Chapter 2: Probability (Part 2)

Example 2.7. ***How accurate is rapid testing for COVID-19?***
From the iHealth® website
https://ihealthlabs.com/pages/ihealth-covid-19-antigen-rapid-test-details:
"Based on the results of a clinical study where the iHealth® COVID-19 Antigen Rapid Test was compared to an FDA authorized molecular SARS-CoV-2 test, iHealth® COVID-19 Antigen Rapid Test correctly identified 94.3% of positive specimens and 98.1% of negative specimens."

Suppose you take the iHealth® rapid test.

(1) *What is the probability of a positive test result?*
(2) *What is the probability of having COVID-19 if you get a positive test result?*
(3) *What is the probability of not having COVID-19 if you get a negative test result?*

What information were we given?
First, let's define our events of interest:

- D = *event one has disease (COVID-19)*
- D^c = *event one does not have disease*
- T^+ = *event one tests positive for disease*
- T^- = *event one tests negative for disease*

Translate given information into mathematical notation:

- *Test correctly gives a positive result 94.3% of the time:*
  Sensitivity = $\mathbb{P}(T^+|D) = 0.943$
  (aka PPA = positive percent agreement)

- *Test correctly gives a negative result 98.1% of the time:*
  Specificity = $\mathbb{P}(T^-|D^c) = 0.981$
  (aka NPA = negative percent agreement)

- *What are all the possible scenarios of test results? Make 2-way table of test results (rows) vs. disease status (columns)*

	D	D^c	*Total*
T^+	*sensitivity*	*false positive*	$\neq 1$
T^-	*false negative*	*specificity*	$\neq 1$
Total	*1*	*1*	

Given: $\mathbb{P}(T^+|D) = 0.943, \ \ \mathbb{P}(T^-|D^c) = 0.981$

Solutions to questions

(1) *What is the probability of a positive test result?*

- *Sample space with* D *vs.* D^c *and blob for* T^c
- *Law of Total Probability*
- *Need to know prevalence,* $\mathbb{P}(D)$
  Assume $\mathbb{P}(D) = 0.000838 \ \ (0.0838\%)$
- *Tree Diagram for calculating total probability*
- *Answer: 0.01977431*

Given: $\mathbb{P}(T^+|D) = 0.943$, $\mathbb{P}(T^-|D^c) = 0.981$

(2) *What is the probability of having COVID-19 if you get a positive test result?*

- $PPV = \mathbb{P}(D|T^+)$
- *conditional probability*
- *Assume* $\mathbb{P}(D) = 0.000838 \ (0.0838\%)$ *(Multnomah Co on 10/12/2022)*
- *Bayes' rule*
- *Answer: 0.03996265*

$$\frac{0.943 * 0.000838}{0.943 * 0.000838 + (1 - 0.981) * (1 - 0.000838)} = 0.03996265$$

- *If* $\mathbb{P}(D) = 0.01 \ (1\%)$, *then answer = 0.3339235*
- *If* $\mathbb{P}(D) = 0.1 \ (10\%)$, *then answer = 0.8464991*

Given: $\mathbb{P}(T^+|D) = 0.943, \;\; \mathbb{P}(T^-|D^c) = 0.981$

(3) *What is the probability of not having COVID-19 if you get a negative test result?*

- $NPV = \mathbb{P}(D^c|T^-)$
- *conditional probability*
- *Assume* $\mathbb{P}(D) = 0.000838 \;\; (0.0838\%)$
- *Bayes' rule*
- *Answer: 0.9999513*
$$\frac{0.981 * 0.999162}{0.981 * 0.999162 + (1 - 0.943) * 0.000838} = 0.9999513$$
- *If* $\mathbb{P}(D) = 0.01 \;\; (1\%)$, *then answer = 0.9994134*
- *If* $\mathbb{P}(D) = 0.1 \;\; (10\%)$, *then answer = 0.9935854*

Bayes' Theorem (Section 2.2.5)

In the previous examples we derived the formula for Bayes' Theorem.

