9^{th} Semester

School of Information and Communication Technologies

Electronics and IT

Fredrik Bajers Vej 7C

9220 Aalborg

http://www.sict.aau.dk/electronics-and-it

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Trajectory Planning for Cart Pendulum System

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Participants:
Niels Skov Vestergaard

Supervisor:
Anton Shiriaev
John-Josef Leth

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Synopsis

The aim of the project was to investigate a method for trajectory planning for a cart pendulum. To find an interesting trajectory, a task was posed for the system to complete.
The system was modeled and a simulation of the system dynamics created using MATLAB Simulink. To aid in the progress and to be able to show the result in a visual manner a graphical layer was added to the simulation.
The properties of the system was assessed using phase portraits. It was found that the phase portrait can be altered making different trajectories possible by designing the input force as a solution to the integral of the system dynamics given some initial conditions and constraints.

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1 | The System

In this chapter, the system and the problem to be solved is presented. A model is developed along with a simulation presented here in form of a block diagram. Further the nonlinear nature of the system is investigated.

1.1 Problem Description

The system in Figure 1.1, is the well-known cart pendulum system. The system can only be controlled by a the force shown directly on the cart in the horisontal direction. So the system is underactuated, and nonlinear.

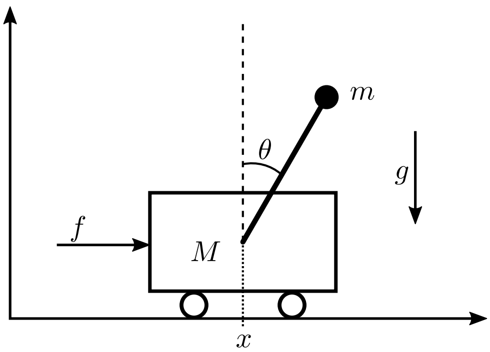

Figure 1.1: The cart pendulum system with generalized coordinates, x and θ, where x is the center of the cart, θ is the angle of the pendulum, m is the point mass of the pendulum, M is the mass of the cart, g is the gravitational acceleration and f is the force of actuation.

The system is presented with a specific task, see Figure 1.2. This aids in understanding the convinience of the presented trajectory planning tools, as the task requires nonlinear operation, where linearization alone is not enough.

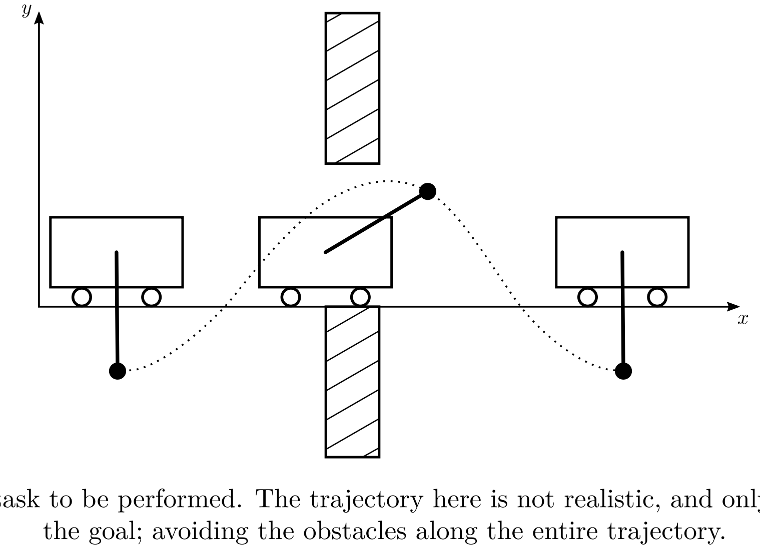

Figure 1.2: The task to be performed. The trajectory here is not realistic, and only shown to ilustrate the goal; avoiding the obstacles along the entire trajectory.

1.2 Model

Here a model is presented using the conventions shown in Figure 1.1 and assuming zero friction in the system.

