Chapter 3: Distributions of Random Variables (Part 1)

Day 6 topics:
Section 3.1: Random variables
Section 3.2: Binomial distribution

3.1. **Random variables.**

Definition 3.1. *A **random variable (r.v.)** assigns numerical values to the outcome of a random phenomenon.*

Notation:
A random variable is usually denoted with a capital letter such as X, Y, or Z.

Example 3.2. ***Data points***
Suppose you have a dataset of size 3 ($n = 3$) with $x_1 = 5, x_2 = 3$, and $x_3 = 6$.

- *Each data point is the outcome of a random phenomenon*
- *Each data point is a numerical value*
- *The data points are examples of values of a sequence of random variables X_1, X_2, and X_3*
- *For datasets, we almost always assume the data points came from random variables that are independent and have the same **distribution**.*
- *To calculate the likelihood of data, we need to know the distribution of the random variable that models the data.*
- *First, let's remind ourselves how to calculate them mean and variance of a dataset:*
  - *What is the mean of the data points?*

$$\bar{x} = 14/3 = 4.66667$$

  - *What are the variance and standard deviation of the data points?*

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$
$$s^2 = 2.3333$$
$$s = 1.527525 \sim 1.53$$

Example 3.3. ***Rolling a die***
Suppose you roll a fair die. Let the random variable (r.v.) X be the outcome of the roll, i.e. the value of the face showing on the die.

(1) *What is the probability distribution of the r.v. X?*

(2) *What is the expected outcome of the r.v. X?*

- $$7/2 = 3.5$$
- *Not a possible outcome!*
- *Do not round.*

(3) *Now suppose the 6-sided die is not fair. How would we calculate the expected outcome?*

x	$\mathbb{P}(X = x)$	$x\mathbb{P}(X = x)$
1	0.10	0.1
2	0.20	0.4
3	0.05	0.15
4	0.05	0.2
5	0.25	1.25
6	0.35	2.10
sum	1	$\mu = 4.2$

(From Textbook § 2.1.5)

Definition 3.4. *A* ***probability distribution*** *consists of all disjoint outcomes and their associated probabilities.*

Rules for a probability distribution

A probability distribution is a list of all possible outcomes and their associated probabilities that satisfies three rules:

(1) The outcomes listed must be disjoint.
(2) Each probability must be between 0 and 1.
(3) The probabilities must total to 1.

Probability distributions are usually either **discrete** or **continuous**, depending on whether the random variable is discrete or continuous.

(Back to Textbook § 3.1.1)

Definition 3.5. *A* ***discrete*** *r.v. X takes on a finite number of values or countably infinite number of possible values.*

Definition 3.6. *A* ***continuous*** *r.v. X can take on any real value in an interval of values or unions of intervals.*

Class notes:

Sketch histogram of discrete values 1, 2, ..., 6 with continuous approximation
X could be values from a die or a waiting time in minutes

§ 3.1.2 Expectation

- We call the mean of a random variable its **expected value**
- The expected value is calculated as a weighted average

Definition 3.7. ***Expected value*** *of a discrete random variable*
If X takes on outcomes x_1, ..., x_k with probabilities $P(X = x_1)$, ..., $P(X = x_k)$, the expected value of X is the sum of each outcome multiplied by its corresponding probability:

$$\begin{aligned}\mu = E(X) &= x_1 P(X = x_1) + \cdots + x_k P(X = x_k) \\ &= \sum_{i=1}^{k} x_i P(X = x_i).\end{aligned}$$

§ 3.1.3 Variability of random variables

- Just like with data, the variability of a r.v. is described with its variance or standard deviation.
- Squared deviations from the mean are weighted by their respective probabilities

Definition 3.8. ***Variance*** *of a discrete random variable*
If X takes on outcomes x_1, ..., x_k with probabilities $P(X = x_1), \ldots, P(X = x_k)$ and expected value $\mu = E(X)$, then the variance of X, denoted by Var(X) or σ^2, is

$$\begin{aligned}Var(X) &= (x_1 - \mu)^2 P(X = x_1) + \cdots + (x_k - \mu)^2 P(X = x_k) \\ &= \sum_{i=1}^{k} (x_i - \mu)^2 P(X = x_i).\end{aligned}$$

