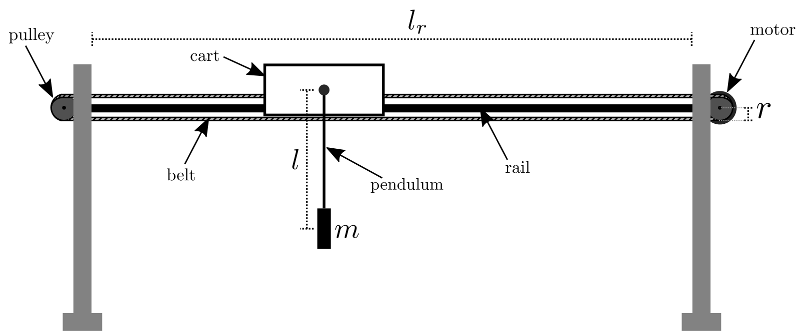
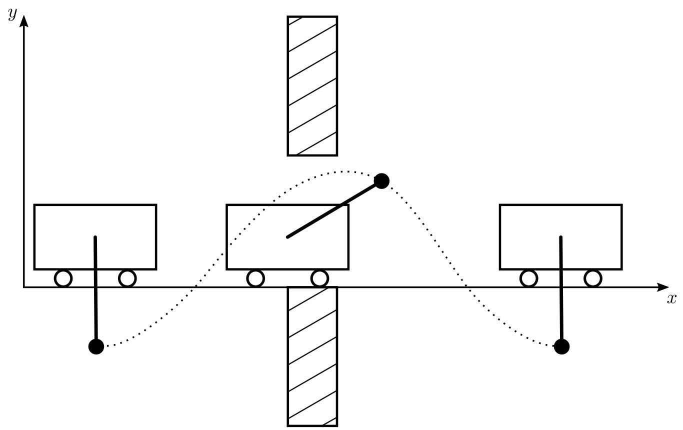
Sliding Mode Stabilization and Phase Plane Trajectory Planning for a Cart Pendulum System

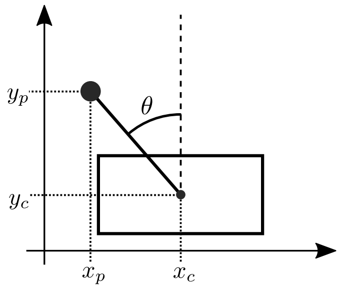
Introduction
The System

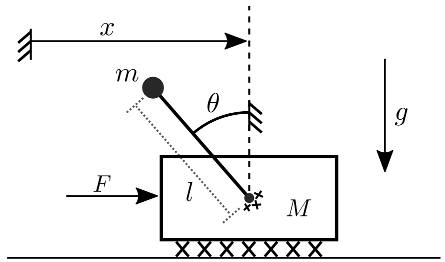


Introduction
Task



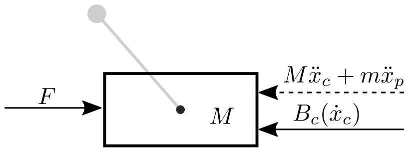
Modeling
Conventions and Assumptions

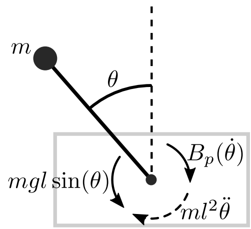




$$\begin{cases} x_c &= x \\ y_c &= 0 \end{cases} \quad \begin{cases} x_p &= x - l\sin\theta \\ y_p &= l\cos\theta \end{cases} \quad \begin{cases} \dot{x}_p &= \dot{x} - l\cos\theta\dot{\theta} \\ \dot{y}_p &= -l\sin\theta\dot{\theta} \end{cases} \quad \begin{cases} \ddot{x}_p = \ddot{x} + l\sin\theta\dot{\theta}^2 - l\cos\theta\ddot{\theta} \\ \ddot{y}_p = -l\cos\theta\dot{\theta}^2 - l\sin\theta\ddot{\theta} \end{cases}$$

