Homework exercise II

Filtering and Identification (sc42025)

Hand in your solutions at the end of the lecture on 29-11-2017 (only hard-copy solutions are accepted). Please highlight your final answer.

Exercise 1: Minimum-Variance Unbiased Estimator

The goal of this exercise is to determine a minimum-variance unbiased estimate of a random vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$, such that:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ \bar{u_2} \end{bmatrix}, P \right), \quad P = \begin{bmatrix} \gamma & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix}$$

The described random vector is related to a set of (noisy) sensor measurements $y \in \mathbb{R}^N$, $N \gg 2$, as follows:

$$y = Fu + L\epsilon, \tag{1}$$

where $F \in \mathbb{R}^{N\times 2}$, $L \in \mathbb{R}^{N\times N}$ and $\epsilon \sim (0, I_N)$.

1. Given the prior information of u (i.e. $\bar{u}$ and P), we can augment (1) to form the following weighted least-squares problem:

$$\begin{aligned} \min_u \quad & w^T w \\ \text{s.t} \quad & \bar{y} = \bar{F}u + \bar{L}w \\ & w \sim (0, I) \end{aligned} \tag{2}$$

   Determine the augmented matrices $\bar{y}$, $\bar{F}$, and $\bar{L}$. When is the solution of (2) unique?

2. Now, let us assume $E[(u - \bar{u})\epsilon^T] = 0$. Determine a linear estimate $\tilde{u}$,

$$\tilde{u} = \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} y \\ \bar{u} \end{bmatrix},$$

   such that $E[(u - \tilde{u})(u - \tilde{u})^T]$ is minimized and $E[\tilde{u}] = \bar{u}$. Give a detailed solution.

Exercise 2: Fusion of two estimates

Two different test laboratories have independently conducted an experiment to retrieve information about an unknown variable $x \in \mathbb{R}^n$. The data received by each of the laboratories can be denoted as an overdetermined set of equations:

$$y_i = A_i x + \epsilon_i$$

for $i = 1, 2$, $y_i \in \mathbb{R}^N$, $E[\epsilon_i \epsilon_i^T] = \sigma_i I_N$, $E[\epsilon_1 \epsilon_2^T] = 0$, $N > n$ and $E[\epsilon_i] = 0$. The full column rank matrices A_i are deterministic and $\sigma_i \in \mathbb{R}$.

The test laboratories ask you to combine this data in order to determine an unbiased minimum variance estimate of x. To help you they provide you with the individual estimates given as:

$$\hat{x}_i = A_i^\dagger y_i$$

with $A_i^\dagger$ the pseudo-inverse of the matrix A_i.

The question to you is to optimally fuse these given estimates. That is to determine the matrices M_1 and M_2 of the fused estimate,

$$\tilde{x} = M_1 \hat{x}_1 + M_2 \hat{x}_2$$

such that $\tilde{x}$ is unbiased with minimum covariance matrix.

1. From the unbiasedness condition derive the relationship between M_1 and M_2.
2. For the condition derived in part (1), determine the matrices M_1 and M_2 such that the covariance matrix of $\tilde{x}$ is minimal. Write the estimate $\tilde{x}$ in terms of the given data σ_i, A_i and y_i (for $i = 1, 2$).
3. Show that the result of 2 equals the estimate of the least squares problem,

   $$\min_x \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x \right\|_2^2$$

   only for the case the laboratories have used identical sensors, such that $\sigma_1 = \sigma_2$.