Curved Koszul Duality for Algebras over Unital Operads

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We develop a curved Koszul duality theory for algebras presented by quadratic-linear-constant relations over binary unital operads. As an application, we study Poisson n-algebras given by polynomial functions on a standard shifted symplectic space. We compute explicit resolutions of these algebras using curved Koszul duality. We use these resolution to compute derived enveloping algebras and factorization homology on parallelized simply connected closed manifolds of these Poisson n-algebras.

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Introduction

Koszul duality was initially developed by Priddy [Pri70] for associative algebras. Given an augmented associative algebra A, there is a "Koszul dual" algebra $A^!$, and there is an equivalence (subject to some conditions) between the derived categories of A and $A^!$. The Koszul dual $A^!$ is actually the linear dual of a certain coalgebra $A^¡$. If the algebra A satisfies a certain condition called "being Koszul", then the cobar construction of $A^¡$ is a quasi-free resolution of the algebra A. In this sense, Koszul duality is a tool to produce resolutions of algebras.

An operad is a kind of combinatorial object which governs categories of "algebras" in a wide sense, for example associative algebras, commutative algebras, Lie algebras… After insights of Kontsevich [Kon93], Koszul duality was then generalized with great success to binary quadratic operads by Ginzburg–Kapranov [GK94] (see also Getzler–Jones [GJ94]), and then to quadratic operads by Getzler [Get95] (see also [Fre04; Mar96]). For example, it was realized that the operad governing commutative algebras and the operad governing Lie algebras are Koszul dual to each other. This duality explains the links between the two approaches of Sullivan [Sul77] and Quillen [Qui69] of rational homotopy theory, which rely respectively on differential graded (dg) commutative algebras and dg-Lie algebras.

Koszul duality of quadratic operads works roughly as follows. Given an augmented quadratic operad P, there is a Koszul dual operad $\mathsf{P}^!$. This Koszul dual is in fact the linear dual of a certain cooperad $\mathsf{P}^¡$, the Koszul dual cooperad of P. If P satisfies the condition of "being Koszul", then the operadic cobar construction of $\mathsf{P}^¡$ is a quasi-free resolution of the operad P. In this sense, operadic Koszul duality provides a tool to produce resolutions of augmented quadratic operads. This is useful when dealing with the homotopy category of algebras over a given operad, which can then be studied as the category of algebras over the Koszul resolution of the operad in question. For example, studying associative algebras up to homotopy is equivalent to studying A_∞-algebras and A_∞-morphisms up to homotopy, and this latter category possesses some interesting properties (e.g. weak equivalences are homotopy equivalences).

Operadic Koszul duality was then generalized to several different settings (see the quick tour in Section 1.1). Two of them will interest us. The first, due to Hirsh–Millès [HM12], is curved Koszul duality, which applies to (pr)operads with quadratic-linear-constant relations (by analogy with curved Koszul duality for associative algebras [Pos93; PP05]). The other, due to Millès [Mil12], is Koszul duality for monogenic algebras over quadratic operads, a generalization of quadratic algebras over binary operads.

Our aim will be, in some sense, to combine these two approaches in order to develop a curved Koszul duality theory for algebras with quadratic-linear-constant relations over unitary binary quadratic operads. Our motivation is the following. If P is a Koszul operad, then there is a functorial way of obtaining resolutions of P-algebras by considering the bar-cobar construction. However, this resolution is somewhat big, and explicit computations are not easy. On the other hand, the theory of Millès [HM12] provides resolutions for Koszul monogenic algebras over Koszul quadratic operads which are much smaller when they exist (see e.g. Remark 4.7). But the construction is unavailable when the operad is not quadratic and/or when the algebra is not monogenic.

We close this gap for algebras with quadratic-linear-relations over unital versions of binary quadratic operads. Our main theorem is the following:

Theorem A (Theorem 3.7)**.** *Let P be a binary quadratic operad and let $u\mathsf{P}$ be a unital version of P (Def. 1.8). Let A be a $u\mathsf{P}$-algebra with quadratic-linear-constant relations (Def. 3.1). Let qA be the P-algebra given by the quadratic reduction of A (Def. 3.3). Finally let $A^{¡} = (qA^{¡}, d_{A^{¡}}, \theta_{A^{¡}})$ be the curved $\mathsf{P}^{¡}$-coalgebra given by the Koszul dual of A (Section 3.2).*

If the P-algebra qA is Koszul in the sense of [Mil12], then the canonical morphism from the curved cobar construction of $A^{¡}$ to A is a quasi-isomorphism:

$$\Omega_\kappa A^{¡} \xrightarrow{\sim} A.$$

By applying this theory to different kinds of operads, we recover some already existing notions of "curved algebras" and "Koszul duality of curved algebras". For example, when applied to associative algebras, we recover the notion of a curved coalgebra from Lyubashenko [Lyu17]. When applied to Lie algebras, we recover (the dual of) curved Lie algebras [CLM16; Mau17].

As an example of application, we study unital Poisson n-algebra which are in some sense the algebra of polynomials functions on a shifted standard symplectic space. Given $n \in \mathbb{Z}$ and $D \geq 0$, the Poisson n-algebra $A_{n;D}$ is free as a unital commutative algebra:

$$A_{n;D} = (\mathbb{R}[x_1, \dots, x_D, \xi_1, \dots, \xi_D], \{\}),$$

where $\deg x_i = 0$, $\deg \xi_j = 1 - n$, and the shifted Lie bracket is given by $\{x_i, \xi_j\} = \delta_{ij}$. We may view x_i as a polynomial function on $\mathbb{R}^D$, ξ_j as the vector field $\partial/\partial x_j$, and $A_{n;D} = \mathrm{Poly}(T^*\mathbb{R}^D[1-n])$.

This algebra is presented by generators and quadratic-linear-constant relations over the operad $u\mathsf{Pois}_n$ governing unital Poisson n-algebra. It is Koszul, hence the cobar construction $\Omega_\kappa A^{¡}$ provides an explicit cofibrant replacement of A. We explicitly describe this cofibrant replacement.

We use $\Omega_\kappa A^{¡}$ to compute the derived enveloping algebra of $A_{n;D}$, which we prove is quasi-isomorphic to the underived enveloping algebra of A. We also compute the factorization homology $\int_M A_{n;D}$ of a simply connected parallelized closed manifold M with coefficients in $A_{n;D}$. We prove that the homology of $\int_M A_{n;D}$ is one-dimensional for such manifolds. This fits in with the physical intuition that the expected value of a quantum observable, which should be a single number, lives in $\int_M A$, see e.g. [CG17] for a broad reference. A computation for a similar object was performed by Markarian [Mar17], and we learned that Döppenschmitt computed the factorization homology of a twisted version of $A_{n;D}$ using physical methods (unpublished), see Remark 4.19.

Outline This paper is organized as follows. In Section 1, we lay out our conventions and notations, as well as background for the rest of the paper. This section does not contain any original result. We give a quick tour of Koszul duality (Section 1.1), recall the definition of "unital version" of a quadratic operad (Section 1.2), and give some background on factorization homology (Section 1.3). In Section 2, we define the objects with which we will be working: curved coalgebras and semi-augmented algebras. We also give the definitions of the bar and cobar constructions, and we prove that they are adjoint to each other. In Section 3, we prove our main theorem. We define algebras with QLC relations and the Koszul dual curved coalgebra of such an algebra. We prove that if the quadratic reduction of the algebra is Koszul, then the cobar construction on the Koszul dual of the algebra is a cofibrant replacement of the algebra. In Section 4, we apply the theory to the symplectic Poisson n-algebras. We explicitly describe the cofibrant replacement obtained by Koszul duality. We use it compute their derived enveloping algebras and factorization homology.

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1 Conventions, background, and recollections

We work with nonnegatively graded chain complexes over some base field $\Bbbk$ of characteristic zero, which we call "dg-modules". Given a dg-module V, its suspension ΣV is given by $(\Sigma V)_n = V_{n-1}$ with a signed differential.

We work extensively with (co)operads and (co)algebras over (co)operads. We refer to e.g. [LV12] or [Fre17, Part I(a)] for a detailed treatment. Briefly, a (symmetric, one-colored) operad P is a collection $\{\mathsf{P}(n)\}_{n\geq 0}$ of dg-modules, with each $\mathsf{P}(n)$ equipped with an action of the symmetric group Σ_n, a unit $\eta \in \mathsf{P}(1)$, and composition maps $\circ_i : \mathsf{P}(k) \otimes \mathsf{P}(l) \to \mathsf{P}(k+l-1)$ for $1 \leq i \leq k$ satisfying the usual equivariance, unit, and associativity axioms. For such an operad P and a dg-module V, we define $\mathsf{P}(V) = \bigoplus_{n\geq 0} \mathsf{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$. A P-algebra is a dg-module A equipped with a structure map $\gamma_A : \mathsf{P}(A) \to A$ satisfying the usual axioms. In particular, for a dg-module V, the algebra $\mathsf{P}(V)$ is the free P-algebra on V. (Conilpotent) cooperads and coalgebras are defined dually.

Example 1.1. Some examples of operads will appear several times:

- the operad Ass governing associative algebras, given by $\mathsf{Ass}(n) = \Bbbk[\Sigma_n]$ for $n \geq 1$;
- the operad Com, governing commutative algebras, given by $\mathsf{Com}(n) = \Bbbk$ for $n \geq 1$;
- the operad Lie, governing Lie algebra, satisfying $\dim \mathsf{Lie}(n) = (n-1)!$;
- the operad Pois_n governing Poisson n-algebras, i.e. algebras with a commutative product and a Lie bracket of degree $n-1$ which is a biderivation with respect to the product.

If $E = \{E(n)\}_{n\geq 0}$ is a symmetric sequence, then we will write $\mathrm{Free}(E)$ for the free operad generated by E. It can be described in terms of rooted trees with internal vertices decorated by elements of E, operadic composition being given by grafting of trees. Moreover if $S \subset \mathsf{P}$ is a subsequence of an operad, then we write $\mathsf{P}/(S)$ for the quotient of the operad P by the operadic ideal generated by S.

We will need at some point the (de)suspension of an operad or a coalgebra. Given an operad P, the suspended operad $\mathscr{S}\mathsf{P}$ is defined such that a $\mathscr{S}\mathsf{P}$-algebra structure on ΣA is the same thing as a P-algebra structure on A. It is also sometimes denoted $\Lambda^{-1}\mathsf{P}$. The suspended cooperad $\mathscr{S}^c\mathsf{C}$ is defined similary. On (co)free (co)algebras, we have:

$$\mathscr{S}\mathsf{P}(\Sigma V) = \Sigma\mathsf{P}(V), \qquad \mathscr{S}^c\mathsf{C}(\Sigma V) = \Sigma\mathsf{C}(V). \tag{1.1}$$

Example 1.2. As a symmetric sequence, the operad Pois_n is given by the composition product $\mathsf{Pois}_n = \mathsf{Com} \circ \mathscr{S}^{1-n}\mathsf{Lie}$. The operad structure map is induced by the structure maps of Com and Lie, as well as a distributive law stating that the bracket is a biderivation.

1.1 Koszul duality for...

We now give a quick tour of some of the various existing incarnations of Koszul duality. We will reuse some of the ideas and notations from these sections.

1.1.1 ...associative algebras

Koszul duality was initially developed for quadratic associative algebras [Pri70], i.e. associative algebras presented by generators and quadratic relations. To a quadratic associative algebra A, there is an associated "Koszul dual" $A^!$, an associative algebra itself. The algebra A is said to be "Koszul" if this algebra is generated by its weight 1 elements. Koszul complex is acyclic. In this case, it can be used to compute the Tor and Ext groups over A.

This can be reformulated as follows. There is an associated Koszul dual coalgebra $A^{¡}$ to A, and $A^!$ is the linear dual of $A^{¡}$. It can be used to define a "Koszul complex" $A \otimes_\kappa A^{¡}$, which is acyclic iff A is Koszul. This coalgebra $A^{¡}$ is a subcoalgebra of the bar construction BA, and is isomorphic to the zeroth cohomology group $H^0(BA)$. There is a canonical morphism from the cobar construction $\Omega A^{¡}$ to A. Then the algebra A is Koszul iff this canonical morphism is a quasi-isomorphism, iff the inclusion $A^{¡} \to BA$ is a quasi-isomorphism. Since the cobar construction $\Omega A^{¡}$ is quasi-free as an associative algebra, this allows to produce an explicit, small, quasi-free resolution of any Koszul algebra.

Example 1.3. Let $A = \mathbb{R}[x_1, \dots, x_k]$ be a free commutative algebra on k variables of degrees zero. Then the Koszul dual $A^{¡} = \Lambda^c(dx_1, \dots, dx_k)$ is the exterior coalgebra on k variables of (homological) degree -1. The complex $(A \otimes A^{¡}, d_\kappa)$ is what was initially called the Koszul complex, with a differential similar to the de Rham differential. It is acyclic, hence A is Koszul.

1.1.2 ...quadratic operads

After insights of Kontsevich [Kon93], Koszul duality was generalized to binary quadratic operads by Ginzburg–Kapranov [GK94] (see also [GJ94]) and then to quadratic operads by Getzler [Get95] (see also [Fre04; Mar96]). We refer to [LV12] for a detailed treatment.