Modeling
Newton's Method





$$M\ddot{x}_c + m\ddot{x}_p = F - B_c(\dot{x}_c) \qquad ml^2\ddot{\theta} = mgl\sin\theta - B_p(\dot{\theta})$$

$$-ml\ddot{x}_p\cos\theta - ml\ddot{y}_p\sin\theta = mgl\sin\theta - B_p(\dot{\theta})$$

$$\begin{cases} ml^2\ddot{\theta} - ml\cos\theta\ddot{x} - mgl\sin\theta & = -B_p(\dot{\theta}) \\ (M+m)\ddot{x} + ml\sin\theta\dot{\theta}^2 - ml\cos\theta\ddot{\theta} & = F - B_c(\dot{x}) \end{cases}$$

Modeling
Energy Method

$$U = mg\overbrace{l(1+\cos\theta)}^{h} + 0 \qquad U = mgl(1+\cos\theta)$$

$$T = \tfrac{1}{2}m\dot{x}_p^2 + \tfrac{1}{2}m\dot{y}_p^2 + \tfrac{1}{2}M\dot{x}_c^2 \qquad T = \frac{1}{2}(M+m)\dot{x}^2 - m\dot{x}l\cos\theta\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2$$

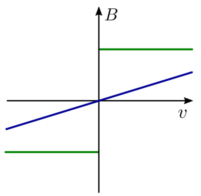
$$\mathcal{L} = T - U$$

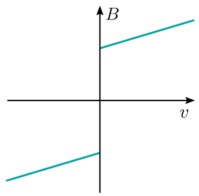
$$\mathcal{L} = \tfrac{1}{2}(M+m)\dot{x}^2 - m\dot{x}l\cos\theta\dot{\theta} + \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl(1+\cos\theta)$$

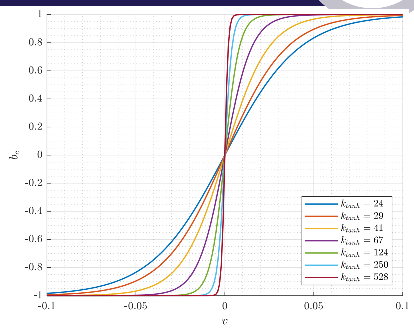
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} - \frac{\partial\mathcal{L}}{\partial\mathbf{q}} = \mathbf{Q}$$

$$\begin{cases} ml^2\ddot{\theta} - ml\cos\theta\ddot{x} - mgl\sin\theta & = -B_p(\dot{\theta}) \\ (M+m)\ddot{x} + ml\sin\theta\dot{\theta}^2 - ml\cos\theta\ddot{\theta} & = F - B_c(\dot{x}) \end{cases}$$

Modeling
Friction Model







$$B_p(\dot{\theta}) = b_{p,v}\dot{\theta} + \mathrm{sgn}(\dot{\theta})b_{p,c}$$
$$B_c(\dot{x}) = b_{c,v}\dot{x} + \mathrm{sgn}(\dot{x})b_{c,c}$$

$$B_p(\dot{\theta}) = b_{p,v}\dot{\theta} + \tanh(k_{\mathrm{tanh}}\dot{\theta})b_{p,c}$$
$$B_c(\dot{x}) = b_{c,v}\dot{x} + \tanh(k_{\mathrm{tanh}}\dot{x})b_{c,c}$$

Nonlinear Control
System Transformation

$$\begin{bmatrix} ml^2 & -ml\cos\theta \\ -ml\cos\theta & M+m \end{bmatrix}\begin{bmatrix}\ddot{\theta} \\ \ddot{x}\end{bmatrix} + \begin{bmatrix} 0 \\ ml\sin\theta\dot{\theta}^2 \end{bmatrix} + \begin{bmatrix} B_p(\dot{\theta}) \\ B_c(\dot{x}) \end{bmatrix} + \begin{bmatrix} -mgl\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{B}(\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{F} \qquad \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} \theta & x & \dot{\theta} & \dot{x} \end{bmatrix}^T$$

