Formalité opéradique et homotopie des espaces de configuration

Operadic Formality and Homotopy of Configuration Spaces

Najib Idrissi Kaïtouni

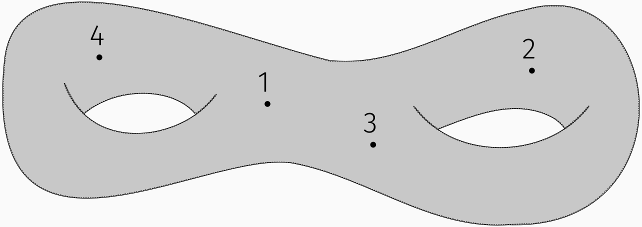
Soutenance de thèse – 17 novembre 2017

Introduction

Overall Goal

Study configuration spaces of manifolds:

$$\mathrm{Conf}_k(M) := \{(x_1, \ldots, x_k) \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}$$



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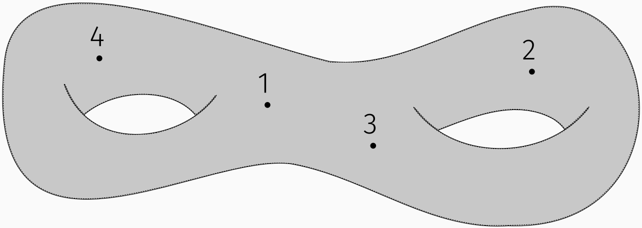

Idea

Use "formality of the little disks operads" = results for $\mathrm{Conf}_k(\mathbb{R}^n)$.

Plan

Little Disks Operads

Swiss-Cheese Operad and Drinfeld Center

The Lambrechts–Stanley Model of Configuration Spaces

Configuration Spaces of Manifolds with Boundary

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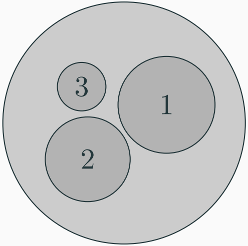

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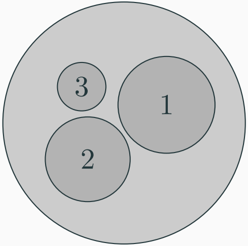
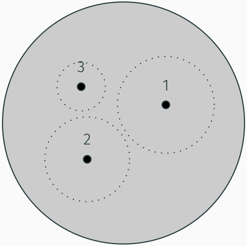


$\in \mathsf{D}_2(3) \mapsto$



$\in \mathrm{Conf}_3(\mathbb{R}^2)$

New structure: insertion

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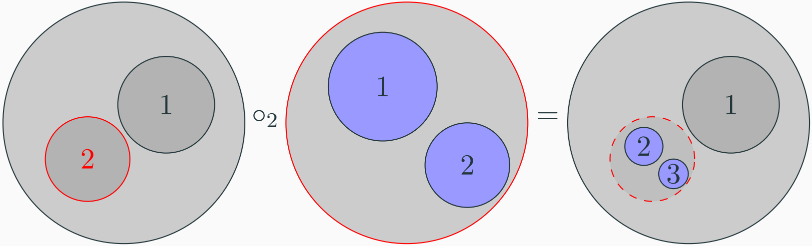

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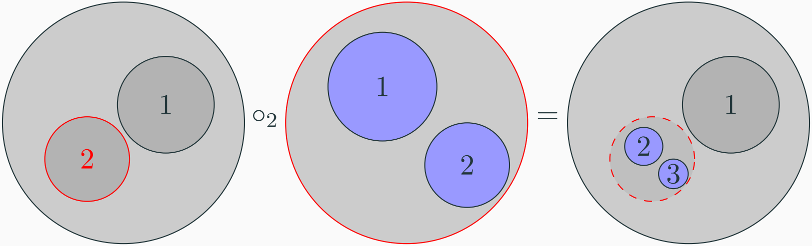
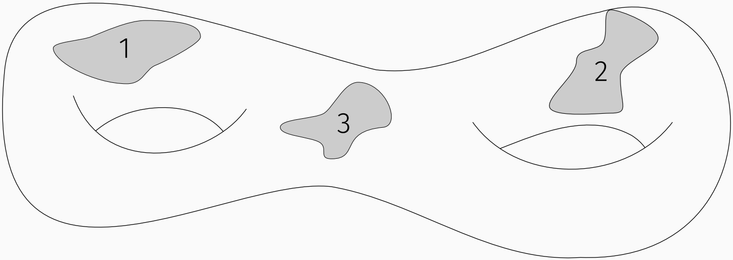


