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# Swing-Up and Stabilization of a Cart Pendulum System and Stabilization of a Twin Pendulum System

Using Nonlinear Control Strategies

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Master Thesis

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# 1 | Introduction

This project is concerned with developing nonlinear control strategies for a cart pendulum system and to apply these to the set-up provided in the Control and Automation Lab at Aalborg University (AAU).

The project is two part. The objective of the first part is to design a swing-up controller along with a stabilizing controller to catch the pendulum at the upright position.

In the second part an additional pendulum is attached to the cart in the setup making it a twin pendulum system. The idea is to estimate the additional state and ultimately stabilize the two pendulums in upright position.

# **Part I**

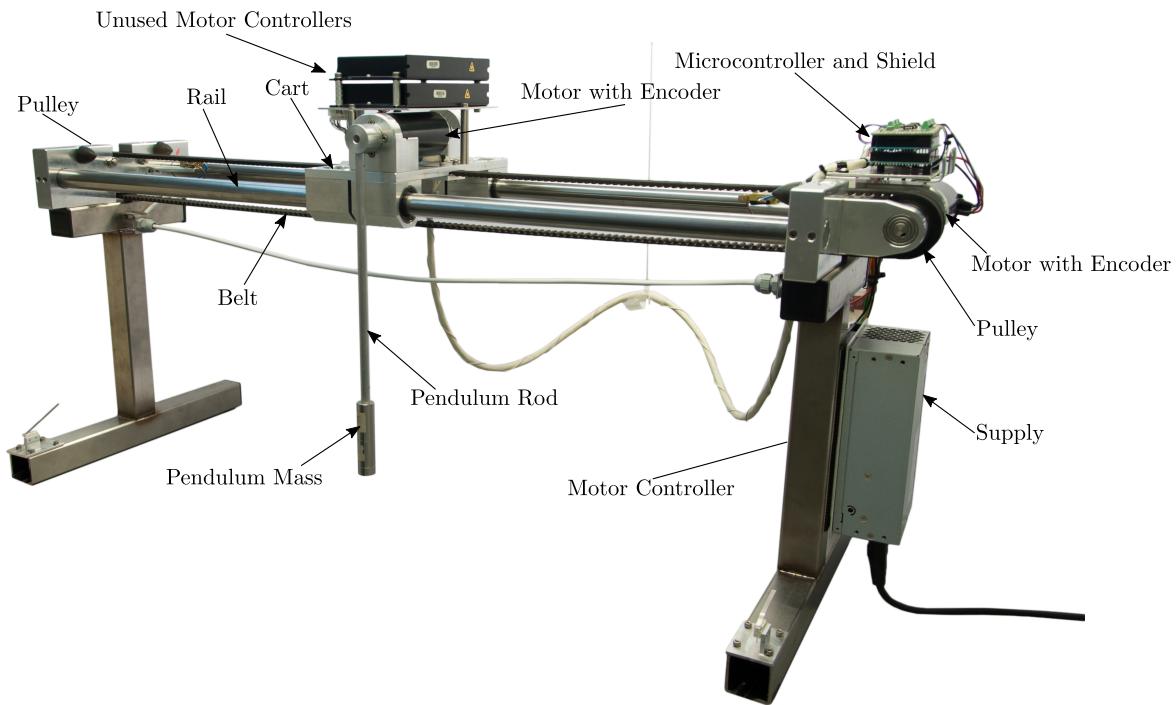
## **Cart Pendulum**

# 2 | System and Model

A brief overview of the relevant system for *Part 1* is presented in this chapter along with a model of the system.

## 2.1 System

A setup is provided by the Control and Automation Department at AAU, see Figure 2.1.



**Figure 2.1:** The setup provided by AAU. The motor controller in use is not directly visible in this picture as it is mounted behind the power supply.

As seen in Figure 2.1 the belt is attracted by pulleys one of which is driven by a brushed Maxon 370356 DC motor [1]. An other of these maxon motors is mounted on the pendulum but is disconnected and just used as a bearing in this project. Both motors are fitted with an HEDS 5540 optical quadrature encoder allowing for relative position and angle of the cart and pendulum respectively [2].

