

CONFIGURATION SPACES AND OPERADS

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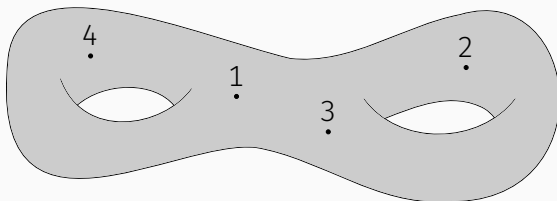
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CONFIGURATION SPACES

M : n -manifold

$$\text{Conf}_r(M) := \{(x_1, \dots, x_r) \in M^r \mid \forall i \neq j, x_i \neq x_j\}$$



- Braid groups
- Loop spaces
- (name dropping) • Moduli spaces of curves
- Particles in movement [physics]
- Motion planning [robotics]

OPEN QUESTION

Question

Does the homotopy type of M determine the homotopy type of $\mathrm{Conf}_r(M)$? How to compute homotopy invariants of $\mathrm{Conf}_r(M)$?

Non-compact manifolds

False: $\mathrm{Conf}_2(\mathbb{R}) \not\sim \mathrm{Conf}_2(\{0\})$ even though $\mathbb{R} \sim \{0\}$.

Closed manifolds

Longoni–Salvatore (2005): counter-example (lens spaces)... but not simply connected.

Simply connected closed manifolds

Homotopy invariance is still open.

We can also localize: $M \simeq_{\mathbb{Q}} N \implies \mathrm{Conf}_r(M) \simeq_{\mathbb{Q}} \mathrm{Conf}_r(N)$?

CONFIGURATIONS IN A EUCLIDEAN SPACES

Presentation of $H^*(\text{Conf}_r(\mathbb{R}^n))$ [Arnold, Cohen]

- Generators: ω_{ij} of degree $n - 1$ (for $1 \leq i \neq j \leq r$)
- Relations:

$$\omega_{ij}^2 = \omega_{ji} - (-1)^n \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$$

Theorem (Arnold 1969)

Formality: $H^*(\text{Conf}_r(\mathbb{C})) \sim_{\mathbb{C}} \Omega_{\text{dR}}^*(\text{Conf}_r(\mathbb{C})), \omega_{ij} \mapsto d \log(z_i - z_j).$

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

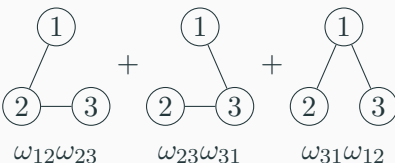
$H^*(\text{Conf}_r(\mathbb{R}^n)) \sim_{\mathbb{R}} \Omega_{\text{dR}}^*(\text{Conf}_r(\mathbb{R}^n))$ for all $r \geq 0$ and $n \geq 2$.

Corollary

The cohomology of $\text{Conf}_r(\mathbb{R}^n)$ determines its rational homotopy type.

KONTSEVICH'S GRAPH COMPLEXES

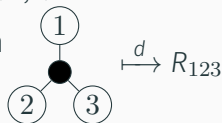
Arnold relations: $R_{123} =$



$$\omega_{12}\omega_{23} + \omega_{23}\omega_{31} + \omega_{31}\omega_{12}$$

$$\implies H^*(\text{Conf}_r(\mathbb{R}^n)) = \mathbb{R}\langle \text{graphs with } r \text{ vertices} \rangle / (R_{ijk})$$

\rightsquigarrow add “internal” vertices and a differential which contracts edges incident to these new vertices:



Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 1)

We get a quasi-free CDGA $\mathbf{Graphs}_n(r)$ and a quasi-isomorphism $\mathbf{Graphs}_n(r) \xrightarrow{\sim} H^*(\text{Conf}_r(\mathbb{R}^n))$.

KONTSEVICH'S INTEGRALS

The relations R_{ijk} are only satisfied up to homotopy in $\Omega^*(\text{Conf}_r(\mathbb{R}^n))$.

How to find representatives to get $\mathbf{Graphs}_n(r) \xrightarrow{\sim} \Omega^*(\text{Conf}_r(\mathbb{R}^n))$?

Let $\varphi \in \Omega^{n-1}(\text{Conf}_2(\mathbb{R}^n))$ be the volume form.

