Configuration spaces of surfaces

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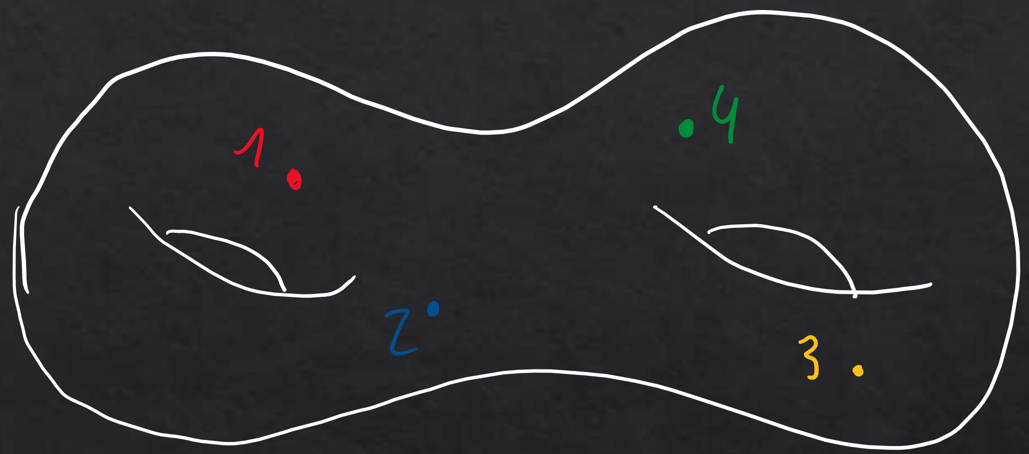
Algebra & Topology Seminar @ University of Copenhagen

May 28th, 2021

Configuration spaces

- Let M be a manifold.
- $\mathrm{Conf}_M(r) :=$
  $\{ (x_1, \dots, x_r) \in M^r \mid \forall i \neq j, x_i \neq x_j \}$
- Classical objects in algebraic topology. Initially used to study braids:
  $$B_r \cong \pi_1(\mathrm{Conf}_{D^2}(r)/\mathfrak{S}_r).$$



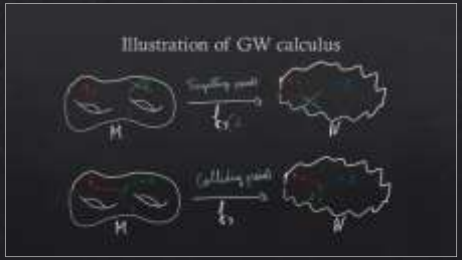
$\in \mathrm{Conf}_{\Sigma_2}(4)$

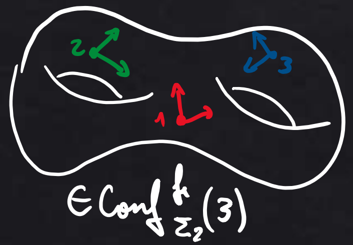
Goodwillie–Weiss manifold calculus

- Want to compute $\mathrm{Emb}(M,N) = \{\, f\colon M \hookrightarrow N \mid f \text{ is an embedding} \,\}$.
- $\mathrm{Emb}(M,N)$ is a subspace of

$$\mathrm{Map}_{\mathfrak{S}}(\mathrm{Conf}_M, \mathrm{Conf}_N) = \prod_{r=0}^{+\infty} \mathrm{Map}_{\mathfrak{S}_r}(\mathrm{Conf}_M(r), \mathrm{Conf}_N(r))\,.$$

