{e}_m^\vee(k) \otimes \mathsf{vsc}_{mn}^\vee(0,l). \tag{3.14}$$

For cosmetic reasons, for $m \geq 2$ we will write $\tilde{\omega}_{ij}$ for the generators of $\mathsf{e}_m^\vee(k)$, to distinguish them from the generators of $\mathsf{vsc}_{mn}^\vee(0,l)$. Recall that for $m = 1$ then $\mathsf{e}_1^\vee(k)$ is simply the algebra of functions on Σ_k. The CDGA $\mathsf{vsc}_{mn}^\vee(k,l)$ is equipped with the obvious action of the product of the symmetric groups $\Sigma_k \times \Sigma_l$. Therefore, we can view $\mathsf{vsc}_{mn}^\vee$ as a bisymmetric collection.

Proposition 3.15. *There is an isomorphism of algebras*

$$\mathsf{vsc}_{mn}^\vee(k,l) \cong H^*(\mathsf{VSC}_{mn}(k,l)) = H^*(\mathrm{Conf}_{\mathbb{R}^m}(k)) \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)). \tag{3.16}$$

Proof. The homology groups of $\mathrm{Conf}_{\mathbb{R}^m}(k)$ and $\mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)$ are free and finitely generated. Hence we may conclude by the Künneth formula and the homotopy equivalence $\mathsf{VSC}_{mn}(k,l) \simeq \mathrm{Conf}_{\mathbb{R}^m}(k) \times \mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)$. □

Next we turn to the cooperad structure of $\mathsf{vsc}_{mn}^\vee$. For reference, the cooperad structure of $\mathsf{e}_n^\vee$ is given as follows. For a pair $W \subset U$ and some $u \in U$, we let $[u] \in U/W$ be its class in the quotient. The cooperad structure is then (for $n \geq 2$):

$$\circ_W^\vee : \mathsf{e}_n^\vee(U) \to \mathsf{e}_n^\vee(U/W) \otimes \mathsf{e}_n^\vee(W), \quad \omega_{uv} \mapsto \begin{cases} 1 \otimes \omega_{uv} & \text{if } u,\, v \in W; \\ \omega_{[u][v]} \otimes 1 & \text{otherwise.} \end{cases} \tag{3.17}$$

On the other hand, if $n = 1$, then $\mathsf{e}_1^\vee(U) = \mathsf{Ass}^\vee(U) = \mathbb{R}[\Sigma_U]^\vee$, where $\Sigma_U = \mathsf{Bij}(U, \{1, \dots, \#U\})$ is the set of linear orders on U. The cocomposition of $\mathsf{e}_1^\vee$ is the dual of the block composition:

$$\Sigma_{U/W} \times \Sigma_W \mapsto \Sigma_U, \quad (\sigma, \tau) \mapsto \sigma \circ_W \tau, \tag{3.18}$$

where $\sigma \circ_W \tau$ is the linear order on U inherited from the linear order of U/W given by σ and the linear order of W given by τ, and positioning the elements of W at the position of $* \in U/W$. For example, $(b < a < c) \circ_a (x < y) = (b < x < y < c)$.

Proposition 3.19. *For $m \geq 2$, the CDGAs $\mathsf{vsc}_{mn}^\vee(U,V)$ assemble into a relative Hopf cooperad over $\mathsf{e}_n^\vee$, with structure maps $\mathsf{vsc}_{mn}^\vee(U,V) \xrightarrow{\circ_T^\vee} \mathsf{vsc}_{mn}^\vee(U,V/T) \otimes \mathsf{e}_n^\vee(T)$ and $\mathsf{vsc}_{mn}^\vee(U, V \sqcup T) \xrightarrow{\circ_{W,T}^\vee} \mathsf{vsc}_{mn}^\vee(U/W,V) \otimes \mathsf{vsc}_{mn}^\vee(W,T)$ given by:*

$$\circ_T^\vee(\tilde{\omega}_{uu'}) = \tilde{\omega}_{uu'} \otimes 1. \qquad \circ_{W,T}^\vee(\tilde{\omega}_{uu'}) = \begin{cases} 1 \otimes \tilde{\omega}_{uu'}, & \text{if } u, u' \in W; \\ \tilde{\omega}_{[u][u']} \otimes 1, & \text{otherwise.} \end{cases}$$

$$\circ_T^\vee(\omega_{vv'}) = \begin{cases} 1 \otimes \omega_{vv'}, & \text{if } v, v' \in T; \\ \omega_{[v][v']} \otimes 1 & \text{otherwise.} \end{cases} \qquad \circ_{W,T}^\vee(\omega_{vv'}) = \begin{cases} 1 \otimes \omega_{vv'}, & \text{if } v, v' \in T; \\ \omega_{vv'} \otimes 1 & \text{if } v, v' \in V; \\ 0 & \text{otherwise.} \end{cases}$$

$$\circ_T^\vee(\eta_v) = \eta_{[v]} \otimes 1. \qquad \circ_{W,T}^\vee(\eta_v) = \begin{cases} \eta_v \otimes 1 & \text{if } v \in V; \\ 1 \otimes \eta_v & \text{if } v \in T. \end{cases}$$

For $m = 1$, the maps $\circ_T^\vee$ and $\circ_{W,T}^\vee$ have the same behavior as above on the generators $\omega_{vv'}$ and η_v. On $\mathsf{e}_1^\vee(U) = \mathsf{Ass}^\vee(U) = \mathbb{R}[\Sigma_U^\vee]$, the map $\circ_T^\vee$ is the identity, and $\circ_{W,T}^\vee$ is defined as the dual of the block composition of Equation (3.18).

Proof. We just need to check the Hopf cooperad axioms (coassociativity, counit, compatibility with the product), which are all straightforward. □

Proposition 3.20. *The maps $(\mathsf{vsc}_{mn}^\vee, \mathsf{e}_n^\vee) \to (H^*(\mathsf{VSC}_{mn}), H^*(\mathsf{D}_n))$ define an isomorphism of relative Hopf cooperads.*

Proof. Our proof is similar to the proof of [Sin13, Theorem 6.3]. We already know that these maps are isomorphisms in each arity, so we just need to check that they are compatible with the cooperad structures on both sides.

Let us first check that $\circ_T^\vee : \mathsf{vsc}_{mn}^\vee(U,V) \to \mathsf{vsc}_{mn}^\vee(U,V/T) \otimes \mathsf{e}_n^\vee(T)$ models $\circ_T : \mathsf{VSC}_{mn}(U,V/T) \times \mathsf{D}_n(T) \to \mathsf{D}_{mn}(U,V)$. To this end, we consider the maps $\theta_{uu'} : \mathsf{VSC}_{mn}(U,V) \to S^{n-1}$, $\theta_{vv'} : \mathsf{VSC}_{mn}(U,V) \to S^{m-1}$, and $\alpha_v : \mathsf{VSC}_{mn}(U,V) \to S^{n-m-1}$ defined in Section 2. Let us check check that, when they are composed with the insertion maps $\circ_T$, we obtain the behavior in Proposition 3.19.

For the maps $\theta_{uu'}$ and $\theta_{vv'}$, this is a computation identical to the one necessary to check that $\mathsf{e}_n^\vee$ (resp. $\mathsf{sc}_m^\vee$) is the cohomology, as a cooperad, of D_n (resp. SC_m). Next we want to determine the homotopy class of the map $\alpha_v(-\circ_T-) : \mathsf{VSC}_{mn}(U,V/T)\times\mathsf{D}_n(T) \to S^{n-m-1}$ for some $v \in V$.

- If $v \notin T$, then the composite map $\alpha_v(- \circ_T -)$ factors as
$$\mathsf{VSC}_{mn}(U,V/T) \times \mathsf{D}_n(T) \xrightarrow{\mathrm{pr}_1} \mathsf{D}_{mn}(U,V/T) \xrightarrow{\alpha_v} S^{n-m-1}. \tag{3.21}$$
We therefore find that
$$\circ_T^\vee(\eta_v) = (\alpha_v(- \circ_T -))^*(\mathrm{vol}_{n-m-1}) = \mathrm{pr}_1^*(\alpha_v^*(\mathrm{vol}_{n-m-1})) = \eta_v \otimes 1. \tag{3.22}$$
- If $v \in T$, let us consider the homotopy which precomposes the embedding indexed by $[v] \in V/T$ in $\mathsf{D}_{mn}(U,V/T)$ by the map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto tx$. At the limit $t = 0$, we find the composite:
$$\mathsf{VSC}_{mn}(U,V/T) \times \mathsf{D}_n(T) \xrightarrow{\mathrm{pr}_1} \mathsf{VSC}_{mn}(U,V/T) \xrightarrow{\alpha_{[v]}} S^{n-m-1}. \tag{3.23}$$
The homotopy class of the map is constant wrt. t, thus $\circ_T^\vee(\eta_v) = \eta_{[v]} \otimes 1$.

Now let us consider the insertion map $\mathsf{VSC}_{mn}(U/W,V) \times \mathsf{VSC}_{mn}(W,T) \to \mathsf{D}_{mn}(U,V \sqcup T)$. We again need to check that the (homotopy classes of the) composite maps agree with the ones prescribed in Proposition 3.19. For the maps $\theta_{uu'}(- \circ_{W,T} -)$, this is again a computation similar to the case of D_n. For the maps $\eta_v(- \circ_{W,T} -)$, it is easy to see that $\eta_v(- \circ_{W,T} -)$ factors by the projection on one of the two factors in the product. For the maps $\theta_{vv'}(- \circ_{W,T} -)$:

- If v, v' are both in V (resp. both in T), then we find that $\theta_{vv'}(- \circ_{W,T} -)$ factors by the projection onto the factor $\mathsf{VSC}_{mn}(U/W,V)$ (resp. $\mathsf{VSC}_{mn}(W,T)$) followed by the appropriate map $\theta_{vv'}$, and thus that $\circ_{W,T}^\vee(\omega_{vv'})$ equals $\omega_{vv'} \otimes 1$ (resp. $1 \otimes \omega_{vv'}$).

- Otherwise, let us assume that $v \in V$ and $v' \in T$ (the other case is symmetrical). If we contract the appropriate disks by a homotopy which linearly decreases the radius as above, then we obtain at the limit a constant map, therefore $\circ^{\vee}_{W,T}(\omega_{vv'}) = 0$. $\square$

Remark 3.24. We can compare $\mathsf{vsc}^{\vee}_{mn}$ with the cohomology of ESC_{mn} (see Remark 2.11). Recall that $\mathsf{ESC}_{mn}(U, V)$ is the fiber product $\mathsf{D}_m(U) \times_{\mathsf{D}_n(U)} \mathsf{D}_n(U \sqcup V)$. The cohomology of ESC_{mn} is the pushout $H^*(\mathsf{ESC}_{mn}(U, V)) = \mathsf{e}^{\vee}_m(U) \otimes_{\mathsf{e}^{\vee}_n(U)} \mathsf{e}^{\vee}_n(U \sqcup V)$, see [Wil17, Proposition 4.1]. Here, the morphism $\mathsf{e}^{\vee}_n(U) \to \mathsf{e}^{\vee}_n(U \sqcup V)$ is the obvious inclusion, while the morphism $\mathsf{e}^{\vee}_n(U) \to \mathsf{e}^{\vee}_m(U)$ sends all the generators $\omega_{uu'}$ to zero. Thus $H^*(\mathsf{ESC}_{mn}(U, V)) = \mathsf{e}^{\vee}_m(U) \otimes \mathsf{e}^{\vee}_n(U \sqcup V)/\mathsf{e}^{\vee}_n(U)$. The cooperadic structure maps are defined by formulas similar to Proposition 3.19 (forgetting the η_v). We also have an inclusion $\mathsf{VSC}_{mn} \subset \mathsf{ESC}_{mn}$, which induces on cohomology the composite $\mathsf{e}^{\vee}_m(U) \otimes \mathsf{e}^{\vee}_n(U \sqcup V)/\mathsf{e}^{\vee}_n(U) \twoheadrightarrow \mathsf{e}^{\vee}_m(U) \otimes \mathsf{e}^{\vee}_n(V) \hookrightarrow \mathsf{vsc}^{\vee}_{mn}(U, V)$.

Remark 3.25. The inclusion of operads $\mathsf{D}_m \subset \mathsf{D}_n$ factors through $\mathsf{VSC}_{mn}(-, \varnothing)$. We thus get an induced left D_m-module structure on $\mathsf{VSC}_{mn}(\varnothing, -)$, which dualizes to the map $\Delta : \mathsf{vsc}^{\vee}_{mn}(\varnothing, \bigsqcup_{u \in U} V_u) \to \mathsf{vsc}^{\vee}_{mn}(U, \varnothing) \otimes \bigotimes_{u \in U} \mathsf{vsc}^{\vee}_{mn}(\varnothing, V_u)$ given by $\Delta(\eta_v) = 1 \otimes \eta_v$ (put in the corresponding $\mathsf{vsc}^{\vee}_{mn}(\varnothing, V_u)$), $\Delta(\omega_{vv'}) = 1 \otimes \omega_{vv'}$ if v and v' are in the same V_u and $\Delta(\omega_{vv'}) = 0$ otherwise.

3.2. Generators and relations for the homology

Although it is not strictly necessary for our purposes, we also describe a presentation of the homology $\mathsf{vsc}_{mn} \coloneqq H_*(\mathsf{VFM}_{mn})$ by generators and relations. It is more convenient to describe this presentation by describing the algebras over the operad. First, let us recall the presentation of $\mathsf{e}_n \coloneqq H_*(\mathsf{FM}_n)$:

Theorem 3.26 (Cohen [Coh76])**.** *An algebra over e_1 is a unital associative algebra. For $n \geq 2$, an algebra over e_n is a unital Poisson n-algebra, i.e. a unital commutative algebra equipped with a Lie bracket of degree $1 - n$ such that the bracket is a biderivation for the product.*

Proposition 3.27. *For $n - 2 \geq m \geq 1$, an algebra over vsc_{mn} is the data (A, B, f, δ) consisting of an e_m-algebra A, an e_n-algebra B, a central morphism of algebras $f : B \to A$, and a central derivation $\delta : B[n - m - 1] \to A$.*

Central means that f and δ land in the center $Z(A) = \{a \in A \mid \forall b \in A,\ [a, b] = 0\}$, where the bracket is the graded commutator ($m = 1$) or the shifted Lie bracket ($m \geq 2$). The map δ is a derivation with respect to f: $\delta(xy) = \delta(x)f(y) \pm f(x)\delta(y)$.

Proof. The proof is an exercise in dualizing the description from Proposition 3.19. The product, bracket, and unit of B are duals to $1 \in \mathsf{e}^{\vee}_n(2)$, $\omega_{12} \in \mathsf{e}^{\vee}_n(2)$, and $1 \in \mathsf{e}^{\vee}_n(0)$. The product and unit of A are duals to $1 \in \mathsf{vsc}^{\vee}_{mn}(2, 0) = \mathsf{e}^{\vee}_m(2)$ and $1 \in \mathsf{vsc}^{\vee}_{mn}(0, 0) = \mathsf{e}^{\vee}_m(0)$; its bracket is dual to $\omega_{12} \in \mathsf{vsc}^{\vee}_{mn}(2, 0) = \mathsf{e}^{\vee}_m(2)$ for $m \geq 2$. The relations between the operations acting exclusively on A (resp. B), i.e. the products, brackets, and units, follow from Theorem 3.26 above.

The morphism f is dual to $1 \in \mathsf{vsc}^\vee_{mn}(0,1)$. The derivation δ is dual to $\eta_1 \in \mathsf{vsc}^\vee_{mn}(0,1)$. The fact that f is a morphism follows from $\circ^\vee_{\{x,y\}}(1) = 1 \otimes 1$ in $\mathsf{vsc}^\vee_{mn}(\varnothing, \{x,y\})$. The fact that δ is a derivation follows from $\circ^\vee_{\{x,y\}}(\eta_x) = \circ^\vee_{\{x,y\}}(\eta_y) = \eta_* \otimes 1$ (so dually we have $\eta^\vee_* \circ 1^\vee = \eta^\vee_x + \eta^\vee_y$). The equations $[x, f(y)] = [x, \delta(y)] = 0$ follow from degrees reasons in $\mathsf{vsc}^\vee_{mn}(\{x\},\{y\})$ (or, if $m = 1$, from an explicit homotopy which shows the centrality).

Finally, the element $\omega_{xy} \in \mathsf{vsc}^\vee_{mn}(\varnothing, \{x,y\})$ is dual to $f([x,y])$, and $\eta_x\omega_{xy} = \eta_y\omega_{xy}$ is dual to $\delta([x,y])$; they do not satisfy any new relation. □

We can rephrase Proposition 3.27 more compactly. Let A be an e_m-algebra. We let $A[\varepsilon] \coloneqq A \otimes \mathbb{R}[\varepsilon]/(\varepsilon^2)$ be the algebra obtained by adjoining a new square-zero variable ε of degree $n - m - 1$. If $m \geq 2$, then there is a Poisson bracket on $A[\varepsilon]$ given by $[x + \varepsilon y, x' + \varepsilon y'] = [x, x'] + \varepsilon([x, y'] \pm [x', y])$. Then an vsc_{mn}-algebra is the data of an e_m-algebra A, an e_n-algebra B, and a central morphism $f + \varepsilon\delta : B \to A[\varepsilon]$.

Compare this result with the ∞-categorical counterparts from [AFT17, Section 4.3]. An algebra over the ∞-categorical version $\mathscr{D}isk^{\mathrm{fr}}_{m\subset n}$ consist of a $\mathscr{D}isk^{\mathrm{fr}}_m$-algebra A, a $\mathscr{D}isk^{\mathrm{fr}}_n$-algebra B, and a morphism of $\mathscr{D}isk^{\mathrm{fr}}_{m+1}$-algebras $\alpha : \int_{S^{n-m-1}} B \to \mathrm{HC}^*_{\mathsf{D}_m}(A)$, where $\int_{S^{n-m-1}} B$ is the "factorization homology" of S^{n-m-1} with coefficients in B, and $\mathrm{HC}^*_{\mathsf{D}_m}$ refers to Hochschild cochains. We view this as a "up to homotopy" version of an vsc_{mn}-algebra, the morphism $f + \varepsilon\delta$ being the part $\int_{S^{n-m-1}} B \to \mathrm{HC}^0_{\mathsf{D}_m}(A)$ and the higher terms being homotopies. It would be interesting to make this observation precise.

