Asymptotic behavior of the nonautonomous Arrow–Hurwicz differential system

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***Abstract*— In a real Hilbert space setting, we investigate the asymptotic behavior of the solutions of the nonautonomous Arrow–Hurwicz differential system. We show that its solutions weakly converge in average towards a saddle point of some limiting closed convex-concave bifunction provided that the associated gap function vanishes sufficiently fast. If, in addition, the limiting saddle function verifies a strict convexity-concavity condition, we find that the solutions of the nonautonomous Arrow–Hurwicz differential system not only converge in an ergodic sense, but in fact admit a weak limit. Numerical experiments illustrate our theoretical findings.**

I. Introduction

Let X and Y be real Hilbert spaces endowed with inner products $\langle\cdot\,,\cdot\rangle_X$, $\langle\cdot\,,\cdot\rangle_Y$ and associated norms $\|\cdot\|_X$, $\|\cdot\|_Y$. Consider the nonautonomous Arrow–Hurwicz differential system

$$\begin{cases} \dot{x} + \nabla_x L_t(x,\lambda) = 0_X \\ \dot{\lambda} - \nabla_\lambda L_t(x,\lambda) = 0_Y, \end{cases} \tag{NAH}$$

where for each $t \geq 0$, $L_t : X \times Y \to \mathbb{R}$ is a convex-concave and continuously differentiable bifunction. We say that $(x,\lambda) : [0,+\infty[\; \to X \times Y$ is a (classical) solution of (NAH) if $(x,\lambda) \in \mathcal{C}^1([0,+\infty[; X\times Y)$ such that (x,λ) satisfies (NAH) on $[0,+\infty[$. We take for granted the existence of the (classical) solutions of (NAH) but refer to Browder [1] and Haraux [2] for the respective results on general nonautonomous evolution equations.

The (NAH) differential system, in its autonomous form, essentially dates back to the early work of Arrow and Hurwicz [3] (see also Arrow et al. [4] and Kose [5]) and is well known to be intimately related to the mini-maximization of the "saddle functions" L_t; see, e.g., Rockafellar [6]. Indeed, we immediately observe that for each $t \geq 0$, the zeros of the operator

$$\begin{aligned} T_t : X \times Y &\longrightarrow X \times Y \\ (x,\lambda) &\longmapsto \big(\nabla_x L_t(x,\lambda), -\nabla_\lambda L_t(x,\lambda)\big) \end{aligned}$$

are precisely the saddle points of L_t. Moreover, the operators T_t are maximally monotone on $X \times Y$ as they are both continuous and monotone; cf. Bauschke and Combettes [7]. As such, the (NAH) differential system falls into the category of nonautonomous evolution equations governed by maximally monotone operators; we refer the reader to Kato [8] (see also Crandall and Pazy [9], Attouch and Damlamian [10]) for a detailed exposition of the subject. We remark that the maximal monotonicity of the operators T_t may also be deduced more elementary from the convexity-concavity properties of the saddle functions L_t; cf. Rockafellar [11].

In this work, we investigate the asymptotic behavior of the solutions of (NAH) in line with the results by Furuya et al. [12] and Attouch et al. [13]. Our convergence results thereby essentially rely on the assumption that there exists a limiting closed convex-concave bifunction $L_\infty : X \times Y \to \mathbb{R}$ such that the associated gap function $\mathrm{GAP}_{L_t - L_\infty} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} \mathrm{GAP}_{L_t - L_\infty}(\xi,\eta) = \sup_{\mu \in Y} &\big(L_t(\xi,\mu) - L_\infty(\xi,\mu)\big) \\ &- \inf_{\nu \in X} \big(L_t(\nu,\eta) - L_\infty(\nu,\eta)\big) \end{aligned}$$

tends to zero "sufficiently fast" as $t \to +\infty$. More precisely, assuming that the bifunction L_∞ admits a non-empty set of saddle points $S \times M$ such that the condition

$$\forall (\xi,\eta) \in X \times Y,\ \int_0^\infty \mathrm{GAP}_{L_\tau - L_\infty}(\xi,\eta)\,\mathrm{d}\tau < +\infty \tag{Γ}$$

is verified, we show that the solutions $(x,\lambda) : [0,+\infty[\; \to X \times Y$ of (NAH) weakly converge in average towards an element of $S \times M$, that is, there exists $(\bar{\sigma},\bar{\omega}) \in S \times M$ such that

$$\frac{1}{t}\int_0^t (x(\tau),\lambda(\tau))\,\mathrm{d}\tau \rightharpoonup (\bar{\sigma},\bar{\omega}) \text{ as } t \to +\infty.$$