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})\left(\mathbf{F} - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) - \mathbf{B}(\dot{\mathbf{q}}) - \mathbf{G}(\mathbf{q})\right)$$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \mathbf{M}^{-1}(x_1)\left(-\mathbf{C}(x_1,x_3) - \mathbf{B}(x_3,x_4) - \mathbf{G}(x_1)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{M}^{-1}(x_1)\begin{bmatrix}0 \\ F\end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4\end{bmatrix} = \underbrace{\begin{bmatrix} x_3 \\ x_4 \\ f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}}_{f(\mathbf{x})} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{\cos x_1}{l(M+m-m\cos^2 x_1)} \\ \frac{1}{M+m-m\cos^2 x_1} \end{bmatrix}}_{g(\mathbf{x})} F$$

Nonlinear Control
System Transformation

$$y = h(\mathbf{x}) = x_1$$
$$\dot{y} = \dot{x}_1 = x_3$$
$$\ddot{y} = \dot{x}_3 = f_1(\mathbf{x}) + \frac{\cos x_1}{l(M+m-m\cos^2 x_1)}F \quad \Rightarrow \quad \rho = 2$$
$$T(\mathbf{x}) = \begin{bmatrix} \phi(\mathbf{x}) \\ \hdashline \psi(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \phi_1(\mathbf{x}) \\ \phi_2(\mathbf{x}) \\ \hdashline h(\mathbf{x}) \\ L_f h(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \phi_1(\mathbf{x}) \\ \phi_2(\mathbf{x}) \\ \hdashline x_1 \\ x_3 \end{bmatrix}$$
$$\frac{\partial \phi_i}{\partial \mathbf{x}} g(\mathbf{x}) = 0\,, \quad \text{for } 1 \le i \le 2$$
$$\frac{\partial \phi_2}{\partial x_3} \cdot \frac{\cos x_1}{l(M+m-m\cos^2 x_1)} + \frac{\partial \phi_2}{\partial x_4} \cdot \frac{l}{l(M+m-m\cos^2 x_1)} = 0$$
$$\frac{\partial \phi_2}{\partial x_3} = l$$
$$\frac{\partial \phi_2}{\partial x_4} = -\cos x_1$$
$$\phi_2 = l\int dx_3 - \cos x_1 \int dx_4$$
$$\phi_2 = l x_3 - \cos x_1 x_4 + \mathrm{C}_1 \quad , \quad \phi(0) = 0 \quad \Rightarrow \quad \mathrm{C}_1 = 0$$

Nonlinear Control
System Transformation

$$T(\mathbf{x}) = \begin{bmatrix} x_2 \\ l x_3 - \cos x_1 x_4 \\ x_1 \\ x_3 \end{bmatrix} \quad ,$$
$$\frac{d}{dt}T(\mathbf{x}) = \begin{bmatrix} \dot{x}_2 \\ l\dot{x}_3 + \sin x_1 x_4 \dot{x}_1 - \cos x_1 \dot{x}_4 \\ \dot{x}_1 \\ \dot{x}_3 \end{bmatrix}$$
$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \hdashline \dot{\xi} \end{bmatrix} = \overbrace{\underbrace{\begin{bmatrix} x_4 \\ \sin x_1 x_4 x_3 + l f_1(\mathbf{x}) - \cos x_1 f_2(\mathbf{x}) \\ x_3 \\ \hdashline f_1(\mathbf{x}) \end{bmatrix}}_{f_b}}^{f_a} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \hdashline \frac{\cos x_1}{l(M+m-m\cos^2 x_1)} \end{bmatrix}}_{g_b} F$$
$$\dot{\boldsymbol{\eta}} = f_a(\boldsymbol{\eta}, \xi)$$