Remark 3.28. An algebra over the homology $\mathsf{sc}_n \coloneqq H_*(\mathsf{SC}_n)$ of the Swiss-Cheese operad is the data of (A, B, f) as in the proposition, see [Liv15, Section 4.3]. However, our computation for $H^*(\mathsf{VSC}_{(n-1)n})$ in Section 3.1 above does not apply (e.g. the class $\omega_{12} \in H^*(\mathsf{VSC}_{(n-1)n}(0,2))$ vanishes on some connected components). Instead, we get that $\mathsf{vsc}_{(n-1)n}$-algebras are given by quadruples (A, B, f, g) where (A, B, f) are as above and $g : B \to A$ is another central morphism (i.e. A is a unitary B-bimodule). There is an embedding $\mathsf{SC}_n \subset \mathsf{VSC}_{(n-1)n}$. On homology, an $\mathsf{vsc}_{(n-1)n}$-algebra (A, B, f, g) viewed as an sc_n-algebra is simply (A, B, f), i.e. we forget the right action. Livernet proved that SC_n is not formal by exhibiting a nontrivial operadic Massey product $\langle \mu_B, f, \lambda_A \rangle$, where μ_B represents the product of B and λ_A represents the Lie bracket of A [Liv15, Theorem 4.3]. This shows that the operad of chains $C_*(\mathsf{VFM}_{(n-1)n}; \mathbb{Q})$ cannot be formal either, because we obtain a nontrivial Massey product there too.

4. Graph complexes

In this section, we define a two-colored Hopf cooperad, whose operations in the second color are given by Kontsevich's cooperad graphs_n [Kon99], and whose operations in the first color will be called $\mathsf{vgraphs}_{mn}$. Our tool of choice to define $\mathsf{vgraphs}_{mn}$ will be "operadic twisting" [Wil14; DW15], just like in [Wil15]. To give an idea of how $\mathsf{vgraphs}_{mn}$ is built, we first recall the steps in the definition of graphs_n. We also refer to [Idr19, Sections 1.5–1.6] for more details with matching notation.

4.1. Recollections: the cooperad graphs_n

In this section, we recall the definition of Kontsevich's graph cooperad graphs_n. We assume that $n \geq 2$ in the whole section.

Remark 4.1. Unlike some earlier works, we use the notation graphs_n for the cooperad rather than its dual operad. Its linear dual $\mathsf{graphs}_n^\vee$ is an operad which is quasi-isomorphic to the homology of the little disks operad. We make this choice because we almost only ever deal with graphs_n, not $\mathsf{graphs}_n^\vee$. Moreover, Kontsevich's graph complex where the differential splits vertices will also be denoted with a dual sign, $\mathrm{GC}_n^\vee$ (dual of the complex where the differential contracts edges). See e.g. [Idr19; Cam+18b] for matching notations.

4.1.1. Untwisted

The first step is to define an untwisted Hopf cooperad Gra_n, given in each arity by the following CDGA, with generators e_{uv} of degree $n-1$:

$$\mathsf{Gra}_n(U) \coloneqq S(e_{uv})_{u,v \in U} / (e_{vu} = (-1)^n e_{uv}). \tag{4.2}$$

We have a graphical interpretation of $\mathsf{Gra}_n(U)$. The CDGA $\mathsf{Gra}_n(U)$ is spanned by graphs on the set of vertices U. The monomial $e_{u_1 v_1} \dots e_{u_r v_r}$ corresponds to the graph with edges $\overrightarrow{u_1 v_1}, \dots, \overrightarrow{u_r v_r}$. The identification $e_{vu} = \pm e_{uv}$ allows us to view the graphs as undirected, although we need directions to define the signs precisely for odd n. Moreover, if n is even then $\deg e_{uv}$ is odd, thus we need to order the edges to get precise signs. Note that we explicitly allow tadpoles (e_{uu}) and double edges (e_{uv}^2). However, for even n, $e_{uv}^2 = 0$ because $\deg e_{uv}$ is odd; and for odd n, $e_{uu} = (-1)^n e_{uu} = -e_{uu}$ thus $e_{uu} = 0$.

The product is given by gluing graphs along their vertices. The cooperadic structure is given by subgraph contraction. Explicitly, the map $\circ_W^\vee : \mathsf{Gra}_n(U) \to \mathsf{Gra}_n(U/W) \otimes \mathsf{Gra}_n(W)$ is given on generators by $\circ_W^\vee(e_{uv}) = e_{**} \otimes 1 + 1 \otimes e_{uv}$ if $u, v \in W$, and by $\circ_W^\vee(e_{uv}) = e_{[u][v]} \otimes 1$ otherwise. One may then produce a first zigzag of Hopf cooperads, defined on generators by:

$$H^*(\mathsf{FM}_n) = \mathsf{e}_n^\vee \leftarrow \mathsf{Gra}_n \xrightarrow{\omega'} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n), \quad \omega_{uv} \leftarrow\!\shortmid e_{uv} \mapsto p_{uv}^*(\varphi_n), \tag{4.3}$$

where $p_{uv} : \mathsf{FM}_n(U) \to \mathsf{FM}_n(2)$ is the projection and φ_n is the "propagator":

$$\varphi_n \coloneqq \mathrm{cst} \cdot \sum_{i=1}^n \pm x_i dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_n \in \Omega^{n-1}_{\mathrm{PA}}(\mathsf{FM}_n(2)) = \Omega^{n-1}_{\mathrm{PA}}(S^{n-1}). \tag{4.4}$$

Given a graph $\Gamma \in \mathsf{Gra}_n(U)$, one may define its *coefficient*:

$$\mu(\Gamma) \coloneqq \int_{\mathsf{FM}_n(U)} \omega'(\Gamma). \tag{4.5}$$

By convention $\mu(\Gamma) \coloneqq 0$ for $\#U \leq 1$. This element μ has a simple description: it vanishes on all graphs, except for the one with exactly two vertices and an edge between them [LV14, Lemma 9.4.3]. In other words, in the dual basis:

$$\mu = \textcircled{1}\!\!-\!\!-\!\!-\!\!\textcircled{2} \in \mathsf{Gra}_n^\vee(2) \subset \prod\nolimits_{i \geq 0} \mathsf{Gra}_n^\vee(i). \tag{4.6}$$

4.1.2. Twist

The second step is to twist the cooperad Gra_n. We refer to Appendix A.1 for the concepts and notations. The element μ above defines a morphism from Lie_n to $\mathsf{Gra}_n^\vee$: we send the generating bracket $\lambda_2 \in \mathsf{Lie}_n(2)$ to the graph appearing in Equation (4.6) (in the dual basis). By composing with the canonical resolution $\mathsf{hoLie}_n \to \mathsf{Lie}_n$, we thus obtain a morphism $\mathsf{hoLie}_n \to \mathsf{Gra}_n^\vee$, which sends the binary bracket to μ and the k-ary brackets to 0 for $k \geq 3$. Cooperadic twisting produces a dg-cooperad $\mathrm{Tw}\,\mathsf{Gra}_n$ from Gra_n and μ.

Let us now describe this twisted cooperad. As a graded module, we have:

$$\mathrm{Tw}\,\mathsf{Gra}_n(r) \coloneqq \Big(\bigoplus_{i\geq 0}\big(\mathsf{Gra}_n(r+i) \otimes \Bbbk[n]^{\otimes i}\big)_{\Sigma_i}, d_\mu\Big). \tag{4.7}$$

The cooperadic structure and the differential are described in Appendix A.1. The CDGA structure is defined using the fact that $\mathsf{Gra}_n(\varnothing) = \mathbb{R}$ (see [Idr19, Lemma 9]) and the fact that μ vanishes on disconnected graphs as well as graphs admitting a cut point (i.e. if the point is removed, then the graph is disconnected).

Let us now give a graphical interpretation of this definition. The CDGA $\mathrm{Tw}\,\mathsf{Gra}_n(U)$ is spanned by graphs with two kinds of vertices: external vertices, which are in bijection with U, and internal vertices, which are indistinguishable (in pictures they will be drawn in black). Given a graph Γ, its differential $d\Gamma = \sum_e \pm\Gamma/e$ is obtained as a sum, over all the edges $e \in E_\Gamma$ connected to at least one internal vertex, of the graphs obtained by contracting these edges. Note that in this differential, edges connected to univalent vertices are not contracted, roughly speaking because the contraction appears twice in $d\Gamma$ and cancels out; but if both endpoints of the edge are univalent, then the edge is contracted with a minus sign, see [Wil14, Appendix I.3] and Section 4.1.3. The product of two graphs is the graph obtained by gluing them along their external vertices. The cooperadic structure is given by subgraph contraction (summing over all choices of whether internal vertices and edges are in the subgraph).

One checks that the zigzag of Equation (4.3) extends to a zigzag:

$$\mathsf{e}_n^\vee \leftarrow \mathrm{Tw}\,\mathsf{Gra}_n \xrightarrow{\omega} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n). \tag{4.8}$$

The left-pointing map sends all graphs with internal vertices to zero. The right-pointing map is given by the following integrals. Given a graph $\Gamma \in \mathsf{Gra}_n(U \sqcup I) \subset \mathrm{Tw}\,\mathsf{Gra}_n(U)$, the form $\omega(\Gamma)$ is the integral of $\omega'(\Gamma)$ along the fiber of the PA bundle $p_U : \mathsf{FM}_n(U \sqcup I) \to \mathsf{FM}_n(U)$ which forgets points in the configuration:

$$\omega(\Gamma) \coloneqq (p_U)_*(\omega'(\Gamma)) = \int_{\mathsf{FM}_n(U\sqcup I)\to\mathsf{FM}_n(U)} \omega'(\Gamma). \tag{4.9}$$

4.1.3. Interlude: Graph complex

We record the following definition for future use.

Definition 4.10. The full graph complex fGC_n is defined to be $\mathrm{Tw}\,\mathsf{Gra}_n(\varnothing)[-n]$. It is spanned by graphs with only internal vertices. The degree of $\gamma \in \mathrm{fGC}_n$ is $\deg\gamma = (n-1)\#E_\gamma - n\#V_\gamma + n$. We describe its differential below.

Remark 4.11. This degree shift by n comes from the fact that we mod out $\mathrm{Conf}_{\mathbb{R}^n}(k)$ by $\mathbb{R}^n \rtimes \mathbb{R}_{>0}$ in the definition of FM_n. Indeed, for $\gamma \in \mathrm{fGC}_n$, in the definition of $\omega(\gamma)$, the form is of degree $(n-1)\#E_\gamma$ and the space is of dimension $\dim \mathsf{FM}_n(V_\gamma) = n\#V_\gamma - n - 1$. In order to obtain a Maurer–Cartan element (of degree 1) out of ω in the Lie algebra $\mathrm{GC}_n^\vee$ defined below, a shift by n is used.

The module fGC_n is a shifted algebra, with a product given by disjoint union of graphs. One can moreover define the subcomplex $\mathrm{GC}_n \subset \mathrm{fGC}_n$ of connected graphs, and there is an isomorphism of shifted algebras:

$$\mathrm{fGC}_n = S(\mathrm{GC}_n[n])[-n]. \tag{4.12}$$

The module GC_n is a (pre)Lie coalgebra. Its cobracket Δ is given by subgraph contraction (i.e. it is inherited from the cooperad structure on Gra_n). Dually, $\mathrm{GC}_n^\vee$ is a (pre)Lie algebra, with a bracket given by graph insertion.

Let us now describe the differential of GC_n. It is given by $(-\mu \otimes 1 + 1 \otimes \mu)\Delta$, where $\mu \in \mathrm{GC}_n^\vee$ is the Maurer–Cartan element defined in Equation (4.6). The summand $(1 \otimes \mu)\Delta$ is given by the sum of contractions of edges, while the summand $(\mu \otimes 1)\Delta$ is given by the sum of contractions of edges attached to a univalent vertex. The contraction of an edge attached to exactly one univalent vertex appears twice in the sum with opposite signs, which thus cancel out. However, if both endpoints of the edge are univalent, then the contractions of this edge appears once with a plus and twice with a minus, which leaves one term with a minus. (See [Wil14, Appendix I.3] for more details)

Note that dually, the differential on $\mathrm{GC}_n^\vee$ is $[\mu, -]$. It is roughly speaking given by vertex splitting (with the caveat about univalent vertices explained above). Finally, note that the space $\mathrm{Tw}\,\mathsf{Gra}_n(U)$ is a module over the shifted CDGA fGC_n, by taking disjoint unions.

4.1.4. Reduction

The next step is to mod out graphs with *internal components*, i.e. connected components with only internal vertices, to obtain a new Hopf cooperad Graphs_n. Formally, we consider the tensor product

$$\mathsf{Graphs}_n(U) \coloneqq \mathrm{Tw}\,\mathsf{Gra}_n(U) \otimes_{\mathrm{fGC}_n} \mathbb{R}, \tag{4.13}$$

where the fGC_n-module structure on $\mathbb{R}$ is simply given by the augmentation, i.e. the action of any nonempty graph vanishes. In other words, in Graphs_n, a graph with a connected component consisting entirely of internal vertices is set equal to zero. The map ω of Equation (4.9) factors through the quotient defining Graphs_n [LV14, Lemma 9.3.7], and the quotient map $\mathrm{Tw}\,\mathsf{Gra}_n \to \mathsf{e}_n^\vee$ clearly does.

We can moreover reduce further the operad. We consider the quotient graphs_n where we have killed graphs containing: internal vertices that are univalent or bivalent; double edges; or tadpoles. It follows again from the lemmas of [LV14, Section 9.3] that ω factors through this quotient (and the map $\mathsf{Graphs}_n \to \mathsf{e}_n^\vee$ clearly does).

Theorem 4.14 ([Kon99; LV14])**.** *This defines a zigzag of quasi-isomorphisms of Hopf cooperads* $\mathsf{e}_n^\vee \xleftarrow{\sim} \mathsf{graphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$*. Thus the operad* FM_n *is formal over* $\mathbb{R}$*.*

4.2. The cooperad $\mathsf{VGraphs}_{mn}$

Let us now define $\mathsf{VGraphs}_{mn}$, using the same methodology that was used to define Graphs_n. Note that we must take special care of the case $m = 1$. We define the further reduced cooperad $\mathsf{vgraphs}_{mn}$ in the next section.

4.2.1. Untwisted

The first step is the definition of the untwisted graph cooperad.

Definition 4.15. For $n - 2 \geq m \geq 2$, the untwisted graph cooperad is a relative VGra_{mn}-cooperad given in each arity by:

$$\mathsf{VGra}_{mn}(U, V) := \frac{S(\tilde{e}_{uu'})_{u,u' \in U} \otimes S(e_{ij})_{i,j \in U \sqcup V}}{(e_{ji} = (-1)^n e_{ij}, \tilde{e}_{uu'} = (-1)^m \tilde{e}_{u'u})},$$

where $\deg \tilde{e}_{uu'} = m - 1$ and $\deg e_{ij} = n - 1$.

Definition 4.16. For $n - 2 \geq m = 1$, we define:

$$\mathsf{VGra}_{1n}(U, V) = \frac{S(e_{ij})_{i,j \in U \sqcup V} \otimes \mathbb{R}[\Sigma_U]^\vee}{(e_{ji} = (-1)^n e_{ij})},$$

where $\Sigma_U = \mathsf{Ass}(U) = \mathsf{Bij}(U, \{1, \dots, \#U\})$.

Let us now give a graphical interpretation of $\mathsf{VGra}_{mn}(U, V)$. We will concentrate first on the case $m \geq 2$. As a vector space, it is spanned by graphs with two kinds of vertices: terrestrial (in bijection with U) and aerial (in bijection with V). We will draw the aerial vertices as circles, and the terrestrial vertices as semicircles, below the aerial ones. There are also two kind of edges: full edges (corresponding to the e_{ij}) between any two vertices, and dashed edges (corresponding to the $\tilde{e}_{uu'}$) between two terrestrial vertices Note that we allow tadpoles (edges between a vertex and itself) as well as double edges. See Figure 4.2.1 for an example. If $m = 1$, the interpretation is similar, except that there are no dashed edges, and in addition the set of terrestrial vertices is ordered.

Remark 4.17. In order to say that terrestrial vertices are ordered in the graphs of the basis, we are the dual of the canonical basis of $\mathbb{R}[\Sigma_U]$ (i.e. we consider the basis element of $\mathbb{R}[\Sigma_U]^\vee$ which vanishes on all linear orders except one).

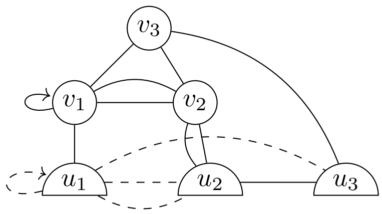


Figure 4.1: Example of a graph in VGra_{mn} (for $m \geq 2$)

The product glues graphs along their vertices, and the differential is zero. These CDGAs assemble to form a relative Hopf Gra_n-cooperad. The cooperadic structure maps act on the generators $e_{vv'}$ and $\tilde{e}_{uu}$ (if $m \geq 2$) just like they do in the formulas of Proposition 3.19, replacing $\omega_{uu'}$ by $e_{uu'}$ and $\tilde{\omega}_{vv'}$ by $\tilde{e}_{vv'}$. If $m = 1$, then $\circ_T^\vee$ is the identity on $\mathbb{R}[\Sigma_U]^\vee$, and $\circ_{W,T}^\vee$ is the dual of block composition from Equation (3.18). Moreover, we have (with the convention $u, u' \in U$, $v \in V$):

$$\circ_T^\vee(e_{uu'}) = e_{uu'}, \qquad \circ_{W,T}^\vee(e_{uu'}) = \begin{cases} e_{**} \otimes 1 + 1 \otimes e_{uu'}, & \text{if } u, u' \in W, \\ e_{[u][u']} \otimes 1, & \text{otherwise,} \end{cases}$$

$$\circ_T^\vee(e_{uv}) = e_{u[v]}, \qquad \circ_{W,T}^\vee(e_{uv}) = \begin{cases} 1 \otimes e_{uv}, & \text{if } u \in W, v \in T \\ 0, & \text{if } u \notin W, v \in T, \\ e_{[u]v} & \text{otherwise.} \end{cases}$$

Graphically, the cooperadic structure map $\circ_{W,T}$ is given by subgraph contraction, which we now explain.