If condition (Γ) is satisfied and, in addition, the closed convex-concave bifunction L_∞ is such that for all $(\bar{x},\bar{\lambda}) \in S \times M$ and $(\xi,\eta) \notin S \times M$, it holds that

$$L_\infty(\bar{x},\eta) < L_\infty(\bar{x},\bar{\lambda}) < L_\infty(\xi,\bar{\lambda}), \tag{Σ}$$

we prove that the solutions $(x,\lambda) : [0,+\infty[\; \to X \times Y$ of (NAH) in fact admit a weak limit in $S \times M$, i.e., there exists $(\bar{x},\bar{\lambda}) \in S \times M$ such that

$$(x(t),\lambda(t)) \rightharpoonup (\bar{x},\bar{\lambda}) \text{ as } t \to +\infty.$$

In view of the above results, we conclude that whenever the saddle functions L_t tend sufficiently fast (in the sense of condition (Γ)) towards some limiting closed convex-concave bifunction L_∞, the asymptotic behavior of the solutions of (NAH) is essentially characterized by the one of the solutions of the autonomous Arrow–Hurwicz differential system; see, e.g., Venets [14], Flåm and Ben-Israel [15], Chbani and Riahi [16], and Niederländer [17], [18]. Our numerical experiments support this observation.

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II. Preliminary results

Let $X \times Y$ be equipped with the Hilbertian product structure $\langle \cdot\,,\cdot \rangle = \langle \cdot\,,\cdot \rangle_X + \langle \cdot\,,\cdot \rangle_Y$ and induced norm $\|\cdot\|$. Consider again the nonautonomous Arrow–Hurwicz differential system

$$\begin{cases} \dot{x} + \nabla_x L_t(x,\lambda) = 0_X \\ \dot{\lambda} - \nabla_\lambda L_t(x,\lambda) = 0_Y \end{cases} \tag{NAH}$$

and recall that $(x,\lambda) : [0,+\infty[\to X \times Y$ is a (classical) solution of (NAH) if $(x,\lambda) \in \mathcal{C}^1([0,+\infty[; X \times Y)$ such that (NAH) is verified on $[0,+\infty[$. Henceforth, we take for granted the existence of the solutions of (NAH).

In view of the convexity-concavity of the saddle functions L_t, we immediately observe that for all $(x,\lambda), (\xi,\eta) \in X \times Y$ and $t \geq 0$, it holds that

$$\langle T_t(x,\lambda), (x,\lambda) - (\xi,\eta) \rangle \geq L_t(x,\eta) - L_t(\xi,\lambda). \tag{1}$$

Utilizing the previous inequality relative to (NAH) gives the following preliminary estimate.

Proposition II.1. *Let $(x,\lambda) : [0,+\infty[\to X \times Y$ be a solution of* (NAH)*. Then, for all $(\xi,\eta) \in X \times Y$, it holds that*

$$\limsup_{t \to +\infty} \int_0^t L_\tau(x(\tau),\eta) - L_\tau(\xi,\lambda(\tau))\,\mathrm{d}\tau < +\infty.$$

Proof. Let $(\xi,\eta) \in X \times Y$. Taking the inner product with $(x(t),\lambda(t)) - (\xi,\eta)$ in (NAH) and subsequently applying the chain rule gives, for every $t \geq 0$,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 \\ &+ \langle T_t(x(t),\lambda(t)), (x(t),\lambda(t)) - (\xi,\eta) \rangle = 0. \end{aligned}$$

In view of inequality (1), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 \\ &+ L_t(x(t),\eta) - L_t(\xi,\lambda(t)) \leq 0. \end{aligned} \tag{2}$$

Integration over $[0,t]$ yields[1]

$$\begin{aligned} &\frac{1}{2}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 - \frac{1}{2}\|(x(0),\lambda(0)) - (\xi,\eta)\|^2 \\ &+ \int_0^t L_\tau(x(\tau),\eta) - L_\tau(\xi,\lambda(\tau))\,\mathrm{d}\tau \leq 0. \end{aligned}$$

Taking into account that $\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 \geq 0$, we deduce

$$\begin{aligned} &\int_0^t L_\tau(x(\tau),\eta) - L_\tau(\xi,\lambda(\tau))\,\mathrm{d}\tau \\ &\leq \frac{1}{2}\|(x(0),\lambda(0)) - (\xi,\eta)\|^2. \end{aligned} \tag{3}$$

Passing to the upper limit as $t \to +\infty$ entails

$$\limsup_{t \to +\infty} \int_0^t L_\tau(x(\tau),\eta) - L_\tau(\xi,\lambda(\tau))\,\mathrm{d}\tau < +\infty,$$

concluding the desired estimate. □

[1]Throughout the section, we assume that $t \longmapsto L_t(\cdot\,,\eta) - L_t(\xi,\cdot\,)$ is continuous for each $(\xi,\eta) \in X \times Y$.

Remark II.2 ("No-regret condition"). Let us define, for every $t \geq 0$, the "regret function" $\mathrm{Regret}_t : X \times Y \to \mathbb{R}$ by

$$\mathrm{Regret}_t(\xi,\eta) = \int_0^t L_\tau(x(\tau),\eta) - L_\tau(\xi,\lambda(\tau))\,\mathrm{d}\tau\,.$$

Given this definition, Proposition II.1 asserts that the solutions of (NAH) verify, for all $(\xi,\eta) \in X \times Y$, the so-called "no-regret condition"

$$\mathrm{Regret}_t(\xi,\eta) \leq \mathcal{O}(t) \text{ as } t \to +\infty,$$

suggesting that the average regret $\mathrm{Regret}_t(\xi,\eta)/t$ is less than or equal to zero as $t \to +\infty$; we refer to Sorin [19] for a recent survey on no-regret algorithms.

