Statistical Modeling
BANA 7042

Lecture 5: Modeling multinomial data

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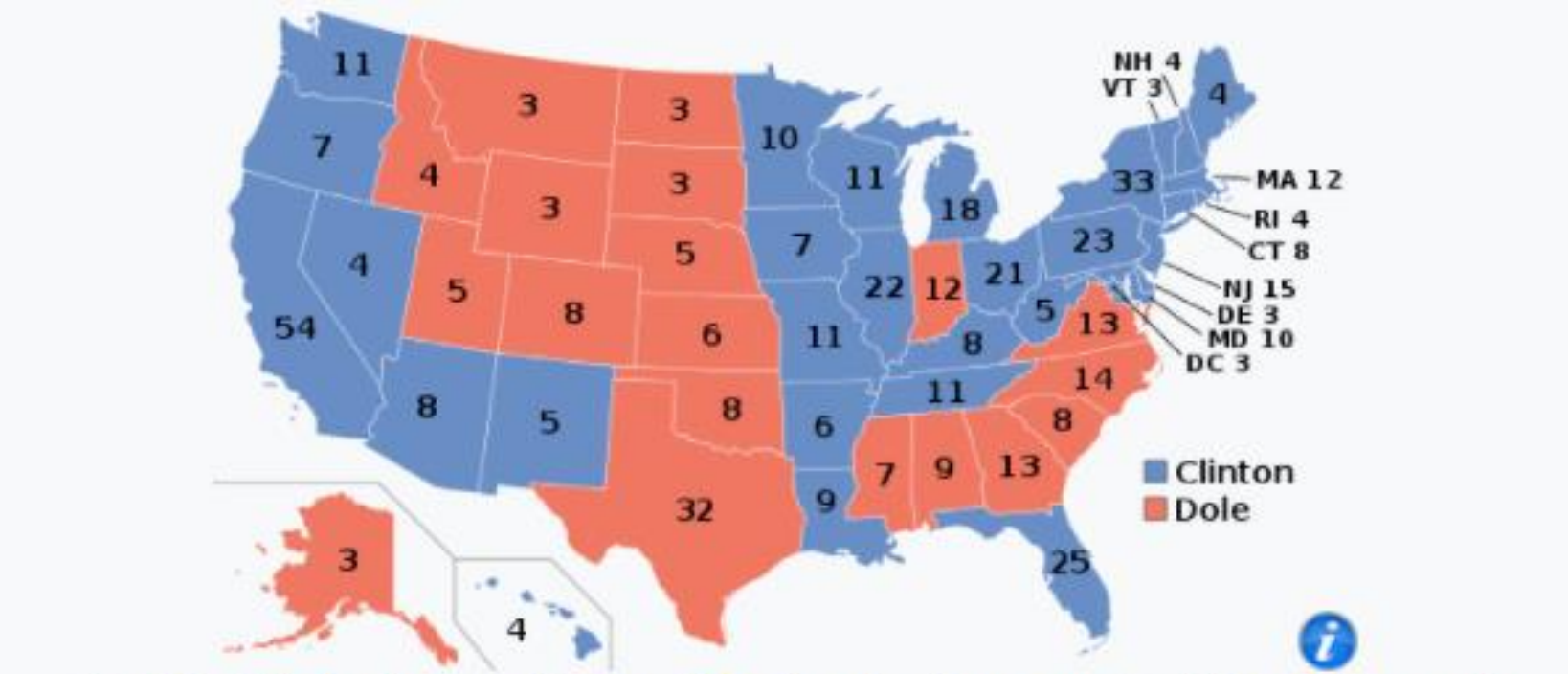
1996 US Presidential Election

← 1992	November 5, 1996	2000 →
All 538 electoral votes of the Electoral College 270 electoral votes needed to win		
Turnout	49.0%[1] ▼ 6.2 pp	
		
Nominee	Bill Clinton	Bob Dole
Party	Democratic	Republican
Home state	Arkansas	Kansas
Running mate	Al Gore	Jack Kemp
Electoral vote	379	159
States carried	31 + DC	19
Popular vote	47,401,185	39,197,469
Percentage	49.2%	40.7%

Nominee	Ross Perot
Party	Reform
Home state	Texas
Running mate	Pat Choate
Electoral vote	0
States carried	0
Popular vote	8,085,294
Percentage	8.4%



Presidential election results map. Blue denotes those won by Clinton/Gore, red denotes states won by Dole/Kemp. Numbers indicate electoral votes allotted to the winner of each state.