Definition 4.18 (Quotient graph, case $m \geq 2$)**.** Let $\Gamma \in \mathsf{VGra}_{mn}(U, V)$ be a graph. Let $\Gamma' \subset \Gamma$ be a subgraph, not necessarily full. We define the quotient graph Γ/Γ' as follows. The set of vertices of Γ/Γ' is the quotient set $V_\Gamma/V_{\Gamma'}$, identifying all the vertices of Γ' to produce a new vertex $[\Gamma']$, always terrestrial even if Γ' contains no terrestrial vertices. The set of edges of Γ/Γ' is the difference $E_\Gamma \setminus E_{\Gamma'}$, and we will informally view the edges of Γ' as having been "contracted". If $e \in E_\Gamma \setminus E_{\Gamma'}$ has an endpoint in Γ', then the corresponding endpoint of $e \in E_{\Gamma/\Gamma'}$ is the new terrestrial vertex $[\Gamma']$. In particular, if an edge is not in Γ' but both of its endpoints are in Γ', then in Γ/Γ' this edge becomes a tadpole on the vertex $[\Gamma']$.

Example 4.19. The quotient $\Gamma/\varnothing$ is Γ with a new isolated external terrestrial vertex.

Definition 4.20 (Quotient graph, case $m = 1$)**.** For $m = 1$, the definition of the quotient graph Γ/Γ' given above has to be clarified to deal with the linear order on the set of terrestrial vertices. As before, suppose that $\Gamma' \subset \Gamma$ is a subgraph. Then Γ/Γ' is a linear combination of graphs, given as follows:

- If Γ' has no terrestrial vertices, then Γ/Γ' is the sum of the quotients as in Definition 4.18, with the new terrestrial vertex $[\Gamma']$ in all possible positions.
- If the terrestrial vertices of Γ' are consecutive, then Γ/Γ' is defined as in Definition 4.18, and the linear order on the terrestrial vertices of Γ/Γ' is inherited from Γ, with the new terrestrial vertex $[\Gamma']$ in the position of the vertices of Γ'.
- If the terrestrial vertices of Γ' are not consecutive, then $\Gamma/\Gamma' = 0$.

Remark 4.21. This definition of the linear order of the terrestrial vertices of Γ/Γ' is dual to the block composition of linear orders from Equation (3.18).

We can interpret $\circ_{W,T}(\Gamma)$ as the sum of the $\Gamma/\Gamma' \otimes \Gamma'$ for all (not necessarily full) subgraphs Γ' with set of vertices (W, T).

Let us now define a zigzag of Hopf cooperad maps

$$H^*(\mathsf{VFM}_{mn}) = \mathsf{vsc}_{mn}^\vee \leftarrow \mathsf{VGra}_{mn} \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}). \tag{4.22}$$

The left-pointing map is defined by $e_{uu} \mapsto 0$, $\tilde{e}_{uu'} \mapsto \tilde{\omega}_{uu'}$ for $u \neq u' \in U$, $e_{vv} \mapsto 0$, $e_{vv'} \mapsto \omega_{vv'}$ for $v \neq v' \in V$, and $e_{ij} \mapsto 0$ if $i \in U$ or $j \in U$. The right-pointing map is defined using the following three "propagators":

- We have the identification $\mathsf{VFM}_{mn}(2,0) = \mathsf{FM}_m(2) \cong S^{m-1}$, for which we can use the propagator $\varphi_m \in \Omega^{m-1}_{\mathrm{PA}}(\mathsf{FM}_m(2))$ of Equation (4.4).
- We have the map $\theta_{12} : \mathsf{VFM}_{mn}(1,1) \to S^{n-1}$ which records the direction from the aerial point to the terrestrial point. We then define the propagator to be the pullback of the volume form of S^{n-1} along θ_{12}:

$$\psi^{\partial}_{mn} \coloneqq \theta^*_{12}(\mathrm{vol}_{S^{n-1}}) \in \Omega^{n-1}_{\mathrm{PA}}(\mathsf{VFM}_{mn}(1,1)). \tag{4.23}$$

- Similarly, there is the map $\theta_{12} : \mathsf{VFM}_{mn}(0,2) \to S^{n-1}$ which records the direction from the second point to the first point. Then the propagator is:

$$\psi_{mn} \coloneqq \theta^*_{12}(\mathrm{vol}_{S^{n-1}}) \in \Omega^{n-1}_{\mathrm{PA}}(\mathsf{VFM}_{mn}(0,2)). \tag{4.24}$$

Remark 4.25. By construction, these propagators are all minimal forms on VFM_{mn}, because the volume form on S^d is $\mathrm{vol}_{S^d} = \mathrm{cst} \cdot \sum_i (-1)^i x_i dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_d$. Hence they can be pushed forward (once) along PA bundles [Har+11].

We may then define a morphism

$$\omega' : \mathsf{VGra}_{mn}(U,V) \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}(U,V)) \tag{4.26}$$

as follows, using the convention that u, $u' \in U$ and v, $v' \in V$, including if $u = u'$ and $v = v'$:

$$\begin{aligned} \omega'(\tilde{e}_{uu'}) &\coloneqq p^*_{uu'}(\varphi_m); & \omega'(e_{vv'}) &\coloneqq p^*_{vv'}(\psi_{mn}); \\ \omega'(e_{uu'}) &\coloneqq 0; & \omega'(e_{vu}) &\coloneqq p^*_{vu}(\psi^{\partial}_{mn}). \end{aligned} \tag{4.27}$$

Here, $p_{\bullet\bullet}$ is the map which forget all but two points in a configuration (for p_{uu} and p_{vv} we also apply the diagonal). The following lemma is immediate:

Lemma 4.28. *This defines a zigzag* $\mathsf{vsc}^{\vee}_{mn} \leftarrow \mathsf{VGra}_{mn} \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$. □

Let us define some more notation. Given a graph $\Gamma \in \mathsf{VGra}_{mn}(U,V)$, we define $V_\Gamma = V^t_\Gamma \cup V^a_\Gamma = U \cup V$ to be its set of vertices, partitioned into terrestrial and aerial ones. Similarly, $E_\Gamma = E^f_\Gamma \cup E^d_\Gamma$ is its set of edges, split into full edges and dashed edges. The graph Γ induces a map

$$\Phi_\Gamma : \mathsf{VFM}_{mn}(U,V) \to (S^{m-1})^{E^d_\Gamma} \times (S^{n-1})^{E^f_\Gamma}, \tag{4.29}$$

obtained using the maps θ_{ij} from Section 2. We also define

$$\mathrm{vol}_\Gamma \in \Omega^{\deg \Gamma}_{\mathrm{PA}}((S^{m-1})^{E^d_\Gamma} \times (S^{n-1})^{E^f_\Gamma}) \tag{4.30}$$

to be the product of the volume forms. Then by definition, $\omega'(\Gamma) = \Phi^*_\Gamma(\mathrm{vol}_\Gamma)$.

Given a graph $\Gamma \in \mathsf{VGra}_{mn}(U,V)$ with $\#U + 2\#V \geq 2$, we may define its *coefficient* $c(\Gamma)$ by

$$c(\Gamma) := \int_{\mathsf{VFM}_{mn}(U,V)} \omega'(\Gamma). \tag{4.31}$$

If $\#U+2\#V \leq 1$ then we just set $c(\Gamma) = 0$ (this is due to the special case in the definition of $\mathcal{BF}''$, see Proposition 2.12).

Remark 4.32. In the case $(n,m) = (2,1)$, these are analogous to the coefficients in Kontsevich's universal formality morphism $T_{\mathrm{poly}} \to D_{\mathrm{poly}}$ from [Kon03].

4.2.2. Twisted

Together with μ from Equation (4.5), this collection of coefficients c defines a morphism of colored operads $(\mathsf{Lie}_n, \mathsf{holie}_{mn}) \to (\mathsf{Gra}_n^\vee, \mathsf{VGra}_{mn}^\vee)$, where hoLie_{mn} was defined in Appendix A.3. Concretely, the generating bracket $\lambda_{U,V} \in \mathsf{hoLie}_{mn}(U,V)$ is sent to $\Gamma \in \mathsf{VGra}_{mn}^\vee(U,V) \mapsto c(\Gamma) \in \mathbb{R}$. We can therefore apply the formalism of operadic twisting from Appendix A.3.

Definition 4.33. The twisted graph cooperad $\mathrm{Tw}\,\mathsf{VGra}_{mn}$ is the relative $(\mathrm{Tw}\,\mathsf{Gra}_n)$-cooperad obtained by twisting VGra_{mn} with respect to μ and c.

Since VGra_{mn} is a Hopf cooperad satisfying $\mathsf{VGra}_{mn}(\varnothing,\varnothing) = \Bbbk$, and since c vanishes on disconnected graphs (see Lemma 4.37) and graphs with a terrestrial cut point (by a simple degree argument: $\dim \mathsf{VFM}_{mn}(i+i'+1, j+j') > \dim \mathsf{VFM}_{mn}(i+1,j) \times \mathsf{VFM}_{mn}(i'+1,j')$), we find again that $\mathrm{Tw}\,\mathsf{VGra}_{mn}$ inherits a Hopf cooperad structure, similarly to the uncolored case.

This twisted cooperad has a graphical description. The CDGA $\mathrm{Tw}\,\mathsf{VGra}_{mn}(U,V)$ is spanned by graphs with four types of vertices: external terrestrial vertices, in bijection with U (drawn as semicircles); external aerial vertices, in bijection with V (drawn as circles); internal terrestrial vertices, indistinguishable among themselves, of degree $-m$ (drawn as black semicircles); internal aerial vertices, indistinguishable among themselves, of degree $-n$ (drawn as black circles).

There are two kinds of edges: aerial (full) edges of degree $n-1$, and terrestrial (dashed) edges of degree $m-1$. If $m = 1$, then there are no dashed edges, and the whole set of terrestrial vertices is ordered.

Description 4.34. The differential is given by the twisting procedure (Appendix A.3). Recall the subgraph contractions defined in Definitions 4.18, 4.20. Given a graph Γ, its differential $d\Gamma$ is given as a sum (with signs in Section A.3):

1. contractions of full edges between an aerial internal vertex and an aerial vertex of any kind, including edges connected to a univalent internal vertex (this uses the simple description of μ in Equation (4.6));
2. contractions of subgraphs Γ' containing at most one external vertex, necessarily terrestrial, with result $c(\Gamma') \cdot \Gamma/\Gamma'$;
3. forgetting of all vertices outside a subgraph Γ'' which contains all the external vertices, with result $c(\Gamma/\Gamma'') \cdot \Gamma''$.

Lemma 4.35. *For $n-2 \geq m \geq 1$, the morphism ω' extends to a morphism of Hopf cooperads*

$$\omega : \operatorname{Tw} \mathsf{VGra}_{mn}(U,V) \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$$

by setting, for $\Gamma \in \mathsf{VGra}_{mn}(U \sqcup I, V \sqcup J) \subset \operatorname{Tw} \mathsf{VGra}_{mn}(U,V)$ and $\#U + 2\#V \geq 2$:

$$\omega(\Gamma) \coloneqq (p_{U,V})_*(\omega'(\Gamma)) = \int_{\mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U,V)} \omega'(\Gamma).$$

If $\#U + 2\#V \leq 1$ (no aerial vertices and ≤ 1 terrestrial vertex), then $\omega(\Gamma) = 1$ if Γ is the graph with no edges and no internal vertices, and $\omega(\Gamma) = 0$ otherwise.

Proof. We use the double pushforward formula [Har+11, Proposition 8.13], Stokes' formula [Har+11, Proposition 8.12] and the decomposition of the fiberwise boundary $\mathsf{VFM}_{mn}(U,V)$ from Section 2.1 (compare with [LV14, Section 9.4]).

We deal separately with the case $\#U + 2\#V \leq 1$ because the dimension of the fibers of $p_{U,V}$ is smaller than expected (cf. [LV14, Section 9.1]). In general, the fiber of $p_{U,V} : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U,V)$ has dimension $m\#I + n\#J$. However, if $\#U + 2\#V \leq 1$, then $\dim \mathsf{VFM}_{mn}(U,V) = 0$ instead of the expected -1 (for $(\#U, \#V) = (1,0)$) or $-1-m$ (for $(\#U,\#V) = (0,0)$) from the general formula. Hence the dimension of the fiber of $p_{U,V}$ is respectively $m\#I + n\#J + 1$ and $m\#I + n\#J + m + 1$, and the integration map would not preserve the degrees. Nevertheless, in these cases $\mathsf{VFM}_{mn}(U,V)$ is a singleton, so we can just define $\omega(\Gamma)$ as above. We have to check that any cocycle is mapped to zero. This is clear if $U = V = \varnothing$, as the differential of a nonempty graph is nonempty. For $(\#U,\#V) = 1$ and $\Gamma \in \operatorname{Tw} \mathsf{VGra}_{mn}(U,V)$ with at least one edge, the graph with no edges appears twice with opposite signs in $d\Gamma$: for $\Gamma' = \Gamma$ in $c(\Gamma')\Gamma/\Gamma'$, and for Γ'' containing only the external vertex in $c(\Gamma/\Gamma'')\Gamma''$ (see Description 4.34). ☐

4.2.3. Interlude: Graph complex

Before turning to the final step of the construction, we will need the following definition, which mimicks the definition of fGC_n from Section 4.1.3:

Definition 4.36. For $n-2 \geq m \geq 1$, the full Swiss-Cheese graph complex is:

$$\mathrm{fVGC}_{mn} = \operatorname{Tw} \mathsf{VGra}_{mn}(\varnothing, \varnothing)[-m]$$

This shifted CDGA is free and generated by its submodule of connected graphs. Thanks to the following lemma, this submodule is a subcomplex when $m \geq 2$:

Lemma 4.37. *Let $n-2 \geq m \geq 1$. Given a disconnected graph $\gamma \in \mathrm{fVGC}_{mn}$, the coefficient $c(\gamma)$ vanishes, unless $m = 1$ and γ is the graph with two terrestrial vertices and no edges.*

Proof. Let us first assume that γ has no isolated terrestrial vertices. Then γ factors as a product $\gamma = \gamma_1 \cdot \gamma_2$, with corresponding sets of vertices (I_1, J_1) and (I_2, J_2). Thanks to our assumption, we have $\#I_\bullet + 2\#J_\bullet \geq 2$ for $\bullet \in \{1,2\}$. The coefficient $c(\gamma) =$

$\int_{\mathsf{VFM}_{mn}(I,J)} \omega'(\gamma)$ is defined as an integral, and the degree of $\omega'(\gamma)$ must be equal to the dimension of $\mathsf{VFM}_{mn}(I,J)$ for the integral to be nonzero. Let us assume that this is the case (otherwise we are done).

We have $\omega'(\gamma) = \Phi_\gamma^*(\mathrm{vol}_\gamma)$. The map Φ_γ factors through the product $\mathsf{VFM}_{mn}(I_1, J_1) \times \mathsf{VFM}_{mn}(I_2, J_2)$. But we know that:

$$\dim \mathsf{VFM}_{mn}(I,J) - \dim \mathsf{VFM}_{mn}(I_1,J_1) \times \mathsf{VFM}_{mn}(I_2,J_2) = m+1 > 0, \tag{4.38}$$

therefore $\deg \mathrm{vol}_\gamma > \dim \mathsf{VFM}_{mn}(I_1,J_1) \times \mathsf{VFM}_{mn}(I_2,J_2)$. It follows that $\omega'(\gamma) = 0$ and therefore $c(\gamma) = 0$.

Let us now assume that γ has an isolated terrestrial vertex, say $i \in I$. As before, $\omega'(\gamma) = \Phi_\gamma^*(\mathrm{vol}_\gamma)$, and Φ_γ factors through $\mathsf{VFM}_{mn}(I \setminus \{i\}, J)$. If $(\#I - 1) + 2\#J \geq 2$, in other words if the graph is not the graph with two terrestrial vertices and no aerial ones, then

$$\dim \mathsf{VFM}_{mn}(I,J) - \dim \mathsf{VFM}_{mn}(I \setminus \{i\}, J) = m > 0 \tag{4.39}$$

and we may again conclude that $\omega'(\gamma) = 0 \implies c(\gamma) = 0$. However, if Γ is the graph with two terrestrial vertices and no aerial ones, i.e. $(\#I, \#J) = (2,0)$, then the codimension is $m-1$, because $\dim \mathsf{VFM}_{mn}(1,0) = 0$ and not -1 as the general formula would give. Thus in the case $m > 1$, we may still conclude that $\omega'(\gamma) = 0$, because the codimension is positive. $\square$

Remark 4.40. In the case $m = 1$, we have that $\mathsf{VFM}_{1n}(2,0) \cong \mathsf{FM}_1(2) \cong S^0$. Hence we find that for $\gamma =$  $\in \mathrm{fVGC}_{1n}$, we have $c(\gamma) = 1$.

Definition 4.41. For $n - 2 \geq m \geq 2$, the connected Swiss-Cheese graph complex VGC_{mn} as the quotient of fVGC_{mn} by disconnected graphs (i.e. VGC_{mn} is given by the indecomposables of fVGC_{mn}).

Just like in [Wil15], the case $m = 1$ is special. Mimicking [Wil15, Section 4], we will define an alternate basis of $\mathrm{Tw}\,\mathsf{Gra}_{1n}(U,V)$. For motivation, recall that $\mathsf{Gra}_{1n}(U,\varnothing)$ contains $\mathsf{Ass}^\vee(U)$, where Ass is the operad encoding associative algebras. The symmetric sequence Ass is isomorphic to the symmetric sequence $\mathsf{Pois} = \mathsf{Com} \circ \mathsf{Lie}$ [LV12, Section 13.3] (essentially a reformulation of the Poincaré–Birkhoff–Witt (PBW) theorem applied to a free Lie algebra $S(L(V)) \cong T(V)$). Roughly speaking, we can thus decree that terrestrial internal vertices are in the same “Lie component” as another vertex if they are connected by brackets.

Definition 4.42 (Cf. [Wil15, Section 4])**.** A Lie-decorated graph is a graph Γ defined similarly as an element of $\mathrm{Tw}\,\mathsf{Gra}_{1n}(U,V)$ (Definition 4.33), but instead of ordering all terrestrial vertices, we plug terrestrial vertices in a commutative product of Lie words, i.e. an element of $\mathsf{Pois}(U \sqcup I)$ (where Pois is the Poisson operad and I is the set of internal terrestrial vertices). We moreover require that there is only one external vertex per Lie word. See Figure 4.2 for an example.