The following preparatory result further characterizes the limiting behavior of the convergent solutions of (NAH).

Proposition II.3. *Let $(x,\lambda) : [0,+\infty[\to X \times Y$ be a solution of* (NAH) *and suppose that there exists a closed convex-concave bifunction $L_\infty : X \times Y \to \mathbb{R}$ such that for every $(\xi,\eta) \in X \times Y$,*

$$L_t(\cdot\,,\eta) - L_t(\xi,\cdot\,) \to L_\infty(\cdot\,,\eta) - L_\infty(\xi,\cdot\,)$$
$$\textit{uniformly on } X \times Y \textit{ as } t \to +\infty.$$

If $(\bar{x},\bar{\lambda}) \in X \times Y$ is such that $(x(t),\lambda(t)) \to (\bar{x},\bar{\lambda})$ strongly in $X \times Y$ as $t \to +\infty$, then, for all $(\xi,\eta) \in X \times Y$,

$$L_\infty(\bar{x},\eta) \leq L_\infty(\bar{x},\bar{\lambda}) \leq L_\infty(\xi,\bar{\lambda}).$$

Proof. Let $(\xi,\eta) \in X \times Y$. Successively dividing inequality (3) by $t > 0$ and taking the upper limit as $t \to +\infty$ gives

$$\limsup_{t \to +\infty} \frac{1}{t}\int_0^t L_\tau(x(\tau),\eta) - L_\tau(\xi,\lambda(\tau))\,\mathrm{d}\tau \leq 0.$$

On the other hand, since $L_t(\cdot\,,\eta) - L_t(\xi,\cdot\,) \to L_\infty(\cdot\,,\eta) - L_\infty(\xi,\cdot\,)$ uniformly on $X \times Y$ and $(x(t),\lambda(t)) \to (\bar{x},\bar{\lambda})$ strongly in $X \times Y$ as $t \to +\infty$, we immediately get

$$\begin{aligned} &\lim_{t \to +\infty} \big(L_t(x(t),\eta) - L_t(\xi,\lambda(t))\big) \\ &= L_\infty(\bar{x},\eta) - L_\infty(\xi,\bar{\lambda}). \end{aligned}$$

In view of the classical Cesàro property, we obtain

$$\begin{aligned} &\lim_{t \to +\infty} \frac{1}{t}\int_0^t L_\tau(x(\tau),\eta) - L_\tau(\xi,\lambda(\tau))\,\mathrm{d}\tau \\ &= L_\infty(\bar{x},\eta) - L_\infty(\xi,\bar{\lambda}) \end{aligned}$$

and thus,

$$L_\infty(\bar{x},\eta) - L_\infty(\xi,\bar{\lambda}) \leq 0.$$

The above derivations being true for every $(\xi,\eta) \in X \times Y$, we conclude that $(\bar{x},\bar{\lambda})$ is a saddle point of L_∞. □

The previous result asserts that whenever there exists a closed convex-concave bifunction L_∞ such that the saddle functions L_t tend towards L_∞ (in the above sense) as $t \to +\infty$, the limit of a convergent solution of (NAH) is necessarily a saddle point of L_∞. In the following sections, we investigate the weak (ergodic) convergence properties of the solutions of (NAH) in the case where the saddle functions L_t tend towards their limit L_∞ (in a sense to be made precise) sufficiently fast as $t \to +\infty$.

III. Weak ergodic convergence

In this section, we investigate the weak ergodic convergence properties of the solutions of (NAH). To this end, let us consider the Cesàro average of a solution $(x,\lambda) : [0,+\infty[\to X \times Y$ of (NAH) defined by

$$\begin{aligned}(\sigma,\omega) : \ ]0,+\infty[&\longrightarrow X \times Y\\ t &\longmapsto \frac{1}{t}\int_0^t (x(\tau),\lambda(\tau))\,\mathrm{d}\tau\,.\end{aligned}$$

Assuming that there exists a closed convex-concave bifunction $L_\infty : X\times Y \to \mathbb{R}$ with a non-empty set of saddle points $S \times M$ such that for all $(\xi,\eta) \in X \times Y$, the gap function

$$\begin{aligned}\mathrm{GAP}_{L_t - L_\infty}(\xi,\eta) = \sup_{\mu\in Y}&\big(L_t(\xi,\mu) - L_\infty(\xi,\mu)\big)\\ &- \inf_{\nu\in X}\big(L_t(\nu,\eta) - L_\infty(\nu,\eta)\big)\end{aligned}$$

vanishes sufficiently fast as $t \to +\infty$, the following result asserts that the Cesàro average $(\sigma(t),\omega(t))$ of a solution of (NAH) weakly converges, as $t \to +\infty$, towards an element of $S \times M$. The arguments we use to prove this fact rely on the Opial–Passty lemma; cf. Lemma A.2.