1996 American National Election Study

The data set "nes96" is a 10 variable subset of the 1996 American National Election Study. It contains information of 944 voters.

```
library("faraway")
data(nes96)
str(nes96)
?nes96
summary(nes96)
```

The data structure

```
> str(nes96)
'data.frame':   944 obs. of  10 variables:
 $ popul : int  0 190 31 83 640 110 100 31 180 2800 ...
 $ TVnews: int  7 1 7 4 7 3 7 1 7 0 ...
 $ selfLR: Ord.factor w/ 7 levels "extLib"<"Lib"<..: 7 3 2 3 5 3 5 5 4 3 ...
 $ ClinLR: Ord.factor w/ 7 levels "extLib"<"Lib"<..: 1 3 2 4 6 4 6 4 6 3 ...
 $ DoleLR: Ord.factor w/ 7 levels "extLib"<"Lib"<..: 6 5 6 5 4 6 4 5 3 7 ...
 $ PID   : Ord.factor w/ 7 levels "strDem"<"weakDem"<..: 7 2 2 2 1 2 2 5 4 1
...
 $ age   : int  36 20 24 28 68 21 77 21 31 39 ...
 $ educ  : Ord.factor w/ 7 levels "MS"<"HSdrop"<..: 3 4 6 6 6 4 4 4 4 3 ...
 $ income: Ord.factor w/ 24 levels "$3Kminus"<"$3K-$5K"<..: 1 1 1 1 1 1 1 1 1 1
...
 $ vote  : Factor w/ 2 levels "Clinton","Dole": 2 1 1 1 1 1 1 1 1 1 ...
```

Variable dictionary

popul

population of respondent's location in 1000s of people

TVnews

days in the past week spent watching news on TV

selfLR

Left-Right self-placement of respondent: an ordered factor with levels extremely liberal, `extLib` < liberal, `Lib` < slightly liberal, `sliLib` < moderate, `Mod` < slightly conservative, `sliCon` < conservative, `Con` < extremely conservative, `extCon`

ClinLR

Left-Right placement of Bill Clinton (same scale as selfLR): an ordered factor with levels `extLib` < `Lib` < `sliLib` < `Mod` < `sliCon` < `Con` < `extCon`

DoleLR

Left-Right placement of Bob Dole (same scale as selfLR): an ordered factor with levels `extLib` < `Lib` < `sliLib` < `Mod` < `sliCon` < `Con` < `extCon`

PID

Party identification: an ordered factor with levels strong Democrat, `strDem` < weak Democrat, `weakDem` < independent Democrat, `indDem` < independent independent`indind` < indepedent Republican, `indRep` < waek Republican, `weakRep` < strong Republican, `strRep`

Variable dictionary (cont'd)

age

Respondent's age in years

educ

Respondent's education: an ordered factor with levels 8 years or less, `MS` < high school dropout, `HSdrop` < high school diploma or GED, `HS` < some College, `Coll` < Community or junior College degree, `CCdeg` < BA degree, `BAdeg` < postgraduate degree, `MAdeg`

income

Respondent's family income: an ordered factor with levels `$3Kminus` < `$3K-$5K` < `$5K-$7K` < `$7K-$9K` < `$9K-$10K` < `$10K-$11K` < `$11K-$12K` < `$12K-$13K` < `$13K-$14K` < `$14K-$15K` < `$15K-$17K` < `$17K-$20K` < `$20K-$22K` < `$22K-$25K` < `$25K-$30K` < `$30K-$35K` < `$35K-$40K` < `$40K-$45K` < `$45K-$50K` < `$50K-$60K` < `$60K-$75K` < `$75K-$90K` < `$90K-$105K` < `$105Kplus`

vote

Expected vote in 1996 presidential election: a factor with levels `Clinton` and `Dole`