Lemma 4.43. *The vector space spanned by Lie-decorated graphs with external vertices* (U,V) *is isomorphic to* $\mathrm{Tw}\,\mathsf{Gra}_{1n}(U,V)$.

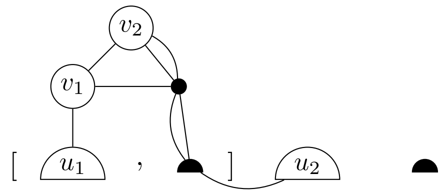


Figure 4.2: Lie-decorated graph

Proof. This follows from the PBW theorem. We use the dual of the PBW isomorphism $\mathsf{Pois}^\vee(U \sqcup I) \cong \mathsf{Ass}^\vee(U \sqcup I)$ (which turns a commutative product of Lie words into a shuffle product of commutators) to identify linear combinations of Lie-decorated graphs with elements of $\mathrm{Tw}\,\mathsf{Gra}_{1n}(U, V)$. For the signs, we consider internal terrestrial vertices as odd objects (as they have degree -1). $\square$

Definition 4.44. Two vertices in a Lie-decorated graph are said to be in the same component if they are connected by full edges or if they are both terrestrial and in the same Lie word; we then extend this relation transitively. A Lie-decorated graph is Lie-connected if all its vertices are in the same Lie component, and Lie-disconnected otherwise.

Lemma 4.45. *The coefficient c vanishes on the image of a Lie-disconnected graph under the isomorphism of Lemma 4.43.*

Proof. This follows from Lemma 4.37. $\square$

Definition 4.46. For $n \geq 3$, the Lie-connected Swiss-Cheese graph complex VGC_{1n} as the quotient of fVGC_{1n} by the image of the Lie-disconnected graph under the isomorphism of Lemma 4.43.

Lemma 4.47. *For $n - 2 \geq m \geq 1$, the coefficient $c : \mathrm{fVGC}_{mn} \to \mathbb{R}$ factors through the quotient defining VGC_{mn}. Abusing notation, we denote by c the induced map $\mathrm{VGC}_{mn} \to \mathbb{R}$.*

Proof. This also follows from Lemma 4.37, in both cases $m = 1$ and $m \geq 2$. $\square$

Remark 4.48. The dual $\mathrm{VGC}_{mn}^\vee$ is a (pre-)Lie algebra and the Lie algebra $\mathrm{GC}_n^\vee$ acts on it by derivations, in both cases using insertion of graphs (respectively at terrestrial and aerial vertices). The differential of $\mathrm{VGC}_{mn}^\vee$ is given by $[\mu + c, -]$ where c represents the coefficients from Equation (4.31). The elements μ and c satisfy the Maurer–Cartan equation, i.e. $[\mu, \mu] = 0$ and $[\mu, c] + \frac{1}{2}[c, c] = 0$.

4.2.4. Reduction

Definition 4.49. For $n - 2 \geq m \geq 2$, we define the graph cooperad $\mathsf{VGraphs}_{mn}$ to be the relative Graphs_n-cooperad given by quotient of $\mathrm{Tw}\,\mathsf{VGra}_{mn}$ by the Hopf cooperadic ideal of graphs with internal components. For $n - 2 \geq m = 1$, we define $\mathsf{VGraphs}_{1n}$ to be the quotient of $\mathrm{Tw}\,\mathsf{VGra}_{1n}$ by the Hopf cooperadic ideal of Lie-disconnected graphs (Definition 4.44).

Remark 4.50. Willwacher [Wil15] defined a model $\mathsf{SGraphs}_n$ for the Swiss-Cheese operad. In $\mathsf{SGraphs}_n$, there are no "dashed" edges, the full edges are oriented, and their source is always aerial.

Proposition 4.51. *The morphism ω factors through the quotient and defines a morphism $\omega : \mathsf{VGraphs}_{mn} \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$.*

Proof. The proof is identical to the proof of [LV14, Lemma 9.3.7]. For $m \geq 2$, we must check that ω vanishes on any graph with a connected component consisting only of internal vertices. As ω is a morphism of CDGAs, it is enough to show that ω vanishes on graphs where all the edges are between internal vertices. Let $\gamma \in \mathrm{Tw}\,\mathsf{VGra}_{mn}(U, V)$ be such a graph, and let I and J be the sets of terrestrial (resp. aerial) internal vertices of γ. The claim is obvious by definition of ω if $\#U + 2\#V \leq 1$ (see Lemma 4.35), so let us assume that we are not in this case.

Thanks to our hypothesis on γ, we can fill the diagram, where ρ is the product of two projections:

$$\begin{tikzcd}
\mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \arrow[rr, "\Phi_\gamma"] \arrow[d, "p_{U,V}"] \arrow[dr, "\rho"] & & (S^{m-1})^{E^t_\Gamma} \times (S^{n-1})^{E^a_\Gamma} \\
\mathsf{VFM}_{mn}(U, V) & \mathsf{VFM}_{mn}(U, V) \times \mathsf{VFM}_{mn}(I, J) \arrow[l, "\mathrm{pr}_1"] \arrow[ur, dashed, "\exists \Phi'_\gamma"] &
\end{tikzcd}.$$

By definition, $\omega'(\Gamma) = \Phi^*_\gamma(\mathrm{vol}_\Gamma)$. Thus we find that:

$$\omega(\Gamma) = (p_{U,V})_*(\omega'(\Gamma)) = (p_{U,V})_*(\rho^*(\Phi'^*_\gamma \mathrm{vol}_\Gamma)). \tag{4.52}$$

The dimension of the fiber of $p_{U,V}$ is $\dim \mathrm{fib}\, p_{U,V} = m\#I + n\#J$. We find that

$$\dim \mathrm{fib}\, \mathrm{pr}_1 = \dim \mathsf{VFM}_{mn}(I, J) \leq m\#I + n\#J - m < \dim \mathrm{fib}\, p_{U,V}, \tag{4.53}$$

therefore $\omega(\Gamma) = 0$ by [Har+11, Proposition 5.1.2].

For $m = 1$, we note that the above argument still applies, and that a Lie-disconnected graph is in particular disconnected. □

4.3. Vanishing lemmas and $\mathsf{vgraphs}_{mn}$

We now prove some vanishing lemmas about ω and c, which allows us to define the further reduced cooperad $\mathsf{vgraphs}_{mn}$. This will be useful to prove that the quotient map $\mathsf{VGra}_{mn} \to \mathsf{vsc}^\vee_{mn}$ extends to $\mathsf{vgraphs}_{mn}$ in Section 5.

Lemma 4.54. *The morphism ω' vanishes on graphs with loops or double edges.*

Proof. This follows from simple dimension arguments, cf. [LV14, Section 9.3]. □

Lemma 4.55 (Cf. [LV14, Lemma 9.3.9])**.** *The morphism ω vanishes on graphs containing univalent aerial internal vertices, or univalent terrestrial internal vertices connected to another terrestrial vertex.*

Proof. The lemma is trivial if the graph has no external aerial vertices and has at most one external terrestrial vertex, i.e. if $\#U + 2\#V \leq 1$ (see the definition of ω in Lemma 4.35), so let us assume that we are not in this case.

Let us first deal with univalent aerial internal vertices connected to another aerial vertex. The graph $\Gamma_{\text{uni}} = \textcircled{1}\!\!-\!\!\!-\!\!\!-\!\!\bullet$ is of negative degree, thus its image under ω must vanish.

Now suppose that $\Gamma \in \mathrm{Tw}\,\mathsf{Gra}_{mn}(U,V)$ is a general graph, with (I,J) as internal vertices, containing a univalent aerial internal vertex ($j \in J$) connected to another aerial vertex ($v \in V$). (See a similar argument spelled out in detail in the proof of [Idr19, Corollary 44], see also the proof of [LV14, Lemma 9.3.9].) Consider the product of projections $\rho : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(\varnothing, \{v,j\}) \times \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J \setminus \{j\})$. Since j is univalent, the map Φ_Γ (4.29) factors as $\Phi'_\Gamma \circ \rho$, where

$$\Phi'_\Gamma : \mathsf{VFM}_{mn}(\varnothing, \{v,j\}) \times \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J \setminus \{j\}) \to (S^{m-1})^{E^d_\Gamma} \times (S^{n-1})^{E^f_\Gamma}.$$

Since j is internal, $p_{U,V} : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U,V)$ factors as $q \circ \rho$ where $q : \mathsf{VFM}_{mn}(\varnothing, \{v,j\}) \times \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J \setminus \{j\}) \to \mathsf{VFM}_{mn}(U,V)$ is the projection on the second factor followed by the canonical projection (note that q is thus an SA bundle). Thanks to [Har+11, Proposition 8.10], we have $\omega(\Gamma) = (p_{U,V})_*(\Phi^*_\Gamma(\mathrm{vol}_\Gamma)) = q_*((\Phi'_\Gamma)^*(\mathrm{vol}_\Gamma))$. We can then apply [Har+11, Proposition 8.13] and the vanishing of $\omega(\Gamma_{\text{uni}}) = \int_{\mathsf{VFM}_{mn}(\varnothing,\{v,j\})} \psi_{mn}$ to conclude that $\omega(\Gamma) = 0$.

Let us now assume that Γ is a graph with a univalent internal vertex connected to a terrestrial vertex. If that univalent vertex is terrestrial and the edge is full, then the integral is zero by definition (see Equation (4.27)), so let us assume that we are not in that case, i.e. either the univalent vertex is aerial, or the incident edge is dashed.

Let $(U \sqcup I, V \sqcup J)$ be the sets of vertices of Γ. Let i be the univalent internal vertex, and let u be the terrestrial vertex to which it is connected. We have a commutative diagram:

$$\begin{tikzcd}
\mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \arrow[r, "\rho"] \arrow[d, "\Phi_\Gamma"] \arrow[rr, bend left, "p_{U,V}"] & X \arrow[r, "q"] \arrow[dl, dashed, "\exists \Phi'_\Gamma"] & \mathsf{VFM}_{mn}(U,V) \\
(S^{m-1})^{E^d_\Gamma} \times (S^{n-1})^{E^f_\Gamma} & &
\end{tikzcd}, \tag{4.56}$$

where

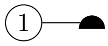
- either $X = \mathsf{VFM}_{mn}(\{u\}, \{i\}) \times \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J \setminus \{i\})$ if i is aerial,
- or $X = \mathsf{VFM}_{mn}(\{u,i\}, \varnothing) \times \mathsf{VFM}_{mn}(U \sqcup I \setminus \{i\}, V \sqcup J)$ if i is terrestrial.

The map ρ is in both cases the product of two projections, q is the projection, and Φ'_Γ exists thanks to the hypothesis that i is univalent and connected to u.

The dimension of the fiber of $p_{U,V}$ is $m\#I + n\#J$. The dimension of the fiber of q is $m\#I + n\#J - 1$ in both cases. Hence, by [Har+11, Proposition 8.14],

$$\omega(\Gamma) = (p_{U,V})_*(\Phi_\Gamma(\omega'(\Gamma))) = (q \circ \rho)_*(\rho^*(\Phi'_\Gamma(\omega'(\Gamma)))) = 0. \qquad \square$$

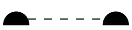
Remark 4.57. If a graph contains a univalent internal terrestrial vertex connected to an aerial vertex, then the argument fails. In fact, if we consider the following graph, then we find that $\omega(\Gamma) \in \Omega^{n-m-1}_{\mathrm{PA}}(\mathsf{VFM}_{mn}(0,1))$ represents the fundamental class of $\mathsf{VFM}_{mn}(0,1) \cong S^{n-m-1}$, i.e. the class η from Section 3.1:

$\Gamma =$  $\in \mathsf{vgraphs}_{mn}(0,1), \qquad (4.58)$

Indeed, by the definition of ψ^∂_{mn} in (4.23) and by [Har+11, Proposition 8.10], we find that $\int_{\mathsf{VFM}_{mn}(1,1)} \psi^\partial_{mn} = \int_{S^{n-1}} \mathrm{vol}_{n-1} = 1$. Therefore $\int_{\mathsf{VFM}_{mn}(0,1)} \omega(\Gamma) = 1$ by [Har+11, Proposition 8.13]. This also implies that if γ is obtained from the Γ above by making the external vertex internal, then we get $c(\gamma) = 1$.

Lemma 4.59. *The coefficient c vanishes on graphs with more than two vertices and which either contain a univalent aerial vertex, or which contain a univalent terrestrial vertex connected to another terrestrial vertex. It also vanishes on the graph with exactly two vertices, both aerial, and a (full) edge between the two.*

Proof. We can reuse the proof of Lemma 4.55 almost verbatim. There is one caveat: the graph γ with the univalent vertex removed must still satisfy the hypothesis $\#I + 2\#J \geq 2$. Otherwise, a degree shift occurs, to deal with the fact that $\dim \mathsf{VFM}_{mn}(I,J) = 0$ and not -1 or $-m-1$ as the general formula would give. This is the case unless γ has exactly two vertices with at most one aerial. If γ has exactly two aerial vertices and one edge, then $c(\gamma)$ vanishes for degree reasons ($\dim \mathsf{VFM}_{mn}(0,2) = 2n - m - 1 > n - 1$). $\square$

Remark 4.60. The restriction about the number of vertices is necessary. Indeed, for $\gamma =$  $\in \mathrm{VGC}_{mn}$. we find $c(\gamma) = \int_{\mathsf{VFM}_{mn}(2,0)} \varphi_m = \int_{S^{m-1}} \mathrm{vol}_{S^{m-1}} = 1$.

Lemma 4.61. *The morphism ω and the coefficient c vanish on graphs with bivalent internal terrestrial vertices connected to two terrestrial vertices.*

Proof. Let us first consider the case of the graph

$\Gamma =$  $\qquad (4.62)$

with three terrestrial vertices and two edges as indicated. Using the identification $\mathsf{VFM}_{mn}(U, \varnothing) \cong \mathsf{FM}_m(U)$ and the fact that the terrestrial propagator is the same as the one used in the definition of the map $\mathsf{Graphs}_m \to \Omega^*_{\mathrm{PA}}(\mathsf{FM}_m)$, we deduce that $\omega(\Gamma) = 0$ from [LV14, Lemma 9.3.9] (see also [Kon94, Lemma 2.1] for the origin of this "trick"). Briefly, the proof uses symmetry: there is an involution on $\mathsf{FM}_m(U)$ which leaves $\omega'(\Gamma)$ invariant but reverses the orientation, thus $\omega(\Gamma) = -\omega(\Gamma)$.

Now, if a graph contains Γ as a subgraph, then we use the theorems for pushforwards of PA forms just like in Lemma 4.55 to show that ω vanishes on it. Finally, simply note that ω and c vanish by definitions on graphs with full edges between terrestrial vertices. $\square$

Definition 4.63. Let $I(U,V) \subset \mathsf{vgraphs}_{mn}(U,V)$ be the vector space spanned by graphs containing loops, double edges, univalent aerial internal vertices, univalent terrestrial vertices connected to another terrestrial vertex, or bivalent internal terrestrial vertices connected to two terrestrial vertices by dashed edges.

Lemma 4.64. *The subspace $I(U,V)$ define a CDGA ideal in $\mathsf{vgraphs}_{mn}(U,V)$, and the subcollection $I \subset \mathsf{vgraphs}_{mn}$ define a cooperadic coideal.*

Proof. It is clear that $I(U,V)$ is an algebra ideal. Let us show that it is a differential ideal. The fact that I defines a cooperadic coideal can also be checked easily case-by-case. We deal with $m \geq 2$, and the case $m = 1$ is also checked by a similar case-by-case argument. Let $\Gamma \in I(U,V)$ be a graph. Let us show that $d\Gamma \in I(U,V)$. The three summands in Description 4.34 are called $d_1 = (-) \cdot (\mu - \mu_1)$, $d_2 = (c - c_1) \cdot (-)$, and $d_3 = (-) \cdot c_1$.

- If Γ contains a loop, then all the summands in $d\Gamma$ contain a loop, as both μ and c vanish on loops.
- Suppose Γ contains a univalent aerial internal vertex i, with incident edge e. In $d_2\Gamma = \sum_{\Gamma'} c(\Gamma')\Gamma/\Gamma'$, if Γ' contains e then $c(\Gamma')$ vanishes by Lemma 4.59, and otherwise Γ/Γ' contains either a loop (if $i \in \Gamma'$) or a univalent aerial internal vertex (if $i \notin \Gamma'$). Similarly, all terms in $d_3\Gamma = \sum_{\Gamma''} c(\Gamma/\Gamma'')\Gamma''$ either vanish, contain a loop, or contain a univalent aerial internal vertex. The only problem in $d_3\Gamma$ is for $\Gamma'' = \Gamma \setminus e$: then $c(\Gamma/\Gamma'') = 1$ (see Equation (5.2)) and Γ'' does not have a univalent vertex anymore. But this term cancels with the contraction of e in $d_1\Gamma$, and all the other summands of $d_1\Gamma$ contain a univalent aerial internal vertex.
- If Γ contains a univalent terrestrial vertices connected to another terrestrial vertex, then Lemma 4.59 and an argument similar to the previous one show that all summands of $d\Gamma$ either vanish, contain a loop, or contain a univalent terrestrial vertices connected to another terrestrial vertex.
- Suppose that Γ contains a bivalent internal terrestrial vertices connected to two terrestrial vertices by dashed edges. Let i be the internal terrestrial vertex, and e, e' its incident edges. All terms in $d_1\Gamma$ contain a similar subgraph. In $d_2\Gamma = \sum_{\Gamma'} c(\Gamma')\Gamma/\Gamma'$:
  - If $i \notin \Gamma'$, then Γ/Γ' still contains a bivalent internal terrestrial vertices connected to two terrestrial vertices.
  - If i is in Γ' but its incident edges are not, then Γ' contains an isolated vertex and $c(\Gamma') = 0$ by Lemma 4.59.
  - The term with $\Gamma' = e$ is cancelled with the terms with $\Gamma' = e'$.
  - If $e \in \Gamma'$ but $e' \notin \Gamma'$ and $\Gamma' \neq e$, then $c(\Gamma') = 0$ by Lemma 4.59. The case $e' \in \Gamma'$ and $e \notin \Gamma'$ is symmetric.
  - Finally if both $e, e' \in \Gamma'$, then $c(\Gamma') = 0$ by Lemma 4.61.