Theorem III.1. *Let $(\sigma,\omega) :]0,+\infty[\to X\times Y$ be the Cesàro average of a solution of* (NAH) *and suppose that there exists a closed convex-concave bifunction $L_\infty : X \times Y \to \mathbb{R}$ with a non-empty set of saddle points $S \times M$ such that*

$$\forall(\xi,\eta) \in X \times Y,\ \int_0^\infty \mathrm{GAP}_{L_\tau - L_\infty}(\xi,\eta)\,\mathrm{d}\tau < +\infty.$$

Then $(\sigma(t),\omega(t))$ converges weakly, as $t \to +\infty$, to some element $(\bar\sigma,\bar\lambda) \in S \times M$.

Proof. In order to apply the Opial–Passty lemma, let us first show that for every $(\xi,\eta) \in S\times M$, $\lim_{t\to+\infty}\|(x(t),\lambda(t)) - (\xi,\eta)\|$ exists. Let $(\xi,\eta) \in S \times M$ and recall from inequality (2) that for all $t \geq 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 \leq L_t(\xi,\lambda(t)) - L_t(x(t),\eta).$$

Upon defining $\mathrm{GAP}_{L_t} : X \times Y \to \mathbb{R}\cup\{+\infty\}$ by

$$\mathrm{GAP}_{L_t}(\xi,\eta) = \sup_{\mu\in Y} L_t(\xi,\mu) - \inf_{\nu\in X} L_t(\nu,\eta),$$

we immediately obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 \leq \mathrm{GAP}_{L_t}(\xi,\eta).$$

Integration over $[s,t]$, for $t \geq s \geq 0$, yields

$$\begin{aligned}\frac{1}{2}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 \leq \frac{1}{2}&\|(x(s),\lambda(s)) - (\xi,\eta)\|^2\\ &+ \int_s^t \mathrm{GAP}_{L_\tau}(\xi,\eta)\,\mathrm{d}\tau\,.\end{aligned}$$

Consequently, the mapping

$$t \longmapsto \frac{1}{2}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 - \int_0^t \mathrm{GAP}_{L_\tau}(\xi,\eta)\,\mathrm{d}\tau$$

is non-increasing and bounded from below, implying that it admits a limit as $t \to +\infty$. The latter is an immediate consequence of the fact that for $(\xi,\eta) \in S \times M$,

$$\mathrm{GAP}_{L_t}(\xi,\eta) \leq \mathrm{GAP}_{L_t - L_\infty}(\xi,\eta),$$

which, together with condition (Γ), implies that

$$\int_0^\infty \mathrm{GAP}_{L_\tau}(\xi,\eta)\,\mathrm{d}\tau < +\infty.$$

In view of the above derivations,

$$\lim_{t\to+\infty}\|(x(t),\lambda(t)) - (\xi,\eta)\| \text{ exists.}$$

Let us now show that every weak sequential cluster point of $(\sigma(t),\omega(t))_{t>0}$ belongs to the set $S \times M$. Let $(\xi,\eta) \in X \times Y$ and observe again from inequality (2) that for every $t \geq 0$, it holds that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 + L_\infty(x(t),\eta) - L_\infty(\xi,\lambda(t))&\\ \leq \mathrm{GAP}_{L_t - L_\infty}(\xi,\eta).&\end{aligned}$$

Integration over $[0,t]$ gives

$$\begin{aligned}\frac{1}{2}\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 - \frac{1}{2}\|(x(0),\lambda(0)) - (\xi,\eta)\|^2&\\ + \int_0^t L_\infty(x(\tau),\eta) - L_\infty(\xi,\lambda(\tau))\,\mathrm{d}\tau&\\ \leq \int_0^t \mathrm{GAP}_{L_\tau - L_\infty}(\xi,\eta)\,\mathrm{d}\tau\,.&\end{aligned} \tag{4}$$

Taking into account that $\|(x(t),\lambda(t)) - (\xi,\eta)\|^2 \geq 0$ and subsequently dividing by $t > 0$ yields

$$\begin{aligned}\frac{1}{t}\int_0^t &L_\infty(x(\tau),\eta) - L_\infty(\xi,\lambda(\tau))\,\mathrm{d}\tau\\ &\leq \frac{1}{2t}\|(x(0),\lambda(0)) - (\xi,\eta)\|^2\\ &+ \frac{1}{t}\int_0^t \mathrm{GAP}_{L_\tau - L_\infty}(\xi,\eta)\,\mathrm{d}\tau\,.\end{aligned}$$

Applying Jensen's inequality, as $L_\infty(\,\cdot\,,\eta)$ and $-L_\infty(\xi,\,\cdot\,)$ are both convex, it follows with condition (Γ) that

$$L_\infty(\sigma(t),\eta) - L_\infty(\xi,\omega(t)) \leq \frac{C}{t},$$

where

$$\begin{aligned}C = \frac{1}{2}&\|(x(0),\lambda(0)) - (\xi,\eta)\|^2\\ &+ \int_0^\infty \mathrm{GAP}_{L_\tau - L_\infty}(\xi,\eta)\,\mathrm{d}\tau\,.\end{aligned}$$

