Homework exercise IV

Filtering and Identification (SC42025)

Hand in your solutions at the end of the lecture on 13-12-2017 (only hard-copy solutions are accepted). Please highlight your final answer.

Exercise 1

We are observing a disturbed sinusoidal scalar signal of the form,

$$y(k) = \alpha\cos(\omega k + \phi) + v(k) \qquad E[v(k)v(\ell)] = \sigma^2\Delta(k-\ell) \quad E[v(k)] = 0 \quad v(k) \ \text{ergodic}$$

with the parameters α, ϕ and ω unknown.
An analysis of signals of this form has shown that the signal can be considered as the output of the autonomous state space model,

$$\begin{aligned} x(k+1) &= Ax(k) \\ y(k) &= Cx(k) + v(k) \qquad E[v(k)v(\ell)] = \sigma^2\Delta(k-\ell) \end{aligned}$$

with A and C given as,

$$A = \begin{bmatrix} 2\cos\omega & -1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The goal is to find the frequency ω from the observations $y(k)$ by means of subspace identification. For that purpose we start by constructing the Hankel matrix H_N as,

$$H_N = \begin{bmatrix} y(1) & y(2) & \cdots & y(N) \\ y(2) & y(3) & \cdots & y(j+1) \\ \vdots & & \ddots & \\ y(s) & y(s+1) & \cdots & y(N+s-1) \end{bmatrix}$$

Then do the following:

1. Let $P = \lim_{N\to\infty} \frac{1}{N} X_N X_N^T$, with $X_N = \begin{bmatrix} x(1) & x(2) & \cdots & x(N) \end{bmatrix}$, denoted by P, and let $\mathcal{O}_s$ denote the extended observability matrix $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{bmatrix}$, show that

$$\lim_{N\to\infty} \frac{1}{N} H_N H_N^T = \mathcal{O}_s P \mathcal{O}_s^T + \sigma^2 I_s$$

2. Show that when $P > 0$, we can use the SVD of the matrix H_N to derive a consistent estimate of the column space of $\mathcal{O}_s$. Derive the condition on s that is necessary for retrieving this consistent estimate.

3. From a consistent estimate of the column space of $\mathcal{O}_s$ we can estimate a matrix A_T that is similarly equivalent to the matrix A. That is,

$$A_T = TAT^{-1} \quad \text{for} \quad T \quad \text{invertible}$$

Derive the condition on s that is necessary to calculate this matrix A_T.

4. Show that the eigenvalues of the matrix A_T provide a consistent estimate of the unknown frequency ω.

Exercise 2

We consider a LTI SISO state-space model of order n be written as:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

Let α, β be two integers such that $0 \leq \alpha \leq \beta$. The input sequence is such that:

$$u(\alpha) = 1, \qquad u(\beta) = 1, \qquad \forall k \notin \{\alpha, \beta\}, u(k) = 0$$

For all $k \leq \alpha$, $x(k) = 0$. The output data is available for $k = 0, \ldots, N$. Let $s > n$.

1. Propose an algorithm to estimate the matrices A, B, C if $\alpha = \beta$.
2. We now assume $\alpha < \beta$. What are the conditions on α, β such that the Hankel matrix $U_{0,s,N}$ is full row rank? Write the data equation in the general case, $\alpha \neq \beta$.
3. Let $\gamma \in \mathbb{N}$ such that $\alpha < \gamma < \beta$. The measurement matrix is now equal to C_1 for all $k < \gamma$ and equal to C_2 when $k \geq \gamma$. Write the i-th column of a data equation (similarly as in (9.4) in [1]).
4. Select α, β and N with respect to γ, s such that the method described in Question 1 can be used to identify the pairs (C_1, A) and (C_2, A).

References

[1] M. Verhaegen and V. Verdult, “Filtering and System Identification: A Least Squares Approach”, Cambridge University Press, 2007.