For $\Gamma \in \mathbf{Graphs}_n(r)$ with i internal vertices:

$$\omega(\Gamma) := \int_{\text{Conf}_{r+i}(\mathbb{R}^n) \rightarrow \text{Conf}_r(\mathbb{R}^n)} \bigwedge_{(ij) \in E_\Gamma} \varphi_{ij}.$$

Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 2)

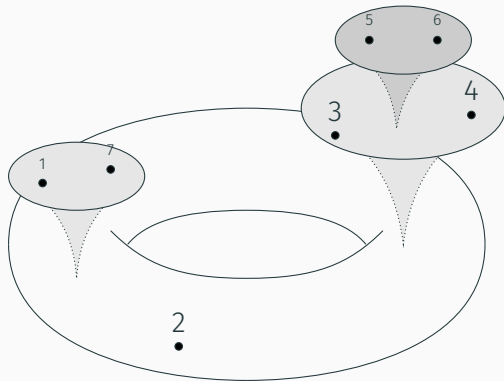
We get a quasi-isomorphism $\omega : \mathbf{Graphs}_n(r) \xrightarrow{\sim} \Omega(\text{Conf}_r(\mathbb{R}^n))$.

△ I'm cheating! We have to compactify $\text{Conf}_r(\mathbb{R}^n)$ to make sure \int converges and to apply the Stokes formula correctly.

COMPACTIFICATION

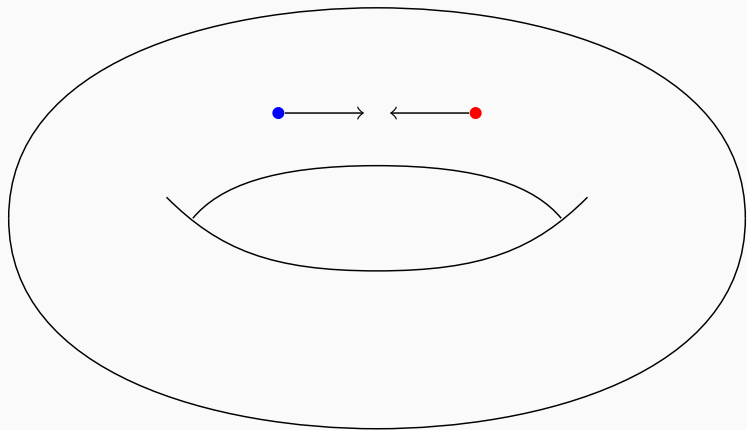
Problem: $\text{Conf}_r(\mathbb{R}^n)$ is not compact.

Fulton–MacPherson compactification $\text{Conf}_r(M) \xrightarrow{\sim} \text{FM}_M(r)$