The motor driving the belt is controlled using a Maxon ADS 50/10 motor controller configured in current control mode. The motor controller takes a  $\pm 10\text{ V}$  input signal which then determines the armature current,  $i_a$ , see [3].

The primary control unit is a Teensy 3.6 microcontroller board. To program the board

through the onboard USB connection a bootloader is used along with the Teensyduino add-on for the Arduino IDE [4].

The encoders are decoded on a shield using Avago HCTL-2021-PLC decoders and read through an 8 bit parallel data bus on the microcontroller board resulting in 2000 tics pr. revolution. This ensures a resolution for the pendulum angle,  $\theta$ , of  $2\pi/2000 = \pi \times 10^{-3}$  rad/tic and  $2\pi r/2000 = 2\pi \cdot 0.028/2000 \approx 0.088 \times 10^{-3}$  m/tic for the cart position,  $x$ , see [5].

The supply circuit on the microcontroller board is powered by 5V which is regulated to 3.3 V resulting in a 0–3.3 V range for the 12 bit analog output [6]. This output is used to provide the motor controller with an armature current reference, thus, the microcontroller analog output is amplified through the shield to meet the ±10 V input requirement of the motor controller [7].

The following relation between analog 12 bit output values,  $\text{bit}_{\text{DAC}}$ , from the microcontroller and armature current in the motor was found by a previous project group [7],

$$(2.1) \quad \text{bit}_{\text{DAC}} = 105.78 \cdot i_a + 1970 ,$$

and as a result of a force test, see [8], Equation 2.1 was corrected to,

$$(2.2) \quad \text{bit}_{\text{DAC}} = 111.9 \cdot i_a + 1970 ,$$

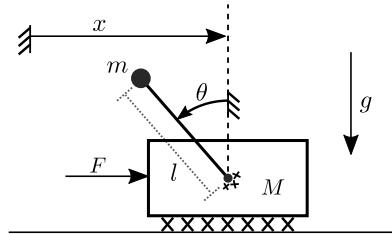
which is the relation used in this project. All the system parameters used in the design are listed in Table 2.1. It is assumed that all frictions in the system can be modeled as a combination of Coulomb and viscous frictions. Wires hanging from the cart are unmodeled and their weight along with that of the belt are contained in the estimation of the cart mass.

Parameter	Notation	Quantity	Unit
Nominal current (max. continuous current)	$I_N$	4.58	A
Torque constant	$\tau_m$	$93.4 \times 10^{-3}$	$\text{N} \cdot \text{m} \cdot \text{A}^{-1}$
Rod Length	$l$	0.3235	m
Rail Length	$l_r$	0.89	m
Pulley Radius	$r$	0.028	m
Pendulum Mass	$m$	0.201	kg
Cart Mass	$M$	5.273	kg
Cart Coulomb Friction	$b_{c,c}$	2.884	N
Cart Viscous Friction	$b_{c,v}$	1.680	$\text{N} \cdot \text{m}^{-1} \cdot \text{s}$
Pendulum Coulomb Friction	$b_{p,c}$	0.004	N·m
Pendulum Viscous Friction	$b_{p,v}$	$0.4 \times 10^{-3}$	$\text{N} \cdot \text{m} \cdot \text{s}$

**Table 2.1:** The motor parameters,  $I_N$  and  $\tau_m$ , are given by maxon in [1]. The rod length is measured from the pendulum pivot point to the geometrical center of the pendulum. Pendulum mass, rod length, pulley radius and rail length are measured parameters, while cart mass is estimated same as all frictions. The estimations are performed by a previous project group [7].

## 2.2 Model

The model is based on the generalized coordinates presented in Figure 2.2.



**Figure 2.2:** Mechanical drawing of the system, where  $\theta$  is the angle of the pendulum,  $x$  is the position of the center of the cart along the rail,  $F$  is the applied force and  $g$  is the gravitational acceleration. It is indicated that friction is modeled between cart and rail as well as in the pendulum joint.