  The fact that $d_3\Gamma \in I(U,V)$ follows by similar arguments. □

Definition 4.65. The reduced graph cooperad $\mathsf{vgraphs}_{mn}$ is the relative graphs_n-cooperad given in each arity by the quotient of $\mathsf{VGraphs}_{mn}(U,V)$ by $I(U,V)$.

Proposition 4.66. *The CDGA $\mathsf{vgraphs}_{mn}(U,V)$ is well-defined, and ω factors through the quotient to define $\mathsf{vgraphs}_{mn}(U,V) \xrightarrow{\omega} \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}(U,V))$.*

Proof. This follows from Lemmas 4.54, 4.55, 4.59, and 4.61. $\square$

5. Proof of the formality

In this section we complete the proof of the formality of the operad VFM_{mn}. We first show that, up to homotopy, the differential of $\mathsf{vgraphs}_{mn}$ can be simplified. Then we construct a map from our graph complex to the cohomology of VFM_{mn}, and we prove that this map, as well as ω, are quasi-isomorphisms.

5.1. Change of Maurer–Cartan element and $\mathsf{vgraphs}^0_{mn}$

We would like to define a morphism $\mathsf{vgraphs}_{mn} \to \mathsf{vsc}^\vee_{mn}$. However, $\mathsf{vgraphs}_{mn}$ depends on the Maurer–Cartan element $c \in \mathrm{VGC}^\vee_{mn}$ from Equation (4.31), and we do not know the precise form of c. We just know its leading terms:

Proposition 5.1. *For $n-2 \geq m \geq 1$, we have $c = c_0 + (\dots)$, where $(\dots)$ denotes terms where $\#\{$terr. vert.$\} + 2\#\{$aer. vert.$\} > 3$, and:*

$c_0 :=$

 $\in \mathrm{VGC}^\vee_{mn}$, *if* $m \geq 2$;

 $\in \mathrm{VGC}^\vee_{1n}$, *if* $m = 1$. (5.2)

Proof. This follows from Remark 4.60, Remark 4.57, and Remark 4.40 for $m = 1$. Note that c vanishes by definition on the graph with just one internal terrestrial vertex (see after Equation (4.31)). There are no other (Lie-)connected graphs that satisfy $\#\{\text{terr. vert.}\} + 2\#\{\text{aer. vert.}\} \leq 3$, with no loops, double edges, full edges between terrestrial vertices, or bivalent terrestrial vertices connected by dashed edges to other terrestrial vertices (see Lemmas 4.54 and 4.61). $\square$

If we knew that $c = c_0$, then we would be able to build a map $\mathsf{vgraphs}_{mn} \to \mathsf{vsc}^\vee_{mn}$ easily (see Section 5.2). In this section, we show that c is gauge equivalent to c_0. For this we show that the cohomology of $\mathrm{VGC}^\vee_{mn}$ twisted by c_0 vanishes in the right degree. Obstruction theory then shows that c is gauge equivalent to c_0. As we will see, the cohomology of the graph complex

$$\mathrm{VGC}^{\vee,c_0}_{mn} := (\mathrm{VGC}^\vee_{mn}, d + [c_0, -]) \tag{5.3}$$

is related to the cohomology of $\mathrm{GC}^\vee_n$ and the cohomology of the "hairy graph complex" $\mathrm{HGC}^\vee_{mn}$ that will be defined below (see e.g. [FW15, Section 2.2.6] or [AT15]). This allows us to use known vanishing results about $\mathrm{GC}^\vee_n$ and HGC_{mn}.

Proposition 5.4 ([FW15, Proposition 2.2.3])**.** *The cohomology of $\mathrm{GC}^\vee_n$ splits as a direct sum $H^*(\mathrm{GC}^{\vee,\geq 3}_n) \oplus \bigoplus_{l \equiv 2n+1 \pmod 4} \Bbbk\gamma_l$, where $\deg \gamma_l = l - n$ and $\mathrm{GC}^{\vee,\geq 3}_n$ is the Lie subalgebra of graphs whose vertices are all at least trivalent. The class γ_l is represented by the loop with l vertices. Moreover $H^{>-n}(\mathrm{GC}^{\vee,\geq 3}_n) = 0$.*

Definition 5.5. For $k \geq 0$, let $\mathsf{Graphs}'_n(k)$ be the quotient of $\mathsf{Graphs}_n(k)$ by the ideal spanned by graphs which are disconnected or whose external vertices are not univalent. The hairy graph complex is (with differential from Graphs'_n):

$$\mathrm{HGC}^\vee_{mn} := \prod_{k\geq 1} \left(\mathsf{Graphs}'_n(k)^\vee \otimes (\mathbb{R}[m])^{\otimes k}\right)^{\Sigma_k}[-m]. \tag{5.6}$$

The full hairy graph complex is the dual of the shifted CDGA $S(\mathrm{HGC}_{mn}[m])[-m]$.

The complex $\mathrm{HGC}^\vee_{mn}$ is spanned by (infinite sums of) graphs whose external vertices are exactly univalent and indistinguishable. The differential is given by vertex splitting. Each external vertex, together with its only incident edge, can be seen as a "hair", which justifies the terminology.

Proposition 5.7 ([FW15, Proposition 2.2.7])**.** *When $n - m \geq 2$, the cohomology of the hairy graph complex $\mathrm{HGC}^\vee_{mn}$ vanishes in degrees > -1.*

Proof. Note that our definition of the hairy graph complex (denoted by HGC_{mn} without the dual in [FW15]) is slightly different, as we allow bivalent and univalent internal vertices. However, we can reuse their arguments to show that the inclusion of their complex into ours is a quasi-isomorphism (see also [Wil14, Proposition 3.4] for a similar argument). Briefly, we can filter both complexes by the number of internal vertices of valence ≥ 3. Both spectral sequences collapse starting on page E^2, and the inclusion induces an isomorphism on this page. We can then use [FW15, Proposition 2.2.7] to show the vanishing of the homology in degrees > -1 (note that in the reference, homologically graded complexes are used, so we just use the natural correspondence that reverse degrees). $\square$

There is a natural preLie product on $\mathrm{HGC}^\vee_{mn}$, induced by the operad structure of $\mathsf{Graphs}^\vee_n$. Roughly speaking, $\Gamma \circ \Gamma'$ is obtained by inserting Γ' in an external vertex of Γ and reconnecting the incident edge to a vertex of Γ', in all possible ways. Moreover, there is a natural action of the Lie algebra $\mathrm{GC}^\vee_n$ (see Definition 4.10 and the discussion that follows) on $\mathrm{HGC}^\vee_{mn}$. Given $\Gamma \in \mathrm{HGC}^\vee_{mn}$ and $\gamma \in \mathrm{GC}^\vee_n$, the action $\Gamma \cdot \gamma$ is given by inserting γ at a vertex of Γ in all possible ways.

Definition 5.8. Let $\mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ be the submodule of $\mathrm{fVGC}^{\vee,c_0}_{mn}$ spanned by graphs whose connected components all have at least one terrestrial vertex. Let $\mathrm{VGC}^{\vee,c_0,\mathrm{terr}}_{mn} \subset \mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ be the submodule of connected graphs. Then clearly:

Lemma 5.9. *The submodule $\mathrm{VGC}^{\vee,c_0,\mathrm{terr}}_{mn} \subset \mathrm{VGC}^{\vee,c_0}_{mn}$ is a dg-Lie subalgebra and a Lie $\mathrm{GC}^\vee_n$-submodule.* $\square$

Lemma 5.10. *There is an inclusion of dg-modules $\mathrm{fHGC}^\vee_{mn} \subset \mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ obtained by considering all external vertices as terrestrial, with no dashed edges. On the connected parts, this inclusion is compatible with the (pre)Lie algebra structure and the action of the Lie algebra $\mathrm{GC}^\vee_n$ on both sides.*

Proof. Simple inspection shows that the inclusion is well-defined, and that it is compatible with the differential and with the algebraic structures. □

Proposition 5.11. *The inclusion* $\mathrm{fHGC}^{\vee}_{mn} \subset \mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ *is a quasi-isomorphism.*

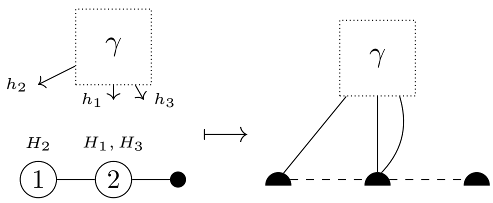
Proof. The proof is similar to the proof of [Wil14, Lemma 4.4]. Indeed, the graph complex $\mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ is very close to the deformation complex of the morphism of dg-operads $\mathsf{Graphs}^{\vee}_m \to \mathsf{Graphs}^{\vee}_n$ denoted by $\mathrm{Def}(\mathsf{hoe}_m \to \mathsf{Graphs}^{\vee}_n)$ in [Wil14] (without the dual sign there due to differing conventions).

We first filter both complexes by the number of full edges, which is the only kind of edges in $\mathrm{fHGC}^{\vee}_{mn}$. The differential of $\mathrm{fHGC}^{\vee}_{mn}$ always increases this number strictly by 1. Let us write $c_0 = c_0' + c_0''$, where c_0' is the part with two terrestrial vertices, and c_0'' with one vertex of each kind (see Equation (5.2)). The differential of $\mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ increases the filtration number by 1 (for the action of $\mu + c_0''$) or keeps it constant (for the action of c_0'). Hence on the associated spectral sequences, the differential of $E^0\mathrm{fHGC}^{\vee}_{mn}$ vanishes, while the differential of $E^0\mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ is just the bracket $[c_0', -]$.

We now check that the inclusion induces a quasi-isomorphism on these E^0 pages, from which the proposition follows. Let us first assume that $m \geq 2$. Given $\Gamma \in \mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$, define its character $[\Gamma] \in \mathrm{fHGC}^{\vee}_{mn}$ defined as follow: remove all terrestrial vertices and dashed edges, and call the full edges that used to be connected to terrestrial vertices "dangling", then make the dangling edges into hairs (see [Wil14, Lemma 4.4] for an analogous definition). The differential $[c_0', -]$ does not change the character of a graph. Hence $E^0\mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn}$ splits:

$$E^0\mathrm{fVGC}^{\vee,c_0,\mathrm{terr}}_{mn} = \prod_{\gamma \in \mathrm{fHGC}^{\vee}_{mn}} \underbrace{\{\Gamma \in E^0\mathrm{fVGC}^{\vee,c_0}_{mn} \mid [\Gamma] = \gamma\}}_{=:C_\gamma}. \tag{5.12}$$

Let $\gamma \in \mathrm{fHGC}^{\vee}_{mn}$ be a graph with hairs $\{h_1, \dots, h_k\}$. Let G be the group of permutations of hairs. Then C_γ is isomorphic to $C'_\gamma = (\mathsf{Graphs}^{\vee}_m \circ \bar{S}^c(H_1, \dots, H_k)^{(1,\dots,1)})^G$, where $\bar{S}^c(H_1, \dots, H_k)$ is the (non counital) cofree cocommutative coalgebra on variables H_i of degree $-m$, $\mathsf{Graphs}^{\vee}_m \circ -$ is the free $\mathsf{Graphs}^{\vee}_m$-algebra functor, and $(-)^{(1,\dots,1)}$ is the subcomplex where each H_i appears exactly once. Indeed, we can view $\Xi \in C'_\gamma$ as a linear combination of graphs from $\mathsf{Graphs}^{\vee}_m(r)$ with each external vertex decorated by one or more H_i, with each H_i appearing once. We can identify Ξ with an element of C_γ by making its edges dashed, its vertices terrestrial, and we glue γ to the graph obtained, connecting the hair h_i to the vertex decorated by H_i. The hairs are indistinguishable, but Ξ is invariant under G so this is well-defined. This is illustrated by (with Ξ at the bottom):



$\in C_\gamma$. (5.13)

The differential $[c_0', -]$ replicates the differential of $\mathsf{Graphs}^{\vee}_m(k)$ (i.e. vertex splitting), thus this is an isomorphism of dg-modules.

The homology of $\mathsf{Graphs}_m^\vee$ is the m-Poisson operad (Theorem 4.14). Checking the degrees and the induced differential $[\mu + c_0'', -]$, we can identify the page $E^1\mathrm{fVGC}_{mn}^{\vee,c_0,\mathrm{terr}}$ with (a shift of) the deformation complex $\mathrm{Def}(\mathsf{hoe}_m \xrightarrow{*} \mathsf{Graphs}_n^\vee)$ considered in [Wil14] (note that there, the case $n = m$ is considered and so the map $\mathsf{hoe}_n \to \mathsf{Graphs}_n^\vee$ sends the Lie bracket to a nonzero element; however, in [Wil14, Lemma 4.4], the part of the differential induced by this nonzero element is discarded and so the complex considered is $\mathrm{Def}(\mathsf{hoe}_m \to \mathsf{Graphs}_n^\vee)$ up to shifts; compare also with the results of [AT15, Section 5], where the full hairy graph complex is called $HH^{m,n}$). Since the differential of $\mathrm{fHGC}_{mn}^\vee$ raises the number of edges by exactly 1, the page $E^1\mathrm{fHGC}_{mn}^\vee$ is just $\mathrm{fHGC}_{mn}^\vee$. We can then conclude using the result of [Wil14, Lemma 4.4].

For $m = 1$, the proof is similar. The difference is that instead there are no dashed edges but Lie clusters instead. We get that $E^1\mathrm{fVGC}_{1n}^{\vee,c_0,\mathrm{terr}}$ is the (chains) deformation complex of $\mathsf{hoe}_1 \to \mathsf{Graphs}_n^\vee(k)$, whose homology is the full hairy graph complex fHGC_{1n}. The induced morphism on the E^2 pages is the identity and we can conclude. □

Corollary 5.14. *The inclusion* $\mathrm{HGC}_{mn}^\vee \subset \mathrm{VGC}_{mn}^{\vee,c_0,\mathrm{terr}}$ *is a quasi-isomorphism.*

Proof. Both CDGAs $\mathrm{fHGC}_{mn}^\vee$ and $\mathrm{fVGC}_{mn}^{\vee,c_0,\mathrm{terr}}$ are free as CDGAs, so they are in particular cofibrant. The functor of indecomposables is a left Quillen adjoint (see e.g. [LV12, Section 12.1.3]), therefore it preserves quasi-isomorphisms between cofibrant objects. Since the indecomposables of the two CDGAs mentioned above are respectively $\mathrm{HGC}_{mn}^\vee$ and $\mathrm{VGC}_{mn}^{\vee,c_0,\mathrm{terr}}$, the corollary follows from Lemma 5.10 and Proposition 5.11. □

Corollary 5.15. *The Maurer–Cartan element* $c - c_0 \in \mathrm{VGC}_{mn}^{\vee,c_0}$ *is gauge equivalent to zero; equivalently,* c *and* c_0 *are gauge equivalent.*

Proof. Let $C \subset \mathrm{VGC}_{mn}^{\vee,c_0}$ be the subalgebra spanned by graphs which are not the loops γ_l from Proposition 5.4. Similarly let $C' \subset \mathrm{GC}_n^\vee$ be the subalgebra spanned by graphs with are not the loops. We have a short exact sequence $0 \to \mathrm{VGC}_{mn}^{\vee,c_0,\mathrm{terr}} \to C \to C' \to 0$. Since the coefficient c vanishes on the loops γ_l by degree reasons, dually, we can say that c belongs to C. We can then combine Propositions 5.4, 5.7 and 5.11 to get that C vanishes in degrees > -1. We conclude by applying the Goldman–Millson theorem to the inclusion of the truncation $\tau_{<0}\mathrm{VGC}_{mn}^{\vee,c_0} \subset \mathrm{VGC}_{mn}^{\vee,c_0}$. □

Definition 5.16. Let $\mathsf{vgraphs}_{mn}^0$ be the variant of $\mathsf{vgraphs}_{mn}$ where we use c_0 instead of c to twist the Hopf cooperad VGra_{mn} in the step of Definition 4.33.

Corollary 5.17. *The Hopf cooperads* $\mathsf{vgraphs}_{mn}$ *and* $\mathsf{vgraphs}_{mn}^0$ *are quasi-isomorphic.*

Proof. This follows from the same general arguments of [Cam+18b, Section 5.4]. Let us briefly describe them. Let $S(t, dt)$ be the algebra of polynomial forms on the interval $[0, 1]$, with $\deg t = 0$ and $\deg dt = 1$. Let $\mathrm{VGC}_{mn}^{\vee,\sim}$ be the Lie algebra with differential $[\mu, -]$, i.e. we are only allowed to split aerial vertices. Both c and c_0 are Maurer–Cartan elements, i.e. they satisfy $[\mu, c] + \frac{1}{2}[c, c] = [\mu, c_0] + \frac{1}{2}[c_0, c_0] = 0$. The Lie algebra $\mathrm{VGC}_{mn}^\vee$ is the twist of $\mathrm{VGC}_{mn}^{\vee,\sim}$ with respect to c.

The gauge equivalence between c and c_0 can be seen as a Maurer Cartan element $c_t \in \mathrm{VGC}^{\vee,\sim}_{mn} \otimes S(t, dt)$ whose restriction at $t = 1$ (resp. $t = 0$) is c (resp. c_0). We can use this element c_t to produce a differential on $\mathsf{vgraphs}_{mn} \otimes S(t, dt)$ such that restriction at $t = 1$ (resp. $t = 0$) gives $\mathsf{vgraphs}_{mn}$ (resp. $\mathsf{vgraphs}^0_{mn}$). In other words, we have a zigzag:

$$\mathsf{vgraphs}_{mn} \xleftarrow{\mathrm{ev}_{t=1}} \mathsf{vgraphs}_{mn} \otimes S(t, dt) \xrightarrow{\mathrm{ev}_{t=0}} \mathsf{vgraphs}^0_{mn}. \tag{5.18}$$

The evaluation maps $\mathrm{ev}_{t=0}, \mathrm{ev}_{t=1} : S(t, dt) \to \mathbb{R}$ are quasi-isomorphisms of CDGAs. This implies that the two maps in the zigzag above are quasi-isomorphisms too. □

5.2. Connecting the graphs to the cohomology

The goal of this section is to describe a quasi-isomorphism of Hopf cooperads $\pi : \mathsf{vgraphs}^0_{mn} \to \mathsf{vsc}^\vee_{mn}$, where $\mathsf{vsc}^\vee_{mn} = H^*(\mathsf{VFM}_{mn})$ was obtained in Section 3.1.