$\implies$ operad structure, cannot be seen on $\mathrm{Conf}_\bullet(\mathbb{R}^n)$

If M is "framed":

$$\mathsf{D}_M(k) := \mathrm{Emb}^{\mathrm{fr}}\left(\mathbb{D}^n \sqcup \cdots \sqcup \mathbb{D}^n, M\right)$$



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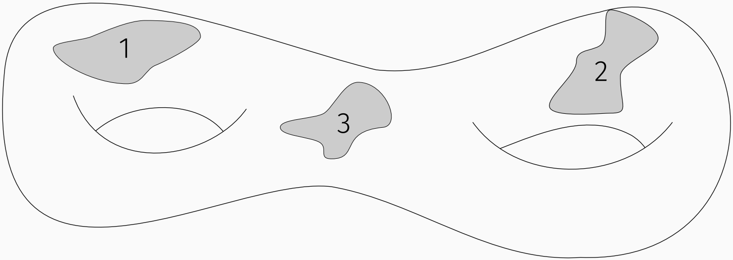

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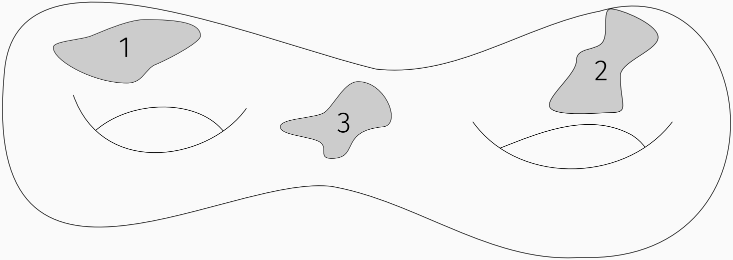

$\implies \mathsf{D}_M = \{\mathsf{D}_M(k)\}_{k\geq 0}$ is a "right module" over D_n

Idea

Use this extra structure to study $\mathrm{Conf}_k(M)$.

Algebras over D_n in the topological world

An algebra over D_n is a space on which D_n “acts”:

$$\mathsf{D}_n(k) \times X^k \to X$$

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$$\mathsf{D}_n(k) \times X^k \to X$$

Theorem (Boardmann–Vogt, May 1972)

- If $X = \Omega^n Y$, then D_n acts on X;
- if D_n acts on X (+ grouplike), then $X \simeq \Omega^n Y$ for some Y.

Algebraic world

Operad $\mathsf{D}_n \mapsto$ homology $H_*(\mathsf{D}_n)$ (⚠ lose info) -

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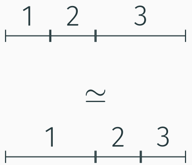
Associativity for $n \geq 1$:



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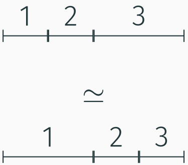
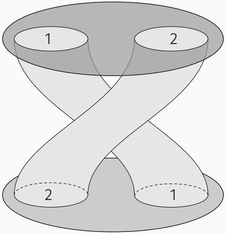


Commutativity for $n \geq 2$:



Swiss-Cheese Operad and Drinfeld Center

Operad $\mathsf{D}_n \mapsto$ fundamental groupoid $\pi\mathsf{D}_n$

Proposition

For $n \in \{1, 2\}$, no loss of information: $\mathsf{D}_n \xrightarrow{\sim} \mathrm{B}(\pi\mathsf{D}_n)$.