Passing to the upper limit as $t \to +\infty$ entails

$$\limsup_{t\to+\infty}\big(L_\infty(\sigma(t),\eta) - L_\infty(\xi,\omega(t))\big) \leq 0.$$

Suppose now that $(\sigma(t_n),\omega(t_n)) \rightharpoonup (\bar\sigma,\bar\omega)$ weakly in $X \times Y$, as $n \to +\infty$, for a sequence $t_n \to +\infty$. Substituting t by t_n in the above inequality gives

$$\begin{aligned}0 &\geq \limsup_{n\to+\infty}\big(L_\infty(\sigma(t_n),\eta) - L_\infty(\xi,\omega(t_n))\big)\\ &\geq \liminf_{n\to+\infty} L_\infty(\sigma(t_n),\eta) + \liminf_{n\to+\infty}\big(-L_\infty(\xi,\omega(t_n))\big)\\ &\geq L_\infty(\bar\sigma,\eta) - L_\infty(\xi,\bar\omega)\end{aligned}$$

thanks to the weak lower semi-continuity of $L_\infty(\,\cdot\,,\eta)$ and $-L_\infty(\xi,\,\cdot\,)$ as L_∞ is a closed convex-concave bifunction so that $L_\infty(\,\cdot\,,\eta)$ and $-L_\infty(\xi,\,\cdot\,)$ are both convex and lower semi-continuous. The above inequalities being true for every $(\xi,\eta)\in X\times Y$, we conclude that $(\bar\sigma,\bar\omega)\in S\times M$.

The weak convergence of $(\sigma(t),\omega(t))$ as $t\to+\infty$ is now readily deduced from the Opial–Passty lemma, cf. Lemma A.2, applied to the set $S\times M$. ☐

Remark III.2. Under the hypotheses of Theorem III.1, we observe that the Cesàro average $(\sigma,\omega):\,]0,+\infty[\,\to X\times Y$ of a solution of (NAH) obeys, for every $(\xi,\eta)\in S\times M$, the asymptotic estimate

$$L_\infty(\sigma(t),\eta)-L_\infty(\xi,\omega(t))=\mathcal{O}\Big(\frac{1}{t}\Big)\text{ as }t\to+\infty.$$

This estimate is particularly well known in the study of the autonomous Arrow–Hurwicz differential system; see, e.g., Niederländer [18].

Remark III.3. We note that condition (Γ) is strongly related to the summability estimate

$$\forall(\xi,\eta)\in\operatorname{gph}T_\infty,\ \int_0^\infty G_{T_\tau}(\xi,\eta)\,\mathrm{d}\tau<+\infty$$

proposed by Attouch et al. [13] for some maximally monotone operator T_∞. The above estimate thereby relies on the so-called Brézis–Haraux function $G_T:X\times X\to\mathbb{R}\cup\{+\infty\}$ associated with a maximally monotone operator T which is defined by

$$G_T(\xi,\eta)=\sup_{(\mu,\nu)\in\operatorname{gph}T}\langle\xi-\mu,\nu-\eta\rangle_X.$$

We also refer to Furuya et al. [12] for yet another summability condition introduced in the context of nonautonomous evolution equations governed by subdifferential operators.

IV. Weak convergence

In this section, we focus on the weak convergence properties of the solutions of (NAH). In addition to the integrability estimate (Γ), let us assume that the limiting closed convex-concave bifunction $L_\infty:X\times Y\to\mathbb{R}$ is such that for all $(\bar x,\bar\lambda)\in S\times M$ and $(\xi,\eta)\notin S\times M$,

$$L_\infty(\bar x,\eta)<L_\infty(\bar x,\bar\lambda)<L_\infty(\xi,\bar\lambda).$$

Utilizing the above inequalities, the following result states that the solutions $(x(t),\lambda(t))$ of (NAH) admit a weak limit as $t\to+\infty$. The weak convergence result essentially relies on the Opial lemma; cf. Lemma A.1.

Theorem IV.1. *Let $(x,\lambda):[0,+\infty[\,\to X\times Y$ be a solution of* (NAH) *and suppose that there exists a closed convex-concave bifunction $L_\infty:X\times Y\to\mathbb{R}$ with a non-empty set of saddle points $S\times M$ such that*

$$\forall(\xi,\eta)\in X\times Y,\ \int_0^\infty \mathrm{GAP}_{L_\tau-L_\infty}(\xi,\eta)\,\mathrm{d}\tau<+\infty.$$

Assume, in addition, that the bifunction L_∞ is such that for all $(\bar x,\bar\lambda)\in S\times M$ and $(\xi,\eta)\notin S\times M$,

$$L_\infty(\bar x,\eta)<L_\infty(\bar x,\bar\lambda)<L_\infty(\xi,\bar\lambda).$$

Then $(x(t),\lambda(t))$ converges weakly, as $t\to+\infty$, to some element $(\bar x,\bar\lambda)\in S\times M$.