Source

Sapiro, Virginia, Steven J. Rosenstone, Donald R. Kinder, Warren E. Miller, and the National Election Studies. AMERICAN NATIONAL ELECTION STUDIES, 1992-1997: COMBINED FILE [Computer file]. 2nd ICPSR version. Ann Arbor, MI: University of Michigan, Center for Political Studies [producer], 1999. Ann Arbor, MI: Inter-university Consortium for Political and Social Research [distributor], 1999.

Summary statistics

```
> summary(nes96)
     popul              TVnews          selfLR       ClinLR       DoleLR
 Min.   :   0.0   Min.   :0.000   extLib: 16   extLib:109   extLib: 13
 1st Qu.:   1.0   1st Qu.:1.000   Lib   :103   Lib   :317   Lib   : 31
 Median :  22.0   Median :3.000   sliLib:147   sliLib:236   sliLib: 43
 Mean   : 306.4   Mean   :3.728   Mod   :256   Mod   :160   Mod   : 87
 3rd Qu.: 110.0   3rd Qu.:7.000   sliCon:170   sliCon: 67   sliCon:195
 Max.   :7300.0   Max.   :7.000   Con   :218   Con   : 36   Con   :460
                                  extCon: 34   extCon: 19   extCon:115
      PID             age              educ             income          vote
 strDem :200   Min.   :19.00   MS    : 13   $60K-$75K:103   Clinton:551
 weakDem:180   1st Qu.:34.00   HSdrop: 52   $50K-$60K:100   Dole   :393
 indDem :108   Median :44.00   HS    :248   $30K-$35K: 70
 indind : 37   Mean   :47.04   Coll  :187   $25K-$30K: 68
 indRep : 94   3rd Qu.:58.00   CCdeg : 90   $105Kplus: 68
 weakRep:150   Max.   :91.00   BAdeg :227   $35K-$40K: 62
 strRep :175                   MAdeg :127   (Other)  :473
```

What question could be answered?

- What are the underlying factors that would influence an individual's
  - Vote? --- A binary variable
  - Party Identification? --- A multinomial variable
- Are there any common factors?
- Generally speaking, Democrats voted for Clinton, and Republican voted for Dole. --- Is this true?
- Let us focus on 5 variables: "vote", "PID", "income", "educ", and "age".

Initial data processing

- Aggregate the levels of "PID" such that we have 3 categories: "Democrat", "Independent", and "Republican".

```
party<-nes96$PID
levels(party)<-c("Democrat", "Democrat", "Independent", "Independent", "Independent",
"Republican", "Republican")
```

- Convert "income" to a numeric variable by taking the midpoint of each income range.

```
inca<-
c(1.5,4,6,8,9.5,10.5,11.5,12.5,13.5,14.5,16,18.5,21,23.5,27.5,32.5,37.5,42.5,47.5,
55,67.5,82.5,97.5,115)

income<-inca[unclass(nes96$income)]
```