The energy method is applied to find the dynamic equations using generalized coordinates, x and θ, as defined in Figure 1.1. The kinetic energy T is given by,

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} + l\dot{\theta}\cos\theta)^2 + \frac{1}{2}m(-l\dot{\theta}\sin\theta)^2$$
$$T = \frac{1}{2}(M+m)\dot{x}^2 + m\dot{x}l\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 \quad . \qquad [\mathrm{N \cdot m}] \quad (1.1)$$

Where:

T	is the kinetic energy	$[\mathrm{N \cdot m}]$
M	is the mass of the cart	$[\mathrm{kg}]$
$\dot{x}$	is the velocity of the cart	$[\mathrm{m \cdot s^{-1}}]$
m	is the mass of the pendulum	$[\mathrm{kg}]$
l	is the length of the pendulum	$[\mathrm{m}]$
θ	is the angle of the pendulum	$[\mathrm{rad}]$
$\dot{\theta}$	is the angular velocity of the	$[\mathrm{rad \cdot s^{-1}}]$

The potential energy is given by,

$$V = Mgl\cos\theta \quad . \qquad [\mathrm{N \cdot m}] \quad (1.2)$$

Where:

V	is the potential energy	$[\mathrm{N \cdot m}]$
g	is the gravitational acceleration	$[\mathrm{m \cdot s^{-2}}]$

By Equation 1.1 and Equation 1.2 the Lagrangian becomes,

$$\mathcal{L} = T - V$$
$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + m\dot{x}l\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 - Mgl\cos\theta \quad . \qquad [\mathrm{N \cdot m}] \quad (1.3)$$

Where:

$\mathcal{L}$	is the Lagrangian	$[\mathrm{N \cdot m}]$

From the energy method we find the dynamic equations corresponding to each of the generalized coordinates, θ and x, by,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{1.4}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = f \quad . \tag{1.5}$$

Where:

f is the actuation force, see Figure 1.1 $[\mathrm{N} \cdot \mathrm{m}]$

Inserting the Lagrangian and reducing the expressions finally yields the dynamic equations,

$$ml\cos\theta\ddot{x} + ml^2\ddot{\theta} - mlg\sin\theta = 0 \tag{1.6}$$

$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = f \quad . \tag{1.7}$$

A block diagram is derived in Figure 1.3 from Equation 1.6 and Equation 1.7.

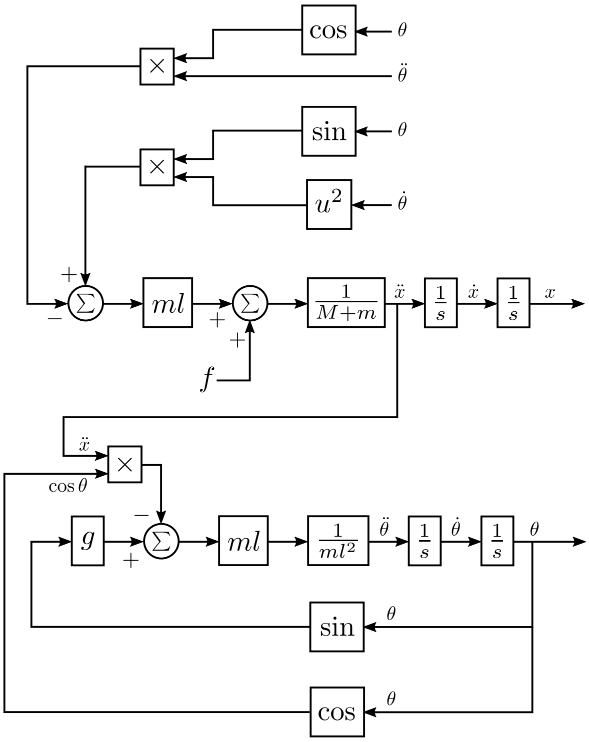

Figure 1.3: A block diagram derived from the dynamic equations, later used for simulation. The four signals in the top, θ, $\dot{\theta}$ and $\ddot{\theta}$, are drawn without explicit connection only to keep the figure clear.

This is implemented in Matlab Simulink to be able to simulate the system. A graphical layer, including the obstacles, is added to the simulation to show the system in action.

