CS 5135/6035 Learning Probabilistic Models

Exercise Questions for Lecture 14: Conjugacy, Parameter Summarization

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Question

1. Number of crimes reported in the Clifton neighborhood in Cincinnati during each of the first 10 days of October 2018 are shown in the table below:

Date	# Crimes
Oct 1, 2018	2
Oct 2, 2018	6
Oct 3, 2018	2
Oct 4, 2018	2
Oct 5, 2018	0
Oct 6, 2018	6
Oct 7, 2018	1
Oct 8, 2018	0
Oct 9, 2018	1
Oct 10, 2018	2

Data Source: PDI (Police Data Initiative) Crime Incidents http://data.cincinnati-oh.gov

a. Assuming that these number of crimes are generated from a Poisson distribution ($y \sim Poisson(\lambda)$), derive the expression for the data likelihood. **[3 points]**

b. Let us use an uninformative prior $p(\lambda) = k$ for the parameter of interest λ (mean number of crime incidents in Clifton). What is the integral of this prior over all possible values of λ ($0 \leq \lambda \leq \infty$)? Is it even a 'proper' prior? **[2 points]**

c. Using an uninformative prior $p(\lambda) = k$, derive the expression for the Posterior (using the easy way). **[6 points]**

[*Hint:* Priors that do not integrate to 1 are called *improper priors*. It turns out that even when a prior does not integrate to 1, posteriors are bound to integrate to 1 and so it is fine to use an 'improper' prior for parameter estimation.]

d. Write Julia code to compute mean, median, and the MAP estimates for the posterior distribution obtained in (c). Write Julia code to plot the prior, posterior, and the point estimates. **[3 points]**

e. Using the informative prior $p(\lambda) = Gamma(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\lambda^{\alpha-1}e^{-\lambda/\beta}$, derive the expression for the Posterior (using the easy way). The values of hyperparameters are $\alpha = 20$ and $\beta = 0.1$. **[5 points]**

[*Hint:* Gamma distribution is a conjugate of the Poisson distribution.]

f. Comparing the prior $p(\lambda) = Gamma(20, 0.1)$ with the posterior from (c), discuss how you the information in the *Gamma* prior can be interpreted. **[4 points]**

g. Using the MAP estimate of the posterior obtained in (e), compute the probability that there will be 4 or more crimes reported today. **[2 points]**

[*Hint:* You may write Julia code to answer this part.]

Bonus questions

1. Write an algorithm (and a Julia function) to compute the one-sided (left) and the one-sided (right) 95% credible intervals for a given probability distribution.
2. Write an algorithm (and a Julia function) to compute the highest-posterior density (95% credible interval) for a given probability distribution.
3. Write Julia code to plot the posterior distribution in 1(d) above along with four interval estimates (equal-tailed, one-sided (left), one-sided (right), and the highest-posterior density).

Sample code

1. Computing point estimates

```
using Distributions, Gadfly;
d = Beta(3,9);
[Base.mean(d) Base.median(d) Distributions.modes(d)]
```

```
## 1×3 Array{Float64,2}:
##  0.25  0.235786  0.2
```

2. Plotting posterior, and point estimates

```
x = collect(0:0.001:10);
prior = ones(length(x));
d = Beta(3,9);
median_val = Base.median(d);
mean_val = Base.mean(d);
mode_val = Distributions.modes(d);
posterior1 = pdf.(d,x);
myplot = Gadfly.plot(
        layer(x=x,y=posterior1,Geom.line,Theme(default_color=colorant"purple")),
        layer(xintercept=[mean_val],Geom.vline(color=colorant"blue")),
        layer(xintercept=[median_val],Geom.vline(color=colorant"red")),
        layer(xintercept=[mode_val[1]],Geom.vline(color=colorant"green")),
        Coord.Cartesian(xmin=0, xmax=1,ymax=4), Guide.ylabel("probability"),
        Guide.xlabel(" "), Guide.manual_color_key("", ["p( |y) NH=2,NT=8",
        "Mean", "Median", "Mode/MAP"], ["purple","blue","red","green"]));
draw(PNG("./figs/point_estimate.png", 8inch, 4inch), myplot);
```