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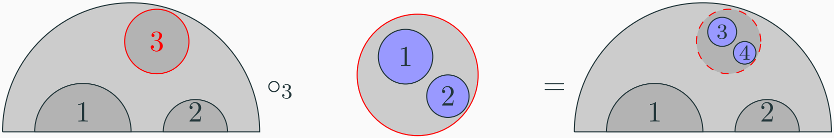

$\mathsf{SC}^{\mathfrak{o}}(2,1)$ $\quad$ $\mathsf{SC}^{\mathfrak{c}}(0,2) = \mathsf{D}_2(2)$ $\quad$ $\mathsf{SC}^{\mathfrak{o}}(2,2)$

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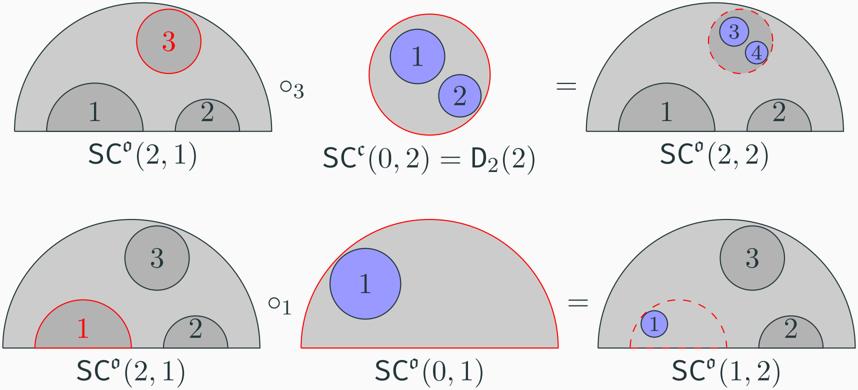

Homology vs fundamental groupoid of SC

Theorem (Voronov 1999, Hoefel 2009)

An algebra over $H_*(\mathsf{SC})$ is a triplet (A, B, f) where:

- $(A, \cdot)$ is an associative algebra;
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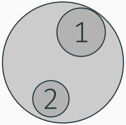
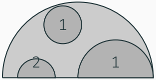
- $(\mathsf{M}, \otimes)$ is a monoidal category;
- $(\mathsf{N}, \otimes, \tau)$ is a braided monoidal category;
- $F : \mathsf{N} \to \mathcal{Z}(\mathsf{M})$ is a braided functor to the "Drinfeld center"

Recap

	Topological	Algebraical $H_*(-)$	Categorical $\pi(-)$
D_1		associative $(A, \cdot)$	monoidal $(\mathsf{M}, \otimes)$
D_2		Gerstenhaber $(B, \wedge, [,])$	braided $(\mathsf{N}, \otimes, \tau)$
SC		$(B, \wedge, [,]) \xrightarrow{f} Z(A, \cdot)$	$(\mathsf{N}, \otimes, \tau) \xrightarrow{F} \mathcal{Z}(\mathsf{M}, \otimes)$

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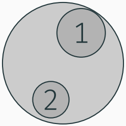
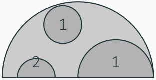
D_2		Gerstenhaber $(B, \wedge, [,])$	braided $(\mathsf{N}, \otimes, \tau)$
SC		$(B, \wedge, [,]) \xrightarrow{f} Z(A, \cdot)$	$(\mathsf{N}, \otimes, \tau) \xrightarrow{F} \mathcal{Z}(\mathsf{M}, \otimes)$

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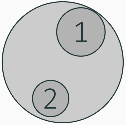
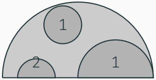
D_2		Gerstenhaber $(B, \wedge, [,])$	braided $(\mathsf{N}, \otimes, \tau)$
SC		$(B, \wedge, [,]) \xrightarrow{f} Z(A, \cdot)$	$(\mathsf{N}, \otimes, \tau) \xrightarrow{F} \mathcal{Z}(\mathsf{M}, \otimes)$

Remark

I also build a model $\mathsf{PaP}\widehat{\mathsf{CD}}_+^{\phi} =$ "$\mathsf{PaP} \rtimes_\phi \widehat{\mathsf{CD}}_+$" out of a Drinfeld associator ϕ, following Tamarkin's proof of the formality of D_2.