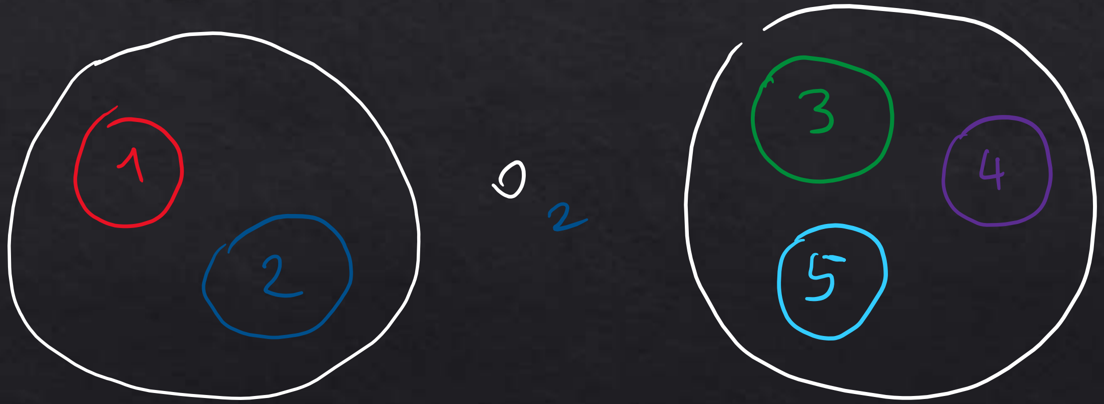
- GW calculus "approximates" $\mathrm{Emb}(M,N)$ by a more easily computable subspace:
  - Forgetting in the source maps to forgetting in the target *up to homotopy*;
  - Proximity in the source maps to proximity in the target *up to homotopy*.
- Restrict to $\mathrm{Map}_{\mathfrak{S}}(\mathrm{Conf}_M^{\mathrm{fr}}, \mathrm{Conf}_N^{m-\mathrm{fr}})$ to have a correct homotopy type.
- If $\dim N - \dim M \geq 3 \Rightarrow$ recover the homotopy type of $\mathrm{Emb}(M,N)$.





Operadic structure

- We want to clarify what "compatible up to homotopy" means
- $\Rightarrow$ we need operads!
- Let $\mathrm{D}_M^{\mathrm{fr}}(r) \coloneqq \mathrm{Emb}(\coprod_{i=1}^{r} \mathbb{D}^m, M)$ and $\mathrm{D}_n^{\mathrm{fr}}(\mathrm{r}) \coloneqq \mathrm{Emb}(\coprod_{i=1}^{r} \mathbb{D}^m, \mathbb{D}^m)$
- $\mathrm{D}_m^{\mathrm{fr}} \coloneqq \{\mathrm{D}_m^{\mathrm{fr}}(r)\}_{r\geq 0}$ is the (framed) **little disks operad**:



Extra structure:
$\mathrm{D}_m^{\mathrm{fr}}(r) \times \mathrm{D}_m^{\mathrm{fr}}(s) \to \mathrm{D}_m^{\mathrm{fr}}(\mathrm{r}+\mathrm{s}-1)$

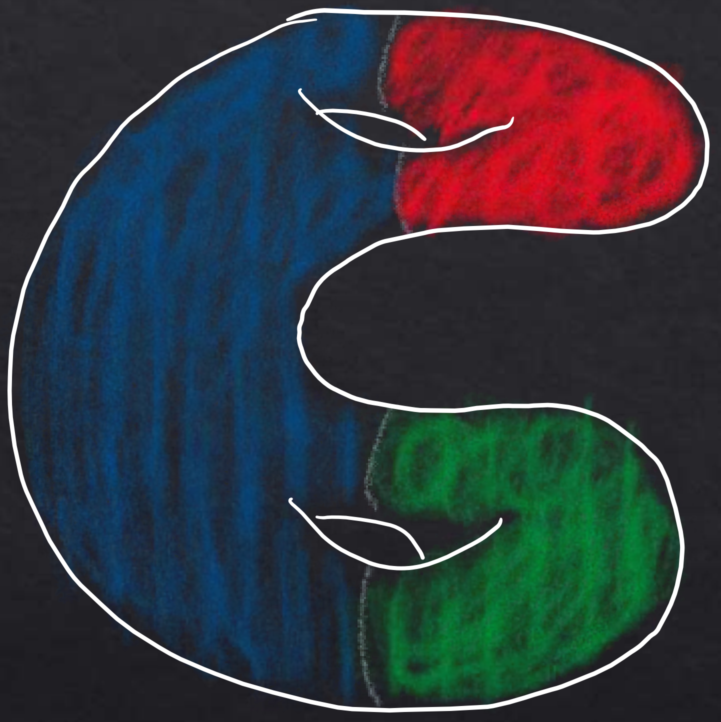
- $\mathrm{D}_M^{\mathrm{fr}} \coloneqq \{\mathrm{D}_M^{\mathrm{fr}}(r)\}_{r\geq 0}$ is a **right module** over $\mathrm{D}_m^{\mathrm{fr}}$ via $\mathrm{D}_M^{\mathrm{fr}}(r) \times \mathrm{D}_m^{\mathrm{fr}}(s) \to \mathrm{D}_M^{\mathrm{fr}}(\mathrm{r}+\mathrm{s}-1)$