Let $\mathsf{P} = \mathrm{Free}(E)/(R)$ be an operad presented by generators $E = \{E(n)\}_{n\geq 0}$ and relations $R \subset \mathrm{Free}(E)$. The presentation is said to be quadratic if the relations R are quadratic, i.e. they form a subsequence of the weight 2 component $\mathrm{Free}(E)^{(2)}$ of the free operad $\mathrm{Free}(E)$. A quadratic operad is an operad equipped with a presentation with quadratic relations (not unique in general). To this quadratic operad, there is an associated Koszul dual cooperad $\mathsf{P}^{\text{¡}}$, obtained by the cofree cooperad on "cogenerators" ΣE subject to the "corelations" $\Sigma^2 R$. One can also define the Koszul dual operad $\mathsf{P}^!$ as the shifted linear dual of $\mathsf{P}^{\text{¡}}$.

Example 1.4. The operad governing associative algebras (without unit) Ass is auto-dual, i.e. $\mathsf{Ass}^! = \mathsf{Ass}$. The operad governing commutative algebras (without unit) Com is Koszul dual to the operad Lie governing Lie algebras, i.e. $\mathsf{Com}^! = \mathsf{Lie}$ and $\mathsf{Lie}^! = \mathsf{Com}$. The operad $\mathsf{Pois}_n = \mathsf{Com} \circ \mathscr{S}^{1-n}\mathsf{Lie}$ governing Poisson n-algebras is auto-dual up to suspension, i.e. $\mathsf{Pois}_n^! = \mathscr{S}^n \mathsf{Pois}_n$.

Just like for algebras, $\mathsf{P}^{\text{¡}}$ is a suboperad of $B\mathsf{P}$, the operadic bar construction of P. This implies that there is a canonical "twisting morphism" κ of degree -1:

$$\kappa : \mathsf{P}^{\text{¡}} \twoheadrightarrow \Sigma E \xrightarrow{\Sigma^{-1}} E \hookrightarrow \mathsf{P}, \tag{1.2}$$

This is an element that satisfies the Maurer–Cartan equation $\partial\kappa + \kappa \star \kappa$, where $\star$ is the preLie convolution product on $\mathrm{Hom}(\mathsf{P}^{\text{¡}}, \mathsf{P})$. It thus induces a canonical morphism from the operadic cobar construction $\Omega\mathsf{P}^{\text{¡}}$ to P. The operad is said to be Koszul if this canonical morphism is a quasi-isomorphism. Just like for associative algebras, this is equivalent to the inclusion $\mathsf{P}^{\text{¡}} \to B\mathsf{P}$ being a quasi-isomorphism, and it's also equivalent to a certain "Koszul complex" ($\mathsf{P}^{\text{¡}} \circ_\kappa \mathsf{P}$) being acyclic.

The operads Ass, Com, Lie, and Pois_n are all Koszul. This allows to define small, explicit quasi-free resolutions of these operads, for example the A_∞, C_∞, or L_∞ operads. This also allows to produce functorial resolutions of algebras over these operads, using the bar/cobar constructions. Indeed, the twisting morphism $\kappa : \mathsf{P}^{\text{¡}} \to \mathsf{P}$ induces an adjunction $\Omega_\kappa \dashv B_\kappa$ between the categories of $\mathsf{P}^{\text{¡}}$-coalgebras and P-algebras, and if P is Koszul, then $\Omega_\kappa B_\kappa(-)$ is a functorial cofibrant replacement functor.

For example, for the operad Ass, this gives the usual bar/cobar resolution of an algebra. For the operad Com, the resolution obtained is (up to degree shift) the free commutative algebra on the cofree Lie coalgebra on the initial commutative algebras, together with some differential.

1.1.3 ...monogenic algebras over operads

Millès [Mil12] extended Koszul duality to "monogenic" algebras over quadratic operads, which generalize quadratic associative algebras. Given a quadratic operad

$\mathsf{P} = \mathrm{Free}(E)/(R)$, a monogenic P-algebra A is an algebra equipped with a presentation $A = \mathsf{P}(V)/(S)$, where V is some set of generators, and $S \subset E(V)$ is a set of relations. The differential of A is zero.

To this monogenic algebra $A = \mathsf{P}(V)/(S)$, there is an associated Koszul dual $\mathsf{P}^{¡}$-coalgebra, defined by the suspension of the cofree $\mathsf{P}^{!}$-coalgebra on V subject to the corelations ΣS. There is a canonical "algebra-twisting morphism" $\varkappa$ of degree -1,

$$\varkappa : A^{¡} \twoheadrightarrow \Sigma V \xrightarrow{\Sigma^{-1}} V \hookrightarrow A, \tag{1.3}$$

Let $\kappa : \mathsf{P}^{¡} \to \mathsf{P}$ be the operad-twisting morphism introduced in Equation (1.2). The κ-star product $\star_\kappa(\varkappa)$ is given by the composition:

$$\star_\kappa(\varkappa) : \Sigma A^{¡} \xrightarrow{\Delta_{A^{¡}}} \Sigma \mathsf{P}^{¡}(A^{¡}) \xrightarrow{\kappa \circ \varkappa} \mathsf{P}(A) \to A. \tag{1.4}$$

The element $\varkappa$ satisfies the Maurer–Cartan equation $\star_\kappa(\varkappa) = 0$ (the differential vanishes). It thus naturally defines a morphism of P-algebras $f_\varkappa : \Omega_\kappa A^{¡} \to A$. Finally, the algebra A is said to be Koszul if this morphism is a quasi-isomorphism. In this case, the algebra $\Omega_\kappa A^{¡}$ is an explicit, small resolution of A. For example, if $\mathsf{P} = \mathsf{Ass}$, this recovers the usual Koszul duality/resolution of associative algebras.

Our goal in this paper is to generalize these ideas to *unital* algebras, using the ideas presented in the next subsection.

1.1.4 ...operads with QLC relations (curved Koszul duality)

Curved Koszul duality is a generalization of Koszul duality for *unital* associative algebras [Pos93; PP05]. This was generalized by Hirsh–Millès [HM12] for (pr)operads with quadratic-linear-constant (QLC) relations (see also Lyubashenko [Lyu11] for the case of the unital associative operad). Examples include the operads governing unital associative algebras, unital commutative algebras, and Lie algebras equipped with a central element. (See also [GTV12] for operads with quadratic-linear relations.)

For simplicity, let us just deal with operads (and not properads). Let I be the unit operad, i.e. $\mathsf{I}(1) = \Bbbk$ and $\mathsf{I}(n) = 0$ for $n \neq 1$. A QLC presentation of an operad is a presentation $\mathsf{P} = \mathrm{Free}(E)/(R)$, where E is some module of generators and $R \subset \mathsf{I} \oplus E \oplus \mathrm{Free}(E)^{(2)}$ is some module of relation with constant (i.e. multiple of id_{P}), linear, and quadratic terms.

Example 1.5. The operad governing unital associative algebras has a QLC presentation: if $\mathbb{1}$ is a nullary generator and μ a binary generator, we have:

$$u\mathsf{Ass} = \mathrm{Free}(\mathbb{1}, \mu)/(\underbrace{\mu \circ_1 \mu - \mu \circ_2 \mu}_{\text{quadr.}}, \underbrace{\mu \circ_1 \mathbb{1} - \mathrm{id}, \mu \circ_2 \mathbb{1} - \mathrm{id}}_{\text{quadr. and. const.}}). \tag{1.5}$$

The quadratic reduction $q\mathsf{P}$ is the quadratic operad $\mathrm{Free}(E)/(qR)$, where qR is the projection of R onto $\mathrm{Free}(E)^{(2)}$. Hirsh and Millès impose some conditions on this presentation:

- the space of generators is minimal, i.e. $R \cap \mathsf{I} \oplus E = 0$;
- the space of relations is maximal, i.e. $R = (R) \cap \mathsf{I} \oplus E \oplus \mathrm{Free}(E)^{(2)}$.

Therefore R is the graph of some map $\varphi = (\varphi_0 + \varphi_1) : qR \to \mathsf{I} \oplus E$, i.e. $R = \{X + \varphi(X) \mid X \in qR\}$.

From this data they define the Koszul dual cooperad P^{i}, which is a *curved* cooperad. This curved cooperad is a triplet $(q\mathsf{P}^{\mathsf{i}}, d_{\mathsf{P}^{\mathsf{i}}}, \theta_{\mathsf{P}^{\mathsf{i}}})$, where:

- the cooperad $q\mathsf{P}^{\mathsf{i}}$ is the Koszul dual cooperad of the quadratic cooperad $q\mathsf{P}$ (in the sense of Section 1.1.2);
- the predifferential $d_{\mathsf{P}^{\mathsf{i}}}$ is the unique degree -1 coderivation of $q\mathsf{P}^{\mathsf{i}}$ whose corestriction (composition with the projection) onto ΣE is given by

$$d_{\mathsf{P}^{\mathsf{i}}}|^{\Sigma E} : q\mathsf{P}^{\mathsf{i}} \twoheadrightarrow \Sigma^2 qR \xrightarrow{\varphi_1} \Sigma E; \tag{1.6}$$

- the curvature $\theta_{\mathsf{P}^{\mathsf{i}}}$ is the map of degree -2 obtained by:

$$\theta_{\mathsf{P}^{\mathsf{i}}} : q\mathsf{P}^{\mathsf{i}} \twoheadrightarrow \Sigma^2 qR \xrightarrow{\varphi_0} I. \tag{1.7}$$

This data satisfies some axioms. These axioms imply that the cobar construction $\Omega(q\mathsf{P}^{\mathsf{i}}) = \big(\mathrm{coFree}(q\mathsf{P}^{\mathsf{i}}), d_2\big)$ is equipped with an extra differential $d_0 + d_1$, defined respectively from $\theta_{\mathsf{P}^{\mathsf{i}}}$ and $d_{\mathsf{P}^{\mathsf{i}}}$. The canonical morphism $\Omega(q\mathsf{P}^{\mathsf{i}}) \to q\mathsf{P}$ extends to a canonical morphism:

$$\Omega\mathsf{P}^{\mathsf{i}} := \big(\Omega(q\mathsf{P}^{\mathsf{i}}), d_0 + d_1\big) \to \mathsf{P}. \tag{1.8}$$

The main theorem of [HM12] (Theorem 4.3.1) states in particular that if the quadratic operad $q\mathsf{P}$ is Koszul, then the canonical morphism $\Omega\mathsf{P}^{\mathsf{i}} \to \mathsf{P}$ is a quasi-isomorphism. This justifies the following definition: the operad P is said to be Koszul if the quadratic operad $q\mathsf{P}$ is Koszul in the usual sense.

Our aim, in this paper, is to reuse these ideas to define a curved Koszul duality theory for "unital" algebras (in some sense) over "unital" operads. The general philosophy is as follows: operads are monoids in the category of symmetric sequence. Hence, the results of Hirsh–Millès above are in some sense results about the associative operad and its Koszul dual, which is itself This explains why the Koszul dual of an operad is a cooperad, i.e. a comonoid; Koszul duality is "hidden" is the implicit identification of the weight 2 part of the free operad on some generators with the weight two part of the cofree cooperad on these generators. With this point of view, we reuse the ideas of Millès in Section 1.1.3 to define curved Koszul duality for algebras over any operad.

Remark 1.6. Le Grignou [LeG17] defined a model category structure on the category of curved cooperads, which is Quillen equivalent to the model category of operads using the bar/cobar adjunction.

1.2 Unital versions of operads

In what follows, we will only work with algebras over special kinds of operads, namely unital versions of binary quadratic operads.

Let
$$\mathsf{P} = \mathrm{Free}(E)/(R) \tag{1.9}$$
be a an operad presented by binary generators $E = \{0, 0, E(2), 0, \ldots\}$ and (homogeneous) quadratic relations $R \subset \left(E(2)^{\otimes 2}\right)_{\Sigma_2}$. Moreover we assume that the differential of P is zero.

Remark 1.7. While most of this paper could be carried out without the assumption that P is binary, we need this assumption to be able to show Proposition 3.4.

Recall from [HM12, Definition 6.5.4] the definition of a *unital version* of P. It is obtained by adjoining a generator of arity zero, which we denote ⫯ as in *op. cit.* and of which we think as the unit.

Definition 1.8 (Adapted from [HM12, Definition 6.5.4])**.** A unital version $u\mathsf{P}$ of P is given by
$$u\mathsf{P} = \mathrm{Free}(E \oplus \text{⫯})/(R + R'), \tag{1.10}$$
where ⫯ is a generator of arity 0 and degree 0, and such that:

- the inclusion $E \subset E \oplus \text{⫯}$ induces an injective morphism of operads $\mathsf{P} \to u\mathsf{P}$;
- we have an isomorphism of operads $\text{⫯} \oplus \mathsf{P} \cong qu\mathsf{P}$ induced by the inclusions $\mathsf{P} \subset u\mathsf{P}$ and $\text{⫯} \subset u\mathsf{P}$;
- the inhomogeneous quadratic relations (in R') have no linear terms, only quadratic and constant terms.

Example 1.9. Examples include:

- the operads $u\mathsf{Ass}$ of unital associative algebras and $u\mathsf{Com}$ of unital commutative algebras;
- the operad $c\mathsf{Lie}$ encoding Lie algebras equipped with a central element;
- the operad $u\mathsf{Pois}_n$ encoding Poisson n-algebras equipped with an element which is a unit for the product and a central element for the shifted Lie bracket.