The Lambrechts–Stanley Model of Configuration Spaces

Models

We are interested in rational/real models

$$A \simeq \Omega^*(M) \text{ “forms on } M\text{”}$$ (e.g. de Rham, piecewise polynomial...)

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We're looking for a CDGA $\simeq \Omega^*(\mathrm{Conf}_k(M))$ built from A

Formality of $\mathrm{Conf}_k(\mathbb{R}^n)$

$\mathrm{Conf}_k(\mathbb{R}^n)$ is a formal space, i.e. [Kontsevich]:

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \simeq \Omega^*(\mathrm{Conf}_k(\mathbb{R}^n))$$

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Theorem (Arnold 1969, Cohen 1976)

- $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) = S(\omega_{ij})_{1 \le i \ne j \le k}/I$
- $\deg \omega_{ij} = n - 1$
- $I = \left(\omega_{ji} = \pm\omega_{ij},\ \omega_{ij}^2 = 0,\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0\right)$

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Theorem (Lambrechts–Stanley 2004)

Any simply connected manifold has such a model

$$\begin{tikzcd} \Omega^*(M) \arrow[dr, "\int_M"'] & \cdot \arrow[l, "\sim"'] \arrow[r, "\sim"] & \exists A \arrow[dl, "\exists\varepsilon"] \\ & \Bbbk & \end{tikzcd}$$

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Remark

By a result of Longoni–Salvatore (2005), $\exists$ non simply-connected $L \simeq L'$ but $\mathrm{Conf}_k(L) \not\simeq \mathrm{Conf}_k(L')$

The Lambrechts–Stanley model

$\mathsf{G}_A(k)$ conjectured model of $\mathrm{Conf}_k(M) = M^{\times k} \setminus \bigcup_{i \neq j} \Delta_{ij}$

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  - $p_i^*(a) \cdot \omega_{ij} = p_j^*(a) \cdot \omega_{ij}$. $(p_i^*(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1)$

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Theorem (Lambrechts–Stanley 2008)

$$\dim_{\mathbb{Q}} H^i(\mathrm{Conf}_k(M)) = \dim_{\mathbb{Q}} H^i(\mathsf{G}_A(k))$$

First part of the theorem

$\mathsf{G}_A(k)$ was known to be a rational model of $\mathrm{Conf}_k(M)$ in a few cases:

- M smooth projective complex variety [Kriz];
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Corollary

The real homotopy type of $\mathrm{Conf}_k(M)$ only depends on the real homotopy type of M:

$$M \simeq_{\mathbb{R}} N \implies \mathrm{Conf}_k(M) \simeq_{\mathbb{R}} \mathrm{Conf}_k(N).$$

Operads

Ideas & Goals

Adapt the construction for D_n & keep track of the D_n-action whenever it exists

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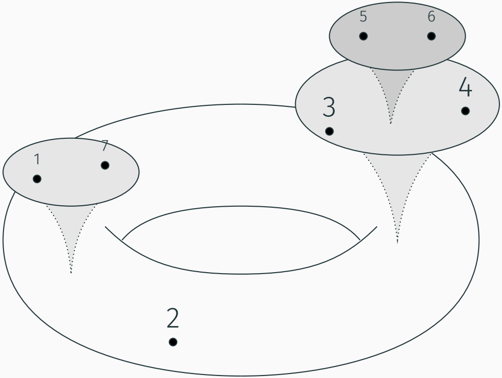
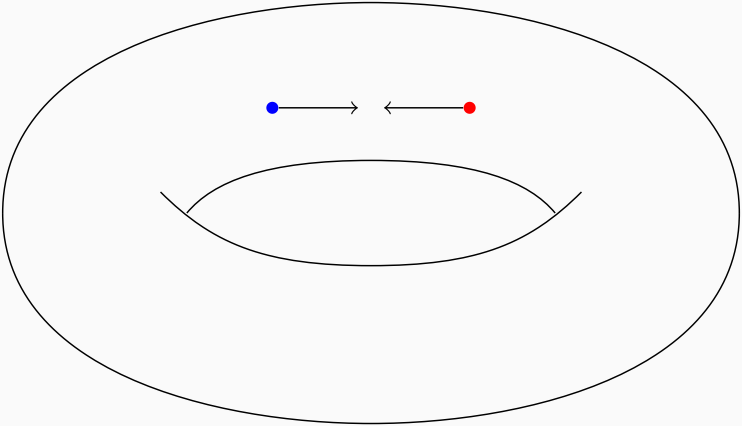
Fulton–MacPherson compactification $\mathrm{Conf}_k(M) \xrightarrow{\sim} \mathsf{FM}_M(k)$