Proof. In view of condition (Γ), we immediately obtain from Theorem III.1 that for all $(\xi,\eta)\in S\times M$,

$$\lim_{t\to+\infty}\|(x(t),\lambda(t))-(\xi,\eta)\|\text{ exists.}$$

Hence, in order to apply the Opial lemma, it suffices to show that every weak sequential cluster point of $(x(t),\lambda(t))_{t\geq0}$ belongs to the set $S\times M$. Let $(\xi,\eta)\in S\times M$ and recall from inequality (4) that for all $t\geq0$, we have

$$\begin{aligned}\frac12\|(x(t),\lambda(t))-(\xi,\eta)\|^2-\frac12\|(x(0),\lambda(0))-(\xi,\eta)\|^2&\\+\int_0^t L_\infty(x(\tau),\eta)-L_\infty(\xi,\lambda(\tau))\,\mathrm{d}\tau&\\\leq\int_0^t \mathrm{GAP}_{L_\tau-L_\infty}(\xi,\eta)\,\mathrm{d}\tau\,.&\end{aligned}$$

Utilizing again the fact that $\|(x(t),\lambda(t))-(\xi,\eta)\|^2\geq0$, we obtain

$$\begin{aligned}\int_0^t L_\infty(x(\tau),\eta)-L_\infty(\xi,\lambda(\tau))\,\mathrm{d}\tau&\\\leq\frac12\|(x(0),\lambda(0))-(\xi,\eta)\|^2&\\+\int_0^t \mathrm{GAP}_{L_\tau-L_\infty}(\xi,\eta)\,\mathrm{d}\tau\,.&\end{aligned}$$

In view of condition (Γ), it follows that

$$\int_0^t L_\infty(x(\tau),\eta)-L_\infty(\xi,\lambda(\tau))\,\mathrm{d}\tau\leq C,$$

where

$$\begin{aligned}C=\frac12\|(x(0),\lambda(0))-(\xi,\eta)\|^2&\\+\int_0^\infty \mathrm{GAP}_{L_\tau-L_\infty}(\xi,\eta)\,\mathrm{d}\tau\,.&\end{aligned}$$

The above majorization being valid for every $t\geq0$, taking the supremum with respect to t gives

$$\int_0^\infty L_\infty(x(\tau),\eta)-L_\infty(\xi,\lambda(\tau))\,\mathrm{d}\tau<+\infty.$$

Suppose now that $(x(t_n),\lambda(t_n))\rightharpoonup(\bar x,\bar\lambda)$ weakly in $X\times Y$, as $n\to+\infty$, for a sequence $t_n\to+\infty$. Since

$$L_\infty(x,\eta)-L_\infty(\xi,\lambda)\in\mathcal{L}^1([0,+\infty[;\mathbb{R}),$$

there must be a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{k\to+\infty}\big(L_\infty(x(t_{n_k}),\eta)-L_\infty(\xi,\lambda(t_{n_k}))\big)=0$. Consequently,

$$\begin{aligned}0&=\lim_{k\to+\infty}\big(L_\infty(x(t_{n_k}),\eta)-L_\infty(\xi,\lambda(t_{n_k}))\big)\\&\geq\liminf_{k\to+\infty}L_\infty(x(t_{n_k}),\eta)+\liminf_{k\to+\infty}\big(-L_\infty(\xi,\lambda(t_{n_k}))\big)\\&\geq L_\infty(\bar x,\eta)-L_\infty(\xi,\bar\lambda),\end{aligned}$$

where we again utilized the weak lower semi-continuity of $L_\infty(\,\cdot\,,\eta)$ and $-L_\infty(\xi,\,\cdot\,)$. On the other hand, since $(\xi,\eta)\in S\times M$, it holds that $L_\infty(\bar x,\eta)-L_\infty(\xi,\bar\lambda)\geq0$. Hence,

$(\bar{x}, \bar{\lambda}) \in X \times Y$ is such that $L_\infty(\bar{x}, \eta) - L_\infty(\xi, \bar{\lambda}) = 0$. Owing to the fact that (ξ, η) is a saddle point of L_∞, we get

$$L_\infty(\xi, \bar{\lambda}) = L_\infty(\xi, \eta) = L_\infty(\bar{x}, \eta).$$

In view of condition (Σ), this clearly implies that $(\bar{x}, \bar{\lambda}) \in S \times M$, i.e., $(\bar{x}, \bar{\lambda})$ is a saddle point of L_∞.