1.3 Nonlinear Analysis

In this section the nonlinear nature of the system is studied.[1]

It is possible to map trajectories in the $(\theta,\, \dot{\theta})$-plane assuming no applied force. To construct the phase portrait the dynamic equations, Equation 1.6 and Equation 1.6, are combined. First $\ddot{x}$ is isolated in Equation 1.7,

$$\ddot{x} = -\frac{ml\cos\theta\ddot{\theta}}{M+m} + \frac{ml\sin\theta\dot{\theta}^2}{M+m} + \frac{f}{M+m} \quad . \tag{1.8}$$

This expression for $\ddot{x}$ is then substituted in Equation 1.6 to obtain the dynamics in therms of the angle and its derivatives.

$$ml\cos\theta\left(-\frac{ml\cos\theta\ddot{\theta}}{M+m} + \frac{ml\sin\theta\dot{\theta}^2}{M+m} + \frac{f}{M+m}\right) + ml^2\ddot{\theta} - mlg\sin\theta = 0 \quad . \tag{1.9}$$

Rearranging yields,

$$\left(ml^2 - \frac{m^2l^2}{M+m}\cos^2\theta\right)\ddot{\theta} + \left(\frac{m^2l^2}{M+m}\sin\theta\cos\theta\right)\dot{\theta}^2 + f\frac{ml}{M+m}\cos\theta - mgl\sin\theta = 0 \quad , \tag{1.10}$$

which is the reduced system on the general form,

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0 \quad , \tag{1.11}$$

where, $\alpha(\theta)$, $\beta(\theta)$ and $\gamma(\theta)$ are the scalar functions of θ from Equation 1.10.

Finally the reduced system on state space form, where $x_1 = \theta$ and $x_2 = \dot{\theta}$,

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{\beta(x_1)}{\alpha(x_1)}x_2^2 - \frac{\gamma(x_1)}{\alpha(x_1)} \quad . \end{aligned} \tag{1.12}$$

In Figure 1.4 the phase portrait is generated using pplane8.[2]

[1] FiXme Note: name the tools used in the section
[2] FiXme Note: source of pplane8.m

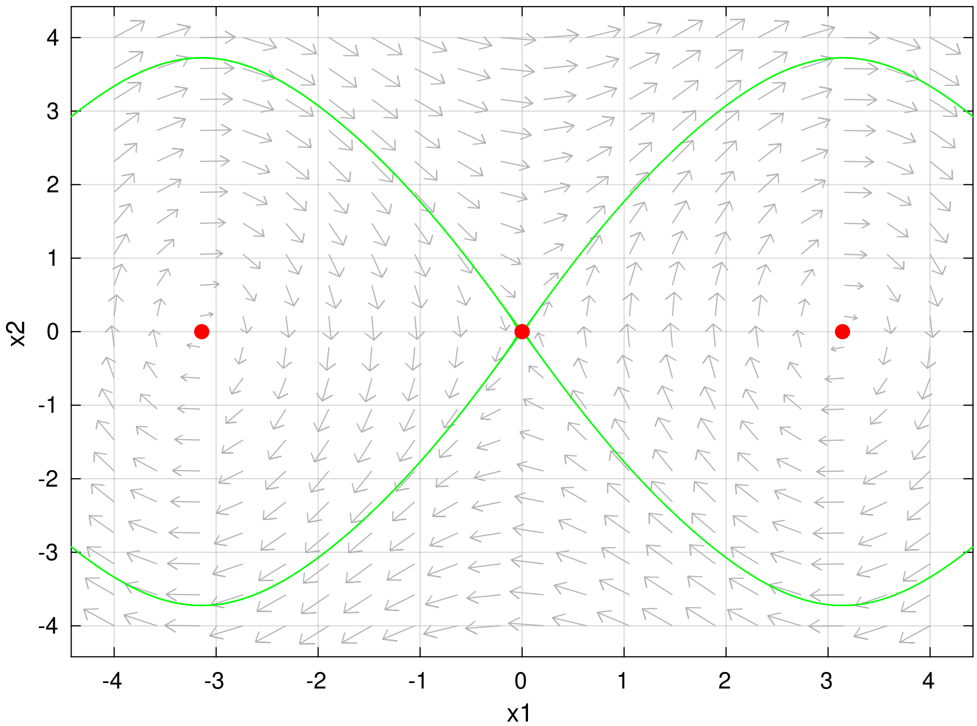

Figure 1.4: Phase portrait of the θ-dynamics, where $x_1 = \theta$ and $x_2 = \dot{\theta}$. The equilibrium points are shown in red and the unstable orbits at the saddle point are indicated in green.

