CS 5135/6035 Learning Probabilistic Models

Exercise Questions for Lecture 15: Natural Conjugacy, Mixture of Priors

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Questions

1. Show that the Gamma distribution is a member of the exponential family of distributions. **[6 points]**

$$Gamma(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

2. Number of crimes reported in the Clifton neighborhood in Cincinnati during each of the first 10 days of October 2018 are shown in the table below:

Date	# Crimes
Oct 1, 2018	2
Oct 2, 2018	6
Oct 3, 2018	2
Oct 4, 2018	2
Oct 5, 2018	0
Oct 6, 2018	6
Oct 7, 2018	1
Oct 8, 2018	0
Oct 9, 2018	1
Oct 10, 2018	2

Data Source: PDI (Police Data Initiative) Crime Incidents http://data.cincinnati-oh.gov
The following questions will lead you through the process of constructing a mixture of Gamma priors and determining the resultant posterior.

   a. Write the expression for a mixture of Gamma priors: $\{Gamma(8, 0.1), Gamma(16, 0.1), Gamma(24, 0.1)\}$ using mixing probabilities $\{0.33, 0.33, 0.34\}$. **[3 points]**

   b. Derive the expression to compute the marginal likelihood $p(y) = \int p(y|\theta)p(\theta)$. For the likelihood, use the likelihood derived in the previous exercise. **[8 points]**

   c. Write the equation for the posterior as a mixture of separate posteriors constructed using individual priors.

   [3 points]

   d. Write equations for computing mixing probabilities for the mixture of posteriors (as well as $p(y_i)$'s).

   [5 points]

Bonus Questions

1. For the problem in Question 2 above
   a. Write Julia code to compute the posterior parameters. Which of the priors did the posterior heavily weigh on?

b. Write Julia code to visualize the mixture of priors and the posterior distributions.

Sample code

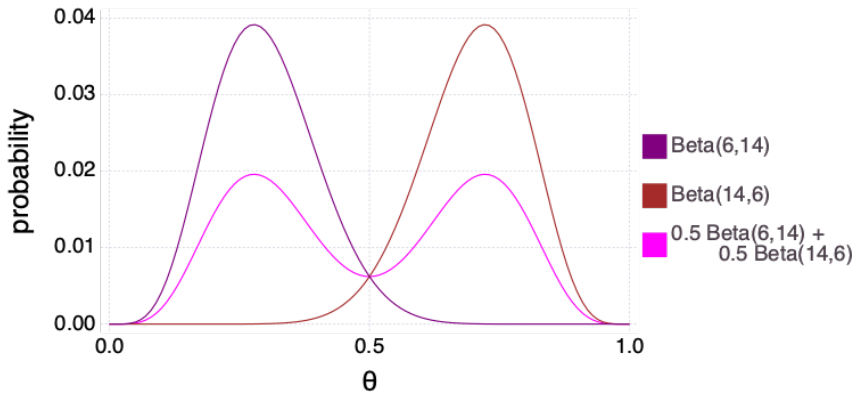
1. Visualizing a mixture of Beta priors

```
using Distributions, Gadfly;
white_panel = Theme(
    panel_fill=colorant"white",
    default_color=colorant"purple",bar_spacing=3mm,
    major_label_font_size=18pt,
    minor_label_font_size=14pt,
    key_title_font_size = 18pt,
    key_label_font_size = 14pt,
    major_label_color=colorant"black",
    minor_label_color=colorant"black"
);

prior1 = Beta(6,14);
prior2 = Beta(14,6);
mix_prior = MixtureModel(Beta, [(6, 14), (14, 6)], [0.5, 0.5]);

x = collect(0:0.01:1);
prior_val1 = pdf.(prior1,x);
prior_val1 = prior_val1./sum(prior_val1);
prior_val2 = pdf.(prior2,x);
prior_val2 = prior_val2./sum(prior_val2);
prior_mix_val = pdf.(mix_prior,x);
prior_mix_val = prior_mix_val./sum(prior_mix_val);

myplot = Gadfly.plot(
        layer(x=x,y=prior_val1,Geom.line,Theme(default_color=colorant"purple")),
        layer(x=x,y=prior_val2,Geom.line,Theme(default_color=colorant"brown")),
        layer(x=x,y=prior_mix_val,Geom.line,Theme(default_color=colorant"magenta")),
        Guide.ylabel("probability"),Guide.xlabel(" "),
        Guide.manual_color_key("", ["Beta(6,14)", "Beta(14,6)", "0.5 Beta(6,14) +
        0.5 Beta(14,6)"], ["purple","brown","magenta"]), white_panel);
draw(PNG("./figs/beta_mix.png", 8inch, 4inch), myplot);
```



2. Computing posterior parameters

```
mix_prior = MixtureModel(Beta, [(6, 14), (14, 6)], [0.5, 0.5]);
x = collect(0:0.01:1);
prior_mix_val = pdf.(mix_prior,x);
prior_mix_val = prior_mix_val./sum(prior_mix_val);

p1_y = (factorial(19)*factorial(7)*factorial(BigInt(21)));
p2_y = (factorial(19)*factorial(15)*factorial(13));
pi_1 = p1_y/(p1_y+p2_y);
pi_2 = p2_y/(p1_y+p2_y);
# Using observation that there were 2 Heads and 8 Tails
posterior_mix = MixtureModel(Beta, [(8, 22), (16, 14)], [Float64(pi_1), Float64(pi_2)])
```

```
## MixtureModel{Distributions.Beta}(K = 2)
## components[1] (prior = 1.0000): Distributions.Beta{Float64}( =8.0,  =22.0)
## components[2] (prior = 0.0000): Distributions.Beta{Float64}( =16.0,  =14.0)
```

3. Visualizing mixture of Beta priors and posterior

```
post_mix_val = pdf.(posterior_mix,x);
post_mix_val = post_mix_val./sum(post_mix_val);

myplot = Gadfly.plot(
        layer(x=x,y=prior_mix_val,Geom.line,Theme(default_color=colorant"black")),
        layer(x=x,y=post_mix_val,Geom.line,Theme(default_color=colorant"purple")),
        Guide.ylabel("probability"),Guide.xlabel(" "),
        Guide.manual_color_key("", ["p( )", "p( |y)"], ["black","purple"]), white_panel);

draw(PNG("./figs/beta_mix_post.png", 8inch, 4inch), myplot);
```