CS 5135/6035 Learning Probabilistic Models

Exercise Questions for Lecture 8: MLE, Gradient Descent, Multivariate Gaussian

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2/6/2020

Questions

1. For a Beta Distribution
$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$
    a. Write the likelihood function. **[3 points]**
    b. Write the log-likelihood function and negative log-likelihood functions. **[2 points]**
    c. Write the gradient descent update equations for the parameters. **[6 points]**
    d. Write the gradient descent algorithm to estimate the parameters. **[5 points]**
    e. Write Julia code to draw 1000 random samples from a $Beta(2,2)$ and to compute $\hat{\alpha}, \hat{\beta}$, the parameter estimates, using gradient descent approach? **[5 points]**
    f. Write Julia code to test the consistency of the α, β. Based on your observations, comment on the consistency of the estimates. **[4 points]**

Bonus Questions

1. Write Julia code to
    a. Plot the heatmap of density function for a bivariate normal distribution with mean $\mu = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and covariance matrix $\Sigma = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}$. Comment on the dependence between the two random variables in the bivariate distribution based on the density plot and the covariance matrix.
    b. Sample 1000 points from the above bivariate normal distribution and generate a scatter plot of the two variables.
    c. Sample 1000 points each from the two normal distributions $a \sim Normal(\mu = 3, \sigma^2 = 25)$ and $b \sim Normal(\mu = 2, \sigma^2 = 9)$.
    d. Generate a scatter plot of the two univariate normally distributed variables (a, b). Comment on the dependence between a and b based on the scatter plot.
2. Write Julia code to
    a. Plot the heatmap of density function for a bivariate normal distribution with mean $\mu = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and covariance matrix $\Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$. Comment on the dependence between the two random variables in the bivariate distribution based on the density plot and the covariance matrix.
    b. Sample 1000 points from the above bivariate normal distribution and generate a scatter plot of the two variables.
    c. Sample 1000 points each from the two normal distributions $a \sim Normal(\mu = 3, \sigma^2 = 10)$ and $b \sim Normal(\mu = 2, \sigma^2 = 5)$.
    d. Generate a scatter plot of the two univariate normally distributed variables (a, b). Comment on the dependence between a and b based on the scatter plot.
3. For an exponential distribution ($f(x) = \lambda e^{-\lambda x}$), use gradient descent approach to compute MLE estimate for λ.

4. A particular gene occurs as one of two alleles ('A' and 'a'), where allele 'A' has frequency θ in the population. That is, a random copy of the gene is 'A' with probability θ and 'a' with probability $1-\theta$. Since a diploid genotype consists of two genes, the probability of each genotype is given by:

genotype	AA	Aa	aa
probability	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

When a random sample of 100 people are tested, 'AA' genotype was found in 20 people, 'Aa' genotype was found in 10 people, and 'aa' genotype was found in 70 people. Use gradient descent approach to compute MLE estimate for θ.