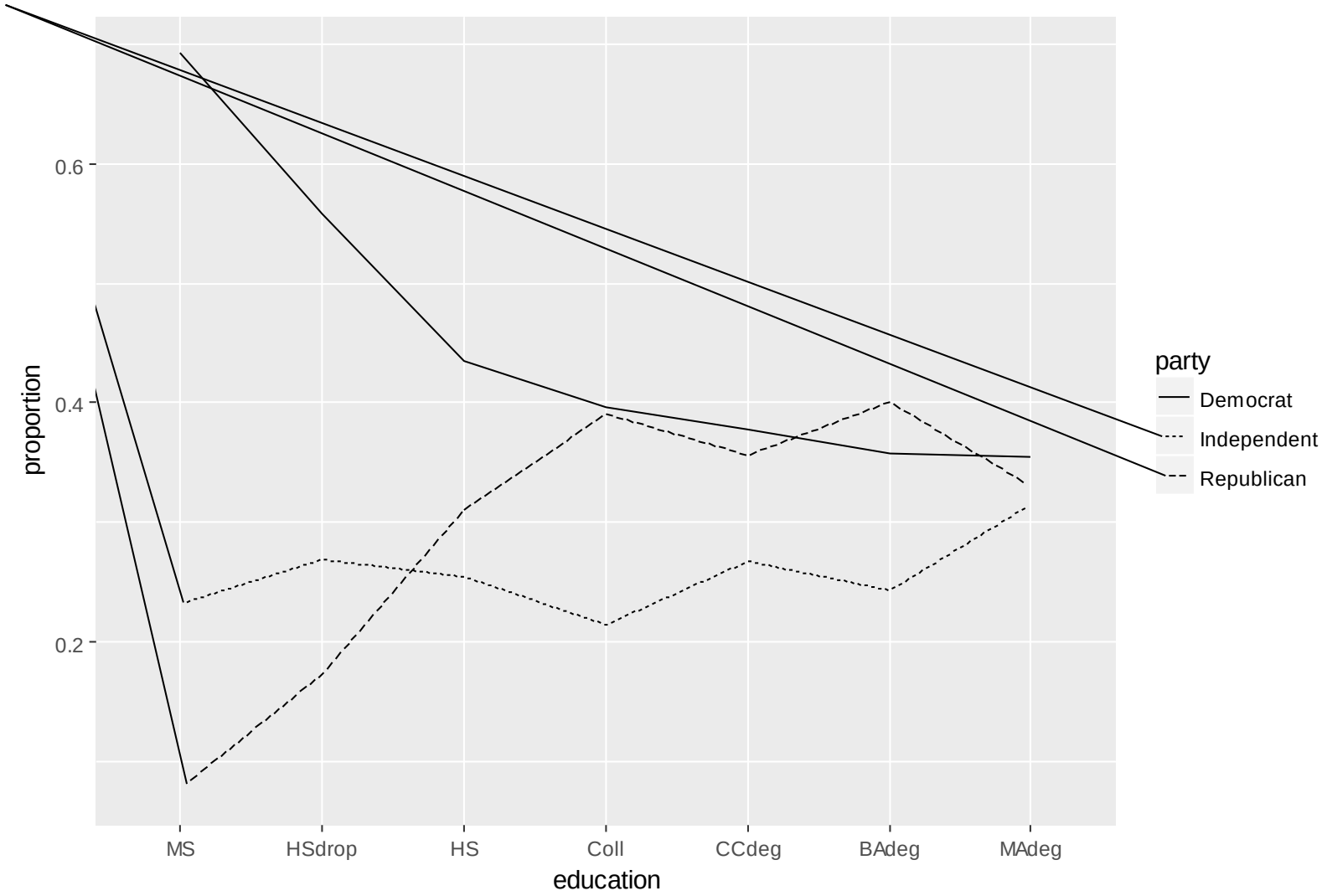
Summary of the processed data

```
rnes96<-data.frame(party, income, education=nes96$educ, age=nes96$age)
summary(rnes96)
```

```
         party          income          education         age        
 Democrat   :380   Min.   :  1.50   MS     : 13   Min.   :19.00  
 Independent:239   1st Qu.: 23.50   HSdrop: 52   1st Qu.:34.00  
 Republican :325   Median : 37.50   HS     :248   Median :44.00  
                   Mean   : 46.58   Coll   :187   Mean   :47.04  
                   3rd Qu.: 67.50   CCdeg  : 90   3rd Qu.:58.00  
                   Max.   :115.00   BAdeg  :227   Max.   :91.00  
                                    MAdeg  :127                  
```