Definition 3.9. ***Standard deviation*** *of a discrete random variable*
The standard deviation of X, labeled $SD(X)$ or σ, is

$$SD(X) = \sigma = \sqrt{\sigma^2}$$

Example 3.10. ***Rolling a fair die: variance***
Suppose you roll a fair 6-sided die. Let the random variable (r.v.) X be the outcome of the roll, i.e. the value of the face showing on the die.
Find the variance and standard deviation of X.

$$\mu = 3.5$$

x	$\mathbb{P}(X=x)$	$x-\mu$	$(x-\mu)^2$	$\mathbb{P}(X=x)(x-\mu)^2$ *R*
1	*1/6*	*-2.5*	*6.25*	*1.04166667*
2	*1/6*	*-1.5*	*2.25*	*0.3750*
3	*1/6*	*-0.5*	*0.25*	*0.04166667*
4	*1/6*	*0.5*	*0.25*	*0.04166667*
5	*1/6*	*1.5*	*2.25*	*0.3750*
6	*1/6*	*2.5*	*6.25*	*1.04166667*
sum	*1*			$\sigma^2 = 2.916667$

$$\sigma = 1.707825$$

R commands

x	$\mathbb{P}(X=x)$	$x-\mu$	$(x-\mu)^2$	$\mathbb{P}(X=x)(x-\mu)^2$ *R*
$x < -1 : 6$	$px < -1/6$	$x - mu$	$(x-mu)^2$	$px * (x-mu)^2$
sum	*1*			$sum(px * (x-mu)^2)$

Example 3.11. ***Vaccinated people testing positive for Covid-19***
About 25% of people that test positive for Covid-19 are vaccinated for Covid-19. Define the r.v. X to be 1 if someone that tests positive is vaccinated and 0 if they are not vaccinated.

(1) *Make a table for the probability distribution for the r.v. X*

(2) *What is the expected value of X?*

$$\mathbb{E}[X] = p = 0.25$$

(3) *What is the variance of X?*

$$Var[X] = pq = 0.25 \cdot 0.75 = 0.1875$$
$$SD[X] = 0.4330127$$

§ 3.1.4 Linear combinations of random variables

Seems abstract, but actually fundamental to basic statistical theory - using it all the time!

Definition 3.12. ***Linear combinations*** *of random variables.*
If X and Y are random variables and a and b are constants, then

$$aX + bY$$

is a linear combination of the random variables.

Theorem 3.13. ***Expected value of a linear combination*** *of random variables.*
If X and Y are random variables and a and b are constants, then