1.3 Factorization homology

1.3.1 General definition

Factorization homology [AF15], also known as topological chiral homology [BD04], is an invariant of manifolds with coefficients in a homotopy commutative algebra, just like standard homology is an invariant of topological spaces with coefficients in a commutative ring. One possible definition of factorization homology is the following [Fra13], which we give for parallelized manifold for simplicity (but a similar definition exists for unparallelized manifold). Fix some integer $n \geq 0$. There is an operad Disk_n, obtained as the operad of endomorphisms of $\mathbb{R}^n$ in the category of parallelized manifolds and parallelized embeddings. In each arity,

$$\mathsf{Disk}_n(k) \coloneqq \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \ldots \sqcup \mathbb{R}^n}_{k\times}, \mathbb{R}^n), \tag{1.11}$$

and operadic composition is given by composition of embeddings. This operad is weakly equivalent to the usual little n-disks operad, i.e. it is an E_n-operad. In particular, its homology $H_*(\mathsf{Disk}_n)$ is the unital associative operad $u\mathsf{Ass}$ for $n = 1$, and the unital n-Poisson operad $u\mathsf{Pois}_n$ for $n \geq 2$.

Moreover, given a parallelized n-manifold M, there is a right Disk_n-module given by:

$$\mathsf{Disk}_M(k) \coloneqq \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \ldots \sqcup \mathbb{R}^n}_{k\times}, M). \tag{1.12}$$

Then for a Disk_n-algebra A (i.e. an E_n-algebra), the factorization homology of M with coefficients in A, denoted by $\int_M A$, is given by the derived composition product:

$$\int_M A \coloneqq \mathsf{Disk}_M \circ^{\mathbb{L}}_{\mathsf{Disk}_n} A = \mathrm{hocoeq}\big(\mathsf{Disk}_M \circ \mathsf{Disk}_n \circ A \rightrightarrows \mathsf{Disk}_M \circ A\big). \tag{1.13}$$

This definition also makes sense in the category of chain complexes, replacing Disk_n and Disk_M by their respective chain complexes. The operad Disk_n is formal over the rationals [Kon99; Tam03; LV14; Pet14; FW15], hence up to homotopy we may replace $C_*(\mathsf{Disk}_n; \mathbb{Q})$ by $H_*(\mathsf{Disk}_n; \mathbb{Q})$, which is $u\mathsf{Pois}_n$ for $n \geq 2$.

1.3.2 Small model to compute it over $\mathbb{R}$

In [Idr16], given a simply connected closed manifold M of dimension ≥ 4, we found an explicit model for $C_*(\mathsf{Disk}_M; \mathbb{R})$. If moreover M is parallelized then our model is a right module over the operad $H_*(\mathsf{Disk}_n) = u\mathsf{Pois}_n$, and this is compatible with the action of Disk_n on Disk_M in an appropriate sense. We called it the Lambrechts–Stanley model as it had been conjectured in [LS08a] (without mention of operads). This allowed us to compute factorization homology of such manifolds by replacing $C_*(\mathsf{Disk}_M)$ with our explicit model.

This explicit model for $C_*(\mathsf{Disk}_M; \mathbb{R})$ depends on a Poincaré duality model P of M. This is a (upper-graded) commutative differential graded algebra equipped with a linear form $\varepsilon : P^n \to \mathbb{Q}$ satisfying $\varepsilon \circ d = 0$ and inducing a non-degenerate pairing $P^k \otimes P^{n-k} \to \mathbb{Q}$, $x \otimes y \mapsto \varepsilon(xy)$ for all $k \in \mathbb{Z}$. It is moreover a rational model for M in the sense of Sullivan rational homotopy theory. These are known to exist for all simply connected manifold thanks to [LS08b].

We will not give the original definition of our explicit model for $C_*(\mathsf{Disk}_M; \mathbb{R})$, which denote $\mathsf{G}_P^\vee$ (it is actually the dual of our cohomological model). Instead we give an alternative description as explained in [Idr16, Section 5]. Recall the operad Lie governing Lie algebras, and let $\mathsf{Lie}_n = \mathscr{S}^{1-n}\mathsf{Lie}$ be its suspension governing shifted Lie algebras. For convenience, define a Lie algebra in the category of Lie_n-right modules by $\mathsf{L}_n(k) := \Sigma^{n-1}\mathsf{Lie}_n(k)$, satisfying $\mathsf{L}_n \circ V = \mathsf{Lie} \circ (\Sigma^{n-1}V)$. Given a Lie algebra $\mathfrak{g}$, its Chevalley–Eilenberg chain complex (with trivial coefficients) is a quasi-cofree cocommutative coalgebra on the suspension of $\mathfrak{g}$:

$$C_*^{CE}(\mathfrak{g}) := (\bar{S}^c(\Sigma\mathfrak{g}), d_{CE}), \tag{1.14}$$

with differential defined by $d_{CE}(x_1 \dots x_k) = \sum_{i<j} \pm x_1 \dots [x_i, x_j] \dots \widehat{x_j} \dots x_k$.

Then, as a right Lie_n-module, we have an isomorphism [Idr16, Lemma 5.2]:

$$\mathsf{G}_P^\vee \cong_{\mathsf{Lie}_n-\mathrm{RMod}} C_*^{CE}(P^{-*} \otimes \mathsf{L}_n), \tag{1.15}$$

where P^{-*} is P with grading reversed. There is also the right $u\mathsf{Com}$-module structure, which we describe in Section 4.4 below (it is not given in [Idr16]).

Finally, using theorems about preservations of weak equivalences by the relative operadic composition product, we find that given a $u\mathsf{Pois}_n$-algebra A, the factorization homology of M with coefficients in A over $\mathbb{R}$ is quasi-isomorphic to, under the hypotheses stated above:

$$\int_M A \simeq \mathsf{G}_P^\vee \circ^{\mathbb{L}}_{u\mathsf{Pois}_n} A. \tag{1.16}$$

As an explicit example, if $A = S(\Sigma^{1-n}\mathfrak{g})$ is the "universal enveloping n-algebra" of $\mathfrak{g}$, we recovered a theorem of Knudsen [Knu16]: $\int_M S(\Sigma^{1-n}\mathfrak{g}) \simeq C_*^{CE}(P^{n-*} \otimes \mathfrak{g})$.

2 Curved coalgebras and semi-augmented unital algebras

From now on, we fix a binary, homogeneous quadratic operad $\mathsf{P} = \mathrm{Free}(E)/(R)$ and a unital version $u\mathsf{P} = \mathrm{Free}(\mathbb{1} \oplus E)/(R + R')$ as in Section 1.2, with the same notations. Note that the operad P is automatically augmented.

2.1 Definitions

Let C be a coaugmented conilpotent cooperad, with zero differential. Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism [LV12, Section 6.4.3], i.e. a map satisfying the Maurer–Cartan equation

$$\varphi \star \varphi = 0, \tag{2.1}$$

where $\star$ is the convolution product (the differential of φ vanishes because d_{C} and d_{P} do). Suppose moreover that $\operatorname{im}\varphi \subset E = \mathsf{P}(2)$, the space of binary generators of P. This twisting morphism φ induces a twisting morphism

$$\bar{\varphi} : \mathbb{1} * \mathsf{C} \to u\mathsf{P}, \tag{2.2}$$

where $\mathbb{1} * \mathsf{C}$ is the free product of C with the cooperad on one generator $\mathbb{1}$ of arity 0.

Given a C-coalgebra C and a map $\theta : C \to \Bbbk\mathbb{1}$, we denote by

$$\Theta : C \xrightarrow{\theta} \Bbbk\mathbb{1} \subset u\mathsf{P}(C) \tag{2.3}$$

the composition, where $u\mathsf{P}(C)$ is the free $u\mathsf{P}$-algebra on C.

Definition 2.1. The $\bar{\varphi}$-star product of θ is the composition:

$$\star_{\bar{\varphi}}(\theta) : C \xrightarrow{\Delta_C} \mathsf{C}(C) \xrightarrow{\bar{\varphi}\circ_{(1)}\Theta} u\mathsf{P}(u\mathsf{P}(C)) \xrightarrow{\gamma_{u\mathsf{P}(C)}} u\mathsf{P}(C). \tag{2.4}$$

Thanks to our hypotheses that $\operatorname{im}\varphi \subset \mathsf{P}(2)$ and that the inhomogeneous relations of $u\mathsf{P}$ have no linear terms (see Section 1.2), we then check that the image of $\star_{\bar{\varphi}}(\theta)$ is included in $C \subset u\mathsf{P}(C)$.

Example 2.2. Let $u\mathsf{P} = u\mathsf{Ass}$ be the operad governing unital associative operads and $\kappa : \mathsf{Ass}^{\mathsf{i}} \to \mathsf{Ass}$ the Koszul twisting morphism. Then given a coalgebra C and a map $\theta : C \to \mathbb{k}$, the $\bar{\kappa}$-star product of θ is given by:

$$\star_{\bar{\kappa}}(\theta) = (\theta \otimes \mathrm{id} \mp \mathrm{id} \otimes \theta)\Delta_C : C \to C. \tag{2.5}$$

We then have the following definition, inspired by the definition of a curved coproperad in [HM12, Section 3.2.1] (where implicitly the twisting morphism is the Koszul morphism from the colored operad of operads to its Koszul dual):

Definition 2.3. A φ-curved C-coalgebra is a triple (C, d_C, θ_C) where:

- C is a C-coalgebra (with no differential);
- $d_C : C \to C$ is a degree -1 coderivation (the "predifferential");
- $\theta_C : C \to \mathbb{k}^{\uparrow}$ is a degree -2 linear map (the "curvature");

satisfying:

$$d_C^2 = \star_{\bar{\varphi}}(\theta_C), \qquad \theta_C d_C = 0. \tag{2.6}$$

Remark 2.4. This notion looks similar to the notion of coalgebra over a curved cooperad from [HM12, Section 5.2.1]. A coalgebra over a curved cooperad $(\mathsf{C}, d_{\mathsf{C}}, \theta_{\mathsf{C}})$ is a pair (C, d_C) where C is a C-coalgebra, d_C is a coderivation of C, and $d_C^2 = (\theta_{\mathsf{C}} \circ \mathrm{id}_C)\Delta_C$. In our setting, the curvature is part of the data of the coalgebra, rather than the cooperad itself, and we have an extra condition $\theta_C d_C = 0$. Moreover our notion of curved coalgebra depends on the data of a twisting morphism $\varphi : \mathsf{C} \to \mathsf{P}$, whereas in the other setting this is extra data required to define a bar/cobar adjunction. Le Grignou [LeG16] endowed the category of coalgebras over a curved cooperad with a model category structure, such that the bar/cobar adjunction defines a Quillen equivalence with the category of unital algebras.

Example 2.5. Consider the case $u\mathsf{P} = u\mathsf{Ass}$ is the operad governing unital associative algebras, $\mathsf{C} = \mathsf{Ass}^{\mathsf{i}} = \mathscr{S}^c\mathsf{Ass}^*$ is the Koszul dual of Ass (the suspension of its linear dual), and $\varphi = \kappa$ is the twisting morphism of Koszul duality. Then a κ-curved $\mathsf{Ass}^{\mathsf{i}}$-coalgebra is a (shifted) coassociative coalgebra C, together with a predifferential d_C and a "curvature" $\theta_C : C \to \mathbb{k}$, such that $\theta_C d_C = 0$ and d_C^2 is the cocommutator of θ:

$$d_C^2 = \star_{\bar{\kappa}}(\theta) : C \xrightarrow{\Delta} C \otimes C \xrightarrow{\theta\otimes\mathrm{id} \mp \mathrm{id}\otimes\theta} C. \tag{2.7}$$

This recovers the notion of curved homotopy coalgebra from Lyubashenko [Lyu17].

Example 2.6. For Lie algebras and the Koszul twisting morphism $\varphi = \kappa : \mathsf{Lie}^{\text{i}} \to \mathsf{Lie}$, this definition recovers (the dual of) the notion of "curved Lie algebra" [CLM16; Mau17]. A curved Lie algebra is a Lie algebra $\mathfrak{g}$ equipped with a derivation d of degree -1 and an element ω of degree -2 such that $d^2 = [\omega, -]$.

We also define the notion of a "semi-augmented" algebra over $u\mathsf{P}$ (the terminology is adapted from [HM12]). This is necessary because, in general, the bar construction of an algebra is not a curved coalgebra.

Definition 2.7. A semi-augmented $u\mathsf{P}$-algebra is a dg-$u\mathsf{P}$-algebra A equipped with an linear map $\varepsilon_A : A \to \Bbbk$ (not necessarily compatible with the dg-algebra structure), such that $\varepsilon_A(\uparrow) = 1$. Given such a semi-augmented $u\mathsf{P}$-algebra, we let $\bar{A}$ be the kernel of ε_A.

Given a semi-augmented $u\mathsf{P}$-algebra (A, ε_A), the exact sequence $0 \to \bar{A} \to A \xrightarrow{\varepsilon} \Bbbk\uparrow \to 0$ defines an isomorphism of graded modules $A \cong \bar{A} \oplus \Bbbk\uparrow$. This isomorphism is not compatible with the differential or the algebra structure in general. This allows to define a "composition" $\bar{\gamma}_A : u\mathsf{P}(\bar{A}) \to \bar{A}$ (which is generally not associative or a chain map) and a "differential" $d_{\bar{A}} : \bar{A} \to \bar{A}$ (which does not square to zero in general), by using the inclusion and projection $\bar{A} \to \bar{A} \oplus \Bbbk\uparrow \cong A \to \bar{A}$.