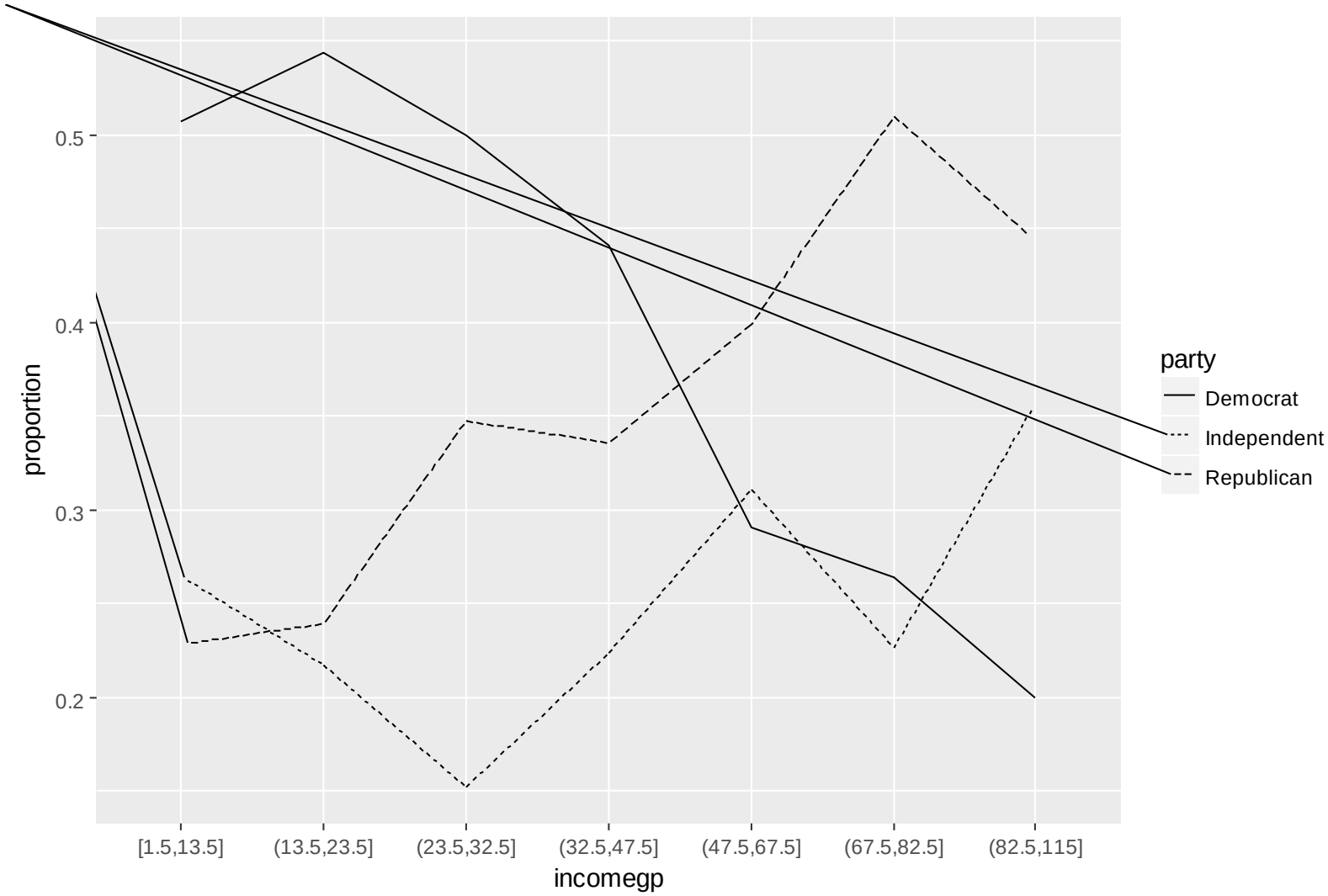
Visualization: Party versus Education

```
install.packages("dplyr")
library("dplyr")

egp<-group_by(rnes96, education, party) %>%
summarise(count=n()) %>% group_by(education) %>%
mutate(etotal=sum(count), proportion=count/etotal)

library("ggplot2")
ggplot(egp, aes(x=education, y=proportion,
group=party, linetype=party)) + geom_line()
```