The pendulum mass center is positioned at zero height at rest s.t. all energies in the system are positive. It is assumed that the pendulum rod is rigid and massless and that the pendulum weights are a point mass at the geometrical center of the weights.

The motor torque is given by direct relation to the armature current by the motor constant,  $\tau_m = k_\tau i_a$ , such that,

$$(2.3) \quad F = \frac{1}{r} k_\tau i_a \quad [\text{N}] .$$

To avoid excessive notation  $F$  is considered to be the control input in the remaining of this thesis, while keeping in mind the relation in Equation 2.3 along with the knowledge that  $F$  must be converted to armature current in implementation.

It is well known that the potential energy,  $U$ , and the kinetic energy,  $T$ , are given by, [9]

$$(2.4) \quad U = mgl(1 + \cos \theta) \quad [\text{J}]$$

$$(2.5) \quad T = \frac{1}{2}(M + m)\dot{x}^2 - m\dot{x}l \cos \theta \dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 \quad [\text{J}]$$

The frictions, indicated in Figure 2.2, are, as mentioned, comprised of Coulomb and viscous frictions with values stated in Table 2.1. The viscous frictions are modeled as linear functions of velocities, [10, 11]

$$(2.6) \quad b_{p,v}\dot{\theta} \quad , \quad b_{c,v}\dot{x} \quad ,$$

for the rotational and linear case respectively. The coulomb frictions are modeled as a constant with its sign depending on the signs of the velocities, such that, [10, 11]

$$(2.7) \quad \operatorname{sgn}(\dot{\theta})b_{p,c} \quad , \quad \operatorname{sgn}(\dot{x})b_{c,c} \quad .$$

This, however, introduces discontinuities at zero velocities. Thus, tanh-functions are used to obtain a continues approximation of the sign-functions,

$$(2.8) \quad \tanh(k_{\tanh}\dot{\theta})b_{p,c} \quad , \quad b_{c,v}\dot{x} - \tanh(k_{\tanh}\dot{x})b_{c,c} \quad ,$$

where  $k_{\tanh} = 250$  to increase the steepness of the tanh-functions thereby obtaining a closer approximation of the sign-functions. Finally, by use of the Lagrange-d'Alembert Principle, [9]

$$(2.9) \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q} \quad ,$$

where,

$$(2.10) \quad \mathbf{q} = \begin{bmatrix} \theta \\ x \end{bmatrix} \quad , \quad \mathbf{Q} = \begin{bmatrix} -b_{p,v}\dot{\theta} - \tanh(k_{\tanh}\dot{\theta})b_{p,c} \\ \frac{1}{r}k_{\tau}i_a - b_{c,v}\dot{x} - \tanh(k_{\tanh}\dot{x})b_{c,c} \end{bmatrix} \quad ,$$

and  $\mathcal{L} = T - U$ , the dynamics of the system are found,

$$(2.11) \quad ml^2\ddot{\theta} - ml \cos \theta \ddot{x} - mgl \sin \theta = -b_{p,v}\dot{\theta} - \tanh(k_{\tanh}\dot{\theta})b_{p,c} \quad [\text{N} \cdot \text{m}]$$

$$(2.12) \quad (M + m)\ddot{x} + ml \sin \theta \dot{\theta}^2 - ml \cos \theta \ddot{\theta} = F - b_{c,v}\dot{x} - \tanh(k_{\tanh}\dot{x})b_{c,c} \quad . \quad [\text{N}]$$

By setting up the dynamic equations, Equation 2.12 and 2.11, in the following manner,

$$(2.13) \quad \begin{aligned} & \begin{bmatrix} ml^2 & -ml \cos \theta \\ -ml \cos \theta & M + m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ ml \sin \theta \dot{\theta}^2 \end{bmatrix} + \\ & + \begin{bmatrix} b_{p,v}\dot{\theta} + \tanh(k_{\tanh}\dot{\theta})b_{p,c} \\ b_{c,v}\dot{x} + \tanh(k_{\tanh}\dot{x})b_{c,c} \end{bmatrix} + \begin{bmatrix} -mgl \sin \theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix} \quad , \end{aligned}$$

