CS 5135/6035 Learning Probabilistic Models

Exercise Questions for Lecture 22: Markon Chain Monte Carlo Methods II

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Question

1. Gibbs sampling over a Binomial-Beta-Poisson joint distribution.

$$\text{Let } \theta \sim Beta(a,b),\ n \sim Poisson(\lambda),\ x \sim Binomial(n,\theta),\ \text{ where } a=5, b=5, \lambda=4$$

$$Beta(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1};\ Poisson(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!};\ Binomial(n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

   a. Write the joint distribution of θ, n, x, i.e., $f(\theta, n, x)$. **[3 points]**

   b. If our goal is to draw samples from a marginal $f(x)$, how would you have accomplished it prior to knowing MCMC methods? What challenges do you encounter if you were to pursue it? **[3 points]**

   c. If our goal is to draw samples from a marginal $f(x)$, how would you have accomplished it prior to knowing Gibbs sampling? What challenges do you encounter if you were to pursue it? [**1 point**]

   d. Write the expression for the conditional $f(x|\theta, n)$. **[3 points]**

   e. Write the expression for the conditional $f(\theta|x, n)$. **[3 points]**
   [*Hint:* this will have a functional form of a Beta distribution.]

   f. Write the expression for the conditional $f(n|\theta, x)$. **[3 points]**
   [*Hint:* the conditional distribution of a tranformed version of n will have a functional form of a Poisson distribution. First derive the Poisson expression and then write the necessary transformation of n.]

   g. Write the algorithm for Gibbs sampling using these conditionals. **[3 points]**

   h. Write the Julia code to generate samples from the joint distribution. **[3 points]**

   i. Plot the histogram of the samples from the marginal $f(x)$. **[3 points]**

Bonus Questions

1. Draw samples from the above joint distribution using a Random-walk Metropolis-Hastings algorithm.
   a. Write the proposal distribution.
   b. Write the equation for acceptance probability.
   c. Write the Random-walk Metropolis-Hastings algorithm.
   d. Write the Julia code and plot the histogram of the samples.
   e. Plot the trace in 3D.
2. Draw samples from the above joint distribution using an Independent Metropolis-Hastings algorithm.
   a. Write the proposal distribution.
   b. Write the equation for acceptance probability.
   c. Write the Random-walk Metropolis-Hastings algorithm.
   d. Write the Julia code and plot the histogram of the samples.
   e. Plot the trace in 3D.

Sample code

1. Julia code for Gibbs sampling

```
N = 10000; n = 10; a = 5; b =5;
theta = zeros(N);
x = zeros(N);
theta[1] = rand(Beta(a,b));
x[1] = rand(Binomial(n,theta[1]));
for i=2:N
  theta[i] = rand(Beta(x[i-1]+a,n-x[i-1]+b));
  x[i] = rand(Binomial(n,theta[i-1]));
end
```

2. Julia code for plotting histogram of samples

```
hist_x = [fit(Histogram,x,collect(0:1:10)).weights; 0]./10150;
hist_theta = [fit(Histogram,theta,collect(0:0.01:1)).weights; 0]./100;

myplot1 = plot(layer(x=collect(0:1:10),y=pdf.(BetaBinomial(10,5,5),collect(0:1:10)),
                     Geom.line,Theme(default_color=colorant"red")),
               layer(x=collect(0:1:10),y=hist_x, Geom.bar),
               Guide.ylabel("probability"),Guide.xlabel("x"),
               Coord.Cartesian(xmin=0,xmax=10),white_panel);

myplot2 = plot(layer(x=collect(0:0.01:1),y=pdf.(Beta(5,5),collect(0:0.01:1)),Geom.line,
                     Theme(default_color=colorant"red")),
               layer(x=collect(0:0.01:1),y=hist_theta, Geom.bar),
               Guide.ylabel("probability"), Guide.xlabel("theta"),
               Coord.Cartesian(xmin=0,xmax=1),white_panel);

myplot = vstack(myplot1,myplot2);
draw(PNG("./figs/gibbs_marginals.png", 5inch, 10inch), myplot);
```