The weak convergence of $(x(t), \lambda(t))$ as $t \to +\infty$ is now an immediate consequence of the Opial lemma, cf. Lemma A.1, applied to the set $S \times M$. □

Remark IV.2. We note that condition (Σ) is, of course, trivially satisfied whenever the set $S \times M$ is non-empty such that the bifunction L_∞ is strictly convex in its first argument and strictly concave in its second argument. A less conservative assumption that may be used to deduce the weak convergence of the solutions of (NAH) is the following one proposed by Chbani and Riahi [16]:

$$\begin{cases} \text{If } (\bar{x}, \bar{\lambda}) \in X \times X \text{ is a saddle point of } L_\infty \text{ such that} \\ \qquad L_\infty(\bar{x}, \eta) = L_\infty(\bar{x}, \bar{\lambda}) = L_\infty(\xi, \bar{\lambda}) \\ \text{for some } (\xi, \eta) \in X \times X, \text{ then } (\xi, \eta) \text{ is a saddle} \\ \text{point of } L_\infty. \end{cases}$$

We also refer to Rockafellar [6] and Venets [14] for similar conditions that appeared in the study of the autonomous Arrow–Hurwicz differential system.

Remark IV.3. By Theorem IV.1, in order to prove the weak convergence of the solutions of (NAH), it suffices to show that the weak sequential cluster points of $(x(t), \lambda(t))_{t \geq 0}$ belong to the set $S \times M$. A key tool to ensure this is the concept of demipositivity, first developed by Bruck [20] for monotone operators, and later extended by Chbani and Riahi [16] for monotone bifunctions. We refer the reader to Pazy [21], [22] (see also Peypouquet and Sorin [23]) for an exposition of conditions that imply demipositivity.

V. Numerical experiments

In this section, we provide two simple, yet representative, numerical experiments that allow for a direct exposition of our main results.

Example 1 (Weak ergodic convergence). Let $X, Y = \mathbb{R}$ and consider, for every $t \geq 0$, the convex-concave and continuously differentiable saddle function

$$\begin{aligned} L_t : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, \lambda) &\longmapsto \mathrm{e}^{-t}(x^2 - \lambda^2)/2 + (\lambda - 1)(x - 1). \end{aligned}$$

Clearly, the limiting closed convex-concave bifunction reads $L_\infty(x, \lambda) = (\lambda - 1)(x - 1)$ with $S \times M = \{(1, 1)\}$. The associated gap function reduces to

$$\mathrm{GAP}_{L_t - L_\infty}(\xi, \eta) = \mathrm{e}^{-t}(\xi^2 + \eta^2)/2.$$

Hence, for every $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$, we obtain

$$\int_0^\infty \mathrm{GAP}_{L_\tau - L_\infty}(\xi, \eta)\, \mathrm{d}\tau < +\infty$$

so that condition (Γ) is satisfied. Figure 1 illustrates the trajectories of a solution $(x(t), \lambda(t))$ of (NAH) together with its Cesàro average $(\sigma(t), \omega(t))$. The initial data of the (NAH) solution is chosen as $(x(0), \lambda(0)) = (2, 1)$.

Example 2 (Weak convergence). Let $X, Y = \mathbb{R}$ and consider now, for every $t \geq 0$, the convex-concave and continuously differentiable bifunction

$$\begin{aligned} K_t : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, \lambda) &\longmapsto L_t(x, \lambda) + (x - 1)^2/2 - (\lambda - 1)^2/2 \end{aligned}$$

with $K_\infty(x, \lambda) = L_\infty(x, \lambda) + (x - 1)^2/2 - (\lambda - 1)^2/2$ and $S \times M = \{(1, 1)\}$. For every $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$, we have

$$\mathrm{GAP}_{K_t - K_\infty}(\xi, \eta) = \mathrm{GAP}_{L_t - L_\infty}(\xi, \eta)$$

so that condition (Γ) holds. In addition, as K_∞ is strongly convex-concave, condition (Σ) is verified. The trajectories of a solution $(x(t), \lambda(t))$ of (NAH) along with its Cesàro average $(\sigma(t), \omega(t))$ are depicted in Figure 2. The initial data is again set to $(x(0), \lambda(0)) = (2, 1)$.

Appendix

We present here some auxiliary results which are used in the asymptotic analysis of the (NAH) differential system.

Let us first recall the continuous version of the classical Opial lemma; cf. Opial [24].

Lemma A.1 (Opial)**.** *Let X be a real Hilbert space and let $x : [0, +\infty[\to X$ be such that there exists a non-empty subset S of X which verifies*

(i) for all $\xi \in S$, $\lim_{t\to+\infty} \|x(t) - \xi\|_X$ exists;
(ii) $\forall t_n \to +\infty$ such that $x(t_n) \rightharpoonup \bar{x}$ weakly in X, it holds that $\bar{x} \in S$.

Then $x(t)$ converges weakly, as $t \to +\infty$, to some element $\bar{x} \in S$.

For the following ergodic variant of the Opial lemma, the reader is referred to Passty [25].

Lemma A.2 (Opial–Passty)**.** *Let X be a real Hilbert space, let S be a non-empty subset of X, and let $x : [0, +\infty[\to X$ be continuous. For every $t > 0$, set*

$$\sigma(t) = \frac{1}{t} \int_0^t x(\tau)\, \mathrm{d}\tau$$

and assume that

(i) for all $\xi \in S$, $\lim_{t\to+\infty} \|x(t) - \xi\|_X$ exists;
(ii) $\forall t_n \to +\infty$ such that $\sigma(t_n) \rightharpoonup \bar{\sigma}$ weakly in X, it holds that $\bar{\sigma} \in S$.