For simplicity, we will describe this map on generators. The CDGA $\mathsf{vgraphs}^0_{mn}(U, V)$ is free as an algebra. Its generators are the "internally connected" graphs, i.e. the graphs which stay connected when all the external vertices are removed (keeping dangling edges). For example, if Γ has no internal vertices, then it is internally connected iff it has exactly one edge (an empty graph is not connected).

If Γ is such an internally connected graph, then $\pi(\Gamma) \in \mathsf{vsc}^\vee_{mn}(U, V)$ is given by:

- If $\Gamma = 1$ is the graph with no edges then $\pi(\Gamma) = 1$. If $m = 1$ then this is extended Σ_U-equivariantly to $\mathsf{vsc}^\vee_{mn}(U, \varnothing)$.
- If $\Gamma = e_{vv'}$ has no internal vertices and one full edge between v, $v' \in V$, then $\pi(\Gamma) = \omega_{vv'}$.
- If $\Gamma = \tilde{e}_{uu'}$ (for $m \geq 2$) has no internal vertices and one dashed edge between u, $u' \in U$, then $\pi(\Gamma) = \tilde{\omega}_{uu'}$.
- If Γ is the graph from Remark 4.57, i.e. it has exactly one internal vertex, which is terrestrial, and exactly one edge connecting that internal vertex to some aerial vertex u, then $\pi(\Gamma) = \eta_u$.
- In all other cases, $\pi(\Gamma) = 0$.

Proposition 5.19. *The map π is a quasi-isomorphism of Hopf cooperads $\mathsf{vgraphs}^0_{mn} \to \mathsf{vsc}^\vee_{mn}$.*

The proof of this proposition is split in a series of lemmas, which occupies the rest of this section (until the conclusion, Theorem 5.31).

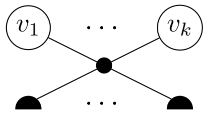
Lemma 5.20. *The map π is a well-defined algebra map and is equivariant with the symmetric group actions.*

Proof. Since we defined π on the generators of a free algebra (forgetting about the differential), it is well-defined. It is moreover clearly equivariant. □

Lemma 5.21. *The map π commutes with the differentials, i.e. $\pi d = 0$.*

Proof. Since π is an algebra map and the differential is a derivation, it is sufficient to check this on generators. Let Γ be an internally connected graph.

- If Γ has no internal vertices, then $d\Gamma = 0$ thus $\pi d\Gamma = 0$.
- If Γ has one internal vertex, then $\pi d\Gamma = 0$ follows from the Arnold relations in $\mathsf{e}_n^\vee$ (if the vertex is aerial) or in $\mathsf{e}_m^\vee$ (if the vertex is terrestrial) and the fact that full edges incident to terrestrial vertices are mapped to zero.
- Suppose Γ has at least two internal vertices. For a summand in $d\Gamma$ to be nonzero, it must be the case that, after contracting one edge, all remaining edges are between external vertices or between an external aerial vertex and a univalent terrestrial internal one. This implies first that there can be at most one aerial internal vertex. Moreover, since contracting an edge cannot reduce the valence of the remaining vertices (because contracting dead ends is forbidden), there can only be one internal vertex of valence greater than one, necessarily aerial. Using the internal connectedness of Γ, this special vertex must be connected to all the univalent terrestrial vertices by a full edge. In other words, the graph Γ must be of this type (plus some disconnected external vertices):



(5.22)

  Then the Arnold relations in $\mathsf{e}_n^\vee$, the symmetry relation $\eta_v \omega_{vv'} = \eta_{v'} \omega_{vv'}$, and the relation $\eta_v^2 = 0$ (if there is more than one terrestrial vertex) show that $\pi d\Gamma = 0$. □

Lemma 5.23. *The map π commutes with the cooperad structure maps.*

Proof. Once again it is sufficient to check this on generators, i.e. internally connected graphs. The verifications require a large number of cases but do not present any particular difficulty. □

5.3. Proof that π is a quasi-isomorphism

The last step for Proposition 5.19 is proving that π is a quasi-isomorphism. We split this proof in several sub-lemmas which occupy the rest of this section until the conclusion, Theorem 5.31. We deal separately with $m \geq 2$ and $m = 1$.

5.3.1. Case $m \geq 2$

Let us give a rough outline of our strategy. It is clear that π is surjective on cohomology, so we just need to check that $\mathsf{vgraphs}^0_{mn}(k,l)$ has the same Betti numbers as $\mathsf{VFM}_{mn}(k,l)$. In a first step, we prove the case $l = 0$, using an inductive argument inspired by the proof of [LV14, Theorem 8.1]. Then, we prove that we can reduce to the case of "split" graphs, where external aerial vertices and external terrestrial vertices are not in the same connected components. This mirrors the fact that as a space, $\mathsf{VFM}_{mn}(k,l) \simeq \mathrm{Conf}_{\mathbb{R}^m}(k) \times \mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)$. Finally, we prove the case $k = 0$, again using an inductive argument. We conclude using the Künneth formula.

We make an observation that will be useful throughout the proof. A graph $\Gamma \in \mathsf{vgraphs}^0_{mn}(k,l)$ determines a partition of $\{1,\dots,k\} \sqcup \{1,\dots,l\}$, by looking at connected components of Γ. We can define the subcomplex of connected graphs:

$$\mathsf{vgraphs}^0_{mn}(k,l)_{\mathrm{cn}} \subset \mathsf{vgraphs}^0_{mn}(k,l). \tag{5.24}$$

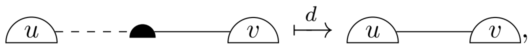
Then the complex $\mathsf{vgraphs}^0_{mn}(k,l)$ splits as a direct sum, over all partitions of $\{1,\dots,k\} \sqcup \{1,\dots,l\}$, of tensor products of complexes fo the type $\mathsf{vgraphs}^0_{mn}(-,-)_{\mathrm{cn}}$, one for each set in the partition (see [LV14, Equation (8.4)] for a similar statement about Graphs_n). For ease of notation, we will abbreviate the Betti numbers of this complex as:

$$\beta^j(k,l) \coloneqq \dim H^j(\mathsf{vgraphs}^0_{mn}(k,l)_{\mathrm{cn}}). \tag{5.25}$$

Note that we will focus on the two cases $k = 0$ and $l = 0$, as these will be the relevant ones for the application of the Künneth formula.

Lemma 5.26. *The map $\pi : \mathsf{vgraphs}^0_{mn}(k,0) \to \mathsf{vsc}^\vee_{mn}(k,0) = \mathsf{e}^\vee_m(k)$ is a quasi-isomorphism for all $k \geq 0$.*

If $\mathsf{vgraphs}^0_{mn}(k,0)$ had no aerial (internal) vertices and no full edges, then it would be equal to $\mathsf{Graphs}_m(k)$, which is quasi-isomorphic to $\mathsf{e}^\vee_m(k)$ by [LV14, Theorem 8.1]. The following proof is the formalization of the intuitive fact that a full edge is killed by:



(5.27)

and that internal vertices, whether aerial or terrestrial, do not produce any homology class and are just here to kill the Arnold relations.

Proof of Lemma 5.26. The case $k = 0$ is obvious: since there are no external vertices and each connected component must contain an external vertex, it follows that the graph must be empty. Therefore we get $\mathsf{vgraphs}^0_{mn}(0,0) = \mathsf{e}_m(0) = \mathbb{R}$, and the map π is the identity.

Let us now assume that $\mathsf{vgraphs}^0_{mn}(k,0) \to \mathsf{e}^\vee_m(k)$ is a quasi-isomorphism for a given $k \geq 0$. We want to prove that the same statement is true for $k+1$. The quotient map is clearly surjective on cohomology ($\tilde{\omega}_{ij}$ is represented by the graph with a single dashed edge between external vertices i and j). Hence it is sufficient to prove that $\mathsf{vgraphs}^0_{mn}(k+1,0)$ and $\mathsf{e}^\vee_m(k+1)$ have the same Betti numbers.

Using the known recurrence relation satisfied by the Betti numbers of $\mathrm{Conf}_{\mathbb{R}^m}$ and the splitting of the complexes in terms of connected components (see [LV14, Section 9]), we can restrict our attention to connected graphs (see Equation (5.25)). It suffices to prove that the Betti numbers of the connected part satisfy the relation $\beta^i(k+1,0) = k \cdot \beta^{i-m+1}(k,0)$.

We split the complex $\mathsf{vgraphs}^0_{mn}(k+1,0)_{\mathrm{cn}}$ in several submodules.

- The submodule U_i ($1 \leq i \leq k$) spanned by graphs such that the $(k+1)$th external vertex is univalent and connected to the external vertex i by a dashed edge. For $k \geq 1$, we set $U \coloneqq \bigoplus_{i=1}^k U_i$. In case $k = 0$, we instead set $U = \mathbb{R}$, the one-dimensional space spanned by the unit graph.

- the submodule V spanned by graphs such that the $(k+1)$th external vertex is univalent and connected by a dashed edge to an internal vertex;
- the submodule W spanned by graphs such that the $(k+1)$th external vertex is at least bivalent, or univalent and connected by a full edge to an internal vertex.

The submodule U is closed under the differential. We moreover have an isomorphism $(U_i, d) \cong \mathsf{vgraphs}^0_{mn}(k,0)_{\mathrm{cn}}[1-m]$ (for all $1 \le i \le k$), by removing the vertex $k+1$ and its incident edge. In case $k=0$, we clearly have $U = \mathbb{R} \cong \mathsf{vgraphs}^0_{mn}(0,0) = \mathsf{vgraphs}^0_{mn}(0,0)_{\mathrm{cn}}$

Let $\mathcal{Q} = \mathsf{vgraphs}^0_{mn}(k+1,0)_{\mathrm{cn}}/U$. As a graded module, we have $\mathcal{Q} = V \oplus W$, with some induced differential. Let us show that $\mathcal{Q}$ is acyclic. We put a filtration $F_\bullet \mathcal{Q}$ on $\mathcal{Q}$. A graph of V is in $F_p\mathcal{Q}$ if it has at most p edges, while a graph of W is in $F_p\mathcal{Q}$ if it has at most $p-1$ edges. On the $E^0\mathcal{Q}$ page of the associated spectral sequence, we see that the differential maps V isomorphically onto W. This shows that $E^1\mathcal{Q} = 0$, thus $\mathcal{Q}$ is acyclic.

Therefore $U \subset \mathsf{vgraphs}^0_{mn}(k+1,0)_{\mathrm{cn}}$ is a quasi-isomorphism. If $k=0$ then we have $U = \mathbb{R}$, which coincides with $\mathsf{e}^\vee_m(1) = \mathbb{R}$. If $k \ge 1$, then, as we observed earlier, we have $U_i \cong \mathsf{vgraphs}^0_{mn}(k,0)_{\mathrm{cn}}[1-m]$. Hence we find that the Betti numbers satisfy $\beta^j(k+1,0) = \sum_{i=1}^k \dim H^j(U_i) = k \cdot \beta^{j-m+1}(k,0)$, as expected. □

Let us now turn to the second step of the proof. We prove that we can, in some sense, "split" our graph complex in two: external aerial and external terrestrial.

Lemma 5.28. *Let $k, l \ge 1$ and let $I_{k,l} \subset \mathsf{vgraphs}^0_{mn}(k,l)$ be the module spanned by graphs where one of the connected components contains an external aerial vertex and an external terrestrial vertex. Then I is an acyclic dg-ideal.*

Proof. Contracting edges does not affect connected components. Therefore the differential cannot change the partition of the set of external vertices induced by some graph Γ, and thus $I_{k,l}$ is closed under the differential. It is an ideal: gluing along external vertices can merge connected components but never split them.

Let us now prove that $I_{k,l}$ is acyclic. We can restrict our attention to connected graphs (the general case follows by the Künneth formula). We use a technique similar to the proof of Lemma 5.26, by induction on $k \ge 1$, for any fixed $l \ge 1$.

For $k=1$ we check directly that $I_{1,l}$ is acyclic. We split $I_{1,l} = I'_{1,l} \oplus I''_{1,l}$ where $I'_{1,l}$ is spanned by graphs where the external terrestrial vertex is univalent with an incident dashed edge, and $I''_{1,l}$ is spanned by all other graphs. We can then filter $I_{1,l}$ (similarly to Lemma 5.26) so that on the E^0 page, $I'_{1,l}$ is mapped isomorphically on $I''_{1,l}$, which shows that $I_{1,l}$ is acyclic.

Let us now assume that the claim is true for a given $k \ge 1$, and let us prove the claim for $k+1$. Just like in the proof of Lemma 5.26, we can split $I_{k+1,l}$ in three summands, depending on whether the last external terrestrial vertex is: univalent, connected by a dashed edge to an external vertex; univalent, connected by a dashed edge to an internal vertex; all other cases. The first subcomplex is isomorphic to k copies of the ideal $I_{k,l}$, thus it is acyclic by the induction hypothesis. On the quotient by this subcomplex, we can put a filtration just like in Lemma 5.26, so that on the E^0 page, the second complex is mapped isomorphically onto the third. This shows that the quotient is acyclic too, and therefore $I_{k+1,l}$ is acyclic. □

Lemma 5.29. *The map* $\pi : \mathsf{vgraphs}^0_{mn}(0,l) \to \mathsf{vsc}^\vee_{mn}(0,l)$ *is a quasi-isomorphism* $\forall l \geq 0$.

Proof. This final lemma is also proved by induction. Once again this map is clearly surjective on cohomology, so it suffices to prove that both complexes have the same Betti numbers. Using the results of Section 3.1, the Poincaré polynomial of $\mathsf{VFM}_{mn}(0,l) \simeq \mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)$ is $\mathscr{P}(\mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)) = \prod_{i=0}^{l-1}(1 + t^{n-m-1} + it^{n-1})$.

We can again work with the connected part of the graph complex $\mathsf{vgraphs}^0_{mn}(0,l)_{\mathrm{cn}}$. Note that the case $l = 0$ is covered by Lemma 5.26. The base case that we need to prove is $\beta^0(0,1) = \beta^{n-m-1}(0,1) = 1$, and $\beta^j(0,1) = 0$ for other j. The recurrence relation is $\beta^j(0,l+1) = l \cdot \beta^{j-n+1}(0,l)$ for all j and all $l \geq 1$.

For $l = 1$, we again have an explicit homotopy. Given a graph Γ, the graph $h(\Gamma)$ is obtained by replacing the external (aerial) vertex of Γ by an internal (aerial) one, adding back an external vertex, and connected the new external vertex to the old one by a (full) edge. We then check that if Γ is the unit graph or the graph from Remark 4.57, then $(dh + hd)(\Gamma) = 0$, while if Γ is another graph, then $(dh + hd)(\Gamma) = \Gamma$ (including if the external vertex of Γ is univalent). To prove the recurrence relation, we split $\mathsf{vgraphs}^0_{mn}(0,l+1)_{\mathrm{cn}}$:

- The submodule U_i where the external vertex $(k+1)$ is univalent, connected to the external vertex i. We also set $U = \bigoplus_{i=1}^k U_i$.
- The submodule V, where the external vertex $(k+1)$ is at least bivalent.
- The submodule V' where the external vertex $(k+1)$ is univalent, connected to an aerial internal vertex.
- The submodule W, where the external vertex $(k+1)$ is univalent, connected to a terrestrial internal vertex; this terrestrial vertex is itself bivalent, and its other incident edge is dashed.
- The submodule W', where the external vertex $(k+1)$ is univalent, connected to a terrestrial internal vertex; this terrestrial vertex is either at least trivalent, or bivalent and both incident edges are full.

Let $\mathcal{Q} = \mathsf{vgraphs}^0_{mn}(0,l+1)/U$ be the quotient. We can set up a spectral sequence just like in the proof of Lemma 5.26, such that on the E^0 page, the differential takes V' isomorphically onto V, and W' isomorphically onto W. This show that $\mathcal{Q}$ is acyclic; thus, the inclusion $U \subset \mathsf{vgraphs}^0_{mn}(0,l+1)$ is a quasi-isomorphism. We have an isomorphism $U_i \cong \mathsf{vgraphs}^0_{mn}(0,l)[1-m]$ being given by removing the last external vertex and its incident edge. It follows that the Betti numbers satisfy the expected recurrence relation, $\beta^j(0,l+1) = l \cdot \beta^{j-n+1}(0,l)$. □

5.3.2. Case $m = 1$

We deal separately with the case $m = 1$, because e_1 is the associative operad and not the Poisson operad. To summarize the differences, recall that: the graphs do not have dashed edges, and the terrestrial vertices are ordered (Definition 4.16); the notion of "disconnected" is replaced by "Lie-disconnected" (Definition 4.44); the differential $[c_0, -]$ merges Lie clusters (5.2).

Proposition 5.30. *The map* $\pi : \mathsf{vgraphs}^0_{1n}(k,l) \to \mathsf{vsc}^\vee_{1n}(k,l)$ *is a quasi-isomorphism for all* $k, l \geq 0$.

Proof. As in the case $m = 2$, the map π is clearly surjective on cohomology, so we just need to check that $\mathsf{vgraphs}^0_{1n}(k,l)$ has the correct Betti numbers.

The proofs of Lemmas 5.26, 5.28, and 5.29 can be adapted in a straightforward manner. We can follow the same proofs, replacing m with 1. The crucial difference will be in the splitting of the complex $\mathsf{vgraphs}^0_{1n}(k,l)_{\mathrm{cn}}$ or of $I_{k,l}$.

- In $\mathsf{vgraphs}^0_{1n}(k,0)_{\mathrm{cn}}$ (for Lemma 5.26), we set U to be the submodule where the $(k+1)$th external vertex is isolated but not adjacent to a terrestrial internal vertex (terrestrial vertices are ordered for $m = 1$), V the submodule where the $(k+1)$th external vertex is isolated and adjacent to a terrestrial internal vertex, and W all other kinds of graphs.
- In $I_{k,l}$ (for Lemma 5.28), we use the same splitting as for $\mathsf{vgraphs}^0_{1n}(k,0)_{\mathrm{cn}}$.
- In $\mathsf{vgraphs}^0_{1n}(0,l)$ (for Lemma 5.29), we keep the same U, V, and V' as in the proof of Lemma 5.29. We change the submodules W and W': in W, we require the last external vertex to be connected to a univalent internal terrestrial vertex, while in W' we put all other graphs.

