Curved Koszul duality for algebras over unital operads

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May 2019 @ Higher Algebras in Topology, MPIM Bonn

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Tool of choice: Koszul duality.

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Quadratic algebras – Koszul duals

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$\implies$ Koszul complex $K_A \coloneqq (A \otimes A^{¡}, d_\kappa)$; A is Koszul if K_A is acyclic

Example

$F(E)$ and $S(E)$ are both Koszul.

Bar/cobar adjunction:

$$\Omega : \{\text{coaug.coalgebras}\} \leftrightarrows \{\text{aug.algebras}\} : B$$

where $BA = (F^c(\Sigma\bar{A}), d_B)$ and $\Omega C = (F(\Sigma^{-1}\bar{C}), d_\Omega)$.

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Much smaller resolution!

Examples

$A = F(E) \implies \Omega A^{¡} = A = F(E)$ versus $\Omega BA = F(F^c(F(E)))$

$A = S(E) \implies \Omega A^{¡} = F(\Lambda^c(E))$ versus $\Omega BA = F(F^c(S(E)))$.

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Goal: do this for more general types of algebras (e.g. Poisson algebras).

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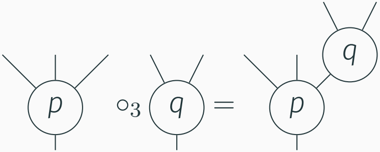

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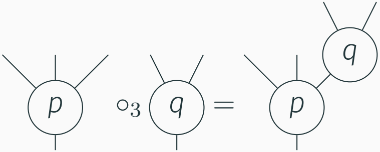

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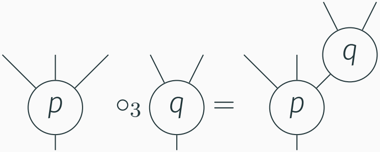

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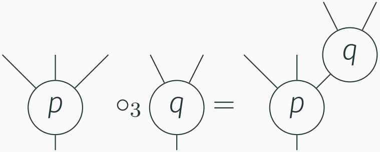

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$\mathsf{e}_n := H_*(\mathsf{E}_n)$, $n \geq 2$ = Poisson n-algebras.

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Formally similar definitions: Koszul dual cooperad $\mathsf{P}^{¡} = \mathrm{FOp}^c(\Sigma E, \Sigma^2 R)$ and its linear dual $\mathsf{P}^! = \mathrm{FOp}(E^*)/(R^\perp)$.

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$\mathtt{Ass}^! = \mathtt{Ass}$; $\mathtt{Com}^! = \mathtt{Lie}$, $\mathtt{Lie}^! = \mathtt{Com}$; $\mathsf{e}_n^! = \mathsf{e}_n\{-n\}$.

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$\mathtt{Ass}_\infty = A_\infty$-algebras, $\mathtt{Com}_\infty = C_\infty$-algebras, $\mathtt{Lie}_\infty = L_\infty$-algebras...

KD for monogenic algebras

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$\mathsf{P} = \mathsf{Ass}$: recovers the classical Koszul duality of associative algebras.

Extension to operads with quadratic-linear-constant relations:

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Bar/cobar extends to the curved setting

Theorem (Hirsh–Millès '12)

If $qu\mathrm{P}$ is Koszul, then $u\mathrm{P}_\infty := \Omega(u\mathrm{P}^{\text{i}}) \xrightarrow{\sim} u\mathrm{P}$: resolution of $u\mathrm{P}$

$u\mathsf{P} = \mathrm{FOp}(E \oplus \text{⫯})/(R)$: "unital version" of a quadratic operad P

Curved KD for algebras over binary unital operads

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Generalization of bar/cobar adjunction:

$$\Omega_\kappa : \{\text{curved } \mathsf{P}^{¡}\text{-coalgebras}\} \leftrightarrows \{\text{semi.aug. } u\mathsf{P}\text{-algebras}\} : B_\kappa$$

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Theorem (I. ’18)

If qA is Koszul then $\Omega_\kappa A^{\text{¡}} \xrightarrow{\sim} A$ is a resolution.