Then $\sigma(t)$ converges weakly, as $t \to +\infty$, to some element $\bar{\sigma} \in S$.

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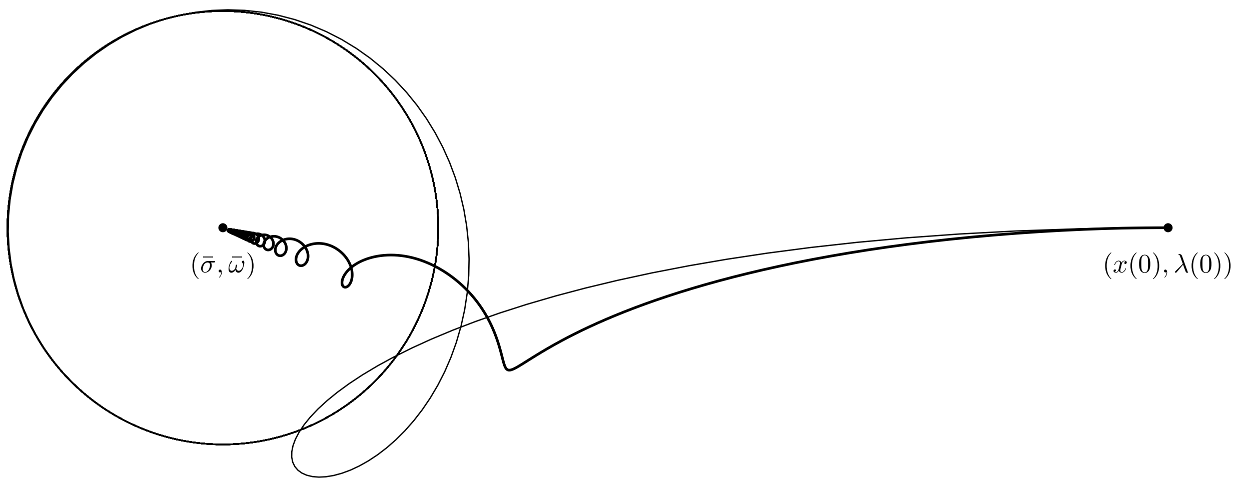

Fig. 1. Graphical view on the trajectories of a solution $(x(t), \lambda(t))$ of (NAH) and its Cesàro average $(\sigma(t), \omega(t))$. Clearly, the solution $(x(t), \lambda(t))$ of (NAH) remains bounded but does not admit a limit as $t \to +\infty$. However, since the saddle functions L_t tend sufficiently fast (in the sense of condition (Γ)) towards the limiting closed convex-concave bifunction L_∞, its Cesàro average $(\sigma(t), \omega(t))$ converges, as $t \to +\infty$, towards the unique saddle point $(\bar{\sigma}, \bar{\omega}) \in S \times M$ of L_∞; cf. Theorem III.1.

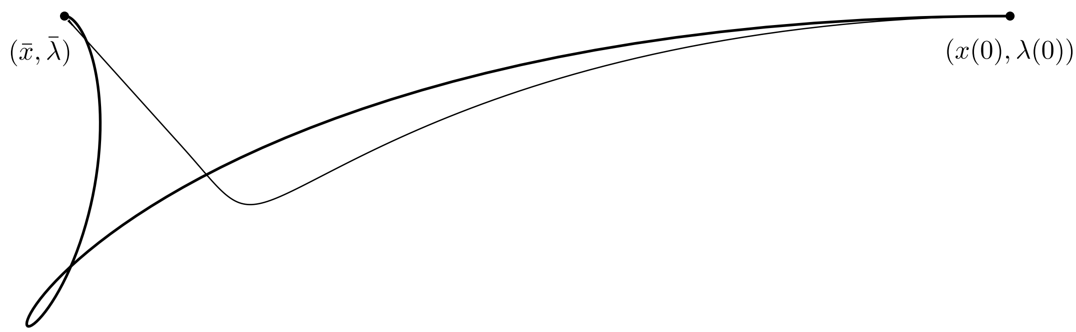

Fig. 2. Graphical view on the trajectories of a solution $(x(t), \lambda(t))$ of (NAH) together with its Cesàro average $(\sigma(t), \omega(t))$. As the gap function $\mathrm{GAP}_{K_t - K_\infty}$ vanishes fast enough (in the sense of condition (Γ)) and, in addition, the limiting closed convex-concave bifunction K_∞ satisfies condition (Σ), the solution $(x(t), \lambda(t))$ of (NAH) and its Cesàro average $(\sigma(t), \omega(t))$ converge, as $t \to +\infty$, towards the unique saddle point $(\bar{x}, \bar{\lambda}) \in S \times M$ of K_∞; cf. Theorem IV.1.

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