As there is no friction in the system the pendulum will maintain one particular orbit around one of the center equilibrium points so long as the initial value is within the confines of the unstable orbit at the saddle point. This means that the pendulum will oscillate without loss of amplitude if the initial angular velocity, $\dot{\theta}$, is zero or at least places the starting point within the saddle.
If on the other hand the initial value of $\dot{\theta}$ is positive and θ is zero (upright position), then the pendulum will do full rotations around the cart, thus continuously increasing the angle.

If a force is imposed on the system it can change the phase portrait, allowing for other possible trajectories. This is further investigated in the following chapter.

2 | Trajectory Planning

In this chapter, the problem posed in section 1.1, Figure 1.2, is broken down into successive tasks. Each of these tasks are then analyzed to find feasible trajectories. These trajectories are finally realized and concatenated to finally accomplish the posed problem of obstacle avoidance.

2.1 Successive Trajectories

The first step in accomplishing the task would be to raise the pendulum high enough that it would be able to pass through the obstacles. It would be possible to clear a straight path through the obstacles by raising the pendulum from the starting position of $\theta = \pi$ to a little less than $\theta = \frac{\pi}{2}$. Further, the velocity would ideally achieve zero angular velocity as it reaches its target angle. This idea is presented in Figure 2.1, without restricting the pendulum to a specific trajectory but rather showing the initial and final conditions.

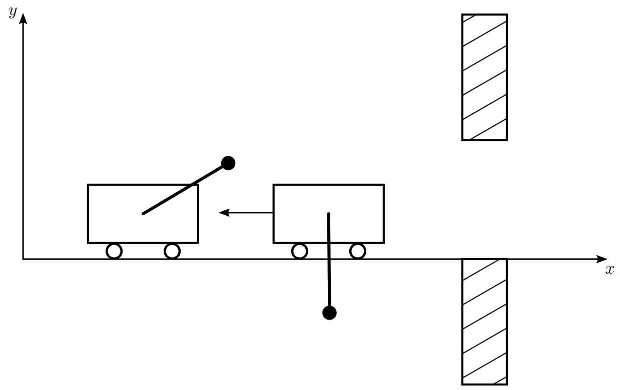

Figure 2.1: The first task is to find a trajectory which raises the pendulum to a position where it is clear of the obstacles in the horizontal direction. Further, to maintain clearance, the angular velocity should hit zero as the target angle is achieved.

The phase portrait in Figure 1.4, showing the natural theta-dynamics, it is possible to get an idea of how the pendulum moves. The starting position would be the right-most equilibrium of the phase portrait. If this task should be achieved without external force, the pendulum would have to be initialized in the orbit containing the desired final values. However, by exerting a force on the cart it is possible to shift the theta-dynamics. This is further investigated in the following section.

Assuming that the pendulum was raised above $\frac{\pi}{2}$ while briefly achieving zero angular velocity, the next task would be to maintain zero angular velocity while moving the cart, passing the pendulum through the obstacles, see Figure 2.2.

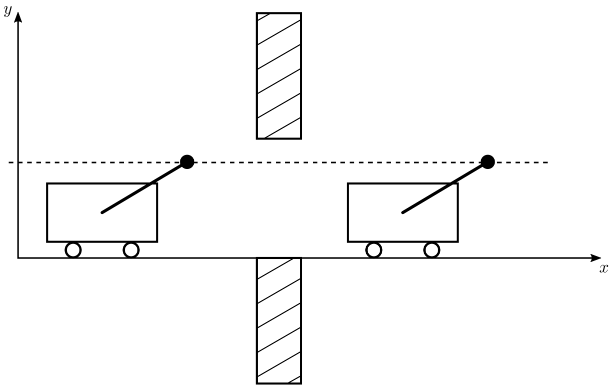

Figure 2.2: The second task is to find a trajectory which keeps the angle and angular velocity at zero as the pendulum is moved with the cart through the obstacles. This can be seen as a virtual constraint where the pendulum is mass is constrained to the dotted line.