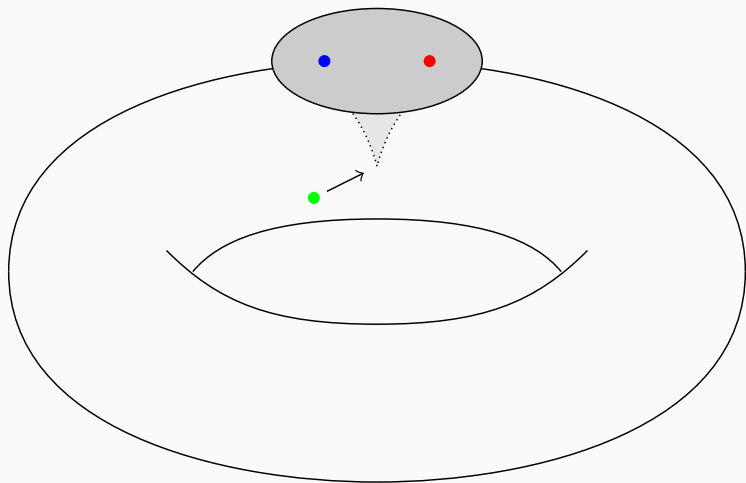


M closed manifold \implies semi-algebraic stratified manifold $\dim = nr$

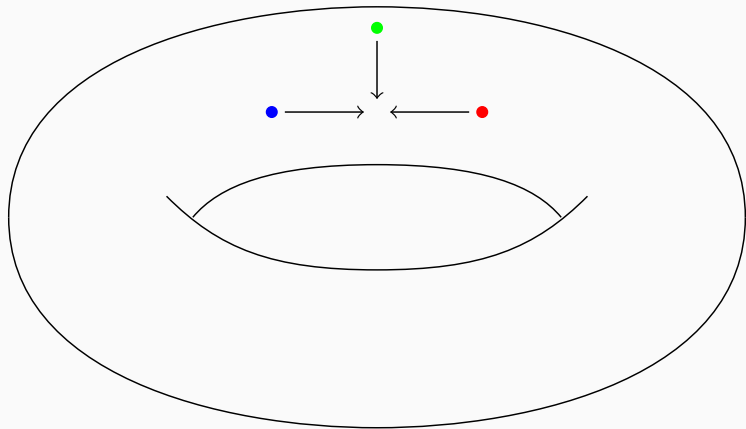
ANIMATION #1



ANIMATION #2



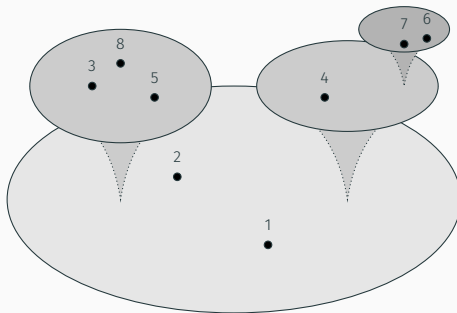
ANIMATION #3



COMPACTIFICATION OF $\text{Conf}_r(\mathbb{R}^n)$

We have to “normalize” $\text{Conf}_r(\mathbb{R}^n)$ to mitigate the non-compactness of \mathbb{R}^n :

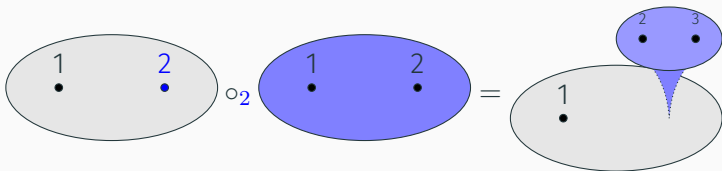
$$\text{Conf}_r(\mathbb{R}^n) \xrightarrow{\sim} \text{Conf}_r(\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_{>0}) \xrightarrow{\sim} \text{FM}_n(r)$$



\implies semi-algebraic stratified manifold $\dim = nr - n - 1$

OPERAD

We see a new structure on \mathbf{FM}_n : an **operad**! We can “insert” an infinitesimal configuration in another one:



$$\mathbf{FM}_n(k) \times \mathbf{FM}_n(l) \xrightarrow{\circ_i} \mathbf{FM}_n(k+l-1), \quad 1 \leq i \leq k$$

Remark

Weakly equivalent to the “little disks operad”.

COMPLETE THEOREM

Functoriality $\implies H^*(\mathbf{FM}_n) = H^*(\mathrm{Conf}_\bullet(\mathbb{R}^n))$ and $\Omega^*(\mathbf{FM}_n)$ are Hopf cooperads; \mathbf{Graphs}_n is one too, and:

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

The operad \mathbf{FM}_n is formal over \mathbb{R} :

$$\Omega^*(\mathbf{FM}_n) \xleftarrow[\omega]{\sim} \mathbf{Graphs}_n \xrightarrow{\sim} H^*(\mathbf{FM}_n).$$

Formality has important applications, e.g. Deligne conjecture, deformation quantization of Poisson manifolds, etc.

(Note: $H_*(\mathbf{FM}_n)$ governs Poisson n -algebras for $n \geq 2$.)

THE LAMBRECHTS-STANLEY MODEL

M : oriented closed manifold

$A \sim \Omega(M)$: Poincaré duality model of M

$\mathbf{G}_A(r)$: (conjectural) model of $\text{Conf}_r(M) = M^{\times r} \setminus \bigcup_{i \neq j} \Delta_{ij}$
 $\Delta_{ij} \rightarrow \{x_i = x_j\}$

- “Generators”: $A^{\otimes r}$ and the ω_{ij} from $\text{Conf}_r(\mathbb{R}^n)$
- Arnold relations + symmetry
- $d\omega_{ij}$ kills the dual of $[\Delta_{ij}]$.