Understanding FM_M (#1)



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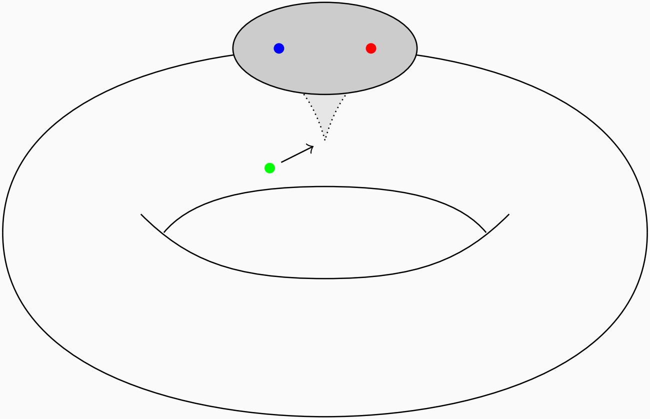
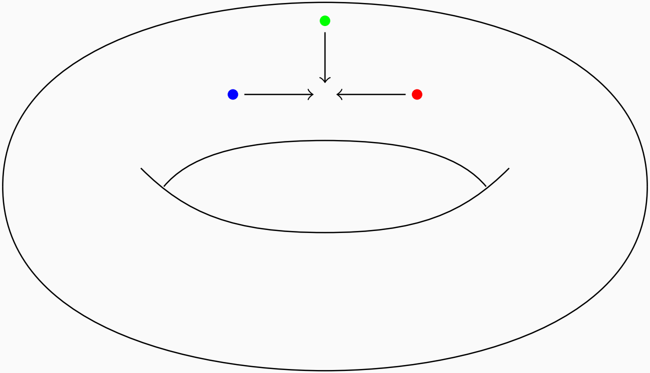
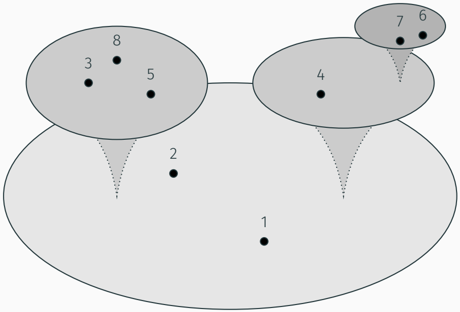


Understanding FM_M (#3)

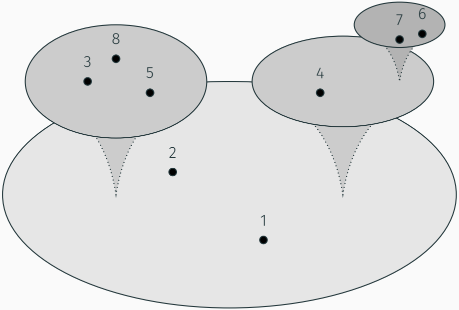


Can also compactify $\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\sim} \mathtt{FM}_n(k)$



Compactifying $\mathrm{Conf}_k(\mathbb{R}^n)$

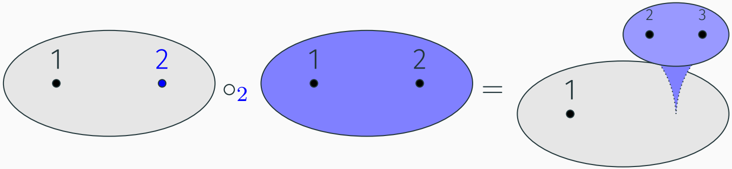
Can also compactify $\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\simeq} \mathrm{Conf}_k(\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_+^*) \xhookrightarrow{\sim} \mathsf{FM}_n(k)$