0.6
0.4
0.2
proportion
MS
HSdrop
HS
Coll
CCdeg
BAdeg
MAdeg
education
party
Democrat
Independent
Republican

Visualization: Party versus Income

```
igp<-mutate(rnes96, incomegp=cut_number(income,7)) %>%
group_by(incomegp, party) %>% summarise(count=n()) %>%
group_by(incomegp) %>% mutate(etotal=sum(count),
proportion=count/etotal)

ggplot(igp, aes(x=incomegp, y=proportion, group=party,
linetype=party)) + geom_line()
```


0.5
0.4
0.3
0.2
proportion
[1.5,13.5]
(13.5,23.5]
(23.5,32.5]
(32.5,47.5]
(47.5,67.5]
(67.5,82.5]
(82.5,115]
incomegp
party
Democrat
Independent
Republican

Are the trends real or by chance?

- As the education level increases, the proportion of democrats decreases and that of republicans increases.
- As the income increases, the proportion of democrats decreases and that of republicans increases.
- Can we build a model to examine the plausible association?

Modeling ordinal data

From "binary" to "ordinal multinomial"

- Unlike the variable "vote" which is binary, "party" has three categories and the categories have an "order".

- How can we extend the idea of modeling "vote" to that of modeling "party"?

- Suppose we use a logistic regression to model "vote". How do we interpret the model using a **latent variable**?

The latent variable for a binary variable

- Suppose there is a latent variable Z for the binary variable "vote".

$$Z \sim \text{logistic}(\mu = -\beta x, \sigma^2 = 1)$$

$$\text{"vote"} = Y = \begin{cases} 0 & \text{if } Z \leq c, \\ 1 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \Pr\{Y = 0\} &= \Pr\{Z \leq c\} = \Pr\{\text{logistic}(\mu, 1) \leq c\} \\ &= \Pr\{\text{logistic}(0, 1) \leq c - \mu\} = F(c + \beta x) \end{aligned}$$

- This is one way to establish a logistic regression model.

The latent variable for an ordinal variable

- Suppose there is a latent variable Z for the ordinal variable "party".

$$Z \sim \text{logistic}(\mu = -\beta x, \sigma^2 = 1)$$

$$\text{"party"} = Y = \begin{cases} 0 & \text{if } Z \leq c_1, \\ 1 & \text{if } Z \leq c_2, \\ 2 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \Pr\{Y \leq 0\} &= \Pr\{Z \leq c_1\} = \Pr\{\text{logistic}(\mu, 1) \leq c_1\} \\ &= \Pr\{\text{logistic}(0, 1) \leq c_1 - \mu\} = F(c_1 + \beta x) \\ \Pr\{Y \leq 1\} &= \Pr\{Z \leq c_2\} = \Pr\{\text{logistic}(\mu, 1) \leq c_2\} \\ &= \Pr\{\text{logistic}(0, 1) \leq c_2 - \mu\} = F(c_2 + \beta x) \end{aligned}$$

Fit a proportional odds model

```
library(VGAM)
nmod<-vglm(party ~ age + education + income,
family=cumulative(parallel=TRUE),rnes96)
summary(nmod)
```

```
vglm(formula = party ~ age + education + income, family = cumulative(parallel = TRUE),
    data = rnes96)

Pearson residuals:
                  Min      1Q  Median     3Q   Max
logit(P[Y<=1]) -2.059 -0.9564 -0.3698 1.0152 1.974
logit(P[Y<=2]) -2.804 -1.1156  0.3450 0.9138 1.707

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  0.644871   0.244989   2.632  0.00848 **
(Intercept):2  1.737347   0.250638   6.932 4.16e-12 ***
age           -0.005775   0.003878  -1.489  0.13649
education.L   -0.724109   0.392387  -1.845  0.06498 .
education.Q    0.781386   0.358405   2.180  0.02924 *
education.C   -0.040190   0.297745  -0.135  0.89263
education^4    0.019942   0.236013   0.084  0.93266
education^5    0.079399   0.193456   0.410  0.68149
education^6    0.061111   0.157335   0.388  0.69771
income        -0.012739   0.002186  -5.826 5.68e-09 ***
---
Number of linear predictors:  2

Names of linear predictors: logit(P[Y<=1]), logit(P[Y<=2])

Residual deviance: 1984.211 on 1878 degrees of freedom

Log-likelihood: -992.1056 on 1878 degrees of freedom


Exponentiated coefficients:
        age education.L education.Q education.C education^4 education^5 education^6      income
  0.9942419   0.4847561   2.1844985   0.9606074   1.0201419   1.0826362   1.0630167   0.9873423
```