With these changes, we obtain the correct recurrence relations on the Betti numbers. □

5.3.3. Conclusion

Theorem 5.31. *The operad* VFM_{mn} *is formal over* $\mathbb{R}$ *for* $n - 2 \geq m \geq 1$.

Proof. We have a zigzag:

$$\mathsf{vsc}^\vee_{mn} \xleftarrow{\pi} \mathsf{vgraphs}^0_{mn} \leftarrow \mathsf{vgraphs}_{mn} \otimes S(t, dt) \to \mathsf{vgraphs}_{mn} \xrightarrow{\omega} \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}), \tag{5.32}$$

where $\mathsf{vsc}^\vee_{mn}$ is defined in Section 3.1, $\mathsf{vgraphs}_{mn}$ in Section 4.3, $\mathsf{vgraphs}^0_{mn}$ in Definition 5.16, and $\mathsf{vgraphs}_{mn} \otimes S(t, dt)$ in Corollary 5.17. The map π is defined at the beginning of Section 5.2, and the map ω is defined in Proposition 4.66.

We proved in Proposition 3.20 that $\mathsf{vsc}^\vee_{mn} \cong H^*(\mathsf{VFM}_{mn})$ as Hopf cooperads. We moreover proved in Corollary 5.17 that the two maps involving the three variants of $\mathsf{vgraphs}_{mn}$ were quasi-isomorphisms of Hopf cooperads. In addition, we proved that π was a quasi-isomorphism of Hopf cooperads in Proposition 5.19 (for $m \geq 2$) and Proposition 5.30 (for $m = 1$). Therefore it just remains to check that ω is a quasi-isomorphism of Hopf cooperads to conclude.

By the previous results, we know that $\mathsf{vgraphs}_{mn}$ and $\Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$ have the same cohomology, namely $\mathsf{vsc}^\vee_{mn}$. Thus it is sufficient to show that the map ω is surjective on cohomology. The generators of $\mathsf{e}^\vee_n$ are represented by full edges between aerial external vertices. Generators of $\mathsf{e}^\vee_m$ are represented by dashed edges between terrestrial external vertices. The classes η_v are represented by graphs of the type seen in Remark 4.57. □

A. Relative cooperadic twisting

Operadic twisting is a tool originally introduced in [Wil14, Appendix I], studied in further detail in [DW15], and generalized to certain types of colored operads in [Wil16, Appendix C]. In this appendix, we quickly recall operadic twisting for cooperads and right comodules, and we combine both to deal with relative cooperads.

A.1. Twisting cooperads

General references for twisting of plain operads are [Wil14, Appendix I] and [DW15]. The dual notion of cooperadic twisting is spelled out in [Idr19, Section 1.5]. Let $\mathsf{Lie}_n = \mathsf{Lie}\{n-1\}$ be the operad governing Lie algebras with a bracket of homological degree $n-1$ (so $\mathsf{Lie}_n \subset \mathsf{e}_n$). Let $\mathsf{hoLie}_n = \Omega(\mathsf{Lie}_n^{\mathsf{i}}) = \Omega(\mathsf{Com}^\vee\{n\})$ be its Koszul resolution. Suppose that C is an operad with finite-dimensional components (so that $\mathsf{C}^\vee$ is a cooperad) equipped with an operad morphism $\mu : \mathsf{hoLie}_n \to \mathsf{C}^\vee$. We consider the following convolution Lie algebra:

$$\mathfrak{g}_{\mathsf{C}} \coloneqq \hom_\Sigma(\mathsf{Com}^\vee\{n\}, \mathsf{C}^\vee) = \Bigl(\prod_{i\geq 0} \bigl(\mathsf{C}^\vee(i) \otimes \mathbb{R}[-n]^{\otimes i}\bigr)^{\Sigma_i}[n], d, [-,-]\Bigr).$$

The differential is induced from C. The Lie bracket of $f, g \in \mathfrak{g}_{\mathsf{C}}$ is $[f,g] = f \star g \mp g \star f$, where $\star$ is the convolution product. Thanks to [LV12, Theorem 6.5.7], the morphism $\mu : \mathsf{hoLie}_n \to \mathsf{C}^\vee$ can equivalently be seen as a Maurer–Cartan element $\mu \in \mathfrak{g}_{\mathsf{C}}$.

The twist of C with respect to μ has as underlying graded module:

$$\mathrm{Tw}\,\mathsf{C}(U) \coloneqq \bigoplus_{i\geq 0} \bigl(\mathsf{C}(U \sqcup \{1,\dots,i\}) \otimes \mathbb{R}[n]^{\otimes i}\bigr)_{\Sigma_i}.$$

The entries labeled by U are called "external", whereas the entries that were labeled by $\{1,\dots,i\}$ before taking coinvariants are called "internal". The cooperadic structure is inherited from C. Let $\mu_1 \in \prod_{j\geq 0} \mathsf{C}^\vee(\{1,\dots,j,*\})^{\Sigma_j}$ (up to shifts and signs) be the element obtained from μ by summing over all possible ways of distinguishing one of the inputs. The differential of $x \in \mathrm{Tw}\,\mathsf{C}$ is $dx = d_{\mathsf{C}}x + x \cdot \mu - x \cdot \mu_1 - \mu_1 \cdot x$, i.e. the sum of the internal differential of C with a threefold action of μ: (i) co-insertion of μ in an internal input of x in all possible way; (ii) co-insertion of $-\mu_1$ in an external input of x in all possible ways; (iii) co-insertion of x in the external input of $-\mu_1$. One checks that $\mu - \mu_1$ defines a Maurer–Cartan element in $\mathfrak{g}_{\mathsf{C}} \ltimes \mathrm{Tw}\,\mathsf{C}^\vee(*)$, so this differential squares to zero. (Note that $\mathrm{Tw}\,\mathsf{C}(*)$ acts on the right and on the left of $\mathrm{Tw}\,\mathsf{C}$, which is explicitly written in the formula for dx above.) The compatibility with the cooperad structure is immediate by coassociativity.

A.2. Twisting right comodules

We now recall twisting of right comodules (see [Wil16, Appendix C.1] for the dual case of right modules). Fix $\mu : \mathsf{hoLie}_n \to \mathsf{C}^\vee$ as in Section A.1. Suppose that M is a right

C-comodule. Then as a graded module,

$$\operatorname{Tw}\mathsf{M}(U) := \bigoplus_{i\geq 0}\left(\mathsf{M}(U\sqcup\{1,\dots,i\})\otimes\mathbb{R}[n]^{\otimes i}\right)_{\Sigma_i}.$$

This inherits a right $(\operatorname{Tw}\mathsf{C})$-comodule structure from the C-comodule structure of M. The differential of $x\in\operatorname{Tw}\mathsf{M}(U)$ is given by $dx = d_{\mathsf{M}}x + x\cdot\mu - x\cdot\mu_1$ (where one uses the comodule structure instead of the cooperad structure). Note that since M is only a right module, there can be no term of the type $\mu_1\cdot x$.

A.3. Twisting relative cooperads

Let us finally deal with relative cooperads (see Section 1.1 for the definition). The definition is inspired by the case of "moperads" (i.e. relative operads which can only admit operations with zero or one terrestrial input) [Wil16, Appendix C.3].

Let Lie_{mn} be the relative Lie_n-operad generated by $\lambda_{2,0}\in\mathsf{Lie}_{mn}(\{u,u'\},\varnothing)^{1-m}$ and $\beta\in\mathsf{Lie}_{mn}(\{u\},\{v\})^{1-n}$. The relations are the shifted Jacobi relation for $\lambda_{2,0}$, and (where $\lambda_2\in\mathsf{Lie}_n(\{v,v'\})^{1-n}$ is the generator):

$$\beta(\lambda_2(x_v,x_{v'}),y_u) = \beta(x_v,\beta(x_{v'},y_u)) \pm \beta(x_{v'},\beta(x_v,y_u)),$$
$$\beta(x_v,\lambda_{2,0}(y_u,y_{u'})) = \lambda_{2,0}(\beta(x_v,y_u),y_{u'}) \pm \lambda_{2,0}(y_u,\beta(x_v,y_{u'})).$$

Remark A.1. This is not the moperad $\mathsf{Lie}_{k,1}$ from [Wil16, Appendix C.3]. We consider an action $\beta\in\mathsf{Lie}_{mn}(\{u\},\{v\})$ instead of a morphism $f\notin\mathsf{Lie}_{mn}(\varnothing,\{v\})$ as otherwise the operad would not be quadratic: $f(\lambda_2)=\lambda_{2,0}(f,f)$ is cubical. (Informally, we view β as being $\lambda_{2,0}(f,\mathrm{id})$.)

Proposition A.2. *The Koszul dual of* Lie_{mn} *is* $\mathsf{Com}^\vee_{mn}$*, the dual of the operad governing triples* (A,B,α) *where* A *is a* $\mathsf{Com}\{n\}$*-algebra,* B *is a* $\mathsf{Com}\{m\}$*-algebra, and* $\alpha: B\otimes A\to B[-m]$ *is an* A*-module structure map satisfying* $\alpha(bb'\otimes a)=\alpha(b\otimes\alpha(b'\otimes a))$*. In particular,* $\mathsf{Com}^\vee_{mn}(U,V)\cong\mathbb{R}[-m]^{\otimes U}\otimes\mathbb{R}[-n]^{\otimes V}\otimes\mathbb{R}[m]$ *(for* $U\neq\varnothing$*) is a one-dimensional representation of* $\Sigma_U\times\Sigma_V$*.*

Proof. This follows from a direct computation of the dual of the space of relations in the presentation of Lie_{mn} above. Compare with [LV12, Proposition 13.1.1] for the proof that $\mathsf{Lie}^{\mathsf{i}}=\mathsf{Com}^\vee\{1\}$ and with [HL13, Lemma 3.1.1] for a computation in the colored case of the Swiss-Cheese operad. □

Let $\mathsf{hoLie}_{mn}=\Omega(\mathsf{Com}^\vee_{mn})$, which is a relative hoLie_n-operad (it seems likely that Lie_{mn} is Koszul but this is not necessary for what follows). Let D be a relative C-cooperad equipped with a morphism $(c,\mu):(\mathsf{hoLie}_{mn},\mathsf{hoLie}_n)\to(\mathsf{D},\mathsf{C})$. This can equivalently be seen as a Maurer–Cartan element in $\mathfrak{g}_{\mathsf{C}}\oplus\mathfrak{g}_{\mathsf{C},\mathsf{D}}$ (i.e. $d\mu+\frac{1}{2}[\mu,\mu]=dc+[\mu,c]+\frac{1}{2}[c,c]=0$), where $\mathfrak{g}_{\mathsf{C},\mathsf{D}}=\hom_\Sigma(\mathsf{Com}^\vee_{mn},\mathsf{D})$.

Let us define the twisted relative $(\operatorname{Tw}\mathsf{C})$-cooperad $\operatorname{Tw}\mathsf{D}$. As a graded module,

$$\operatorname{Tw}\mathsf{D}(U,V) := \bigoplus_{i,j\geq 0}\left(\mathsf{D}(U\sqcup\{1,\dots,i\},V\sqcup\{1,\dots,j\})\otimes\mathbb{R}[m]^{\otimes i}\otimes\mathbb{R}[n]^{\otimes j}\right)_{\Sigma_i\times\Sigma_j}.$$

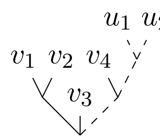
The relative ($\mathrm{Tw}\,\mathsf{C}$)-cooperad structure is inherited from D. Let c_1 be the element obtained from c by summing over all possible ways of distinguishing one of the terrestrial inputs of c (similarly to how μ_1 is defined from μ). Then the differential of $x \in \mathrm{Tw}\,\mathsf{D}(U, V)$ is given by $d_{\mathsf{D}}x + x \cdot \mu + x \cdot (c - c_1) - c_1 \cdot x$.

Proposition A.3. *The collection* $\mathrm{Tw}\,\mathsf{D}$ *defines a relative* ($\mathrm{Tw}\,\mathsf{C}$)*-cooperad.*

Proof. Generalizing the proofs of [Wil14, Appendix I] and [Wil16, Appendix C.3] to this setting is straightforward. One checks that $\mu + c + c_1$ defines a Maurer–Cartan element in $\mathfrak{g}_{\mathsf{C}} \rtimes \mathfrak{g}_{\mathsf{C},\mathsf{D}} \rtimes \mathrm{Tw}\,\mathsf{D}^{\vee}(*, \varnothing)$ – which acts by cooperadic coderivations on $\mathrm{Tw}\,\mathsf{D}$ – so the differential above squares to zero. Compatibility with the cooperad structure follows from the coassociativity of the cooperad structure. $\square$

B. Compactifications and projections

In this appendix, we sketch a proof of Proposition 2.9: $\mathsf{VFM}_{mn}(U, V)$ is a compact SA manifold and a smooth manifold with corners, and the canonical projection maps are SA bundles. Our proofs are heavily inspired by [LV14, Section 5.9].



Let (U, V) be a pair of finite sets. A relative (rooted) tree $\mathcal{T}$ with leaves (U, V) is a rooted tree with dashed and full edges. We require that the leaves with incident full (resp. dashed) edge are in bijection with U (resp. V), that if a vertex has only one incoming edge then this edge is full, and that if an edge is full then all the edges above it are full. An example is on the side.

For a relative tree $\mathcal{T}$, we let $V_{\mathcal{T}}$ be the set of all its vertices, $V^0_{\mathcal{T}} = V_{\mathcal{T}} \setminus \mathrm{root}$, and $V^*_{\mathcal{T}} = V_{\mathcal{T}} \setminus (U \cup V)$. The set $V_{\mathcal{T}}$ is partially ordered by considering that a vertex is smaller than any vertex above it. For $i \in V_{\mathcal{T}}$, we let $\mathrm{in}(i) = \mathrm{in}_t(i) \cup \mathrm{in}_a(i) = \{\text{incoming dashed edges}\} \cup \{\text{incoming full edges}\}$ and $\mathrm{par}(i) \in V_{\mathcal{T}}$ be the immediate predecessor of i. Finally, we let:

$$\underline{\mathrm{Conf}}^{\mathcal{T}}_{mn} \coloneqq \prod\nolimits_{i \in V^*_{\mathcal{T}}} \underline{\mathrm{Conf}}_{mn}(\mathrm{in}_t(i), \mathrm{in}_a(i)).$$

The spaces $\underline{\mathrm{Conf}}^{\mathcal{T}}_{mn}$ will be used to give a decomposition of $\mathsf{VFM}_{mn}(U, V)$ as in [LV14, Section 5.9.1]. Let $\xi = (\xi_i)_{i \in V^*_{\mathcal{T}}} \in \underline{\mathrm{Conf}}^{\mathcal{T}}_{mn}$. We can represent ξ_i by a configuration $\bar{\xi}_i \in \mathrm{Conf}_{mn}(\mathrm{in}_t(v), \mathrm{in}_a(i))$ of radius 1 and whose barycenter is in $\{0\}^m \times \mathbb{R}^{n-m}$. For $i \in V^0_{\mathcal{T}}$, we let $\xi(i) \coloneqq \bar{\xi}_{\mathrm{par}(i)}(i)$. We then define, for $r > 0$ and $i \in V_{\mathcal{T}}$:

$$x(\xi, r, i) \coloneqq \sum\nolimits_{j \in V^0_{\mathcal{T}}, j \leq i} \xi(j) \cdot r^{\mathrm{height}(j)} \in \mathbb{R}^n.$$

Then $(x(\xi, r, i))_{i \in U \sqcup V}$ is a configuration for r small enough. Let use define $h_{\mathcal{T}} : \underline{\mathrm{Conf}}^{\mathcal{T}}_{mn} \hookrightarrow \mathsf{VFM}_{mn}(U, V)$ by $h_{\mathcal{T}}(\xi) = \lim_{r \to 0}(x(\xi, r, i))_{i \in U \sqcup V}$. The map $h_{\mathcal{T}}$ is a homeomorphism onto its image, $\{\mathrm{im}(h_{\mathcal{T}})\}_{\mathcal{T}}$ covers $\mathsf{VFM}_{mn}(U, V)$, and the interior of $\mathsf{VFM}_{mn}(U, V)$ is the stratum corresponding to a corolla.

Now let $x = h_{\mathcal{T}}(\xi) \in \mathsf{VFM}_{mn}(U, V)$. We want to build an SA chart around x. Let

$$r_1 \coloneqq \frac{1}{4} \min\{\|\bar{\xi}_i(a) - \bar{\xi}_i(b)\| \mid i \in V_{\mathcal{T}}^0, a \neq b \in \mathrm{in}(i)\} \cup \{d(\bar{\xi}_i(v), \mathbb{R}^m) \mid i \in V_{\mathcal{T}}^0, v \in \mathrm{in}_a(i)\}.$$

Note that $r_1 \leq \frac{1}{2}$, because $\bar{\xi}_i$ has radius 1. Define a neighborhood of ξ by $W = \{\zeta \in \underline{\mathrm{Conf}}_{mn}^{\mathcal{T}} \mid \forall i, \|\xi(i) - \zeta(i)\| \leq r_1^{\#U+\#V+1}\}$ (thanks to the distance condition, distinct points stay distinct and aerial points stay aerial). For $\tau \in [0, r_1]^{V_{\mathcal{T}}^*}$ with $\tau_{\mathrm{root}} = 0$ and $0 \leq r \leq r_1$, we let $y(\zeta, \tau, r, \mathrm{root}) \coloneqq 0$ and

$$y(\zeta, \tau, r, i) \coloneqq y(\zeta, \tau, r, \mathrm{par}(i)) + \xi(i) \cdot \prod_{j<i} \max(r, \tau_j).$$

We can then define $\Phi : W \times [0, r_1]^{V_{\mathcal{T}}^* \setminus \{\mathrm{root}\}} \to \mathsf{VFM}_{mn}(U, V)$ by $\Phi(\zeta, \tau) \coloneqq \lim_{r \to 0}(y(\xi, \tau, r, i))_{i \in U \sqcup V}$. We also let V be the image of Φ. The proof that Φ is an SA chart onto a compact neighborhood of x is identical to [LV14, Lemma 5.9.3]. This proves the first part of Proposition 2.9.