Operads & GW calculus

- Any embedding $f : M \hookrightarrow N$ induces a **morphism** $\mathrm{D}_M^{\mathrm{fr}} \to \mathrm{D}_N^{\mathrm{fr}}$.
- **Theorem** [Goodwillie-Weiss, Arone-Turchin, Turchin, Boavida-Weiss, Sinha…].
  If $\dim N - \dim M \geq 3$, then
  $$\mathrm{Emb}(M, N) \simeq \mathbb{R}\mathrm{Map}_{\mathrm{D}_m^{\mathrm{fr}}}\left(\mathrm{D}_M^{\mathrm{fr}}, \mathrm{D}_N^{m-\mathrm{fr}}\right).$$
- However, computing $\mathrm{D}_M^{\mathrm{fr}}(r)$ is difficult. For example,
  $$M \simeq M' \not\Rightarrow \mathrm{Conf}_M(r) \simeq \mathrm{Conf}_{M'}(r).$$

Approach: cut the surface

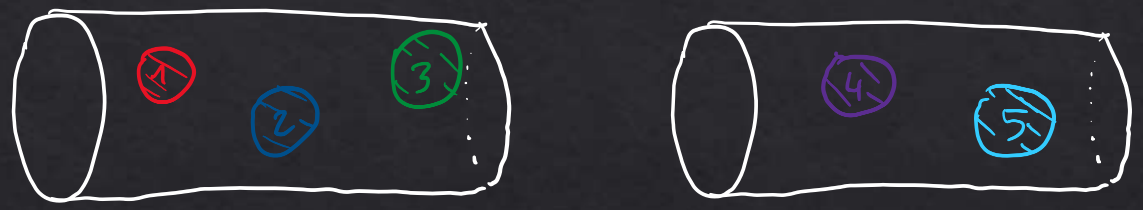
- Take $\Sigma_g = (S^1 \times S^1) \# \ \dots \ \# \ (S^1 \times S^1)$;
- Cut! $\Sigma_g = (S^2 \setminus (D^2)^{\sqcup 2g}) \cup (S^1 \times \mathbb{R})^{\sqcup g}$.
- Each part is $D^2 \setminus \boldsymbol{k}$ for some k.
- We have a fiber bundle:
  $\mathrm{Conf}^{\mathrm{fr}}_{M\setminus *}(r) \to \mathrm{Conf}^{\mathrm{fr}}_{M}(r+1) \to \mathrm{Fr}_M$
  $\to$ computation by **induction.**
- We just need to know $\mathrm{Conf}^{\mathrm{fr}}_{D^2\setminus \boldsymbol{k}}(r)$.



Σ_2

Hochschild complex

- The collection $\mathrm{D}^{\mathrm{fr}}_{N\times\mathbb{R}} = \{\mathrm{D}^{\mathrm{fr}}_{N\times\mathbb{R}}(r)\}_{r\geq 0}$ is a monoid up to homotopy:



$$D^{fr}_{N\times\mathbb{R}}(3)\times D^{fr}_{N\times\mathbb{R}}(2)\longrightarrow D^{fr}_{N\times\mathbb{R}}(5)$$

- If $\partial M = N$, then $\mathrm{D}^{\mathrm{fr}}_{M}$ is a left module over $\mathrm{D}^{\mathrm{fr}}_{N\times\mathbb{R}}$.
- We have:

$$\mathrm{D}^{\mathrm{fr}}_{M\cup_{N\times\mathbb{R}}M'} \simeq \mathrm{D}^{\mathrm{fr}}_{M} \otimes^{\mathbb{L}}_{\mathrm{D}^{\mathrm{fr}}_{N\times\mathbb{R}}} \mathrm{D}^{\mathrm{fr}}_{M'}.$$

- Upshot: $\mathrm{D}^{\mathrm{fr}}_{\Sigma_g}$ is an “iterated Hochschild complex” of the $\left(\mathrm{D}^{\mathrm{fr}}_{S^1\times\mathbb{R}}\right)^{\otimes g}$-bimodule $\mathrm{D}^{\mathrm{fr}}_{S^2\setminus 2g}$.