Ideally this trajectory is a straight horizontal line. This can be seen as a virtual constraint that forces the angle and the angular velocity to stay unchanged. The result will be significantly reduced dynamics, which can be used to directly achieve the desired trajectory.

Finally it is necessary to recover the system on the other side of the obstacles. It is found that this can be achieved, to some extend, by reversing the first trajectory.

2.2 First Trajectory

Firstly the initial and final values of of θ and $\dot{\theta}$ are known. Further, the θ-dynamics are known, also for non-zero forces. In the state space and general equations, Equation 1.12 and Equation 1.11, the input force is contained by the γ-function.
If now Equation 1.11 is seen as an initial value problem. After some manipulation, [1] [2] it is seen that the following function preserves its zero value along the trajectory from initial to final value, assuming the solution exists while requiring α to be non-zero,

$$\dot{\theta}^2 - \psi(\theta_0, \theta)\left(\dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \psi(\theta_0, \theta)\frac{2\gamma(s)}{\alpha(s)}ds\right) = 0 \quad , \tag{2.1}$$

where,

$$\psi(\theta_0, \theta) = \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\} \quad .$$

It turns out, that in this case the problem can be solved analytically, and by requiring

[1] FiXme Note: source Anton's paper(s)
[2] FiXme Note: maybe include further explanation/derivations

the final value of the angular velocity be zero, it boils down to,

$$\left[-f\frac{ml}{M+m}\sin(s) - mgl\cos(s)\right]_{\theta_0}^{\theta} = 0 \quad . \tag{2.2}$$

Inserting initial and final value of θ and solving for f, the force which creates the desired trajectory is obtained. Also note that for this trajectory the applied force is constant and negative, pulling the cart to the left.

Going back to the θ-dynamics, it is possible to construct a phase portrait, see Figure 2.3, including this negative constant force.

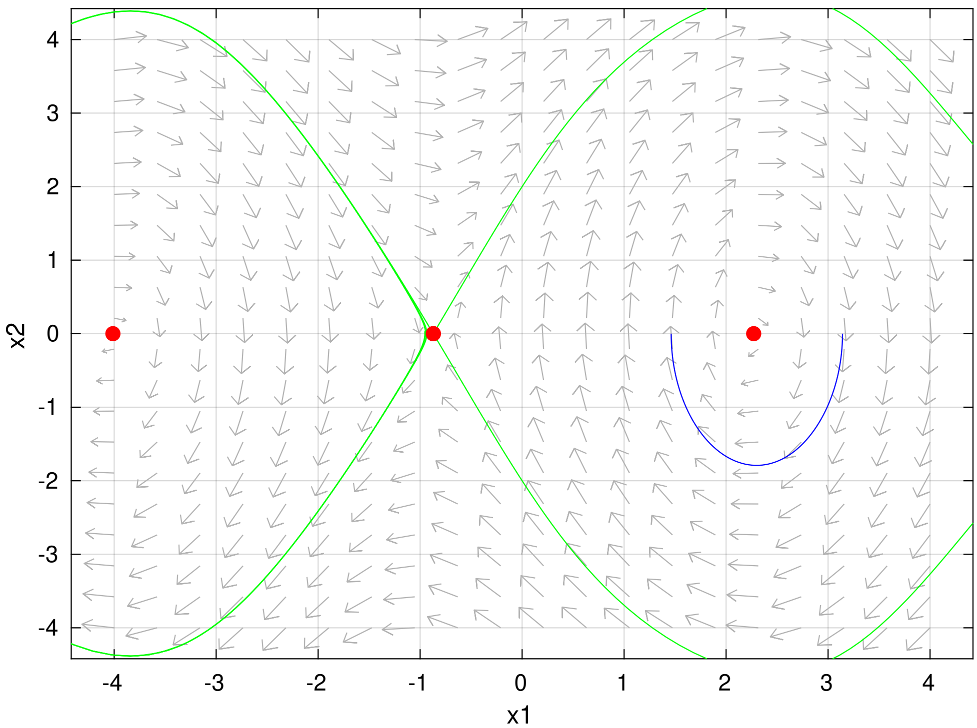

Figure 2.3: Phase portrait of the θ-dynamics including a constant applied force, where $x_1 = \theta$ and $x_2 = \dot{\theta}$. The phase portrait is shifted negatively along θ because of the applied force. This allows the desired trajectory (blue) from the initial downward position.