(+ normalization to deal with $\mathbb{R}^n$ being noncompact)

Operads

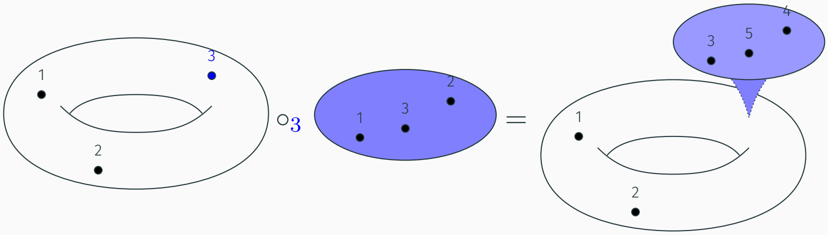
$\mathtt{FM}_n = \{\mathtt{FM}_n(k)\}_{k\geq 0}$ is an operad $\simeq \mathtt{D}_n$



$$\mathtt{FM}_n(k) \times \mathtt{FM}_n(l) \xrightarrow{\circ_i} \mathtt{FM}_n(k+l-1), \quad 1 \leq i \leq k$$

Modules over operads

M framed $\implies$ $\mathsf{FM}_M = \{\mathsf{FM}_M(k)\}_{k \geq 0}$ is a right FM_n-module $\simeq \mathsf{D}_M$



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Cohomology of FM_n and coaction on G_A

$H^*(\mathsf{FM}_n)$ inherits a Hopf cooperad structure

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One can rewrite:

$$\mathsf{G}_A(k) = (A^{\otimes k} \otimes H^*(\mathtt{FM}_n(k))/\text{relations}, d)$$

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Proposition

$\chi(M) = 0 \implies \mathsf{G}_A = \{\mathsf{G}_A(k)\}_{k \geq 0}$ is a Hopf right $H^*(\mathsf{FM}_n)$-comodule

Motivation

We are looking for something to put here:

$$\mathsf{G}_A(k) \xleftarrow{\sim} ? \xrightarrow{\sim} \Omega^*(\mathsf{FM}_M(k))$$

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If true, then hopefully it fits in a diagram like this:

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\mathsf{G}_A \arrow[d, phantom, "\circlearrowleft"] & ? \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & \Omega^*(\mathsf{FM}_M) \arrow[d, phantom, "\circlearrowleft"] \\
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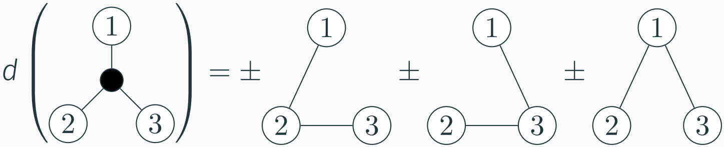
Already known: formality of the little disks operads

Kontsevich’s graph complexes

[Kontsevich] Hopf cooperad $\mathtt{Graphs}_n = \{\mathtt{Graphs}_n(k)\}_{k\geq 0}$



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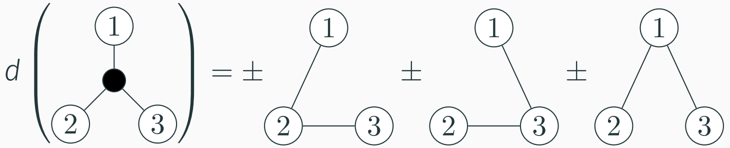

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

$$H^*(\mathsf{FM}_n;\mathbb{R}) \xleftarrow{\sim} \mathtt{Graphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$$

$$\omega_{ij} \longleftarrow\!\shortmid \; \textcircled{i}\!-\!\!-\!\textcircled{j} \; \longmapsto \text{explicit representatives}$$

$$0 \longleftarrow\!\shortmid \; \bullet \; \longmapsto \text{“explicit” integrals}$$

Complete version of the theorem

Idea

Build $\mathtt{Graphs}_R^{z_\varepsilon}$ from $\mathtt{Graphs}_n$ similar to how $\mathtt{G}_A$ is built from $H^*(\mathtt{FM}_n)$

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Theorem (Complete version)

