1)

a)

c)

$P(\theta_1$

$P(\theta_1, \theta_2 | y) \propto P(y | \theta_1, \theta_2)\, P(\theta_1, \theta_2)$

$P(\theta_1, \theta_2) \propto 1$

$P(\theta_1, \theta_2 | y) \propto P(y | \theta_1, \theta_2)$.

$P(\theta_1, \theta_2 | y) \propto P(x_1 | \theta_1; x_2 | \theta_2)$

$P($

$\propto$ Binomial (n_1, θ_1)

$\times$

Binomial

(n_2, θ_2)

$\propto \cancel{\binom{n_1}{x_1}}\, \theta_1^{x_1} (1-\theta_1)^{n_1 - x_1}$

$\cancel{\binom{n_2}{x_2}}\, \theta_2^{x_2} (1-\theta_2)^{n_2 - x_2}$

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b)

$$p(\theta_1 \mid x_1, x_2) = \int p(\theta_1, \theta_2 \mid x_1, x_2)\, d\theta_2$$

$$= \int \theta_1^{x_1} (1-\theta_1)^{n_1 - x_1} \quad \theta_2^{x_2} (1-\theta_2)^{n_2 - x_2}\, d\theta_2$$

$$\propto \theta_1^{x_1} (1-\theta_1)^{n_1 - x_1}$$

c) $$p(\theta_2 \mid x_1, x_2) = \int p(\theta_1, \theta_2 \mid x_1, x_2)\, d\theta_1$$

$$\propto \theta_2^{x_2} (1-\theta_2)^{n_2 - x_2}$$

2)

d)

$$p(\theta_1, \theta_2 \mid x_1, x_2)$$

$$\propto \theta_1^{x_1} (1-\theta_1)^{n_1 - x_1} \theta_2^{x_2} (1-\theta_2)^{n_2 - x_2}$$

$$\propto p(\theta_1 \mid x_1, x_2) \times p(\theta_2 \mid x_1, x_2)$$

2)

a)

$$p(y \mid \mu, \sigma^2) = (2\pi\sigma^2)^{7/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{7} (y_i - \mu)^2\right)$$

b)

$$p(\mu, \sigma^2) \propto p(\mu)\, p(\sigma^2)$$

$$\propto 1 \times \frac{1}{\sigma^2}$$

$$\propto 1/\sigma^2$$

c)

$$p(\mu, \sigma^2 \mid y) \propto p(y \mid \mu, \sigma^2)\, p(\mu, \sigma^2)$$

$$p(\mu, \sigma^2 \mid y) \propto (\sigma^2)^{-7/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{7} (y_i - \mu)^2\right) \times \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-9/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{7} (y_i - \mu)^2\right)$$

$$\propto (\sigma^2)^{-9/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{7} (y_i - \bar{y} + \bar{y} - \mu)^2\right)$$

$$\propto (\sigma^2)^{-9/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{7} (y_i - \bar{y})^2 + 7(\bar{y} - \mu)^2\right)$$

$$\propto (\sigma^2)^{-9/2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)$$

d)

d)

$$p(\sigma^2|y) = \int p(\mu, \sigma^2|y)\, d\mu$$

$$= \int (\sigma^2)^{-9/2} \exp\left(-\frac{1}{2\sigma^2}\left[6s^2 + 7(\bar{y}-\mu)^2\right]\right) d\mu$$

$$\propto (\sigma^2)^{-9/2} \exp\left(-\frac{6s^2}{2\sigma^2}\right)$$

$\propto$ InverseGamma(α, β)

where

$$\alpha = -1 - 9/2$$

$$\beta = \frac{2}{s^2 6} = \frac{1}{3s^2}$$

$\propto$ Inverse Gamma$\left(= 11/2, \frac{1}{3}s^2\right)$

e)

$$p(\mu|y) = \int p(\mu, \sigma^2|y)\, d\sigma^2$$

$$= \int p(\mu|\sigma^2, y)\, p(\sigma^2|y)\, d\sigma^2$$

$$p(\mu \mid \sigma^2, y) = \mathcal{N}(\mu \mid \mu_p, \sigma_p^2)$$

$$= \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left(-\frac{1}{2\sigma_p^2}(\mu - \mu_p)^2\right)$$

$$\mu_p = \sigma_p^2 \left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{n} y_i}{\sigma^2}\right)$$

$$\sigma_p^2 = \frac{1}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}}$$

$$p(\mu \mid y) = \int p(\mu \mid \sigma^2, y)\, p(\sigma^2 \mid y)\, d\sigma^2$$

$$= \int \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left(-\frac{1}{2\sigma^2}(\mu - \mu_p)^2\right) \text{InvGamma}(\alpha, \beta)\, d\sigma^2$$

$$= \text{InvGamma}(\alpha, \beta) \frac{\sigma^2}{(\sqrt{2\pi\sigma_p^2})} \exp\left(-\frac{1}{2\sigma_p^2}(\mu - \mu_p)^2\right)$$

Step 1

1) Compute α & β

2) Sample 1000 values $\sigma^2 \sim \text{InvGamma}(\alpha, \beta)$

Step 2 -

1) Assume prior μ_0 & σ_0^2

2) Repeat for every sample of $\sigma^2|y$

3) Compute posterior parameter μ_p, σ_p.

4) Sample a value of μ from.

$(\mu|\sigma^2, y) \ N(\mu| \mu_p, \sigma_p^2)$

5). Simply draw histogram for values of μ & then

estimate 'μ' (mean, median (or mode) from it