Note how the entire phase portrait is shifted in the negative θ (x_1) direction. This could be expected from Equation 1.10, since the force only scales the magnitude of a term which otherwise only depends on θ. Further, the constant force applied is negative, thus the negative shift in the θ direction.
This shift of the phase portrait means that the initial downward position of the pendulum now is placed in an orbit rather than an equilibrium.
Additionally, the trajectory (blue) reaches zero angular velocity just as it hits the target angle. The trajectory does however not stop there. It oscillates around the new center equilibrium. Therefore the next input force for the next trajectory must be applied the moment zero angular velocity is achieved.

2.3 Second Trajectory

The second trajectory to complete the task seen in Figure 2.2 is different that the first, in that it does not require movement in the $(\theta,\, \dot{\theta})$-plane, in fact, rather the contrary. It does however require the cart to move forward in order to pass the obstacles and hold up the pendulum.
As before, the initial values are known and a final position of the cart can be chosen, which will be the condition for switching to the final task.
Substituting the initial values, $\theta = \theta_t$ (target angle), $\dot{\theta} = 0$ and $\ddot{\theta} = 0$, into the dynamic equations, Equation 1.6 and Equation 1.6, reduces to,

$$ml\cos\theta_t\; \ddot{x} - mlg\sin\theta_t = 0 \tag{2.3}$$
$$(M+m)\ddot{x} = f \quad . \tag{2.4}$$

In Equation 2.4 the force, f, is directly provided as a function of the acceleration of the cart. Feeding back $\ddot{x}$ in this manner will attempt to keep the angle of the pendulum steady while moving forward through the obstacles.
It is interesting to note that the average of the force excreted during this second trajectory changes the θ-dynamics in such a way that the angle and angular velocity are kept in a saddle point equilibrium.

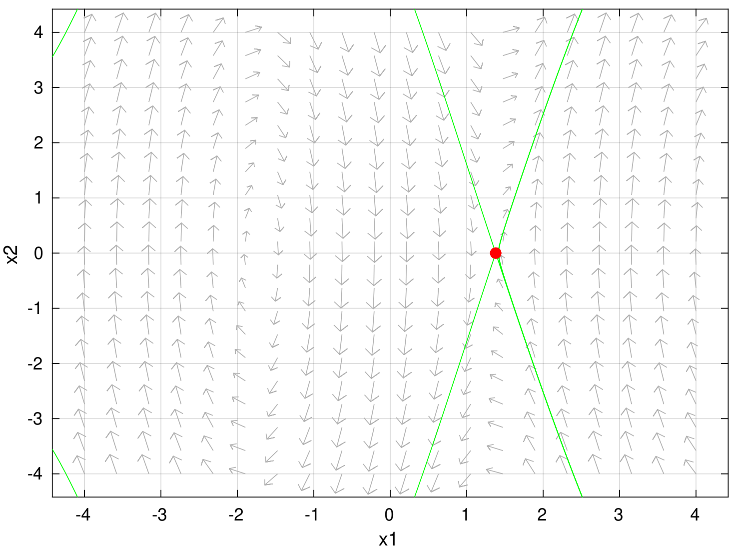

Figure 2.4: Phase portrait showing how the θ-dynamics changes given a constant force averaged from the forces used during the second trajectory. The system is kept near the indicated saddle point equilibrium.

2.4 Third Trajectory

List of Corrections

Note: name the tools used in the section 4

Note: source of pplane8.m 4

Note: source Anton's paper(s) 7

Note: maybe include further explanation/derivations 7