2.2 Cobar construction

Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism, i.e. an element satisfying the Maurer–Cartan equation $\varphi \star \varphi = 0$. Let $C = (C, d_C, \theta_C)$ be a φ-curved C-coalgebra as in Definition 2.3. We adapt the definition of [HM12, Section 5.2.5].

Definition 2.8. The cobar construction of C with respect to φ is given by:

$$\Omega_\varphi(C) := (u\mathsf{P}(\Sigma^{-1}C), d_\Omega = -d_0 + d_1 - d_2), \tag{2.8}$$

where each d_i is a derivation of degree -1 defined on generators by:

$$d_0|_{\Sigma^{-1}C} : \Sigma^{-1}C \xrightarrow{\Sigma\theta_C} \Bbbk\uparrow \hookrightarrow u\mathsf{P}(\Sigma^{-1}C) \tag{2.9}$$

$$d_1|_{\Sigma^{-1}C} : \Sigma^{-1}C \xrightarrow{d_C} \Sigma^{-1}C \hookrightarrow u\mathsf{P}(\Sigma^{-1}C) \tag{2.10}$$

$$d_2|_{\Sigma^{-1}C} : \Sigma^{-1}C \xrightarrow{\Delta} \mathsf{C}(\Sigma^{-1}C) \xrightarrow{\tilde{\varphi}(\mathrm{id})} u\mathsf{P}(\Sigma^{-1}C) \tag{2.11}$$

It is equipped with the semi-augmentation $\varepsilon_\Omega : \Omega_\varphi(C) \to \Bbbk$ given by the projection $u\mathsf{P} \to \uparrow$.

The only thing we need to check is:

Proposition 2.9. *The derivation d_Ω squares to zero: $d_\Omega^2 = 0$.*

Proof. There is a weight decomposition (denoted ω) of $u\mathsf{P}(\Sigma^{-1}C)$ obtained by assigning $\Sigma^{-1}C$ the weight 1. For example, d_0 is of weight -2, d_1 is of weight -1, and d_2 is of weight 0. We may then decompose d_Ω^2 in terms of this weight:

$$d_\Omega^2 = \underbrace{d_0^2}_{\omega=-4} - \underbrace{d_0 d_1 - d_1 d_0}_{\omega=-3} + \underbrace{d_1^2 + d_0 d_2 + d_2 d_0}_{\omega=-2} - (\underbrace{d_1 d_2 + d_2 d_1}_{\omega=-2}) + \underbrace{d_2^2}_{\omega=0}. \tag{2.12}$$

and it suffices to check that each summands vanishes. Each summand is itself a derivation (because $d_\Omega^2 = \frac{1}{2}[d_\Omega, d_\Omega]$ is a derivation thus so are its weight components), so it even suffices to check that they vanish on generators.

- $d_0^2 = 0$: the image of d_0 is included in $\Bbbk\upharpoonleft$, and every derivation vanishes on $\upharpoonleft$;
- $d_1 d_0 + d_0 d_1 = 0$:
  - $d_1 d_0 = 0$ because $d_1(\upharpoonleft) = 0$;
  - $d_0 d_1 = 0$ because $\theta_C d_C = 0$;
- $d_1^2 + d_0 d_2 + d_2 d_0 = 0$:
  - $d_2 d_0 = 0$, again because $d_2(\upharpoonleft) = 0$;
  - $d_1^2 + d_0 d_2 = 0$ comes from $d_C^2 = \star_{\bar\varphi}(\theta_C)$ and the Koszul rule of signs;
- $d_1 d_2 + d_2 d_1 = 0$ comes from the fact that d_C is a derivation (the $\bar\varphi$ appearing in $d_1 d_2$ stays on the outside);
- $d_2 = 0$ comes from the Maurer–Cartan equation $\bar\varphi \star \bar\varphi = 0$ and the commutativity of the following diagram:

$$\begin{tikzcd}
C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"] & \mathsf{C}(C) \arrow[r, "\bar\varphi \circ \mathrm{id}"] \arrow[d, "\mathrm{id} \circ' \Delta_C"] & u\mathsf{P}(C) \arrow[d, "\mathrm{id} \circ' \Delta_C"] & & \\
\mathsf{C}(C) \arrow[r, "\Delta_{(1)} \circ \mathrm{id}"] & \mathsf{C}(C; \mathsf{C}(C)) \arrow[r, "\bar\varphi(\mathrm{id};\mathrm{id})"] & u\mathsf{P}(C; \mathsf{C}(C)) \arrow[r, "\mathrm{id}(\mathrm{id};\bar\varphi(\mathrm{id}))"] & (u\mathsf{P} \circ_{(1)} u\mathsf{P})(C) \arrow[r, "\gamma_{(1)}"] & u\mathsf{P}(C).
\end{tikzcd}$$

□

Corollary 2.10. *Given a φ-curved C-coalgebra $C = (C, d_C, \theta_C)$, the cobar construction $\Omega_\varphi(C)$ is a well-defined semi-augmented $u\mathsf{P}$-algebra.*

2.3 Bar construction

We now define the twin of the cobar construction: the bar construction (see [HM12, Section 3.3.2] for the (pr)operadic case). Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism and let A be a semi-augmented dg-$u\mathsf{P}$-algebra (see Definition 2.7), with augmentation ideal $\bar A$. Again, to avoid signs, we will actually define the bar construction of the $\mathscr{S}^{-1}u\mathsf{P}$-algebra $\Sigma^{-1}A$. Recall that $\varepsilon_A : A \to \Bbbk$ is the semi-augmentation and $\bar A = \ker \varepsilon_A$. There is a "composition map" $\bar\gamma_A : u\mathsf{P}(\bar A) \to \bar A$ and a "differential" $d_{\bar A}$ (which do not satisfy any good property).

We want to define the φ-curved C-algebra (see the analogous definition in [HM12, Section 5.2.3])

$$B_\varphi A \coloneqq (\mathsf{C}(\Sigma \bar A), d_B, \theta_B). \tag{2.13}$$

The underlying C-coalgebra of $B_\varphi(A)$ is merely the cofree coalgebra $\mathsf{C}(\Sigma \bar A)$. The predifferential d_B is the sum $d_1 + d_2$, where

- d_2 is the unique coderivation of $\mathsf{C}(\Sigma\bar{A})$ whose corestriction onto $\Sigma\bar{A}$ is:

$$d_2|^{\Sigma\bar{A}} : \mathsf{C}(\Sigma\bar{A}) \xrightarrow{\varphi\circ\mathrm{id}} \mathsf{P}(\Sigma\bar{A}) \xrightarrow{\bar{\gamma}_A} \Sigma\bar{A}; \tag{2.14}$$

- d_1 is the unique coderivation whose corestriction is:

$$d_1|^{\Sigma\bar{A}} : \mathsf{C}(\Sigma\bar{A}) \twoheadrightarrow \Sigma\bar{A} \xrightarrow{d_{\bar{A}}} \Sigma\bar{A}. \tag{2.15}$$

Let $\varepsilon_C : \mathsf{C} \to I$ be the counit of the cooperad C. The curvature $\theta_B : \mathsf{C}(A) \to \mathbb{k}$ is the map of degree -2 given by:

$$\mathsf{C}(\Sigma\bar{A}) \xrightarrow{(\varepsilon_C\oplus\varphi)(\mathrm{id}_A)} \Sigma\bar{A} \oplus u\mathsf{P}(\Sigma\bar{A}) \xrightarrow{d_A+\gamma_A} \Sigma A \xrightarrow{\varepsilon_A} \mathbb{k}. \tag{2.16}$$

Proposition 2.11. *The predifferential d_B and the curvature θ_B satisfy the equations defining a φ-curved coalgebra structure on $B_\varphi A$.*

Sketch of proof. We need to check that $d_B^2 = \hat{\star}_{\bar{\varphi}}(\theta_B)$ and $\theta_B d_B = 0$. It is enough to check these equalities when projected on cogenerators, as the three maps are coderivations. This then follows from the definitions, the Maurer–Cartan equation $\varphi \star \varphi$, the fact that d_A is a derivation, the associativity of the $u\mathsf{P}$-algebra structure maps on A, and the compatibility of d_A with these structures maps. We refer to [HM12, Lemma 3.3.3] for the case of operads, roughly speaking the case $\mathsf{P} = \mathsf{Ass}$. □

Corollary 2.12. *The data $B_\varphi A = (\mathsf{C}(\Sigma A), d_B, \theta_B)$ defines a φ-curved coalgebra from the semi-augmented $u\mathsf{P}$-algebra A.*

2.4 Adjunction

Definition 2.13. Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism, $C = (C, d_C, \theta_C)$ be a φ-curved C-coalgebra, and A be a semi-augmented $u\mathsf{P}$-algebra. The set of φ-twisting morphisms from C to A is:

$$\mathrm{Tw}_\varphi(C, A) \coloneqq \{\beta : C \to A \mid \partial(\beta) + \hat{\star}_{\bar{\varphi}}(\beta) = \Theta^A\}, \tag{2.17}$$

where $\hat{\star}_{\bar{\varphi}}(\beta)$ and Θ^A are given by:

$$\hat{\star}_{\bar{\varphi}}(\beta) : C \xrightarrow{\Delta_C} \mathsf{C} \circ C \xrightarrow{\bar{\varphi}\circ\beta} u\mathsf{P} \circ A \xrightarrow{\gamma_A}, \tag{2.18}$$

$$\Theta^A : C \xrightarrow{\theta_C} \mathbb{k}\mathbf{1} \hookrightarrow A. \tag{2.19}$$

We then have the following "Rosetta stone", to reuse the terminology of [LV12]:

Proposition 2.14. *Let C be a φ-curved C-coalgebra and A be a semi-augmented $u\mathsf{P}$-algebra. Then there are natural bijections:*

$$\mathrm{Hom}_{\text{sem.aug.}u\mathsf{P}\text{-alg}}(\Omega_\varphi C, A) \cong \mathrm{Tw}_\varphi(C, A) \cong \mathrm{Hom}_{\varphi\text{-curved }\mathsf{C}\text{-coalg}}(C, B_\varphi A). \tag{2.20}$$

In particular, the functor Ω_φ and B_φ are adjoint to each other.

Proof. Let us first prove the existence of the first bijection. Given $\beta \in \mathrm{Tw}_\varphi(C, A)$, we let $f_\beta : u\mathsf{P}(\Sigma^{-1}C) \to A$ be the $u\mathsf{P}$-algebra morphism given on generators by β. We must check that $f_\beta d_\Omega = d_A f_\beta$. As we are working with derivations and morphisms, it is enough to check this on $\Sigma^{-1}C$. The restrictions of the maps involved are:

- $f_\beta d_0|_{\Sigma^{-1}C} = \Theta^A$;
- $f_\beta d_1|_{\Sigma^{-1}C} = \beta d_C$;
- $f_\beta d_2|_{\Sigma^{-1}C} = (\Sigma^{-1}C \xrightarrow{\Delta_C} \mathsf{C} \circ C \xrightarrow{\bar\varphi \circ \beta} u\mathsf{P}(A) \xrightarrow{\gamma_A} A) = \hat\star_{\bar\varphi}\beta$;
- $d_A f_\beta|_{\Sigma^{-1}C} = d_A\beta$.

The Maurer–Cartan equation $\partial(\beta) + \hat\star_{\bar\varphi}(\beta) = \Theta^A$ then implies $f_\beta d_\Omega = d_A f_\beta$.

Conversely, given $f : \Omega_\varphi C \to A$, then we can define $\beta \coloneqq f|_{\Sigma^{-1}C}$. The same proof as above but in the reverse direction shows that the compatibility of f with the differentials imply the Maurer–Cartan equation. Moreover, the two constructions are inverse to each other and we are done.

Checking the existence of the second bijection is extremely similar (see also the proof of [HM12, Theorem 3.4.1] for the case of (co)operads). Checking that the two bijections are natural in terms of C and A is also a simple exercise in commutative diagrams. $\square$

3 Koszul duality of unitary algebras

3.1 Algebras with quadratic-linear-constant relations

We now define the type of algebras for which we will developed a Koszul duality theory, namely algebras with "quadratic-linear-constant" (QLC) relations. For this we adapt the notion of a "monogenic algebra" from [Mil12, Section 4.1] (see Section 1.1.3). We still assume that we are given a unital version $u\mathsf{P} = \mathrm{Free}(\dagger \oplus E)/(R + R')$ of a binary quadratic operad $\mathsf{P} = \mathrm{Free}(E)/(R)$ as in Section 1.2.

Definition 3.1. An $u\mathsf{P}$-algebra with QLC relations is a $u\mathsf{P}$-algebra A with $d_A = 0$, equipped with a presentation by generators and relations:

$$A = u\mathsf{P}(V)/I, \tag{3.1}$$

with $d_A = 0$, satisfying the two conditions:

1. the ideal $I = (S)$ is generated by $S \coloneqq I \cap (\dagger \oplus V \oplus E(V))$ (where $E(V) = E \otimes_{\Sigma_2} V^{\otimes 2}$),
2. the relations in S are least quadratic, $S \cap (\dagger \oplus V) = 0$.