Examples:

- $\mathbf{G}_A(0) = \mathbb{R}$ is a model of $\text{Conf}_0(M) = \{\emptyset\}$ ✓
- $\mathbf{G}_A(1) = A$ is a model of $\text{Conf}_1(M) = M$ ✓
- $\mathbf{G}_A(2) \sim A^{\otimes 2}/(\Delta_A)$ should be a model of $\text{Conf}_2(M) = M^2 \setminus \Delta$?
- $r \geq 3$: more complicated.

BRIEF HISTORY OF G_A

1969 [Arnold, Cohen] $H^*(\text{Conf}_r(\mathbb{R}^n)) = G_{H^*(\mathbb{R}^n)}(r)$

1978 [Cohen–Taylor] spectral sequence $E^2 = G_{H^*(M)}(k) \Rightarrow H^*(\text{Conf}_k(M))$

1994 For smooth projective complex manifolds (\implies Kähler):

- [Kříž] $G_{H^*(M)}(r)$ is a model of $\text{Conf}_r(M)$;
- [Totaro] the Cohen–Taylor SS collapses.

2004 [Lambrechts–Stanley] model for $r = 2$ if $\pi_{\leq 2}(M) = 0$

2004 [Félix–Thomas, Berceanu–Markl–Papadima] relation with Bendersky–Gitler spectral sequence

2008 [Lambrechts–Stanley] $H^i(G_A(r)) \cong_{\Sigma_r\text{-Vect}} H^i(\text{Conf}_r(M))$

2015 [Cordova Bulens] model for $r = 2$ if $\dim M = 2m$

FIRST PART OF THE THEOREM

By generalizing the proof of Kontsevich & Lambrechts–Volić:

Theorem (I.)

Let M be a closed simply connected smooth manifold and A be any Poincaré duality model of M . Then $\mathbf{G}_A(r)$ is a real model of $\mathrm{Conf}_r(M)$.

Corollary (cf. Campos–Willwacher)

$M \sim_{\mathbb{R}} N \implies \mathrm{Conf}_r(M) \sim_{\mathbb{R}} \mathrm{Conf}_r(N)$ for all r .

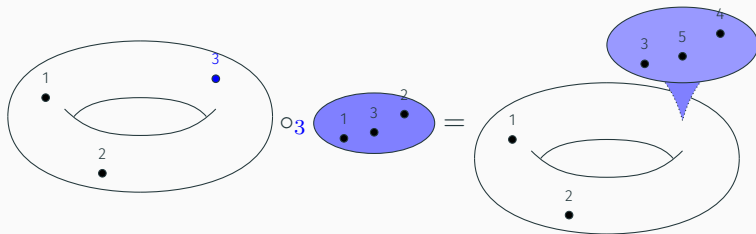
We can “compute everything” over \mathbb{R} for $\mathrm{Conf}_r(M)$.

Remark

$\dim M \leq 3$: only spheres (Poincaré conjecture) and we know that \mathbf{G}_A is a model anyway, but adapting the proof is problematic!

MODULES OVER OPERADS

M parallelized $\implies \mathbf{FM}_M = \{\mathbf{FM}_M(r)\}_{r \geq 0}$ is a right \mathbf{FM}_n -module :



We can rewrite:

$$\mathbf{G}_A(r) = (A^{\otimes r} \otimes H^*(\mathbf{FM}_n(r)) / \text{relations}, d)$$

A bit of abstract nonsense:

Proposition

$\chi(M) = 0 \implies \mathbf{G}_A = \{\mathbf{G}_A(r)\}_{r \geq 0}$ is a Hopf right $H^*(\mathbf{FM}_n)$ -comodule.

COMPLETE VERSION OF THE THEOREM

Theorem (I. 2018)

M : closed simply connected smooth manifold, $\dim M \geq 4$

$$\begin{array}{ccccc} \mathbf{G}_A & \xleftarrow{\sim} & \mathbf{Graphs}_R & \dashrightarrow^{\sim} & \Omega_{PA}^*(\mathbf{FM}_M) \\ \circlearrowleft^\dagger & & \circlearrowleft^\dagger & & \circlearrowleft^\ddagger \\ H^*(\mathbf{FM}_n) & \xleftarrow{\sim} & \mathbf{Graphs}_n & \xrightarrow{\sim} & \Omega_{PA}^*(\mathbf{FM}_n) \end{array}$$

† if $\chi(M) = 0$

‡ if M is parallelized.

$$A \xleftarrow{\sim} R \xrightarrow{\sim} \Omega_{PA}^*(M)$$

Conclusion

Not only do we have a model of each $\mathbf{Conf}_r(M)$, but also of their richer structure if we look at them all at once.