M: framed n-manifold, A: $u\mathsf{E}_n$-algebra

Application 1: factorization homology

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Goal

Compute $\int_M A = \operatorname{hocolim}_{(D^n)^{\sqcup k} \hookrightarrow M} A^{\otimes k}$.

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Theorem (Francis 2015)

$\int_M A \simeq \mathsf{E}_M \circ^{\mathbb{L}}_{u\mathsf{E}_n} A = \operatorname{hocoeq}\big(\mathsf{E}_M \circ u\mathsf{E}_n \circ A \rightrightarrows \mathsf{E}_M \circ A\big)$, where:

$$u\mathsf{E}_n(k) = \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k\times}, \mathbb{R}^n); \quad \mathsf{E}_M(k) = \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k\times}, M).$$

($\exists$ version for unframed manifolds.)

If we work over $\mathbb{R}$ and we just want chains:

$$C_*(\textstyle\int_M A) \simeq C_*(\mathsf{E}_M) \circ^{\mathbb{L}}_{C_*(u\mathsf{E}_n)} C_*(A).$$

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M closed, simply connected, smooth, $\dim M \geq 4 \implies$ explicit model of $C_*(\mathsf{E}_M)$ as a right $C_*(u\mathsf{E}_n)$-module: Lambrechts–Stanley model LS_M.

Chains of factorization homology over $\mathbb{R}$

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$\implies$ we need to resolve A as a $u\mathsf{e}_n$-algebra.

WEYL ALGEBRA $\mathrm{Poly}(T^*\mathbb{R}^d[1-n])$

$A = \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) = S(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)$

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(If we had applied curved KD at the level of operads instead:
$\Omega_\kappa B_\kappa A \supset (\underbrace{SL}_{\text{cobar}} \underbrace{S^c L^c}_{\text{bar}} \underbrace{S(x_i, \xi_j)}_{A}, d)$, + resolution of the unit...)

Computation of $\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$

We can also compute

$$\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) \simeq \mathsf{LS}_M \circ_{u\mathsf{e}_n} (SLS^c(\bar{x}_i, \bar{\xi}_j), d)$$

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A bit of homological algebra + explicit description of LS_M:

Theorem (I. '18, see also Markarian '17, Döppenschmitt '18)

$\int_M \mathrm{Poly}(T^*\mathbb{R}^d[1-n]) \simeq C_*^{CE}(\Omega^{n-*}(M) \otimes \mathbb{R}\langle 1, x_i, \xi_j \rangle) \simeq \mathbb{R}.$

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Intuition: quantum observable with values in $A \rightsquigarrow$ "expectation" lives in $\int_M A$, should be a number.

Operad P + P-algebra $A \implies$ notion of A-modules

Application 2: derived enveloping algebra

Operad $\mathtt{P}$ + $\mathtt{P}$-algebra $A \implies$ notion of A-modules

Examples

$\mathtt{P} = \mathtt{Ass} \to (A, A)$ bimodules; $\mathtt{P} = \mathtt{Com} \to A$-modules; $\mathtt{P} = \mathtt{Lie} \to$ representations of the Lie algebra.

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For $A = \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$, the derived enveloping algebra $U^{\mathbb{L}}_{u\mathsf{e}_n}(A)$ is q.iso to the underived one.

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Proposition

For $A = \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$, the derived enveloping algebra $U^{\mathbb{L}}_{\mathtt{ue}_n}(A)$ is q.iso to the underived one.

Explicit description: if $A = S(\Sigma^{1-n}\mathfrak{g})$, then

$$U_{\mathtt{ue}_n}(A) = A \otimes U_{\mathtt{cLie}_n}(\Sigma^{1-n}\mathfrak{g}),$$

with $X_f \in U_{\mathtt{cLie}_n}(\Sigma^{1-n}\mathfrak{g})$ for $f \in \mathfrak{g}$ satisfying $X_{\mathbb{1}} = 0$, $X_{fg} = fX_g \pm gX_f$, $X_f g = \{f, g\} \pm gX_f$, $X_{\{f,g\}} = [X_f, X_g]$, $dX_f = X_{df}$.

Thank you for your attention!

These slides: `https://idrissi.eu`