Solutions Control Theory Set 7 (2010)

Exercise 1

a) By suitably partitioning we infer

$$\operatorname{eig}(A) = \operatorname{eig}\left(\begin{array}{ccc|cc} 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ \hline 0 & -0.1 & -2 & 0 & -0.1 \\ 0 & 1 & 2 & 0 & -0.2 \end{array}\right) =$$

$$= \operatorname{eig}\left(\begin{array}{c|cc} 0 & -1 & -1 \\ \hline 0 & -1 & 0 \\ 0 & -1 & -1 \end{array}\right) \cup \operatorname{eig}\left(\begin{array}{c|c} 0 & -0.1 \\ \hline 0 & -0.2 \end{array}\right) =$$

$$= \{0\} \cup \operatorname{eig}\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \cup \{0, -0.2\} = \{0, -1, -0.2\}.$$

b) For $p = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix}$ consider

$$p\begin{pmatrix} 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -0.1 & -2 & 0 & -0.1 \\ 0 & 1 & 2 & 0 & -0.2 \end{pmatrix} = \lambda p, \quad p\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0.1 & -0.1 \\ 0 & -0.1 & 0 \end{pmatrix} = 0.$$

The latter three equations imply $p_3 = 0$, $p_4 = 0$ and $-p_1 - 0.1p_5 = 0$. The third of the first five equations hence leads to $-p_1 + 2p_5 = \lambda p_3 = 0$ and thus $p_1 = 0$, $p_5 = 0$. Hence $p = (0\ p_2\ 0\ 0\ 0)$ satisfies the equations with $\lambda = -1$. Therefore the system is not controllable, but it is stabilizable with uncontrollable mode $\{-1\}$.

c) This is clear. By removing a column of the matrix matrix, an uncontrollable system stays uncontrollable.

d) Now we have to consider

$$p\begin{pmatrix} 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -0.1 & -2 & 0 & -0.1 \\ 0 & 1 & 2 & 0 & -0.2 \end{pmatrix} = \lambda p, \quad p\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & -0.1 \\ 0 & 0 \end{pmatrix} = 0.$$

The latter two equations show $p_3 = 0$ and $p_4 = 0$. Let's now consider the first five equations. From the first we infer $0 = \lambda p_1$.

In case that $\lambda = 0$ the fifth implies $-0.2p_5 = 0$ and thus $p_5 = 0$. The third then implies $p_1 = 0$. The second leads to $p_2 = 0$.

In case that $p_1 = 0$ the third shows $p_5 = 0$. We again remain with $p = (0\ p_2\ 0\ 0\ 0)$ and $\lambda = -1$. Hence the set of uncontrollable modes is still just $\{-1\}$.

e) We have already seen how the uncontrollable mode is reflected in the system. Hence it can be suspected that a permutation of the $(2,2)$-element of the matrix A to the $(5,5)$-position might do the job. Choose

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad T^{-1} = T^T.$$

Then

$$\tilde{A} = TAT^{-1} = \left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -0.1 & -0.1 \\ 0 & 2 & 0 & -0.2 & 1 \\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right) \quad \text{and} \quad \tilde{B} = TB_1 = \left(\begin{array}{cc} 0 & 0 \\ 0.5 & 0 \\ 0 & -0.1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right).$$

We already know (we just performed permutations) that

$$\left(\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & -0.1 \\ 0 & 2 & 0 & -0.2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & -0.1 \\ 0 & 0 \end{pmatrix} \right)$$

is controllable. Hence we have found the controllability normal form.

f) Yes, since the uncontrollable modes $\{-1\}$ are all in the open left-half plane.

g) Clearly

$$\tilde{F} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & 0 & 0 \end{pmatrix}$$

does the job for the transformed system since

$$\tilde{A} - \tilde{B}\tilde{F} = \left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & -2 & -1 & -0.1 & -0.1 \\ 0 & 2 & 0 & -0.2 & 1 \\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

is Hurwtiz: Note that the matrix is block-triangular, with two 2×2 blocks on the diagonal that are obviously Hurwitz. Then

$$F = \tilde{F}T = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{pmatrix}$$

does the job for the original system.

h) No, since canceling yet one more column cannot render the system controllable. Also the matrix

$$(A\ B_2) = \left(\begin{array}{ccccc|c} 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -0.1 \\ 0 & -0.1 & -2 & 0 & -0.1 & 0 \\ 0 & 1 & 2 & 0 & -0.2 & 0 \end{array}\right)$$

does not have full row rank, just because

$$\operatorname{rank}\left(\begin{array}{ccccc|c} 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -0.1 \\ 0 & -0.1 & -2 & 0 & -0.1 & 0 \\ 0 & 1 & 2 & 0 & -0.2 & 0 \end{array}\right) = \operatorname{rank}\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -0.1 \\ -0.1 & -2 & -0.1 & 0 \\ 1 & 2 & -0.2 & 0 \end{pmatrix}$$

and the latter has at most rank 4. In conclusion 0 is an uncontrollable mode of $(A\ B_2)$ such that the pair is not stabilizable.