Remark 3.2. Such an algebra is automatically semi-augmented (Definition 2.7).

Definition 3.3. Given a $u\mathsf{P}$-algebra A with QLC relations as above, the quadratic reduction qA of A is the P-algebra obtained by:

$$qA \coloneqq \mathsf{P}(V)/(qS), \tag{3.2}$$

where qS is the projection of S onto $E(V)$.

The second condition in the definition of an algebra with QLC relations implies that S is the graph of some map

$$\alpha = (\alpha_0 \oplus \alpha_1) : qS \to \Bbbk\overset{\bullet}{|} \oplus V, \tag{3.3}$$

i.e. $S = \{x + \alpha(x) \mid x \in qS\}$.

Until the end of this section, A will be a $u\mathsf{P}$-algebra with QLC relations, using the same notations.

3.2 Koszul dual coalgebra

Let P^{i} be the Koszul dual cooperad of P, with an operad-twisting morphism $\kappa : \mathsf{P}^{\text{i}} \to \mathsf{P}$ (see Section 1.1.2). The theory of [Mil12] (see Section 1.1.3) defines a Koszul dual $\mathscr{S}^c\mathsf{P}^{\text{i}}$-coalgebra qA^{i} from the quadratic reduction qA:

$$qA^{\text{i}} := \Sigma\mathsf{P}^{\text{i}}(V, \Sigma qS). \tag{3.4}$$

Using the map α from Equation (3.3), we may define $d_{A^{\text{i}}} : qA^{\text{i}} \to \Sigma\mathsf{P}^{\text{i}}(V)$ to be the unique coderivation whose corestriction is given by:

$$d_{A^{\text{i}}}|^{\Sigma V} : qA^{\text{i}} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-1}\alpha_1} \Sigma V. \tag{3.5}$$

Moreover, define $\theta_{A^{\text{i}}}$ by:

$$\theta_{A^{\text{i}}} : qA^{\text{i}} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-2}\alpha_0} \Bbbk\overset{\bullet}{|}. \tag{3.6}$$

We now define the Koszul dual coalgebra of A by adapting [HM12, Section 4.2]:

Proposition 3.4. *The data*

$$A^{\text{i}} := (qA^{\text{i}}, d_{A^{\text{i}}}, \theta_{A^{\text{i}}}) \tag{3.7}$$

defines a κ-curved P^{i}-coalgebra. We call it the Koszul dual curved coalgebra of A.

Proof. We must show that $d_{A^{\text{i}}}(qA^{\text{i}}) \subset qA^{\text{i}}$, that $\star_{\bar{\kappa}}(\theta_{A^{\text{i}}}) = d^2_{A^{\text{i}}}$, and that $\theta_{A^{\text{i}}} d_{A^{\text{i}}} = 0$. Because we assume that the operad P is binary, it is sufficient to show all three facts about elements of qA^{i} of weight 3 in terms of number of generators. The proof is heavily inspired from [HM12, Lemma 4.1.1].

Let $Y \in (qA^{\text{i}})^{(3)}$ be an element of weight 3. The coproduct $\Delta(Y) \in \mathsf{P}^{\text{i}}(qA^{\text{i}})$ must belong, by definition of qA^{i} as a cofree coalgebra with corelations, to the subspace

$$\Delta(Y) \in E \otimes (\Sigma V \otimes \Sigma^2 qS) \cap E \otimes (\Sigma^2 qS \otimes \Sigma V). \tag{3.8}$$

In other words, we must have two decompositions

$$\Delta(Y) = \sum_i \rho_i(\Sigma v_i, \Sigma^2 X_i) = \sum_j \rho'_j(\Sigma^2 X'_j, \Sigma v'_j), \tag{3.9}$$

where $\rho_i, \rho'_j \in E$, $v_i, v'_j \in V$, and $X_i, X'_j \in qS$. Then the rule of signs shows that:

$$\begin{aligned} d_{A^{\text{i}}}(Y) &= -\sum_i \rho_i(\Sigma v_i, (\Sigma^{-1}\alpha_1)(\Sigma^2 X_i)) + \sum_j \rho'_j((\Sigma^{-1}\alpha_1)(\Sigma^2 X'_j), \Sigma v'_j) \\ &= -\sum_i \rho_i(\Sigma v_i, (\Sigma\alpha_1(X_i) + \sum_j \rho'_j((\Sigma\alpha_1(X'_j), \Sigma v'_j) \in E(\Sigma V). \end{aligned} \tag{3.10}$$

And similarly:

$$\star_{\bar{\kappa}} (\theta_{A^{\text{i}}})(Y) = \sum_i \rho_i(v_i, \alpha_0(X_i)) + \sum_j \rho'_j(\alpha_0(X'_j), v'_j) \in V. \tag{3.11}$$

Recall that S is the graph of α from Equation (3.3): for $X \in qS$, we have that $f(X) := (\mathrm{id} + \alpha_0 + \alpha_1)(X) \in S$. Therefore, if we apply the map

$$g := \gamma \circ (\mathrm{id}_E \otimes (\mathrm{id}_{\Sigma V} \otimes \Sigma^{-1} f)) \tag{3.12}$$

to an element in $E \otimes (\Sigma V \otimes \Sigma^2 qS)$, then the result must belong to $(S) \subset A$. The same holds if we apply

$$g' := \gamma \circ (\mathrm{id}_E \otimes (\Sigma^{-1} f \otimes \mathrm{id}_{\Sigma V})) \tag{3.13}$$

to an element of $E \otimes (\Sigma^2 qS \otimes \Sigma V)$.

We know that $\Delta(Y)$ belongs to these two spaces, thus $g(\Delta(Y))$ and $g'(\Delta(Y))$ must belong to (S). Hence so does $(g + g')(\Delta(Y))$. Signs cancel out in $(g + g')(\Delta(Y))$, and therefore:

$$(g + g')(\Delta(Y)) = d_{A^{\text{i}}}(Y) + \star_{\bar{\kappa}}(\theta_{A^{\text{i}}})(Y) \in \Sigma^2 S. \tag{3.14}$$

We know that S is the graph of α, hence the expression of Equation (3.14) is of the form $x + \alpha_1(x) + \alpha_0(x)$ for some $x \in qS$. The element $d_{A^{\text{i}}}$ is of weight 2, while $\star_{\bar{\kappa}}(\theta_{A^{\text{i}}})(Y)$ is of weight 1. Thus by identifying each weight component:

- $d_{A^{\text{i}}}(Y)$ belongs to $\Sigma^2 qS = (qA^{\text{i}})^{(2)}$;
- the element $d^2_{A^{\text{i}}}(Y)$ is equal to to $(\Sigma^{-1}\alpha_1)(d_{A^{\text{i}}}(Y))$, which we know from Equation (3.14) is equal to $\star_{\bar{\kappa}}(\theta_{A^{\text{i}}})(Y)$;
- similarly, $\theta_{A^{\text{i}}} d_{A^{\text{i}}}(Y)$ is equal to $\Sigma^{-2}\alpha_0(d_{A^{\text{i}}}(Y))$, which vanishes because there is no element of weight 0 in Equation (3.14). □

3.3 Main theorem

Let us now define $\varkappa : qA^{\text{i}} \to A$ of degree -1 by:

$$\varkappa : qA^{\text{i}} \twoheadrightarrow \Sigma V \xrightarrow{\Sigma^{-1}} V \hookrightarrow A. \tag{3.15}$$

Proposition 3.5. *The morphism $\varkappa$ satisfies the curved Maurer–Cartan equation, i.e. it is an element of $\mathrm{Tw}_\varphi(qA^{\text{i}}, A)$ from Proposition 2.14:*

$$\partial(\varkappa) + \hat{\star}_{\bar{\kappa}}(\varkappa) = \Theta^A. \tag{3.16}$$

Proof. We can rewrite the equation as:

$$-\varkappa d_{A^{\text{¡}}} + \gamma_A(\bar{\kappa} \circ \varkappa)\Delta_{qA^{\text{¡}}} = \Theta^A. \tag{3.17}$$

Moreover, by checking the definitions, we see that:

- $\varkappa d_{A^{\text{¡}}}$ is obtained as $qA^{\text{¡}} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-1}\alpha_1} \Sigma V \xrightarrow{\Sigma^{-1}} V \hookrightarrow A$;
- Θ^A is obtained as $qA^{\text{¡}} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-2}\alpha_0} \Bbbk\dagger \hookrightarrow A$;
- $\hat{\star}_{\bar{\kappa}}(\varkappa)$ vanishes everywhere except on $(qA^{\text{¡}})^{(2)} = \Sigma^2 qS$, where it is equal to $\gamma_A \circ \Sigma^{-2}\Delta_{qA^{\text{¡}}}$.

Checking signs carefully, we see that the image of $\hat{\star}_{\bar{\kappa}}(\varkappa) - \varkappa d_{A^{\text{¡}}} - \Theta^A$ is included in the image of the graph of α under γ_A. But this graph is S, and $\gamma_A(S) = 0$ because these are part of the relations of A. $\square$

Using the Rosetta stone (Proposition 2.14), we then obtain that $\varkappa$ defines a morphism $f_\varkappa : \Omega_\kappa A^{\text{¡}} \to A$.

Definition 3.6. The $u\mathsf{P}$-algebra A is said to be Koszul if the P-algebra qA is Koszul in the sense of [Mil12].

Recall that qA is Koszul iff the induced morphism $\Omega_\kappa(qA^{\text{¡}}) \to qA$ is quasi-isomorphism, see [Mil12, Theorem 4.9]. Our definition is justified by the following theorem:

Theorem 3.7. *If qA is Koszul, then $f_\varkappa : \Omega_\kappa A^{\text{¡}} \to A$ is a cofibrant resolution of A in the semi-model category of $u\mathsf{P}$-algebras defined in [Fre09, Theorem 12.3.A].*

Proof. Let us filter A and $\Omega_\kappa A^{\text{¡}}$ by the weight in terms of V. It is clear that $f_\varkappa$ is compatible with this filtration. The summands d_0 and d_1 of d_Ω strictly lower this filtration, while d_2 preserves it. Thus, on the first pages of the associated spectral sequences, be obtain the morphism:

$$\Omega_\kappa(qA^{\text{¡}}) \oplus \Bbbk\dagger \to qA \oplus \Bbbk\dagger. \tag{3.18}$$

Our hypothesis on qA implies that this is a quasi-isomorphism, i.e. we have an isomorphism on the second pages of the spectral sequences. The filtration is exhaustive and bounded below, therefore usual theorems about spectral sequences show that $f_\varkappa$ itself is a quasi-isomorphism.

It remains to check that $\Omega_k A^{\text{¡}}$ is cofibrant in the semi-model category from [Fre09, Theorem 12.3.A] (since we are working over a field of characteristic zero, $u\mathsf{P}$ is automatically Σ_*-cofibrant so the theorem applies). The cobar construction is quasi-free, i.e. free as an algebra if we forget the differential, and it is equipped with a good filtration (by weight in terms of $A^{\text{¡}}$), hence by [Fre09, Proposition 12.3.8] it is cofibrant. $\square$

4 Application: symplectic Poisson n-algebras

4.1 Definition

In this section, with deal with Poisson n-algebras for some integer n. We will abbreviate as Lie_n the operad of Lie algebras shifted by $n-1$, i.e. $\mathsf{Lie}_n \coloneqq \mathscr{S}^{1-n}\mathsf{Lie}$. Then the n-Poisson operad is given by:

$$\mathsf{Pois}_n \coloneqq \mathsf{Com} \circ \mathsf{Lie}_n = \mathrm{Free}(E)/(R). \tag{4.1}$$

Here, $E = \Bbbk\mu \oplus \Bbbk\lambda$ is the space of (binary) generators: μ, the product of degree 0, and λ, the Lie bracket of degree $n-1$. The ideal of relations is generated by the relations of Com, the relations of Lie_n, and the Leibniz relation stating that λ is a biderivation with respect to μ. We also consider the unital version $u\mathsf{Pois}_n$, where the unit satisfies $\mu(\mathbb{1}, -) = \mathrm{id}$ and $\lambda(\mathbb{1}, -) = 0$, i.e. it is a unit for the product and a central element for the Lie bracket.

Remark 4.1. For $n \geq 2$, this operad is isomorphic to the homology of the little n-disks operad D_n. For $n = 1$, the homology of D_1 is given by the associative operad Ass, and Pois_1 is isomorphic to the graded operad associated to a certain filtration of Ass (this is essentially the Poincaré–Birkhoff–Witt theorem).