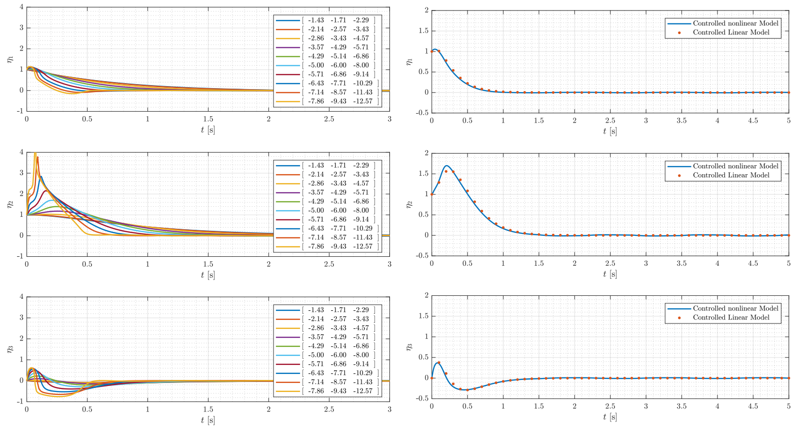
Nonlinear Control
System Transformation

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \xi \end{bmatrix} = \begin{bmatrix} x_2 \\ l x_3 - \cos x_1 x_4 \\ x_1 \\ x_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \eta_3 \\ \eta_1 \\ \xi \\ \frac{l\xi - \eta_2}{\cos \eta_3} \end{bmatrix}$$
$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \frac{l\xi - \eta_2}{\cos \eta_3} \\ \frac{\sin \eta_3}{\cos \eta_3}(l\xi - \eta_2)\xi + l f_1(\boldsymbol{\eta}, \xi) - \cos \eta_3 f_2(\boldsymbol{\eta}, \xi) \\ \xi \\ f_1(\boldsymbol{\eta}, \xi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\cos \eta_3}{l(M+m-m\cos^2 \eta_3)} \end{bmatrix} F$$

Nonlinear Control
Sliding Mode

$$s = \xi - \phi(\boldsymbol{\eta}) = 0$$
$$\dot{\boldsymbol{\eta}} = f_a(\boldsymbol{\eta}, \phi(\boldsymbol{\eta}))$$
$$A = \left.\frac{\partial \dot{\boldsymbol{\eta}}}{\partial \boldsymbol{\eta}}\right|_{\substack{\boldsymbol{\eta}=\mathbf{0} \\ \xi=0 \\ \mathrm{k_{tanh}}=1}} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & \frac{g_{p,c}}{lm} & g \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad B = \left.\frac{\partial \dot{\boldsymbol{\eta}}}{\partial \xi}\right|_{\substack{\boldsymbol{\eta}=\mathbf{0} \\ \xi=0 \\ \mathrm{k_{tanh}}=1}} = \begin{bmatrix} l \\ \frac{-b_{p,v} - b_{p,c}}{lm} \\ 1 \end{bmatrix}$$
$$\phi(\boldsymbol{\eta}) = -\mathbf{k}\boldsymbol{\eta}$$

Nonlinear Control
Sliding Mode



Nonlinear Control
Sliding Mode

$$V = \tfrac{1}{2}s^2$$

$$\dot{V} = s\dot{s}$$

$$\dot{V} = s(\dot{\xi} + \mathbf{k}\dot{\eta})$$

$$\dot{V} = g_b(\eta,\xi)s(\mathbf{k}f_a(\eta,\xi) + f_b(\eta,\xi))g_b^{-1}(\eta,\xi) + g_b(\eta,\xi)sF$$

