# CS 5135/6035 Learning Probabilistic Models

Exercise Questions for Lecture 23: Hierarchical Modeling, Application of Gibbs Sampling

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## Questions

1. Draw plate diagrams for the following models
   a. Model 1 **[2 points]**
      - $\theta \sim Beta(\alpha, \beta)$
      - $y \sim Binomial(n, \theta)$
   
   b. Model 2 **[2 points]**
      - $\theta \sim Beta(\alpha, \beta)$
      - $y_i \sim Binomial(n, \theta)$, where $i = 1, \ldots, N$
   
   c. Model 3 **[3 points]**
      - $\theta_i \sim Beta(\alpha, \beta)$, where $i = 1, \ldots, N$
      - $y_{ij} \sim Binomial(n, \theta_i)$, where $j = 1, \ldots, M$

2. **Change-point Model**
   Suppose we are observing a sequence of counts $y_1, y_2, \ldots, y_n$, where the average of the counts has some value for time steps 1 to $k-1$ and a different value for time steps $k$ to $n$. Given the observed data, the goal is to estimate the timepoint $k$ at which this change happened, and the mean of the counts before and after the change. For example, when the counts are $1, 2, 1, 3, 1, 8, 10, 7, 6, 9$, $k = 6$.
   - Assume that the counts $y_i$ at each time step follow a Poisson distribution.

$$y_i \sim \begin{cases} Poisson(y_i; \lambda_1) & 1 \leq i \leq k-1 \\ Poisson(y_i; \lambda_2) & k \leq i <= n \end{cases}$$

   a. Write the expression for the likelihood. **[3 points]**
   
   b. Assume a Gamma prior over $\lambda_1$ and $\lambda_2$. **[3 points]**

$$\lambda \sim Gamma(\lambda; a, b) = \frac{1}{\Gamma(a)} b^a \lambda^{a-1} e^{-b\lambda}$$

   Assume a Uniform prior over $k$: $k \sim Uniform(1, 2, \ldots, n)$.
   Write the expression for the posterior, assuming a default prior.
   
   c. Write the full conditional for $\lambda_1$. **[3 points]**
   [*Hint:* This will be a Gamma distribution.]
   
   d. Write the full conditional for $\lambda_2$. **[3 points]**
   [*Hint:* This will be a Gamma distribution.]
   
   e. Write the full conditional for $k$. **[3 points]**
   [*Hint:* This will not be a distribution.]
   
   f. Write Julia code to generate data from this above generative model and plot the resultant sequence of counts for 100 time steps. **[3 points]**

## Bonus question

1. Develop a strategy to estimate the parameters for the model in Question 2 (above). Specifically, what sampling algorithm will you use, what choices will you make in terms of proposal densities or how do

you intend to sample from full-conditionals? Your answers may vary depending upon the choice of your sampling algorithm.

2. Write Julia code to sample from the posterior distribution and determine point estimates for $k$, $\lambda_1$, $\lambda_2$.