Theorem 2.8 (Bayes' Theorem). *If the sample space S can be split into disjoint events $A_1, A_2, ..., A_k$ that make up all possible outcomes in S, and if $\mathbb{P}(A_i) > 0$ for $i = 1, \ldots, k$ and $\mathbb{P}(B) > 0$, then*

$$\mathbb{P}(A_1|B) = \frac{\mathbb{P}(B|A_1) \cdot \mathbb{P}(A_1)}{\mathbb{P}(B|A_1) \cdot \mathbb{P}(A_1) + \mathbb{P}(B|A_2) \cdot \mathbb{P}(A_2) + ... + \mathbb{P}(B|A_k) \cdot \mathbb{P}(A_k)}$$

Special case of Bayes' Theorem for sample space being split into A and A^c:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^c) \cdot \mathbb{P}(A^c)}$$

Theorem 2.9 (Law of Total Probability). *(denominator of Bayes' Theorem)*

If the sample space S can be split into disjoint events $A_1, A_2, ..., A_k$ that make up all possible outcomes in S, and if $\mathbb{P}(A_i) > 0$ for $i = 1, \ldots, k$ and $\mathbb{P}(B) > 0$, then

$$\begin{aligned} \mathbb{P}(B) &= \mathbb{P}(B \textit{ and } A_1) + \mathbb{P}(B \textit{ and } A_2) + \ldots + \mathbb{P}(B \textit{ and } A_k) \\ &= \mathbb{P}(B|A_1) \cdot \mathbb{P}(A_1) + \mathbb{P}(B|A_2) \cdot \mathbb{P}(A_2) + \ldots + \mathbb{P}(B|A_k) \cdot \mathbb{P}(A_k) \end{aligned}$$

Special case of Law of Total Probability for sample space being split into A and A^c:

$$\begin{aligned} \mathbb{P}(B) &= \mathbb{P}(B \text{ and } A) + \mathbb{P}(B \text{ and } A^C) \\ &= \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^C) \cdot \mathbb{P}(A^C) \end{aligned}$$

Example 2.10. ***Antibody test for COVID-19***
According to the FDA's EUA Authorized Serology Test Performance website, the Abbott AdviseDx SARS-CoV-2 IgG II (Alinity) antibody test for COVID-19 has sensitivity 98.1% and PPV 98.4% when the prevalence is 20%.

Question: What is the specificity of the antibody test?

What information were we given?
First, let's define our events of interest:

- A *= event one has antibodies for COVID-19*
- A^c *= event one does not have antibodies*
- T^+ *= event one tests positive for antibodies*
- T^- *= event one tests negative for antibodies*

Translate given information into mathematical notation:

- *Sensitivity is 98.1%:*
  Sensitivity = $\mathbb{P}(T^+|A) = 0.981$

- *PPV is 98.4%:*
  PPV = $\mathbb{P}(A|T^+) = 0.984$

- *Prevalence is 20%:*
  $\mathbb{P}(A) = 0.20$

Solution: *Want specificity =* $\mathbb{P}(T^-|A^c) = 1 - \mathbb{P}(T^+|A^c)$

$$\mathbb{P}(A|T^+) = \frac{\mathbb{P}(T^+|A)\cdot\mathbb{P}(A)}{\mathbb{P}(T^+|A)\cdot\mathbb{P}(A) + \mathbb{P}(T^+|A^c)\cdot\mathbb{P}(A^c)}$$
$$0.984 = \frac{0.981\cdot 0.20}{0.981\cdot 0.20 + \mathbb{P}(T^+|A^c)\cdot(1-0.20)}$$
$$0.984 = \frac{0.1962}{0.1962 + \mathbb{P}(T^+|A^c)\cdot 0.80}$$

$$0.1962 = 0.984\cdot(0.1962 + 0.80\mathbb{P}(T^+|A^c))$$
$$0.1962 = 0.1931 + 0.7872\cdot\mathbb{P}(T^+|A^c)$$
$$\mathbb{P}(T^+|A^c) = \frac{0.1962 - 0.984\cdot 0.1962}{0.984\cdot 0.8} = 0.0039$$
$$\mathbb{P}(T^+|A^c) = \frac{0.1962 - 0.1931}{0.7872} = 0.00399$$

specificity = $\mathbb{P}(T^-|A^c) = 1 - \mathbb{P}(T^+|A^c)$ *= 1 - 0.00399 = 0.99601*