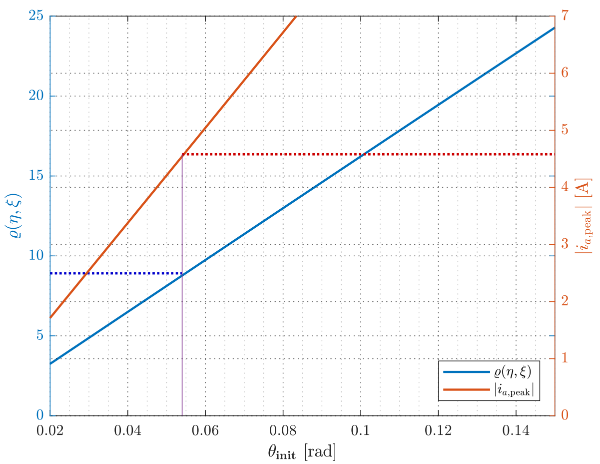
$$\dot{V} \leq g_b(\eta,\xi)|s|\left|\mathbf{k}f_a(\eta,\xi)g_b^{-1}(\eta,\xi) + f_b(\eta,\xi)\right| + g_b(\eta,\xi)sF$$

$$\dot{V} \leq g_b(\eta,\xi)|s|\,|\mathbf{k}f_a(\eta,\xi) + f_b(\eta,\xi)|\,g_b^{-1}(\eta,\xi)$$
$$- g_b(\eta,\xi)\mathrm{sgn}(s)s\,|\mathbf{k}f_a(\eta,\xi) + f_b(\eta,\xi)|\,g_b^{-1}(\eta,\xi)$$

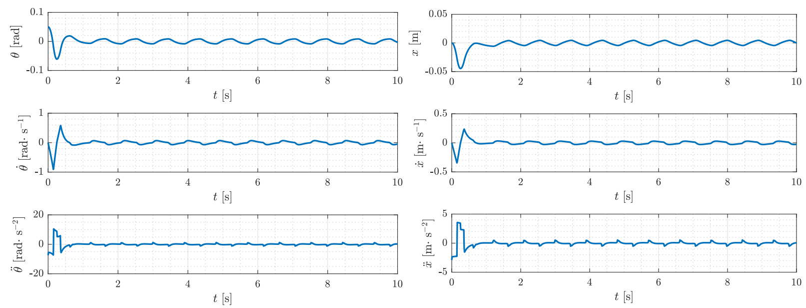
$$F = -\mathrm{sgn}(s)\varrho(\eta,\xi)g_b^{-1}(\eta,\xi) \quad \text{where,} \quad \varrho(\eta,\xi) \geq |\mathbf{k}f_a(\eta,\xi) + f_b(\eta,\xi)|$$

$$F = -\mathrm{sat}(\tfrac{s}{\varepsilon})\beta(\eta,\xi)g_b^{-1}(\eta,\xi) \quad \text{where,} \quad \beta(\eta,\xi) = \varrho(\eta,\xi) + \beta_0$$

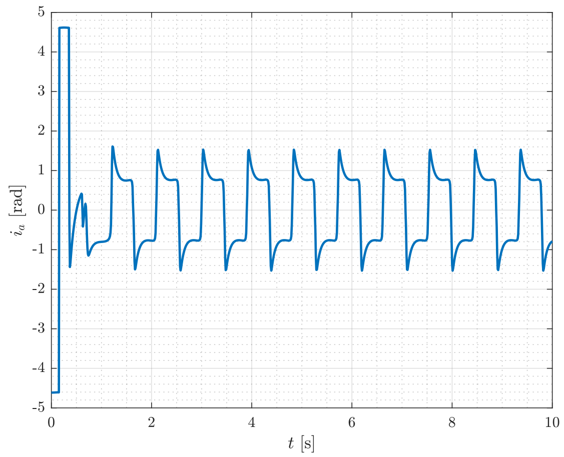
Nonlinear Control
Sliding Mode



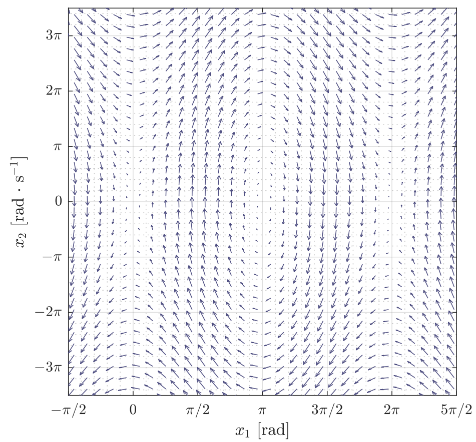
Nonlinear Control
Simulation



Nonlinear Control
Simulation



Trajectory Planning
System



$$\begin{cases} ml^2\ddot{\theta} - ml\cos\theta\ddot{x} - mgl\sin\theta = 0 \\ (M+m)\ddot{x} + ml\sin\theta\dot{\theta}^2 - ml\cos\theta\ddot{\theta} = F \end{cases}$$