## Chapter 2. System and Model

the general form of an m-link robot is obtained, [12, 13]

$$(2.14) \quad \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{B}(\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{F} \quad ,$$

where,

$\mathbf{M}(\mathbf{q})$  is the inertia matrix

$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  is the Coriolis and centrifugal effects

$\mathbf{B}(\dot{\mathbf{q}})$  is the friction

$\mathbf{G}(\mathbf{q})$  is the force due to gravity

$\mathbf{F}$  is the input force

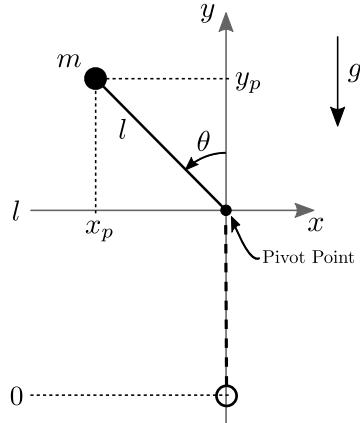
# 3 | Swing-Up Design

In this chapter a swing-up controller is designed based on [14]. The pendulum is started at rest,  $\theta = \pi$ , angle convention is specified in Figure 2.2. The idea of the swing-up controller is to increase the mechanical energy in the system until it matches that of the desired end state, which is the upright position at rest, that is,  $\theta = 0$  and  $\dot{\theta} = 0$ . The minimum energy in the system is the starting position at rest, which is considered to be zero as mentioned in the *Model* section 2.2. So the target energy is  $E_{\text{eq}} = 2mgl$ , that is, the potential energy of the pendulum in the unstable equilibrium.

Consider the pendulum dynamics from Equation 2.12,

$$(3.1) \quad J\ddot{\theta} - ml \cos \theta \ddot{x}_c - mgl \sin \theta = 0 \quad , \quad [\text{N} \cdot \text{m}]$$

where  $J = ml^2$  is the inertia and the pendulum friction is assumed to be zero. This equation captures the behavior of the pendulum corresponding to some acceleration  $\ddot{x}_c$  at the pivot point. This acceleration is viewed as the control input for now. The force needed to achieve this acceleration is considered in the end of the design. It is further convenient to describe the energy of the pendulum with the coordinate frame fixed at its pivot point, see Figure 3.1.



**Figure 3.1:** The energy used in the swing-up controller is described using this convention, where the coordinate frame is fixed at the pivot point of the pendulum. The zero reference is placed as before s.t. all energies are positive.

From Figure 3.1, the conversion from excessive to generalized coordinates is given by,

$$(3.2) \quad x_p = -l \sin \theta \quad , \quad y_p = l(\cos \theta + 1) \quad , \quad \dot{x}_p = -l \cos \theta \dot{\theta} \quad , \quad \dot{y}_p = -l \sin \theta \dot{\theta} \quad .$$

### Chapter 3. Swing-Up Design

The mechanical energy in this coordinate frame is then,

$$(3.3) \quad E_p = mgy_p + \frac{1}{2}m\dot{x}_p^2 + \frac{1}{2}m\dot{y}_p^2 \quad [\text{J}]$$

$$(3.4) \quad E_p = mgl(\cos \theta + 1) + \frac{1}{2}m(-l \cos \theta \dot{\theta})^2 + \frac{1}{2}m(-l \sin \theta \dot{\theta})^2 \quad [\text{J}]$$

$$(3.5) \quad E_p = mgl(\cos \theta + 1) + \frac{1}{2}J(\cos^2 \theta + \sin^2 \theta)\dot{\theta}^2 \quad [\text{J}]$$

$$(3.6) \quad E_p = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos \theta + 1) \quad . \quad [\text{J}]$$

## Energy Control

A Lyapunov function candidate is proposed,

$$(3.7) \quad V = \frac{1}{2}E_\Delta^2 \quad ,$$

where  $E_\Delta$  is the difference in energy in relation to the unstable equilibrium,

$$(3.8) \quad E_\Delta = E_p - E_{\text{eq}} \quad [\text{J}]$$

$$(3.9) \quad E_\Delta = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos \theta + 1) - 2mgl \quad [\text{J}]$$

$$(3.10) \quad E_\Delta = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos \theta - 1) \quad . \quad [\text{J}]$$