Definition 4.2. The Dth symplectic Poisson n-algebra is defined by:

$$A_{n;D} \coloneqq \big(\Bbbk[x_1, \dots, x_D, \xi_1, \dots, \xi_D], \{,\}\big). \tag{4.2}$$

where the generators x_i have degree 0 and the ξ_i have degree $1-n$. The algebra $A_{n;D}$ is free a unital commutative algebra, and the Lie bracket is defined on generators by:

$$\{x_i, x_j\} = 0 \qquad \{\xi_i, \xi_j\} = 0 \qquad \{x_i, \xi_j\} = \delta_{ij}. \tag{4.3}$$

The algebra $A_{n;D} = u\mathsf{Pois}_n(V_{n;D})/(S_{n;D})$ is equipped with a QLC presentation. The space of generators is $V_{n;D} \coloneqq \mathbb{R}\langle x_i, \xi_j\rangle$, a graded vector space of dimension $2D$. We check that the ideal of relations $I_{n;D}$ is generated by the set S given by the three sets of relations fixing the Lie brackets of the generators, namely

$$S_{n;D} = \mathbb{R}\langle \{x_i, x_j\}, \{\xi_i, \xi_j\}, \{x_i, \xi_j\} - \delta_{ij}\mathbb{1}\rangle. \tag{4.4}$$

Remark 4.3. We may view $A_{n;D}$ as the Poisson n-algebra of polynomial functions on the standard shifted symplectic space $T^*\mathbb{R}^D[1-n]$. The element x_i is a polynomial function on the coordinate space $\mathbb{R}^D$, and the element ξ_j can be viewed as the vector field $\partial/\partial x_j$, which is a function on $T^*\mathbb{R}^D[1-n]$.

We will drop the indices n and D from the notation in what follows, taking them as fixed.

4.2 Koszulity and explicit resolution

In this section, we prove:

Proposition 4.4. *The* $u\mathsf{Pois}_n$*-algebra* A *is Koszul.*

Lemma 4.5. *The quadratic reduction* qA *of* A *is a free symmetric algebra with trivial Lie bracket.*

Proof. Let $V = \mathbb{R}\langle x_1, \dots, x_D, \xi_1, \dots, \xi_D \rangle$ be the generators of A. We check that $qS = \lambda(V) = \lambda \otimes_{\Sigma_2} V^{\otimes 2}$, i.e. in the quadratic reduction, all Lie brackets vanish. Therefore, $qA = \mathsf{Pois}_n(V)/(qS) = \mathsf{Pois}_n(V)/(\lambda(V)) = \mathsf{Com}(V)$ is a free symmetric algebra, and the Lie bracket vanishes. □

Let Com^c be the cooperad governing cocommutative coalgebras. Up to a suspension, it is the Koszul dual of the Lie operad. Since we are working over a field of characteristic zero, we can identify the cofree coalgebra $\mathsf{Com}^c(X)$ to

$$\bar{S}^c(X) \coloneqq \bigoplus_{i \geq 1} (X^{\otimes i})_{\Sigma_i} \tag{4.5}$$

with a coproduct given by shuffles. For a shorter notation we will also write $L(X)$ for the free Lie algebra on X, $S(X)$ for the free unital symmetric algebra, and $\bar{S}(X)$ for the free symmetric algebra without unit.

Lemma 4.6. *The Koszul dual coalgebra of* qA *is given by:*

$$qA^{\text{!`}} = \Sigma^{1-n} \bar{S}^c(\Sigma^n V). \tag{4.6}$$

Proof. This follows from the general fact that if $\mathsf{P} = \mathsf{Q}_1 \circ \mathsf{Q}_2$ is obtained by a distributive law and $A = \mathsf{Q}_1(V)$ then $A^{\text{!`}} = \Sigma \mathsf{Q}_2^{\text{!`}}(V)$. Thus we obtain that the Koszul dual of $qA = \mathsf{Com}(V)$ is $qA^{\text{!`}} = \Sigma(\mathsf{Lie}_n)^{\text{!`}}(V)$. Thanks to the Koszul duality between Com and Lie, this is identified with $\Sigma^{1-n}\mathsf{Com}^c(\Sigma^n V)$, and the claim follows. □

Proof of Proposition 4.4. Let $\kappa : \mathsf{Pois}_n^{\text{!`}} \to \mathsf{Pois}_n$ be the twisting morphism of Koszul duality. Then the cobar construction of $qA^{\text{!`}}$ is given by:

$$\Omega_\kappa qA^{\text{!`}} = \big(\underbrace{\bar{S}(\Sigma^{1-n} L(\Sigma^{n-1}}_{=\mathsf{Pois}_n} \Sigma^{-1} \underbrace{\Sigma^{1-n} \bar{S}^c(\Sigma^n V)))}_{=qA^{\text{!`}}}, d_2\big), \tag{4.7}$$

where d_2 is the derivation of Pois_n-algebras whose restriction on $\Sigma^{-n}\bar{S}^c(\Sigma^n V)$ is given by:

$$d_2|_{\Sigma^{-1} qA^{\text{!`}}}(u) = \sum_{(u)} \frac{1}{2} [u_{(1)}, u_{(2)}]. \tag{4.8}$$

This 1/2-factor is due to the identification of Com^c, which is initially defined using invariants under the symmetric group action, with $\bar{S}^c(X)$, which is defined using coinvariants.

The twisting morphism $\varkappa \in \mathrm{Tw}_\kappa(qA^{¡}, qA)$ is given by $\varkappa(\Sigma x_i) = x_i$ and $\varkappa(\Sigma \xi_i) = \xi_i$ on terms of weight 1, and it vanishes on terms of weight ≥ 2. This twisting induces a morphism $\Omega_\kappa(qA^{¡}) \to qA$. We easily see that this morphism is the image under $\bar{S}^c$ of the bar/cobar resolution of the abelian Lie_n algebra V:

$$\left(\Sigma^{1-n} L(\Sigma^{1-n} \bar{S}^c(\Sigma^n V)), d_2\right) \xrightarrow{\sim} V, \tag{4.9}$$

which is indeed a quasi-isomorphism thanks to classical Koszul duality of operads. The functor $\bar{S}^c$ preserves quasi-isomorphisms as we are working over a field, by the Künneth theorem. Therefore we obtain that $\Omega_\kappa(qA^{¡}) \to qA$ is a quasi-isomorphism, therefore by definition A is Koszul. $\square$

We then obtain a small resolution of the $u\mathsf{Pois}_n$-algebra A using the cobar construction of its Koszul dual coalgebra. Let us now describe it. The map $\alpha : qS \to \Bbbk\dagger \oplus V$ from Equation (3.3) is given by $\alpha_1 = 0$ and $\alpha_0(\{x_i, \xi_i\}) = -\dagger$, and $\alpha_0 = 0$ on everything else. Therefore the Koszul dual $A^{¡} = (qA^{¡}, d_{A^{¡}}, \theta_{A^{¡}})$ is such that $d_{A^{¡}} = 0$, and

$$\begin{aligned} \theta : \Sigma^{1-n} \bar{S}^c(\Sigma^n V) &\to \Bbbk\dagger \\ x_i \vee \xi_i &\mapsto -\dagger, \\ \text{everything else} &\mapsto 0. \end{aligned}$$

Here, as a shorthand, we write

$$v_1 \vee \ldots \vee v_k \coloneqq \frac{1}{k!} \sum_{\sigma \in \Sigma_k} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)} \in \Sigma^{1-n} \bar{S}^c(\Sigma^n V). \tag{4.10}$$

We then obtain

$$\Omega_{\bar{\kappa}} A^{¡} = (S(\Sigma^{1-n} L(\Sigma^{-1} \bar{S}^c(\Sigma^n V))), d_0 + d_2) \xrightarrow{\sim} A, \tag{4.11}$$

where (by abuse of notation) we still denote by d_2 the Chevalley–Eilenberg differential from before, satisfying $d_2(\dagger) = 0$. The derivation d_0 is the one whose restriction to $\Sigma^{-1} A^{¡}$ is given by $\Sigma\theta$:

$$d_0|_{\Sigma^{-1} qA^{¡}} : \Sigma^{-n} \bar{S}^c(\Sigma^n V) \xrightarrow{\Sigma\theta} \Bbbk\dagger \hookrightarrow u\mathsf{Pois}_n(\Sigma^{-1} qA^{¡}). \tag{4.12}$$

Remark 4.7. Let us compare the resolution we get with the resolution that would be obtained if one applied curved Koszul duality at the level of operads (see Section 1.1.4) with the bar/cobar resolution, using the curved Koszul duality of the $u\mathsf{Pois}_n$ operad from the theory of [HM12]. Without the suspensions, this second resolution would look like $S(L(\bar{S}^c(L^c(S(V)))))$, i.e. it is the free symmetric algebra on the free Lie algebra on the cofree symmetric coalgebra on the cofree Lie coalgebra on A, which is itself a free symmetric algebra on V. Moreover the differential would be rather complicated (although explicit). Without the suspension, our resolution is just $S(L(\bar{S}^c(V)))$, which is much smaller.

4.3 Derived enveloping algebras

4.3.1 General constructions

Given an operad P and a P-algebra A, the enveloping algebra $\mathcal{U}_{\mathsf{P}}(A)$ is a unital associative algebra such that the left modules of $\mathcal{U}_{\mathsf{P}}(A)$ are precisely the operadic left modules of A (see e.g. [Fre09, Section 4.3]). Let $\mathsf{P}[1]$ be the operadic right P-module given by $\mathsf{P}[1] = \{\mathsf{P}(n+1)\}_{n\geq 0}$. Then the enveloping algebra $\mathcal{U}(A)$ can obtained by the relative composition product:

$$\mathcal{U}_{\mathsf{P}}(A) \cong \mathsf{P}[1] \circ_{\mathsf{P}} A = \operatorname{coeq}\big(\mathsf{P}[1] \circ \mathsf{P} \circ A \rightrightarrows \mathsf{P}[1] \circ A\big). \tag{4.13}$$

We will need the following examples later.

Example 4.8. The enveloping algebra $\mathcal{U}_{\mathsf{Lie}}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the usual universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. We view it as a free associative algebra on symbols X_f, for $f \in \mathfrak{g}$, subject to the relations $X_{[f,g]} = X_f X_g - \pm X_g X_f$. The universal enveloping algebra $\mathcal{U}_{\mathsf{cLie}}(\mathfrak{g})$ of a Lie algebra equipped with a central element $\dagger \in \mathfrak{g}$ is the quotient $\mathcal{U}(\mathfrak{g})/(X_{\dagger})$.

Example 4.9. The enveloping algebra $\mathcal{U}_{\mathsf{Com}}(B)$ of a commutative algebra B is $B_+ = \Bbbk 1 \oplus B$, where 1 is an extra unit. The enveloping algebra $\mathcal{U}_{u\mathsf{Com}}(B)$ of a unital commutative algebra B is B itself – formally, the quotient of $\Bbbk 1 \oplus B$ by $1 - 1_B$.

Suppose now that P is a Koszul operad (potentially unital, e.g. we could take $\mathsf{P} = u\mathsf{Pois}_n$) and let $\mathsf{P}_\infty = \Omega \mathsf{P}^{\text{¡}}$ be its Koszul resolution. The canonical morphism $\mathsf{P}_\infty \xrightarrow{\sim} \mathsf{P}$ induces a Quillen equivalence between the semi-model categories of P- and P_∞-algebras. In particular there is a forgetful functor from P-algebras to P_∞-algebras, allowing us to view A as a P_∞-algebra.

Proposition 4.10. *Let $R_\infty \xrightarrow{\sim} A$ be a cofibrant resolution of A as a P_∞-algebra, and let $R \coloneqq \mathsf{P} \circ_{\mathsf{P}_\infty} R_\infty$. Then there is an equivalence*

$$\mathcal{U}_{\mathsf{P}_\infty}(A) \simeq \mathcal{U}_{\mathsf{P}}(R). \tag{4.14}$$

The slogan is as follows: from the derived point of view, we can either resolve the operad or resolve the algebra. In what follows, we will choose to resolve the algebra using the cobar construction.

Proof. The proposition follows from the following diagram:

$$\begin{tikzcd}
\mathcal{U}_{\mathsf{P}_\infty}(A) \cong \mathsf{P}_\infty[1] \circ_{\mathsf{P}_\infty} A & \mathsf{P}_\infty[1] \circ_{\mathsf{P}_\infty} R_\infty \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
\mathsf{P}[1] \circ_{\mathsf{P}} R \cong \mathcal{U}(R) & \mathsf{P}[1] \circ_{\mathsf{P}_\infty} R_\infty \arrow[l, "\cong"']
\end{tikzcd} . \tag{4.15}$$

The upper horizontal equivalence follows from [Fre09, Theorem 17.4.B(b)], and the right vertical one one follows from [Fre09, Theorem 17.4.A(a)]. Finally, the bottom horizontal isomorphism follows from the cancellation rule $(\dots) \circ_{\mathsf{P}} (\mathsf{P} \circ_{\mathsf{Q}} (\dots)) \cong (\dots) \circ_{\mathsf{Q}} (\dots)$ from [Fre09, Theorem 7.2.2]. ☐

4.3.2 Poisson case

We now consider the symplectic $u\mathsf{Pois}_n$-algebra $A = (\mathbb{R}[x_i, \xi_j], \{\})$ from before. From Proposition 4.10, it follows that the derived enveloping algebra $\mathcal{U}_{(u\mathsf{Pois}_n)_\infty}(A)$ is quasi-isomorphic to $\mathcal{U}(\Omega_\kappa A^{\mathrm{i}})$, where $\Omega_\kappa A^{\mathrm{i}}$ is the cobar construction described in Section 4.2.