Trajectory Planning
Integral of System

$$\underbrace{\left(ml^2 - \frac{m^2l^2}{M+m}\cos^2\theta\right)}_{\alpha(\theta)}\ddot{\theta} + \underbrace{\left(\frac{m^2l^2}{M+m}\cos\theta\sin\theta\right)}_{\beta(\theta)}\dot{\theta}^2 \underbrace{-mgl\sin\theta - \frac{ml}{M+m}\cos\theta F}_{\gamma(\theta)} = 0$$

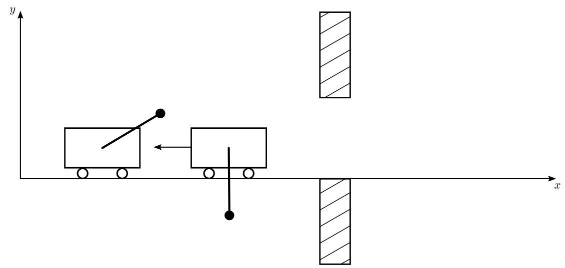
$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \int \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta)\, d\theta$$

$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \dot{\theta}^2 - \exp\left[-2\int_{\theta_0}^{\theta}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right]\left(\dot{\theta}_0^2 - \int_{\theta_0}^{\theta}\exp\left[2\int_{\theta_0}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right]\frac{2\gamma(s)}{\alpha(s)}ds\right)$$

Trajectory Planning
Trajectories

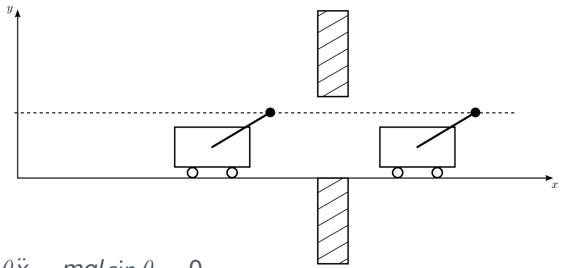
$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \dot{\theta}^2 - \exp\left[-2\int_{\theta_0}^{\theta}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right]\left(\dot{\theta}_0^2 - \int_{\theta_0}^{\theta}\exp\left[2\int_{\theta_0}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right]\frac{2\gamma(s)}{\alpha(s)}ds\right)$$



$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \frac{ml^2 - \frac{m^2l^2}{M+m}}{ml^2 - \frac{m^2l^2}{M+m}\cos^2\theta}\left(-4\left[\frac{Mg + mg - F\tan\frac{s}{2}}{Ml(\tan^2\frac{s}{2} + 1)}\right]_{\theta_0}^{\theta}\right)$$

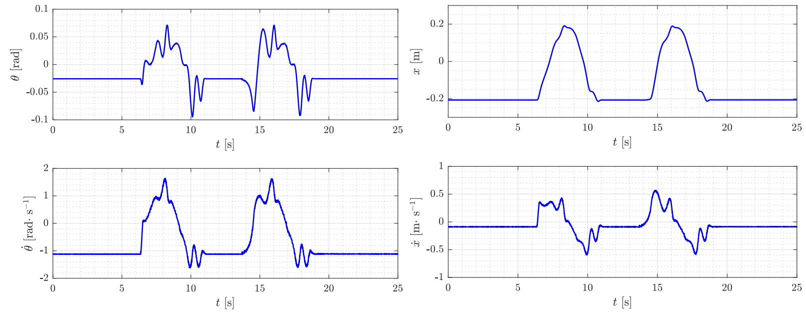
Trajectory Planning
Solution

$$\begin{cases} ml^2\ddot{\theta} - ml\cos\theta\ddot{x} - mgl\sin\theta = 0 \\ (M+m)\ddot{x} + ml\sin\theta\dot{\theta}^2 - ml\cos\theta\ddot{\theta} = F \end{cases}$$