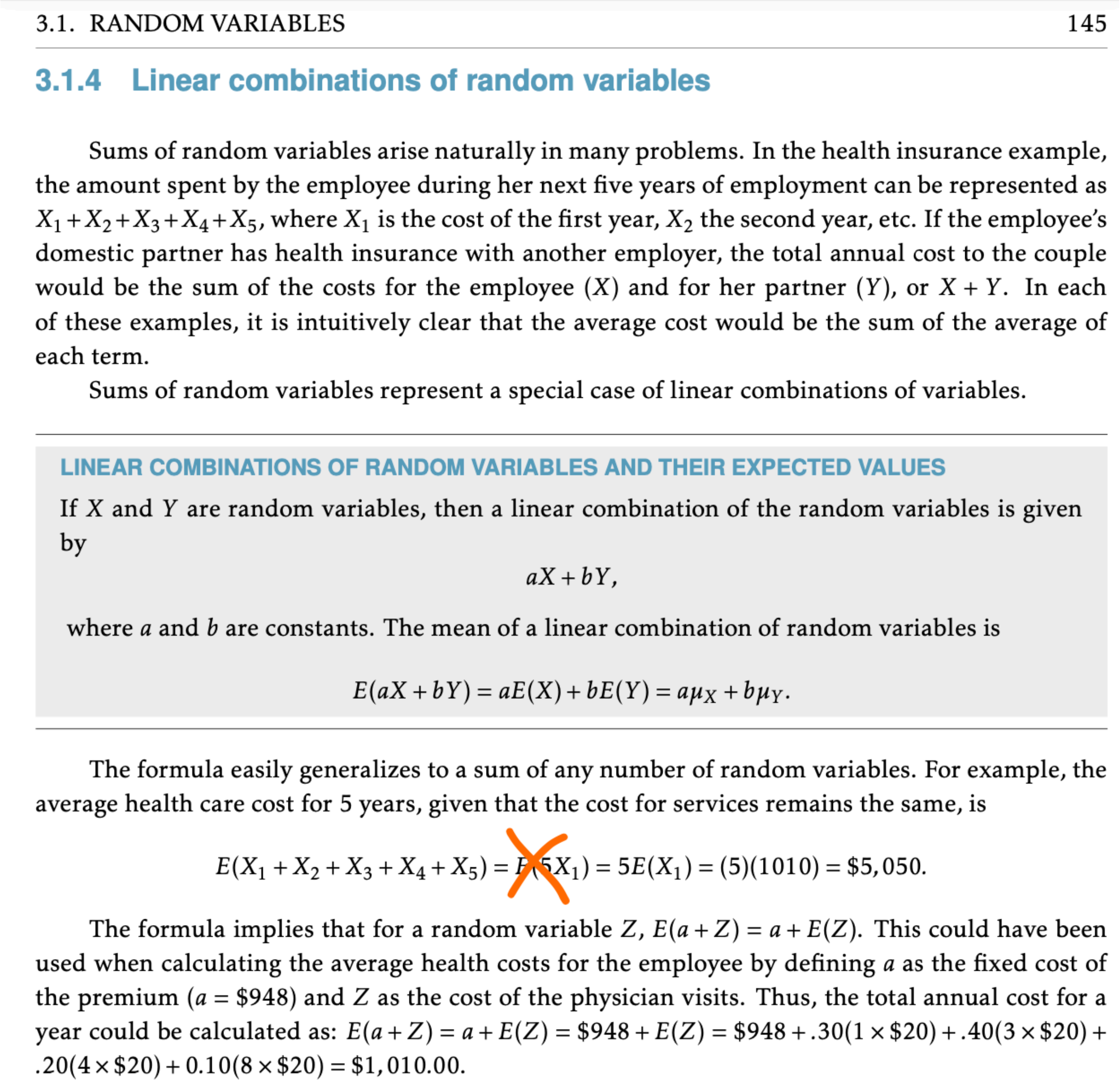
$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y) = a\mu_X + b\mu_Y$$

Example 3.14. ***Expected money for rolling 3 dice***
Let the random variables X_1, X_2, X_3 be the values shown on 3 fair 6-sided dice rolls. Suppose you are given in dollars the amount of the first roll, plus twice the value of the second roll, plus 4 times the value of the third roll.
How much money do you expect to get?

$$\mathbb{E}(X_1 + 2X_2 + 4X_3) = \mathbb{E}(X_1) + 2\mathbb{E}(X_2) + 4\mathbb{E}(X_3) = 7 \cdot 3.5 = 24.5$$

Expected value of the sample mean

Make sure to correct typo in textbook!!!



3.1.4 Linear combinations of random variables

Sums of random variables arise naturally in many problems. In the health insurance example, the amount spent by the employee during her next five years of employment can be represented as $X_1+X_2+X_3+X_4+X_5$, where X_1 is the cost of the first year, X_2 the second year, etc. If the employee's domestic partner has health insurance with another employer, the total annual cost to the couple would be the sum of the costs for the employee (X) and for her partner (Y), or $X+Y$. In each of these examples, it is intuitively clear that the average cost would be the sum of the average of each term.

Sums of random variables represent a special case of linear combinations of variables.