We see that both A and $\Omega_\kappa A^{\mathrm{i}}$ are obtained by considering the relative composition product

$$S(\Sigma^{1-n}\mathfrak{g}) \coloneqq u\mathsf{Pois}_n \circ_{c\mathsf{Lie}_n} \Sigma^{1-n}\mathfrak{g}, \tag{4.16}$$

where $\mathfrak{g}$ is some $c\mathsf{Lie}$-algebra, i.e. a Lie algebra equipped with a central element, and we consider the obvious embedding $c\mathsf{Lie}_n \subset u\mathsf{Pois}_n$. In other words, A and $\Omega_\kappa A^{\mathrm{i}}$ are free as symmetric algebras on a given Lie algebra, with a central element identified with the unit of the symmetric algebra. The differential and the bracket are both extended from the differential and bracket of $\mathfrak{g}$ as (bi)derivations. Recall from Examples 4.8 and 4.9 the descriptions of the enveloping algebras of Lie algebras and commutative algebras.

Proposition 4.11 (Explicit description found in [Fre06, Section 1.1.4])**.** *Let $\mathfrak{g}$ be a $c\mathsf{Lie}$-algebra and $B = S(\Sigma^{1-n}\mathfrak{g})$ the induced $u\mathsf{Pois}_n$-algebra. Then there is an isomorphism of graded modules:*

$$\mathcal{U}_{u\mathsf{Pois}_n}(B) \cong B \otimes \mathcal{U}_{c\mathsf{Lie}_n}(\Sigma^{1-n}\mathfrak{g}). \tag{4.17}$$

The algebra $\mathcal{U}_{c\mathsf{Lie}_n}(\Sigma^{1-n}\mathfrak{g})$ is generated by symbols X_f for $f \in \mathfrak{g}$, with relations:

1. $X_f \cdot g = \{f, g\} + \pm g \cdot X_f$;
2. $X_{fg} = f \cdot X_g + \pm g \cdot X_f$;
3. $X_{\{f,g\}} = X_f X_g - \pm X_g X_f$.

In particular, elements of B and $\mathcal{U}(\Sigma^{1-n}\mathfrak{g})$ do not necessarily commute. The differential is the sum of the differential of B and the differential given by $dX_f \coloneqq X_{df}$, where $df \in B = S(\Sigma^{1-n}\mathfrak{g})$ and we use the relations to get back to $B \otimes \mathcal{U}(\Sigma^{1-n}\mathfrak{g})$.

Proof. The extension of the result from [Fre06, Section 1.1.4] to the unital case is immediate. ☐

Proposition 4.12. *Let $A = (\mathbb{R}[x_i, \xi_j], \{\})$ the symplectic Poisson n-algebra. The derived enveloping algebra $\mathcal{U}_{(u\mathsf{Pois}_n)_\infty}(A)$ is quasi-isomorphic to the underived one $\mathcal{U}_{u\mathsf{Pois}_n}(A)$.*

Proof. We use the cobar resolution $\Omega_\kappa A^{\mathrm{i}}$ and the result of Proposition 4.10 to obtain that this derived enveloping algebra is quasi-isomorphic to $\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\mathrm{i}})$. From the description of Proposition 4.11, as a dg-module, this is isomorphic to

$$\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\mathrm{i}}) \cong \big(\Omega_\kappa A^{\mathrm{i}} \otimes \mathcal{U}_{c\mathsf{Lie}_n}(c\mathsf{Lie}_n(\Sigma^{-1}\bar{S}^c(\Sigma^n V))), d\big), \tag{4.18}$$

where $V = \mathbb{R}\langle x_i, \xi_j\rangle$ is the graded module of generators, the differential d' is explicit, and the product is defined by some explicit relations. By a spectral sequence argument and the fact that $\Omega_\kappa A^{\mathrm{i}} \to A$ is a quasi-isomorphism, we obtain

$$\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\mathrm{i}}) \xrightarrow{\sim} \big(A \otimes \mathcal{U}_{c\mathsf{Lie}_n}(c\mathsf{Lie}_n(\Sigma^{-1}\bar{S}^c(\Sigma^n V))), d\big) \tag{4.19}$$

We now want to explicitly describe this differential. Let X_f be a generator of the universal enveloping algebra, for some $f \in \bar{S}^c(\Sigma^n V)$. Then $dX_f = X_{df} = X_{d_0 f} + X_{d_2 f}$, where d_0 and d_2 were explicitly described in Section 4.2.

Since $d_0 f$ is a multiple of the unit and $X_{\mathbb{1}} = \{\mathbb{1}, -\} = 0$, we obtain that $X_{d_0 f} = 0$. On the other hand,

$$X_{d_2 f} = \frac{1}{2} \sum_{(f)} (X_{f_{(1)}} X_{f_{(2)}} - \pm X_{f_{(2)}} X_{f_{(1)}}), \tag{4.20}$$

where we use the shuffle coproduct of $\bar{S}^c(X)$. Thus we see that the differential stays inside the universal enveloping algebra, and is precisely the one of the bar/cobar resolution of the abelian $c\mathsf{Lie}_n$ algebra $V_+ = V \oplus \mathbb{R}\mathbb{1}$. Thanks to Lemma 4.13 below, we know that $\mathcal{U}_{c\mathsf{Lie}_n}$ preserves quasi-isomorphisms (the unit is freely adjoined and hence is not a boundary), and the universal enveloping algebra of an abelian Lie algebra is just a symmetric algebra, hence:

$$\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\mathsf{i}}) \xrightarrow{\sim} A \otimes \mathcal{U}_{c\mathsf{Lie}_n}(V_+) \cong A \otimes S(\Sigma^{n-1} V). \tag{4.21}$$

This last algebra is simply $\mathcal{U}_{u\mathsf{Pois}_n}(A)$, as claimed. $\square$

We now state the missing lemma in the previous proof. For simplicitly we state it for unshifted Lie algebra; the $c\mathsf{Lie}_n$ case is identical. The functor $\mathcal{U}_{\mathsf{Lie}} = \mathcal{U}$ preserves quasi-isomorphisms (we can filter it by tensor powers and apply Künneth's theorem as we are working over a field). However, in general, the functor $\mathcal{U}_{c\mathsf{Lie}}(-) = \mathcal{U}_{\mathsf{Lie}}(-)/(X_{\mathbb{1}})$ does not preserve quasi-isomorphisms, because of the quotient involved.

Lemma 4.13. *Let $f : \mathfrak{g} \to \mathfrak{h}$ be a quasi-isomorphisms of $c\mathsf{Lie}$-algebras. Suppose that the central elements $\mathbb{1}_{\mathfrak{g}}$, $\mathbb{1}_{\mathfrak{h}}$ are* not *boundaries. Then the induced morphism $\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{g}) \to \mathcal{U}_{c\mathsf{Lie}}(\mathfrak{h})$ is a quasi-isomorphism.*

Proof. The associative algebra $\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{g})$ is given by the presentation:

$$\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - \pm y \otimes x = [x, y], \mathbb{1} = 0), \tag{4.22}$$

where $T(-)$ is the tensor algebra. Let us filter $\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{g})$, $\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{h})$ by tensor powers (increasingly). Then on the first pages of the associated spectral sequences we have the diagram:

$$\begin{tikzcd} S(\mathfrak{g})/(\mathbb{1}) \arrow[r] \arrow[d, "\cong"] & S(\mathfrak{h})/(\mathbb{1}) \arrow[d, "\cong"] \\ S(\mathfrak{g}/\mathbb{1}) \arrow[r] & S(\mathfrak{h}/\mathbb{1}). \end{tikzcd} \tag{4.23}$$

By Künneth's theorem it is thus enough to show that $\mathfrak{g}/\mathbb{1} \to \mathfrak{h}/\mathbb{1}$ is a quasi-isomorphism.

- Surjective on homology: let $\bar{y} \in \mathfrak{h}/\mathbb{1}$ be a cycle for some $y \in \mathfrak{h}$, i.e. $dy \propto \mathbb{1}$. But by hypothesis, $\mathbb{1}$ is not a boundary, hence $dy = 0$ on the nose. Since $f : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism, there exists $x \in \mathfrak{g}$ and $z \in \mathfrak{h}$ such that $dx = 0$ and $f(x) = y + dz$. Then $f(\bar{x}) = \bar{y} + d\bar{z}$ in the quotient.

- Injective on homology: let $\bar{x} \in \mathfrak{g}/\upharpoonleft$ be a cycle (for some $x \in \mathfrak{g}$) such that $f(\bar{x}) \equiv d\bar{y}$ (mod $\upharpoonleft$). In other words, $dx \propto \upharpoonleft_{\mathfrak{g}}$ and $f(x) = dy + \lambda \upharpoonleft_{\mathfrak{h}}$ for some $\lambda \in \Bbbk$. But $\upharpoonleft_{\mathfrak{g}}$ is not a boundary, hence $dx = 0$ on the nose. Thus, in $H_*(\mathfrak{h})$, we get $f_*[x] = \lambda[\upharpoonleft_{\mathfrak{h}}]$, or in other words, $f_*[x - \lambda \upharpoonleft_{\mathfrak{g}}] = 0$. Because f is a quasi-isomorphism, it follows that $x - \lambda \upharpoonleft_{\mathfrak{g}} = dz$ for some z. Hence $\bar{x} = d\bar{z}$ in $\mathfrak{g}/\upharpoonleft_{\mathfrak{g}}$.

Therefore, we obtain an isomorphism on the second page of the associated spectral sequences for $\mathcal{U}_{\mathsf{cLie}}(-)$. Since the filtration is bounded below (the number of tensor powers is clearly nonnegative) and exhaustive, we obtain that the original morphism $\mathcal{U}_{\mathsf{cLie}}(\mathfrak{g}) \to \mathcal{U}_{\mathsf{cLie}}(\mathfrak{h})$ is a quasi-isomorphism. □

4.4 Factorization homology

As another application, let us compute over $\mathbb{R}$ the factorization homology of a parallelized, simply connected, closed manifold of dimension at least 4 with coefficients in A. Until the end of this paper, we let M be such a manifold.

4.4.1 Right $u\mathsf{Com}$-module structure on $\mathsf{G}_P^\vee$

Let P be Poincaré duality model of M. This is a (upper graded) commutative differential-graded algebra equipped with a linear form $\varepsilon : P^n \to \mathbb{R}$ satisfying $\varepsilon \circ d = 0$ and such that the induced pairing $P^k \otimes P^{n-k} \to \mathbb{R}$, $x \otimes y \mapsto \varepsilon(xy)$, is non-degenerate for all k. It it moreover a model of M in the sense of rational homotopy theory. Recall that this means that P is quasi-isomorphic to the CDGA $A^*_{PL}(M)$ of piecewise polynomial forms on M.

Recall from Section 1.3 (or [Idr16]) that we may use our explicit real model $\mathsf{G}_P^\vee$ in order to compute this factorization homology. The object $\mathsf{G}_P^\vee$ is a right $u\mathsf{Pois}_n$-module. As a right Lie_n-module, we have an isomorphism:

$$\mathsf{G}_P^\vee \cong_{\mathsf{Lie}_n} C_*^{CE}(P^{-*} \otimes \Sigma^{n-1}\mathsf{Lie}_n), \tag{4.24}$$

where C_*^{CE} is the Chevalley–Eilenberg chain complex, and $\Sigma^{n-1}\mathsf{Lie}_n = \{\Sigma^{n-1}\mathsf{Lie}_n(k)\}_{k\geq 0}$ is a Lie algebra in the category of right Lie_n-modules.

Moreover, we can also describe the right $u\mathsf{Com}$-module structure. This was not done in [Idr16], but it easily follows from there. Roughly speaking, one needs to use the distributive law $\mathsf{Lie}_n \circ u\mathsf{Com} \to u\mathsf{Com} \circ \mathsf{Lie}_n$ (the bracket is a biderivation with respect to the product, and the unit is a central element for the bracket), then use either $\varepsilon : P^n \to \mathbb{R}$ for the unit of $u\mathsf{Com}$, or the dual of the product of P under Poincaré duality for the product of $u\mathsf{Com}$.

In more detail, given $k \geq 0$, we have an isomorphism of graded modules:

$$\mathsf{G}_P^\vee(k) \cong \bigoplus_{r\geq 0} \Bigl(\bigoplus_{\pi \in \mathrm{Part}_r(k)} (A^{n-*})^{\otimes r} \otimes \mathsf{Lie}_n(\#\pi_1) \otimes \ldots \otimes \mathsf{Lie}_n(\#\pi_r) \Bigr)^{\Sigma_r}, \tag{4.25}$$

where the inner sum runs over all partitions $\pi = \{\pi_1, \ldots, \pi_r\}$ of $\{1, \ldots, k\}$. To describe the right $u\mathsf{Com}$-module structure, we need to say what happens when we insert the two generators, the unit $\mathbb{1}$ and the product μ, at each index $1 \leq i \leq k$, for each summand of the decomposition.

Suppose we are given an element $X = (x_j)_{j=1}^r \otimes \lambda_1 \otimes \ldots \otimes \lambda_r$, where $a_j \in A$ and $\lambda_j \in \mathsf{Lie}_n(\#\pi_j)$. Suppose that $i \in \pi_j$ in the partition. Then:

- $X \circ_i \mathbb{1}$ is obtained by inserting the unit in λ_j; if λ_j has at least one bracket then the result is zero, otherwise the corresponding factor disappears ($c\mathsf{Lie}_n(0) = \mathbb{R}$) and we apply ε to x_j;
- $X \circ_i \mu$ is obtained by inserting the product in λ_j. Using the distributive law for Com and Lie_n, we obtain a sum of products of two elements from Lie_n, splitting π_j in two subsets. We then apply the coproduct $\Delta : P^{n-*} \to (P^{n-*})^{\otimes 2}$, which is Poincaré dual to the product of P, to x_j to obtain a tensor in $A \otimes A$, which we assign to the two subsets of π_j in the corresponding summand.

