Exercises Week 6

Advanced Machine Learning (02460)
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1 Theoretical exercises

Exercise 6.1 (Spherical metric) Let $x_1 \in (0, \pi)$ and $x_2 \in [0, 2\pi)$ be the latent coordinates of the manifold spanned by the function $f : (0, \pi) \times [0, 2\pi) \to \mathbb{R}^3$

$$f(x_1, x_2) = \begin{pmatrix} \sin(x_1)\cos(x_2) \\ \sin(x_1)\sin(x_2) \\ \cos(x_1) \end{pmatrix}. \tag{1}$$

This will span the unit sphere.

1. *Derive the Jacobian matrix of f.*
2. *Derive the metric associated with f according to Eq. 5.21 in the DGGM book.*
3. *Show that the metric is positive definite.*

Exercise 6.2 (Quadratic metric) Consider a two-dimensional abstract manifold with the metric

$$\mathbf{G}_\mathbf{x} = \left(1 + \|\mathbf{x}\|^2\right)\mathbf{I}, \qquad \mathbf{x} \in \mathbb{R}^2. \tag{16}$$

Consider the points

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \tag{17}$$

1. *Compute the local norms of the tangent vector $\mathbf{v}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^\intercal$ assuming the point of tangency is $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, respectively.*
2. *Compute the local angles between $\mathbf{v}_1$ and $\mathbf{v}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^\intercal$ in the same three points of tangency.*

Exercise 6.3 (Euclidean metric) Consider the Euclidean metric of $\mathbb{R}^d$, i.e. $\mathbf{G} = \mathbf{I}$.

1. *Derive the coefficients of the geodesics* ODE *of this metric (Eq. 7.20 in the DGGM book).*
2. *Derive the geodesic* ODE*.*
3. *What is the geodesic that connects points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*?*

Exercise 6.4 (Quadratic metric) Consider a two-dimensional abstract manifold with the metric

$$\mathbf{G}_{\mathbf{x}} = \left(1 + \|\mathbf{x}\|^2\right)\mathbf{I}, \qquad\qquad \mathbf{x} \in \mathbb{R}^2. \tag{33}$$

1. *Derive the coefficients of the geodesics* ODE *of this metric (Eq. 7.20 in the DGGM book). Hint: note that some terms of the metric are zero, which renders some coefficients to be zero as well.*
2. *Derive the geodesic* ODE*.*
3. *Consider a geodesic* c *starting at* $c_0 = \mathbf{0}$ *and initial velocity* $\dot{c}_0 = \mathbf{v}$*. What is the acceleration* $\ddot{c}_0$*?*

2 Programming exercises

Exercise 6.5 (Curve parametrizations) Consider a two-dimensional abstract manifold with the metric

$$\mathbf{G}_{\mathbf{x}} = \left(1 + \|\mathbf{x}\|^2\right)\mathbf{I}, \qquad\qquad \mathbf{x} \in \mathbb{R}^2. \tag{47}$$

1. *Implement direct energy minimization for computing geodesics using piecewise straight lines to parameterize the solution curve.*
2. *Extend the previous implementation to also support third-order polynomials to parametrize the solution curve.*

Exercise 6.6 (Density metrics) Consider the dataset available at

http://hauberg.org/weekendwithbernie/toybanana.npy

This consist of $N = 992$ observations in $\mathbb{R}^2$. Consider the metric over $\mathbb{R}^2$ defined as

$$\mathbf{G}_{\mathbf{x}} = \frac{1}{p(\mathbf{x}) + \epsilon} \tag{48}$$

$$p(\mathbf{x}) = \frac{1}{N}\sum_{n=1}^{N} \mathcal{N}(\mathbf{x}|\mathbf{x}_n, \sigma^2\mathbf{I}), \tag{49}$$

where $\sigma = 0.1$, and $\epsilon = 10^{-4}$ avoids dividing by zero.

1. *Implement direct energy minimization for computing geodesics using piecewise straight lines to parameterize the solution curve.*
2. *Extend the previous implementation to also support third-order polynomials to parametrize the solution curve.*