LINEAR COMBINATIONS OF RANDOM VARIABLES AND THEIR EXPECTED VALUES

If X and Y are random variables, then a linear combination of the random variables is given by

$$aX+bY,$$

where a and b are constants. The mean of a linear combination of random variables is

$$E(aX+bY)=aE(X)+bE(Y)=a\mu_X+b\mu_Y.$$

The formula easily generalizes to a sum of any number of random variables. For example, the average health care cost for 5 years, given that the cost for services remains the same, is

$$E(X_1+X_2+X_3+X_4+X_5)=\xcancel{E(}5X_1)=5E(X_1)=(5)(1010)=\$5,050.$$

The formula implies that for a random variable Z, $E(a+Z)=a+E(Z)$. This could have been used when calculating the average health costs for the employee by defining a as the fixed cost of the premium ($a=\$948$) and Z as the cost of the physician visits. Thus, the total annual cost for a year could be calculated as: $E(a+Z)=a+E(Z)=\$948+E(Z)=\$948+.30(1\times\$20)+.40(3\times\$20)+.20(4\times\$20)+0.10(8\times\$20)=\$1,010.00$.

Theorem 3.15. ***Variance of a linear combination*** *of random variables.*
If X and Y are INDEPENDENT random variables and a and b are constants, then

$$\mathit{Var}(aX + bY) = a^2\mathit{Var}(X) + b^2\mathit{Var}(Y)$$

Example 3.16. ***Variance of money for rolling 3 dice***
Let the random variables X_1, X_2, X_3 be the values shown on 3 fair 6-sided dice rolls. Suppose you are given in dollars the amount of the first roll, plus twice the value of the second roll, plus 4 times the value of the third roll.
What are the variance and standard deviation of the amount you get from the 3 rolls?

$$\mathit{Var}(X_1 + 2X_2 + 4X_3) = \mathit{Var}(X_1) + 4\mathit{Var}(X_2) + 16\mathit{Var}(X_3) = 21 \cdot 17.5 = 367.5$$

$$\mathit{SD}(X_1 + 2X_2 + 4X_3) = \sqrt{367.5} = 19.17029$$

Variance of the sample mean

Example 3.17. ***Vaccinated people testing positive for Covid-19 (revisited)*** *About 25% of people that test positive for Covid-19 are vaccinated for Covid-19. Define the r.v. X_i to be 1 if someone that tests positive is vaccinated and 0 if they are not vaccinated.*

Suppose 3 people have tested positive for Covid-19 (independently of each other). Let T denote the number of people that are vaccinated amongst the 3 that tested positive.

(1) *Using the r.v.'s X_i, write a mathematical equation for calculating T.*

$$T = \sum_{i=1}^{3} X_i$$

(2) *What is the expected value of T?*

$$\mathbb{E}[T] = 3 \cdot p = 3 \cdot 0.25 = 0.75$$

(3) *What is the variance of T?*

$$Var[T] = npq = 3 \cdot 0.25 \cdot 0.75 = 3 \cdot 0.1875 = 0.5625$$

$$SD[T] = 0.5625$$

(4) *What is the probability distribution of T?*

x	$\mathbb{P}(X = x)$
0	0.75^3 = 0.422
1	$3 \cdot 0.25 \cdot 0.75^2$ = 0.422
2	$3 \cdot 0.25^2 \cdot 0.75$ = 0.141
3	0.25^3 = 0.016

3.2. **Binomial distribution.**

- Many situations involve modeling independent random events that have 2 possible outcomes (binary), such as
  - Repeatedly flipping a coin
  - Whether a person that tested positive with Covid-19 is vaccinated or not
- Repeated events are referred to as **trials**
- The 2 possible outcomes are referred to as **successes** and **failures**.
- We denote the probability of a success as p.
- We denote the probability of a failure as $q = 1 - p$.