The derivative of  $E_\Delta$  from Equation 3.10 along the system Equation 3.1 is found to,

$$(3.11) \quad \dot{E}_\Delta = J\dot{\theta}\ddot{\theta} - mgl \sin \theta \dot{\theta}$$

$$(3.12) \quad \dot{E}_\Delta = \dot{\theta}(ml \cos \theta \ddot{x}_c + mgl \sin \theta) - mgl \sin \theta \dot{\theta}$$

$$(3.13) \quad \dot{E}_\Delta = ml \cos \theta \dot{\theta} \ddot{x}_c \quad .$$

The Lyapunov function candidate, Equation 3.7, is continuously differentiable in the entire  $\mathbb{R}^{2 \times 1}$ . Its derivative is evaluated to find a stabilizing controller,

$$(3.14) \quad \dot{V} = E_\Delta \dot{E}_\Delta$$

$$(3.15) \quad \dot{V} = E_\Delta ml \cos \theta \dot{\theta} \ddot{x}_c \leq 0 \quad .$$

The acceleration,  $\ddot{x}_c$ , is then designed to satisfy the stability criterion in Equation 3.15,

$$(3.16) \quad \dot{V} = mlE_\Delta \cos \theta \dot{\theta}(-E_\Delta \cos \theta \dot{\theta})$$

$$(3.17) \quad \dot{V} = -ml(E_\Delta \cos \theta \dot{\theta})^2 \leq 0 \quad ,$$

further a tuning parameter,  $k > 0$ , is introduced such that the control law for acceleration of the pivot point is,

$$(3.18) \quad \ddot{x}_c = -kE_\Delta \cos \theta \dot{\theta} \quad .$$

If this control law is started at zero angular velocity,  $\dot{\theta} = 0$ , in a stable equilibrium, the computed control is maintained at zero and the pendulum never swings up. So for this

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<sup>1</sup>Fixme Note: brush up on stability criteries

control law to work, the pendulum must be started slightly away from a stable equilibrium. The control is also zero when  $\cos \theta = 0$ , however, when this occurs the system is not in a stable equilibrium and the zero value of  $\ddot{x}_c$  is therefore not maintained in these cases.

An extra step is needed to apply this control strategy. So far the control output is an acceleration,  $\ddot{x}_c$ , at the pivot point. in eqREF it is possible to input the desired acceleration,  $\ddot{x}_c$ , and solve for the force,

$F$  dynam eq .

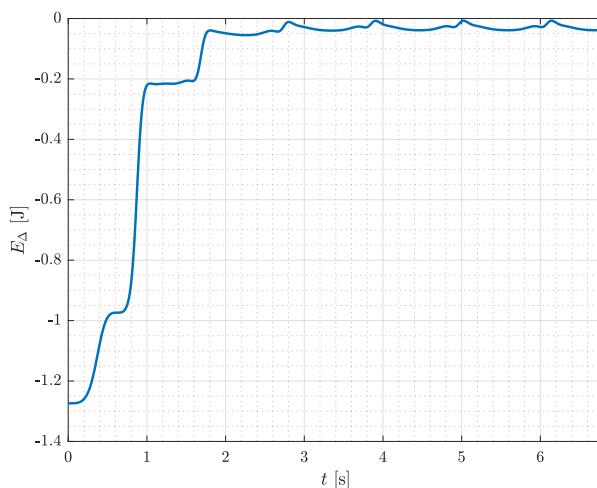
To calculate the force from this expression, eqREF, it is also necessary to know the angular acceleration of the pendulum,  $\ddot{\theta}$ , which can be solved for in the system dynamics, eqREF, inserting known states and control input applied in the previous step,

theta dot dot  $F$  previous eq .

Finally the control law,

$u$  eq .

<sup>2</sup> Initializing the angle,  $\theta$ , at  $\pi - 0.1$  to avoid zero control output, simulation, the energy difference almost reaches its reference at zero, see Figure 3.2.