We would now like to prove that $p_{U,V} : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U, V)$ is an SA bundle. Since the composite of two SA bundles is an SA bundle [Har+11, Proposition 8.5], it is sufficient to check that the following are SA bundles:

$$p : \mathsf{VFM}_{mn}(U \sqcup *, V) \to \mathsf{VFM}_{mn}(U, V), \quad q : \mathsf{VFM}_{mn}(U, V \sqcup *) \to \mathsf{VFM}_{mn}(U, V).$$

We will describe the fibers explicitly as complements of open balls. The fiber of p will be almost identical to the one in [LV14, Section 5.9.4]. However the fiber of q is slightly different, because the new aerial point cannot touch the ground.

Let $x = h_{\mathcal{T}}(\xi)$, $r_1 > 0$, and W as before. For $\zeta \in W$ and $i \in V_{\mathcal{T}}$, define $x_1(\zeta, i) \coloneqq x(\zeta, r_1, i)$ and $\varepsilon(i) \coloneqq 4r_1^{\mathrm{height}(i)+1}$. Let $B_i(\zeta) \coloneqq B(x_1(\zeta, i), \varepsilon(i))$ be the closed ball. Then $B_i(\zeta) \subset 1/3 B_j(\zeta)$ if $i < j$ and $B_i(\zeta) \cap B_j(\zeta) = \varnothing$ otherwise.

Recall $\phi : \mathbb{R}^n \times [0, 1] \times [0, 2] \times \mathbb{R}^n \to \mathbb{R}^n$, $(c, r, \varepsilon, x) \mapsto \phi_r^{c,\varepsilon}(x)$ from [LV14, Lemma 5.9.5]. It is such that $\phi_r^{c,\varepsilon}$ is radial, the identity outside $B(c, \varepsilon)$, and shrinks $B(c, \varepsilon/3)$ by a factor r. Moreover for a configuration $x \in \mathrm{Conf}_{B(c,\varepsilon/3)}(k)$, $\phi_r^{c,\varepsilon}(x)$ is a configuration in $\underline{\mathrm{Conf}}_n(k)$ that does not depend on r, and ϕ behaves well with respect to other points $z \in B(c, \varepsilon)$ (see the reference for details). We note in addition that, thanks to the properties of ϕ, if $c \in \mathbb{R}^m$ then $\phi_r^{c,\varepsilon}(\mathbb{R}^m) \subset \mathbb{R}^m$, and if $c \notin R^m$, then $\phi_r^{c,\varepsilon}(\mathbb{R}^n \setminus B(\mathbb{R}^m, \varepsilon)) \subset \mathbb{R}^n \setminus B(\mathbb{R}^m, \varepsilon)$ (where $B(\mathbb{R}^m, \varepsilon) = \bigcup_{z \in \mathbb{R}^m} B(z, \varepsilon)$).

Now, fix $\zeta \in W$ and $\tau \in [0, r_1]^{V_{\mathcal{T}}}$ s.t. $\tau_{\mathrm{root}} = 0$ and $\tau_i = 0$ for $i \in U \sqcup V$ a leaf. Then for $i \in V_{\mathcal{T}}$ and $0 < r \leq r_1$, let $\phi_r^i = \phi_{\max(r, \tau(i))/r_1}^{x_1(\zeta,i),\varepsilon(i)}$. Moreover, let ϕ_r be the composition (in any order thanks to the disjointness of the balls) of the ϕ_r^i for $i \in U \sqcup V$. (Despite the notation, ϕ_r depends on x, ζ and τ.) We then check, like in [LV14, Lemma 5.9.6], that $\phi_r(x_1(\zeta, i)) = x(\zeta, \tau, r, i)$ for $r > 0$ and $i \in U \sqcup V$.

We can now check the local trivialities of p and q. Let us first deal with $p : \mathsf{VFM}_{mn}(U \sqcup *, V) \to \mathsf{VFM}_{mn}(U, V)$. We define $F_\zeta \coloneqq B(x_1(\zeta, \mathrm{root}), \varepsilon(\mathrm{root})/2) \setminus \bigcup_{u \in U} B(x_1(\zeta, u), \varepsilon(u)/2) \cap \mathbb{R}^m$ for $\zeta \in W$, which will be the fiber of p over $h_{\mathcal{T}}(\zeta)$. We also set $F \coloneqq B(0, \#U + 1) \setminus \bigcup_{i=1}^{\#U} \mathring{B}((i, 0, \dots, 0), 1/4) \cap \mathbb{R}^m$. Note that F is a compact SA manifold (a closed m-ball with U open balls removed). There is an SA-homeomorphism $\Theta_\zeta : F \cong F_\zeta$ since W is small enough. Let

$$\begin{aligned}\widehat{\Phi} : W \times [0,1]^{V^*_{\mathcal{T}} \setminus \mathrm{root}} \times F &\to \mathsf{VFM}_{mn}(U \sqcup *, V),\\ (\zeta, \tau, z) &\mapsto \lim_{r \to 0} \big((\phi_r(x_1(\zeta, i)))_{i \in U \sqcup V}, \phi_r(\Theta_\zeta(z)) \big).\end{aligned}$$

Then $\widehat{\Phi}$ covers Φ. The proof that $\widehat{\Phi}(\zeta, \tau, -)$ maps F_ζ homeomorphically onto $p^{-1}(\Phi(\zeta, \tau))$ is identical to the proof in [LV14, Section 5.9.4].

Now let us deal with $q : \mathsf{VFM}_{mn}(U, V \sqcup *) \to \mathsf{VFM}_{mn}(U, V)$. Its fiber over $h_{\mathcal{T}}(\zeta)$ will be $G_\zeta \coloneqq B(x_1(\zeta, \mathrm{root}), \varepsilon(\mathrm{root})/2) \setminus \bigcup_{v \in V} \mathring{B}(x_1(\zeta, v), \varepsilon(v)/2) \setminus \mathring{B}(\mathbb{R}^m, r_1)$. We also have $G \coloneqq B(0, \#V + 1) \setminus \mathring{B}(\mathbb{R}^m, 1/4) \setminus \bigcup_{i=1}^{\#V} \mathring{B}((i, 0, \dots, 0, 1, 0, \dots, 0), 1/4)$ (where the 1 in the open ball is in position $m+1$). This is a compact SA manifold (a closed n-ball with an open tubular neighborhood of $\mathbb{R}^m$ and V open n-balls removed) and we have an SA-homeomorphism $\Theta_\zeta : G \cong G_\zeta$ since, again, W is small enough. We can then define a chart $\widehat{\Phi} : W \times [0,1]^{V^*_{\mathcal{T}} \setminus \mathrm{root}} \times G \to \mathsf{VFM}_{mn}(U \sqcup *, V)$ with a formula similar to the one above. This map covers Φ, and showing that $\widehat{\Phi}(\zeta, \tau, -)$ maps G_ζ SA-homeomorphically to $q^{-1}(\Phi(\zeta, \tau))$ is a straightforward adaption of the arguments in [LV14, Section 5.9.4]. This completes the proof of the second part of Proposition 2.9.

References

[Arn69] Vladimir I. Arnol'd. "The cohomology ring of the group of dyed braids". In: *Mat. Zametki* 5 (1969), pp. 227–231. ISSN: 0025-567X. DOI: 10.1007/978-3-642-31031-7_18.

[AT15] Gregory Arone and Victor Turchin. "Graph-complexes computing the rational homotopy of high dimensional analogues of spaces of long knots". In: *Ann. Inst. Fourier* 65.1 (2015), pp. 1–62. DOI: 10.5802/aif.2924. arXiv: 1108.1001.

[AS94] Scott Axelrod and I. M. Singer. "Chern–Simons perturbation theory. II". In: *J. Differential Geom.* 39.1 (1994), pp. 173–213. ISSN: 0022-040X. DOI: 10.4310/jdg/1214454681. arXiv: hep-th/9304087.

[AFT17] David Ayala, John Francis, and Hiro Lee Tanaka. "Factorization homology of stratified spaces". In: *Selecta Math. (N.S.)* 23.1 (2017), pp. 293–362. ISSN: 1022-1824. DOI: 10.1007/s00029-016-0242-1. arXiv: 1409.0848v3.

[BV68] J. Michael Boardman and Rainer M. Vogt. "Homotopy-everything H-spaces". In: *Bull. Amer. Math. Soc.* 74 (1968), pp. 1117–1122. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1968-12070-1.

[Cam+18a] Ricardo Campos, Julien Ducoulombier, Najib Idrissi, and Thomas Willwacher. *A model for framed configuration spaces of points*. 2018. arXiv: 1807.08319.

[Cam+18b] Ricardo Campos, Najib Idrissi, Pascal Lambrechts, and Thomas Willwacher. *Configuration Spaces of Manifolds with Boundary*. 2018. arXiv: 1802.00716.

[CW16] Ricardo Campos and Thomas Willwacher. *A model for configuration spaces of points*. 2016. arXiv: 1604.02043.

[Coh76] Frederick R. Cohen. “The homology of $\mathscr{C}_{n+1}$ spaces, $n \geq 0$”. In: Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces*. Lecture Notes in Mathematics 533. Berlin Heidelberg: Springer, 1976. Chap. 3, pp. 207–351. ISBN: 978-3-540-07984-2. DOI: 10.1007/BFb0080467.

[DW15] Vasily Dolgushev and Thomas Willwacher. “Operadic twisting. With an application to Deligne’s conjecture”. In: *J. Pure Appl. Algebra* 219.5 (2015), pp. 1349–1428. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2014.06.010.

[Duc16] Julien Ducoulombier. “Swiss-cheese action on the totalization of operads under the monoid actions actions operad”. In: *Algebr. Geom. Topol.* 16.3 (2016), pp. 1683–1726. DOI: 10.2140/agt.2016.16.1683. arXiv: 1410.3236.

[Fre17] Benoit Fresse. *Homotopy of Operads and Grothendieck–Teichmüller Groups*. Mathematical Surveys and Monographs 217. Providence, RI: Amer. Math. Soc., 2017, pp. xlvi+532. ISBN: 978-1-4704-3481-6.

[Fre18] Benoit Fresse. “The extended rational homotopy theory of operads”. In: *Georgian Math. J.* 25.4 (2018), pp. 493–512. ISSN: 1072-947X. DOI: 10.1515/gmj-2018-0061. arXiv: 1805.00530.

[FW15] Benoit Fresse and Thomas Willwacher. “The intrinsic formality of E_n-operads”. In: *J. Eur. Math. Soc. (to appear)* (2015). arXiv: 1503.08699.

[FM94] William Fulton and Robert MacPherson. “A compactification of configuration spaces”. In: *Ann. of Math. (2)* 139.1 (1994), pp. 183–225. ISSN: 0003-486X. DOI: 10.2307/2946631.

[Har+11] Robert Hardt, Pascal Lambrechts, Victor Turchin, and Ismar Volić. “Real homotopy theory of semi-algebraic sets”. In: *Algebr. Geom. Topol.* 11.5 (2011), pp. 2477–2545. ISSN: 1472-2747. DOI: 10.2140/agt.2011.11.2477. arXiv: 0806.0476.

[HL13] Eduardo Hoefel and Muriel Livernet. “On the spectral sequence of the Swiss-cheese operad”. In: *Algebr. Geom. Topol.* 13.4 (2013), pp. 2039–2060. ISSN: 1472-2747. DOI: 10.2140/agt.2013.13.2039. arXiv: 1209.6122.

[HLS16] Eduardo Hoefel, Muriel Livernet, and Jim Stasheff. “A_∞-actions and recognition of relative loop spaces”. In: *Topology Appl.* 206 (2016), pp. 126–147. ISSN: 0166-8641. DOI: 10.1016/j.topol.2016.03.023. arXiv: 1312.7155.

[Idr19] Najib Idrissi. “The Lambrechts–Stanley Model of Configuration Spaces”. In: *Invent. Math* 216.1 (2019), pp. 1–68. ISSN: 1432-1297. DOI: 10.1007/s00222-018-0842-9. arXiv: 1608.08054.

[KW17] Anton Khoroshkin and Thomas Willwacher. *Real models for the framed little n-disks operads*. 2017. arXiv: 1705.08108.

[Kon94] Maxim Kontsevich. "Feynman diagrams and low-dimensional topology". In: *First European Congress of Mathematics. Paris 1992*. Vol. 120.Invited lectures. Progr. Math. Basel: Birkhäuser, 1994, pp. 97–121. DOI: 10.1007/978-3-0348-9112-7_5.

[Kon99] Maxim Kontsevich. "Operads and motives in deformation quantization". In: *Lett. Math. Phys.* 48.1 (1999), pp. 35–72. ISSN: 0377-9017. DOI: 10.1023/A:1007555725247. arXiv: math/9904055.

[Kon03] Maxim Kontsevich. "Deformation quantization of Poisson manifolds". In: *Lett. Math. Phys.* 66.3 (2003), pp. 157–216. ISSN: 0377-9017. DOI: 10.1023/B:MATH.0000027508.00421.bf. arXiv: q-alg/9709040.

[LV14] Pascal Lambrechts and Ismar Volić. "Formality of the little N-disks operad". In: *Mem. Amer. Math. Soc.* 230.1079 (2014), pp. viii+116. ISSN: 0065-9266. DOI: 10.1090/memo/1079. arXiv: 0808.0457.

[Lee03] John M. Lee. *Introduction to smooth manifolds*. Vol. 218. Graduate Texts in Mathematics. Springer-Verlag, New York, 2003, pp. xviii+628. ISBN: 0-387-95495-3. DOI: 10.1007/978-0-387-21752-9.

[Liv15] Muriel Livernet. "Non-formality of the Swiss-cheese operad". In: *J. Topol.* 8.4 (2015), pp. 1156–1166. DOI: 10.1112/jtopol/jtv018. arXiv: 1404.2484.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 346. Heidelberg: Springer, 2012, pp. xxiv+634. ISBN: 978-3-642-30361-6. DOI: 10.1007/978-3-642-30362-3.

[Lur17] Jacob Lurie. *Higher Algebra*. Sept. 2017. URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Mar99] Martin Markl. "A compactification of the real configuration space as an operadic completion". In: *J. Algebra* 215.1 (1999), pp. 185–204. ISSN: 0021-8693. DOI: 10.1006/jabr.1998.7709. arXiv: hep-th/9608067.

[May72] J. Peter May. *The geometry of iterated loop spaces*. Lectures Notes in Mathematics 271. Berlin: Springer-Verlag, 1972, pp. viii+175. DOI: 10.1007/BFb0067491.

[Pet14] Dan Petersen. "Minimal models, GT-action and formality of the little disk operad". In: *Selecta Math. (N.S.)* 20.3 (2014), pp. 817–822. ISSN: 1022-1824. DOI: 10.1007/s00029-013-0135-5. arXiv: 1303.1448.

[Que15] Alexandre Quesney. *Swiss Cheese type operads and models for relative loop spaces*. 2015. arXiv: 1511.05826.

[Sal01] Paolo Salvatore. “Configuration spaces with summable labels”. In: *Cohomological methods in homotopy theory. Proceedings of the Barcelona conference on algebraic topology*. Bellaterra, Spain, June 4–10, 1998. Ed. by Jaume Aguadé, Carles Broto, and Carles Casacuberta. Progr. Math. 196. Basel: Birkhäuser, 2001, pp. 375–395. ISBN: 978-3-0348-9513-2. DOI: 10.1007/978-3-0348-8312-2. arXiv: math/9907073.

[Sin04] Dev P. Sinha. “Manifold-theoretic compactifications of configuration spaces”. In: *Selecta Math. (N.S.)* 10.3 (2004), pp. 391–428. ISSN: 1022-1824. DOI: 10.1007/s00029-004-0381-7. arXiv: math/0306385.

[Sin13] Dev P. Sinha. “The (non-equivariant) homology of the little disks operad”. In: *Operads 2009. Proceedings of the school and conference*. Luminy, France, April 20–30. Ed. by Jean-Louis Loday and Bruno Vallette. Sémin. Congr. 26. Paris: Soc. Math. France, 2013, pp. 253–279. ISBN: 978-2-85629-363-8. arXiv: math/0610236.

[Tam03] Dmitry E. Tamarkin. “Formality of chain operad of little discs”. In: *Lett. Math. Phys.* 66.1-2 (2003), pp. 65–72. ISSN: 0377-9017. DOI: 10.1023/B:MATH.0000017651.12703.a1.

[TW18] Victor Turchin and Thomas Willwacher. “Relative (non-)formality of the little cubes operads and the algebraic Cerf Lemma”. In: *Am. J. Math.* 140.2 (2018), pp. 277–316. ISSN: 0002-9327; 1080-6377/e. DOI: 10.1353/ajm.2018.0006. arXiv: 1409.0163.

[Vie19] Renato Vasconcello Vieira. “Relative Recognition Principle”. In: *Algebr. Geom. Topol. (to appear)* (2019). arXiv: 1802.01530.

[Vor99] Alexander A. Voronov. “The Swiss-cheese operad”. In: *Homotopy invariant algebraic structures. Baltimore, MD, 1998*. Contemp. Math. 239. Providence, RI: Amer. Math. Soc., 1999, pp. 365–373. DOI: 10.1090/conm/239/03610. arXiv: math/9807037.

[Wil14] Thomas Willwacher. “M. Kontsevich’s graph complex and the Grothendieck–Teichmüller Lie algebra”. In: *Invent. Math.* 200.3 (2014), pp. 671–760. ISSN: 1432-1297. DOI: 10.1007/s00222-014-0528-x. arXiv: 1009.1654.

[Wil15] Thomas Willwacher. *Models for the n-Swiss Cheese operads*. 2015. arXiv: 1506.07021.

[Wil16] Thomas Willwacher. “The Homotopy Braces Formality Morphism”. In: *Duke Math. J.* 165.10 (2016), pp. 1815–1964. DOI: 10.1215/00127094-3450644. arXiv: 1109.3520.

[Wil17] Thomas Willwacher. *(Non-)formality of the extended Swiss Cheese operads*. June 2017. arXiv: 1706.02945.