3.2.1. *Bernoulli distribution.*

Definition 3.18. ***Bernoulli*** *random variable.*
If X is a random variable that takes value 1 with probability of success p and 0 with probability $1 - p$, then X is a Bernoulli random variable.

probability table; ONE trial

- We call the probability of success p the **parameter** of the Bernoulli distribution.
- Each value of p identifies a specific Bernoulli distribution out of the **family** of Bernoulli r.v.'s where p is any value between 0 and 1 (inclusive).
- If a r.v. X is modeled by a Bernoulli distribution, then we write in shorthand

$$X \sim Bern(p)$$

Theorem 3.19. *Mean and SD of a Bernoulli r.v.*
If X is a Bernoulli r.v. with probability of success p, then

$$\mathbb{E}(X) = p$$
$$Var(X) = p(1 - p)$$
$$SD(X) = \sqrt{p(1 - p)}$$

Covid example

3.2.2. *Binomial distribution.*

Recall Example 3.17:

- About 25% of people that test positive for Covid-19 are vaccinated for Covid-19. p
- Define the r.v. X_i to be 1 if someone that tests positive is vaccinated and 0 if they are not vaccinated. S, F
- Suppose 3 people have tested positive for Covid-19 (independently of each other). fixed n; independent trials
- Let T denote the number of people that are vaccinated amongst the 3 that tested positive. total number S

The random variable T above is an example of a Binomial random variable.

In general, a random variable X is **Binomial** if the following hold:

(1) The trials are independent.
(2) The number of trials, n, is fixed.
(3) Each trial outcome can be classified as a *success* or *failure*.
(4) The probability of a success, p, is the same for each trial.
(5) The r.v. X is the total number of successes in the n trials.

Definition 3.20. *Distribution of a* ***Binomial*** *random variable.*
Let X be the total number of successes in n independent trials, each with probability p of a success.
Then probability of observing exactly k successes in n independent trials is

$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}=\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}.$$

- The parameters of a binomial distribution are p and n.
- If a r.v. X is modeled by a binomial distribution, then we write in shorthand

$$X\sim Bin(n,p)$$

Theorem 3.21. *Mean and SD of a Binomial r.v.*
If X is a binomial r.v. with probability of success p, then

$$\mu = \mathbb{E}(X) = np$$
$$\sigma^2 = Var(X) = np(1-p)$$
$$\sigma = SD(X) = \sqrt{np(1-p)}$$

Covid example 3.17

Example 3.22. ***Vaccinated people testing positive for Covid-19 (revisited)*** *About 25% of people that test positive for Covid-19 are vaccinated for Covid-19. Suppose 10 people have tested positive for Covid-19 (independently of each other). Let X denote the number of people that are vaccinated amongst the 10 that tested positive.*

(1) *What is the expected value of X?*

$$\mathbb{E}[X] = 10 \cdot p = 10 \cdot 0.25 = 2.5$$

(2) *What is the SD of X?*

$$Var[X] = npq = 10 \cdot 0.25 \cdot 0.75 = 10 \cdot 0.1875 = 1.875$$

$$SD[X] = 1.369$$

(3) *What is the probability that exactly 4 of the 10 people that tested positive are vaccinated?*

$$P(X = 4) = \binom{10}{4} 0.25^k (0.75)^{10-k} = \frac{10!}{k!(10-k)!} p^k (1-p)^{10-k} = 0.145998$$

```
dbinom(x, size, prob)
dbinom(x= 4, size=10, prob = .25)= 0.145998
```

(4) *What is the probability that at most 3 of the 10 people that tested positive are vaccinated?*

$$P(X \leq 3) =$$

$P(X \leq q) =$

```
pbinom(q, size, prob, lower.tail = TRUE)
pbinom(3, size=10, prob = .25)= 0.2502823
```

(5) *What is the probability that at least 5 of the 10 people that tested positive are vaccinated?*

$$P(X \geq 5) = P(X > 4) = 0.078$$
$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - 0.922$$

$P(X > q) =$

```
pbinom(q, size, prob, lower.tail = FALSE)
pbinom(4, size=10, prob = .25, lower.tail = FALSE)
= 0.07812691
```

$P(X \leq q) =$

```
pbinom(4, size=10, prob = .25, lower.tail = TRUE)
= 0.9218731
```