Rational homotopy theory

- The whole homotopy type is too complex.
- We focus on **characteristic zero**.
- **Definition**: $f : X \to Y$ is a **rational equivalence** if
$$\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{Q} : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$
is an isomorphism.
- **Theorem** [Sullivan]: There is an equivalence $\Omega^* \dashv \langle - \rangle$ between:
  - Simply connected finite-type spaces, up to rational equivalence;
  - Simply connected finite-type commutative differential-graded algebras, up to quasi-isomorphism.
- Upshot: we want to find a **model** of $\Omega^*(\mathrm{D}_M^{\mathrm{fr}})$ with its action of $\Omega^*(\mathrm{D}_m^{\mathrm{fr}})$.

Formality

- **Theorem** [Kontsevich, Tamarkin, Lambrechts–Volić, …….] The operad D_2 is **formal**, i.e.,
$$H^*(\mathrm{D}_2;\mathbb{Q}) \simeq \Omega^*(\mathrm{D}_2).$$
- $\Rightarrow$ we know everything about $\mathrm{D}_2^{\mathbb{Q}}$ from [Arnold, Cohen]:
$$H^*(\mathrm{D}_2(r);\mathbb{Q}) = \frac{S(\omega_{ij})_{1\le i\ne j\le r}}{(\omega_{ij}^2 = \omega_{ji} - \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0)}, \qquad \deg \omega_{ij} = 1.$$
- Many important consequences, e.g., deformation quantization, Deligne conjecture…

Formality: two approaches

- Kontsevich's approach:
  - Replace the 3T relation by "internal vertices";
  - Prove combinatorically that we have a resolution of $H^*(\mathrm{D_n})$;
  - Use integrals to connect with $\Omega^*(\mathrm{D}_n)$.
- [Giansiracusa–Salvatore] formality of $\mathrm{D}_2^{\mathrm{fr}}$.

- Tamarkin's approach:
  - Find a simpler groupoid $\mathrm{PaB} \simeq \pi\mathrm{D}_2$;
  - Find a (Koszul) resolution of $H^*(\mathrm{D}_2)$, the Drinfeld–Kohno Lie algebra;
  - Connect the two with a Drinfeld associator.
- [Ševera] Formality of $\mathrm{D}_2^{\mathrm{fr}}$.
- **Theorem** [CIW] Cyclic formality of $\mathrm{D}_2^{\mathrm{fr}}$. Proof inspired by Ševera's.

The result

Theorem [CIW]. We have a small, explicit model $\mathrm{G}^{\mathrm{fr}}_{\Sigma_g}$ of $\mathrm{D}^{\mathrm{fr}}_{\Sigma_g}$, in arity r:

- Generators: ω_{ij} for $1 \leq i \neq j \leq r$; $\alpha_{1,i}, \dots, \alpha_{g,i}, \beta_{1,i}, \dots, \beta_{g,i}$ for $1 \leq i \leq r$; θ_i for $1 \leq i \leq r$.
- Relations:
  - Same as before: $\omega_{ij}^2 = \omega_{ji} - \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0;$
  - $\alpha_{k,i}\beta_{k,i} = \alpha_{l,i}\beta_{l,i}$ (volume form of Σ_g) and 0 otherwise;
  - Symmetry: $\alpha_{k,i}\omega_{ij} = \alpha_{k,j}\omega_{ij}, \beta_{k,i}\omega_{ij} = \beta_{k,j}\omega_{ij}, \theta_i\omega_{ij} = \theta_j\omega_{ij}$.
- Differential: $d\omega_{ij} = \Delta_{ij}$ and $d\theta_i = (2-2g) \cdot \mathrm{vol}_i$.
- Proof: $\mathrm{G}^{\mathrm{fr}}_{\Sigma_g} \xleftarrow{\text{Combin.}} \mathrm{Graphs}^{\mathrm{fr}}_{\Sigma_g} \xrightarrow{\mathrm{K}} \mathrm{IterHoch}\left(H^*\left(\mathrm{D}^{\mathrm{fr}}_{S^2 \setminus 2g}\right); H^*\left(\mathrm{D}^{\mathrm{fr}}_{S^1 \times \mathbb{R}}\right)\right) \xleftarrow{\mathrm{T}} \Omega^*\left(\mathrm{D}^{\mathrm{fr}}_{\Sigma_g}\right).$

Thank you for your attention!

These slides, the paper: https://idrissi.eu