M: closed, simply connected, smooth manifold with $\dim \geq 4$

$$
\begin{array}{ccccc}
\mathtt{G}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R^{z_\varepsilon} & \overset{\sim}{\dashrightarrow} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M) \\
\circlearrowleft^{\dagger} & & \circlearrowleft^{\dagger} & & \circlearrowleft^{\ddagger} \\
H^*(\mathtt{FM}_n) & \xleftarrow{\sim} & \mathtt{Graphs}_n & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)
\end{array}
$$

† When $\chi(M) = 0$

‡ When M is framed

$A \xleftarrow{\sim} R \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(M)$

Configuration Spaces of Manifolds with Boundary

Poincaré–Lefschetz duality models

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In this case, $A := B/\ker\theta$ is a model of M, and $\theta : A \xrightarrow{\cong} K^\vee[-n]$

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If M is simply connected, ∂M is simply connected, and $\dim M \geq 7$, then $(M, \partial M)$ admits a PLD model.

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Remark

Also true if M admits a “surjective pretty model”, cf. theorems of Cordova Bulens and Cordova Bulens–Lambrechts–Stanley.

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Idea of proof

Combine:

- Techniques of Lambrechts–Stanley to compute homology of spaces of the type $M^k \setminus \bigcup_{i \neq j} \Delta_{ij}$;
- Techniques of Cordova Bulens–L–S to compute homology of $M = N \setminus X$ where N is a closed manifold and $X \subset N$ is a sub-polyhedron.

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$\implies$ must define a "perturbed model" $\tilde{\mathsf{G}}_A(k)$

Proposition

Isomorphism of dg-modules $\mathsf{G}_A(k) \cong \tilde{\mathsf{G}}_A(k)$.

Swiss-Cheese & graphs

M looks like $\mathbb{H}^n$ (locally) $\implies$ Swiss-Cheese operad

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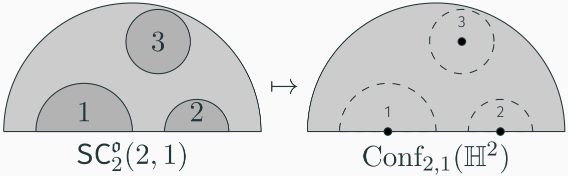

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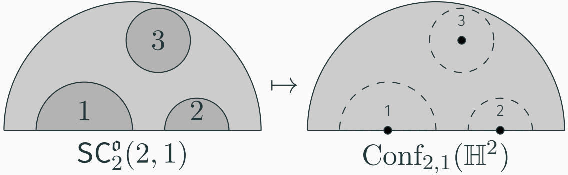
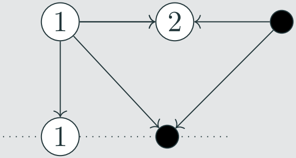


Theorem (Willwacher 2015)

Model $\mathtt{SGraphs}_n$ for $\mathsf{SFM}_n = \overline{\mathrm{Conf}_{\bullet,\bullet}(\mathbb{H}^n)} \simeq \mathsf{SC}_n$:



$\in \mathtt{SGraphs}_n(1,2)$

Theorem for manifolds with boundary

Using similar techniques:

Theorem

For M a smooth, compact manifold of dimension at least ≥ 7, M and ∂M simply connected:

$$
\begin{tikzcd}
\tilde{\mathsf{G}}_A & \mathtt{Graphs}_R^{z_\varepsilon} \arrow[l, "\sim"'] \arrow[r, dashed, "\sim"] & \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(\varnothing, -)) \\
\circlearrowleft & \circlearrowleft & \circlearrowleft^{\dagger} \\
H^*(\mathsf{FM}_n) & \mathtt{Graphs}_n \arrow[l, "\sim"'] \arrow[r, "\sim"] & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)
\end{tikzcd}
$$

Moreover: model $\mathtt{SGraphs}_{R,R_\partial}^{c_M, z_\varphi^S}(k,l)$ of $\mathsf{SFM}_M(k,l)$, compatible with the (co)action of $\mathtt{SGraphs}_n$ / SFM_n

Fin de la présentation