APPLICATION 1: EMBEDDING SPACES

Space of embeddings: $\text{Emb}(M, N) = \{f : M \hookrightarrow N\}$.

Goodwillie–Weiss manifold calculus [Arone, Boavida, Turchin, Weiss...]:
for parallelized manifolds of codimension ≥ 3 ,

$$\text{Emb}(M, N) \simeq \text{Mor}_{\text{Conf}_{\bullet}(\mathbb{R}^n)}^h(\text{Conf}_{\bullet}(M), \text{Conf}_{\bullet}(N)).$$

LS model is small and explicit \implies hope: computations are tractable

Remark

Requires to compare $\text{Mor}_{\text{Conf}_{\bullet}(\mathbb{R}^n)}^h(\text{Conf}_{\bullet}(M), \text{Conf}_{\bullet}(N))^{\mathbb{R}}$ with
 $\text{Mor}_{\text{Conf}_{\bullet}(\mathbb{R}^n)^{\mathbb{R}}}^h(\text{Conf}_{\bullet}(M)^{\mathbb{R}}, \text{Conf}_{\bullet}(N)^{\mathbb{R}})$

APPLICATION 2: FACTORIZATION HOMOLOGY

Factorization homology = homology where \otimes replaces \oplus + homotopy commutative coefficients.

For an E_n -algebra \mathcal{A} ,

$$\int_M \mathcal{A} = \operatorname{hocolim}_{(D^n)^{\sqcup r} \hookrightarrow M} \mathcal{A}^{\otimes r}.$$

Alternate description: $\int_M \mathcal{A} \sim \operatorname{Conf}_\bullet(M) \otimes_{\operatorname{Conf}_\bullet(\mathbb{R}^n)}^h \mathcal{A}$ [Francis].

Theorem (I. 2018, see also Markarian 2017, Döppenschmidt 2018)

M closed simply connected smooth manifold ($\dim \geq 4$),

$$\mathcal{A} := \mathcal{O}_{\text{poly}}(T^*\mathbb{R}^d[1-n]) \implies \int_M \mathcal{A} \sim_{\mathbb{R}} \mathbb{R}.$$

GENERALIZATION 1: MANIFOLDS WITH BOUNDARY

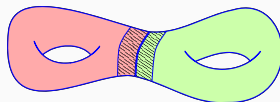
Theorem (Campos–I.–Lambrechts–Willwacher 2018)

For manifolds with boundary: homotopy invariance of $\text{Conf}_r(-)$, generalization of the Lambrechts–Stanley model (and more); under good conditions, including $\dim M \geq \dots$

Remark

Poincaré duality models \rightsquigarrow Poincaré–Lefschetz duality models.

Allows to compute Conf_r by “induction”:



GENERALIZATION 2: ORIENTED MANIFOLDS

M : oriented manifold \rightsquigarrow framed configuration space

$$\mathrm{Conf}_r^{\mathrm{fr}}(M) := \{(x \in \mathrm{Conf}_r(M), B_1, \dots, B_r) \mid B_j: \text{orth. basis of } T_{x_j}M\}.$$

Natural action of the framed little disks operad on $\{\mathrm{Conf}_{\bullet}^{\mathrm{fr}}(M)\}$.

Theorem (Campos–Ducoulombier–I.–Willwacher 2018)

Real model of this module based on graph complexes.

First step towards embedding spaces of non-parallelized manifolds. (Not enough: need partially framed configurations for the larger manifold N .)

WIP: COMPLEMENTS OF SUBMANIFOLDS

Goal: $\text{Conf}(N \setminus M)$ where $\dim N - \dim M \geq 2$.

Motivation: work of Ayala, Francis, Rozenblyum, Tanaka

Knot complement \rightsquigarrow colored Jones polynomial.

There exists an operad \mathbf{VSC}_{mn} which models the local situation $\mathbb{R}^n \setminus \mathbb{R}^m$:



Theorem (I. 2018)

The operad \mathbf{VSC}_{mn} is formal over \mathbb{R} for $n - m \geq 2$.

THANK YOU FOR YOUR ATTENTION!

THESE SLIDES: <https://idrissi.eu>