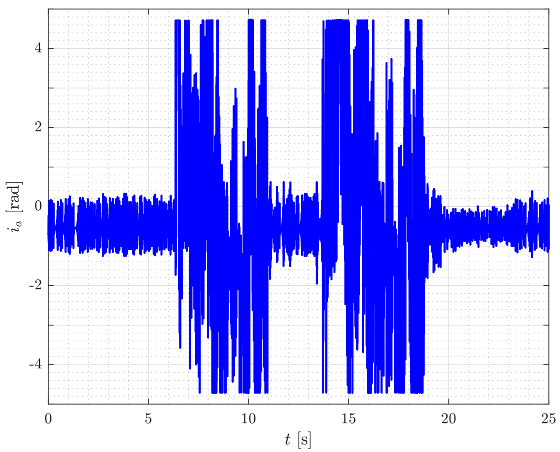


$$\begin{cases} -ml\cos\theta\ddot{x} - mgl\sin\theta = 0 \\ (M+m)\ddot{x} = F \end{cases}$$

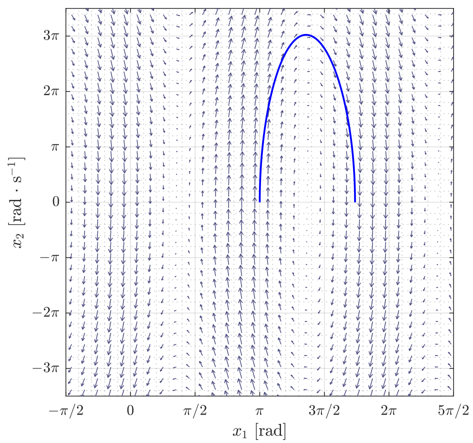
Results
Sliding Mode

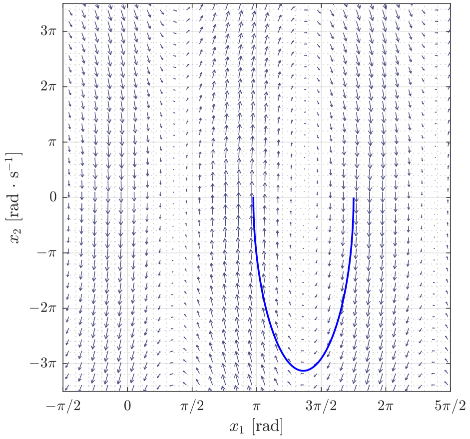


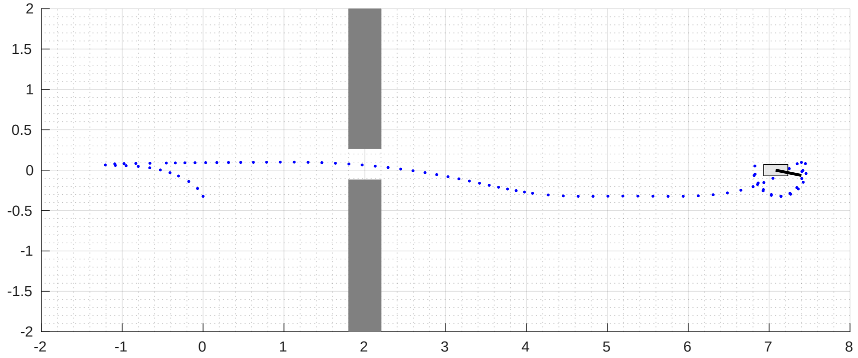
Results
Sliding Mode



Results
Trajectory Planning







Sliding Mode Stabilization and Phase Plane Trajectory Planning for a Cart Pendulum System

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