**Figure 3.2:** Edelta1noConX

The pendulum also almost reaches the heteroclinic orbit, see Figure 3.3. <sup>3</sup> Some energy is lost to friction in the pendulum while the cart is considered to be frictionless as it should be handled by the friction compensation.<sup>4</sup>

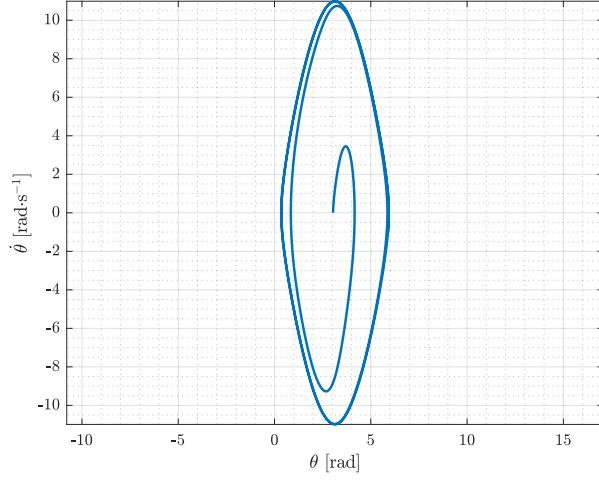
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<sup>2</sup>FiXme Note: formulate properly from here and onward

<sup>3</sup>FiXme Note: include small section on heteroclinic orbit

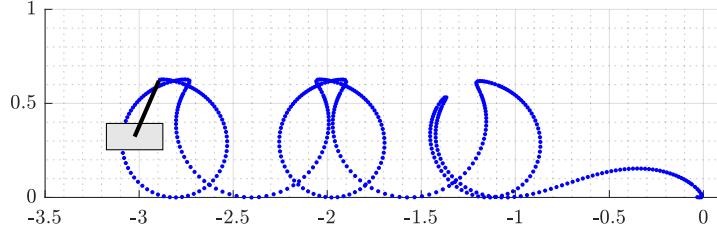
<sup>4</sup>FiXme Note: section on friction compensation

### Chapter 3. Swing-Up Design



**Figure 3.3:** phase1noConX

However, as the energy of the pendulum is not affected by the position or velocity of the cart, this control law, Equation 3.18, is not concerned controlling these. This becomes a problem in the physical setup as it has a rail length of one meter. A traced animation is used to demonstrate this problem in Figure 3.4.



**Figure 3.4:** ani1noConX

An other issue is the actuation which is limited in the real system by the maximum allowed continuous current, see Table 2.1. Figure 3.5 clearly shows how this controller violates the physical limitations of the motor.

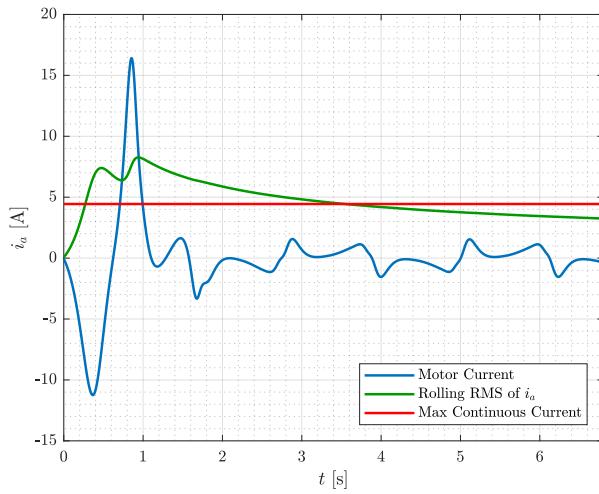


Figure 3.5: ia1noConX

## Sign-Based Energy Control

## Sat-Based Energy Control

# **Part II**

## **Twin Pendulum**

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# List of Corrections

Note: brush up on stability criteries . . . . .	9
Note: friction compensation - to - no friction on cart . . . . .	10
Note: formulate properly from here and onward . . . . .	10
Note: include small section on heteroclinic orbit . . . . .	10