Example 4.14. Given $x \in P$, we can view $x \otimes \mathrm{id}$ as an element of $\mathsf{G}_P(1) = P^{n-*} \otimes \mathsf{Lie}_n(1)$. Inserting the unit gives the element $(x \otimes \mathrm{id}) \circ_1 \mathbb{1} = \varepsilon(x)$ of $\mathsf{G}_P^\vee(0) = \mathbb{R}$, while inserting the product gives

$$(x \otimes \mathrm{id}) \circ_1 \mu = \Delta(x) \otimes \mathrm{id} \otimes \mathrm{id} \;\in\; \left((P^{n-*})^{\otimes 2} \otimes \mathsf{Lie}_n(1)^{\otimes 2}\right)^{\Sigma_2} \subset \mathsf{G}_P^\vee(2).$$

Inserting the bracket $\lambda \in \mathsf{Lie}_n(2)$ simply gives $x \otimes \lambda \in P^{n-*} \otimes \mathsf{Lie}_n(2) \subset \mathsf{G}_P^\vee(2)$.

4.4.2 Computation

Lemma 4.15. *The underived relative composition product $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$ is given by the* unital *Chevalley–Eilenberg homology of the $c\mathsf{Lie}$-algebra $P^{-*} \otimes \Sigma^{n-1} V$.*

This unital Chevalley–Eilenberg complex is given by

$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A = \left(S^c(P^{-*} \otimes \Sigma^n V), d_{CE}\right) \tag{4.26}$$

Here the shifted Lie bracket of V (and hence the differential d_{CE}) can produce a unit. In this case, we apply $\varepsilon : P \to \mathbb{R}$ to the corresponding factor, and we identify this unit is identified with the (co)unit of $S^c(-)$, i.e. the empty tensor.

Proof. This is almost identical to the case of the universal enveloping algebra of a Lie algebra (with no central element) from [Idr16] (see Section 1.3). The Lie bracket cannot produce a product of two elements of V, so we just need to verify what happens to the unit in the isomorphism of [Idr16, Lemma 5.2]. This is what we did in the previous subsection. □

Proposition 4.16. *The factorization homology*

$$\int_M A \simeq \mathsf{G}_P^\vee \circ^{\mathbb{L}}_{u\mathsf{Pois}_n} A \tag{4.27}$$

of the symplectic Poisson n-algebra A is quasi-isomorphic to $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$.

Proof. As we are working with a derived composition product, we take a resolution of A as a $u\mathsf{Pois}_n$-algebra. For this, we use the cobar construction of the Koszul dual algebra, $\Omega_\kappa A^{\mathrm{i}}$, described in Section 4.2. We then have:

$$\int_M A \simeq \mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{\mathrm{i}}. \tag{4.28}$$

The cobar construction $\Omega_\kappa A^{\mathrm{i}}$ is a quasi-free $u\mathsf{Pois}_n$-algebra on the Koszul dual qA^{i}, with some differential. Therefore, by the cancellation rule for relative products over operad ($X \circ_{\mathsf{P}} (\mathsf{P} \circ Y) = X \circ Y$), we obtain that, as a graded module,

$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{\mathrm{i}} \cong \left(\mathsf{G}_P^\vee \circ qA^{\mathrm{i}}, d_\Omega\right), \tag{4.29}$$

with a differential induced by the differential of the cobar construction.

Remember that we write $\bar{S}^c(-)$ for the cofree commutative coalgebra (without counit), $S^c(-)$ for the same but with a counit, and $L(-)$ for the free Lie algebra. The Koszul dual had the explicit form $qA^{\mathrm{i}} = \Sigma^{1-n}\bar{S}^c(\Sigma^n V)$, where $V = \mathbb{R}\langle x_i, \xi_j\rangle$ is the graded vector space of generators. Using the explicit form of the right module $\mathsf{G}_P^\vee$ found in Section 4.4.1, we then find that:

$$\int_M A \simeq \left(S^c(P^{-*} \otimes L\bar{S}^c(\Sigma^n V)), d_{CE} + d_0 + d_2\right). \tag{4.30}$$

Let us now write down explicit formulas for the three summands of the differential. As there are two instances of the cofree cocommutative algebra appearing, we have to be careful with notations. We will write $\wedge$ for the tensor of the outer coalgebra, and $\vee$ for the tensor of the inner coalgebra. Strictly speaking, we need to consider only elements that are invariant under the symmetric group actions. We will consider all elements, and check that formulas are actually well-defined when passing to invariants. The three parts of the differentials are:

- Given $x_1, \dots, x_k \in P$ and $Y_1, \dots, Y_k \in L\bar{S}^c(\Sigma^n V)$, we have

$$d_{CE}(x_1 Y_1 \wedge \ldots \wedge x_k Y_k) = \sum_{i<j} \pm x_1 Y_1 \wedge \ldots \wedge x_i x_j [Y_i, Y_j] \wedge \ldots \wedge \widehat{x_j Y_j} \wedge \ldots \wedge x_k Y_k. \tag{4.31}$$

- The differential d_2 is defined on the inner $\bar{S}^c(\Sigma^n V)$, extended to a derivation of $L\bar{S}^c(V)$, which is itself extended to the full complex as a coderivation:

$$d_2(v_1 \vee \ldots \vee v_k) = \frac{1}{2} \sum_{i+j=k} \sum_{(\mu,\nu)\in \mathrm{Sh}_{i,j}} \pm [v_{\mu(1)} \vee \ldots \vee v_{\mu(i)}, v_{\nu(1)} \vee \ldots \vee v_{\nu(j)}], \tag{4.32}$$

  where the inner sum is over all (i, j)-shuffles. (This is the differential of the bar/cobar resolution of the abelian Lie algebra $\Sigma^{n-1}V$).

- The differential d_0 is similarly defined on $\bar{S}^c(\Sigma^n V)$ and extended to the full complex by:
$$d_0(X) = \begin{cases} -\mathbb{1}, & \text{if } X = \Sigma^n x_i \vee \Sigma^n \xi_i \text{ for some } i; \\ 0 & \text{otherwise.} \end{cases} \tag{4.33}$$
Note that the unit is appearing here. If the unit is inside a Lie bracket, the result is zero ($\mathbb{1}$ is central). Otherwise, we have to apply $\varepsilon : P \to \mathbb{R}$ to the corresponding element of P in the outer $S^c(-)$, and this factor disappears (it is replaced with a real coefficient).

We can project this complex to
$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A = \big(S^c(P^{-*} \otimes \Sigma^n V), d_{CE}\big), \tag{4.34}$$
i.e. the Chevalley–Eilenberg complex (with constant coefficients) of the $c\mathsf{Lie}$ algebra $P^{-*} \otimes \Sigma^{n-1} V$. The projection from $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{\mathsf{i}}$ is compatible with the differential.

Let i be the number of Lie brackets in an element of the complex, and j be the number of inner tensors ($\vee$). We then observe that d_2 preserves the difference $i - j$, while d_{CE} and d_0 increase them by 1. We can thus filter our first complex by this number to obtain what we will call the "first spectral sequence". The second complex $\big(S^c(P^{-*} \otimes \Sigma^{n-1} V), d_{CE}\big)$ is also filtered, with the unit in filtration 0 and the rest in filtration 1. This yields a "second spectral sequence". The projection is compatible with this filtration, hence we obtain a morphism from the first spectral sequence to the second one.

On the E^0 page of the first spectral sequence, only d_2 remains. Recall that d_2 is exactly the differential of the bar/cobar resolution $\Sigma^{-1} L\bar{S}^c(\Sigma^n V) \xrightarrow{\sim} \Sigma^{n-1} V$ of the abelian Lie algebra $\Sigma^{n-1} V$. Thus on the E^1 page of the spectral sequence, we obtain an isomorphism of graded modules from the first spectral sequence to the second. The differential d_{CE} of the first spectral sequence vanishes, and the differential d_0 precisely correspond to the "unital" Chevalley–Eilenberg of the second spectral sequence. Hence we find that the projection $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{\mathsf{i}} \to \mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$ is a quasi-isomorphism. ☐

And finally we can get:

Proposition 4.17. *The homology of $\int_M A$ is one-dimensional.*

Proof. Thanks to Proposition 4.16, we only need to compute the homology of $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$. Let us use the explicit description from Lemma 4.15 as the unital Chevalley–Eilenberg complex of the $c\mathsf{Lie}$-algebra
$$\mathfrak{g}_{P,V} \coloneqq P^{-*} \otimes \Sigma^{n-1} V. \tag{4.35}$$
There is a pairing $\langle -, - \rangle : \mathfrak{g}_{P,V}^{\otimes 2} \to \mathbb{R}$ given by $xv \otimes x'v' \mapsto \varepsilon_P(xx') \cdot \{v, v'\}$. We have the following isomorphism of chain complexes:
$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A \cong \left(\bigoplus_{i \geq 0} \big((\Sigma \mathfrak{g}_{P,V})^{\otimes i} \big)_{\Sigma_i}, d_{CE} \right) \tag{4.36}$$

where

$$d_{CE}(\alpha_1 \wedge \ldots \wedge \alpha_k) = \sum_{i<j} \pm\langle \alpha_i, \alpha_j \rangle \cdot \alpha_1 \wedge \ldots \widehat{\alpha}_i \ldots \widehat{\alpha}_j \ldots \wedge \alpha_k. \tag{4.37}$$

Let $\{x_1, \ldots, x_r\}$ be a basis of $\Sigma\mathfrak{g}_{P,V}$. The non-degeneracy of the Poincaré pairing of P and the explicit description of the pairing of $\Sigma^{n-1}V$ show that the pairing $\langle -, - \rangle$ is non-degenerate. Hence we can find a dual basis $\{x_1^*, \ldots, x_r^*\}$ with respect to $\langle -, - \rangle$, i.e. $\langle x_i, x_j^* \rangle = \delta_{ij}$. We can then identify $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$ with the "algebraic de Rham complex" of $\mathbb{R}^r$:

$$\Omega^*_{adR}(\mathbb{R}^r) = \big(S(x_1, \ldots, x_r) \otimes \Lambda(dx_1, \ldots, dx_r), d_{dR} = \sum_i \frac{\partial}{\partial x_i} \cdot dx_i\big). \tag{4.38}$$

Note that if all the variables x_i had degree zero then this would isomorphic to the algebra $A_{PL}(\Delta^r) \otimes_{\mathbb{Q}} \mathbb{R}$ of piecewise real polynomial forms on Δ^r. There is an isomorphism (up to a degree shift and reversal) given by:

$$\begin{gathered}\Big(\bigoplus_{i \geq 0} \big((\Sigma\mathfrak{g}_P)^{\otimes i}\big)_{\Sigma_i}, d_{CE}\Big) \xrightarrow{\cong} \Omega^*_{adR}(\mathbb{R}^r) \\ x_{i_1} \wedge \ldots \wedge x_{i_k} \wedge x_{j_1}^* \wedge \ldots \wedge x_{j_l}^* \mapsto x_{i_1} \ldots x_{i_k} \cdot \prod_{\substack{1 \leq j \leq r \\ j \notin \{j_1, \ldots, j_l\}}} dx_j\end{gathered} \tag{4.39}$$

For example if $r = 3$, then the isomorphism sends $x_1 \wedge x_2^*$ to $x_1 dx_1 dx_3$.

The algebraic de Rham complex is a particular example of a Koszul complex and is therefore acyclic There is an explicit homotopy given by $h(dx_i) = x_i$, $h(x_i) = 0$ and extended suitably as a derivation. In particular, a representative of the only homology class is the unit of the algebraic de Rham complex, which under our identification is $\bigwedge_{j=1}^r x_j^*$. ☐

Remark 4.18. From a physical point of view, this result is satisfactory: when one wants to compute expected values of observables, one wants a number. The next best thing to a number is a closed element in a complex whose homology is one-dimensional. We thank T. Willwacher for this perspective.

Remark 4.19. This result appears similar to the computation of Markarian [Mar17] for the Weyl n-algebra $\mathcal{W}_n^h(D)$, which is an algebra over the operad $C_*(\mathsf{FM}_n; \mathbb{R}[[h, h^{-1}]])$, where FM_n is the Fulton–MacPherson operad. We do not know the precise relationship between $A_{n;D}$ and $\mathcal{W}_n^h(D)$, though. Curved Koszul duality was conjectured to apply for this computation by Markarian [MT15]. Moreover, while writing this paper, we learned that Döppenschmitt obtained an analogous result (unpublished) for a twisted version of A, using a "physical" approach based on AKSZ theory / Chern–Simons invariants. Our approach is however different from these two approaches. It is also in some sense more general, as we should be able to compute the factorization homology of M with coefficients in any Koszul $u\mathsf{Pois}_n$-algebra, e.g. an algebra of the type $S(\Sigma^{1-n}\mathfrak{g})$ where $\mathfrak{g}$ is a Koszul $c\mathsf{Lie}$-algebra.

Remark 4.20. Using the results from [CILW18], we hope to be able to compute factorization homology of compact manifolds with boundary with coefficients in A.

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