Transform "education" to years

```
edu.years<-c(9,10.5,12,13,14,16,18)
education<-edu.years[unclass(nes96$educ)]
rnes96$education<-education

nmod<-vglm(party ~ age + education + income,
family=cumulative(parallel=TRUE), rnes96)
summary(nmod)
```

```
Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  0.506585   0.441597   1.147 0.251312
(Intercept):2  1.589747   0.444444   3.577 0.000348 ***
age           -0.002902   0.003752  -0.773 0.439242
education     -0.012316   0.028803  -0.428 0.668931
income        -0.012877   0.002181  -5.904 3.55e-09 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Number of linear predictors:  2

Names of linear predictors: logit(P[Y<=1]), logit(P[Y<=2])

Residual deviance: 1994.642 on 1883 degrees of freedom

Log-likelihood: -997.3211 on 1883 degrees of freedom

Number of iterations: 4

No Hauck-Donner effect found in any of the estimates

Exponentiated coefficients:
      age education    income
0.9971018 0.9877591 0.9872058
```

Proportional odds model

- The proportional odds model for "party"

$$\Pr\{Y \leq 1\} = F(\alpha_1 + \beta x) \text{ and } \Pr\{Y \leq 2\} = F(\alpha_2 + \beta x)$$

- Where *F* is the Inverse of the logit function.

- Why do we call it "odds" model?
- Why do we call it "proportional" model?

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept):1  0.506585   0.441597   1.147 0.251312
(Intercept):2  1.589747   0.444444   3.577 0.000348 ***
age           -0.002902   0.003752  -0.773 0.439242
education     -0.012316   0.028803  -0.428 0.668931
income        -0.012877   0.002181  -5.904 3.55e-09 ***
---
Exponentiated coefficients:
      age education    income
0.9971018 0.9877591 0.9872058
```

Non-proportional odds model

- The proportional odds model for "party"

$$\Pr\{Y \leq 1\} = F(\alpha_1 + \beta_1 x) \text{ and } \Pr\{Y \leq 2\} = F(\alpha_2 + \beta_2 x)$$

where *F* is the Inverse of the logit function.

- The beta coefficients rely on the categories.

```
nmod.2<-vglm(party ~ age + education + income,
family=cumulative(parallel=FALSE),rnes96)
summary(nmod.2)
```

```
vglm(formula = party ~ age + education + income, family = cumulative(parallel = FALSE),
    data = rnes96)


Coefficients:
               Estimate Std. Error z value Pr(>|z|)
(Intercept):1  0.527897   0.490515   1.076  0.28183
(Intercept):2  1.511425   0.503245   3.003  0.00267 **
age:1         -0.002214   0.004101  -0.540  0.58931
age:2         -0.003784   0.004323  -0.875  0.38144
education:1   -0.007138   0.032075  -0.223  0.82389
education:2   -0.013991   0.032699  -0.428  0.66875
income:1      -0.016077   0.002550  -6.304 2.90e-10 ***
income:2      -0.010231   0.002408  -4.249 2.15e-05 ***
---

Names of linear predictors: logit(P[Y<=1]), logit(P[Y<=2])

Residual deviance: 1986.59 on 1880 degrees of freedom

Log-likelihood: -993.2951 on 1880 degrees of freedom


Exponentiated coefficients:
      age:1       age:2 education:1 education:2    income:1    income:2
  0.9977883   0.9962231   0.9928875   0.9861065   0.9840517   0.9898212
```

Other link functions

- We can use different link function in the proportional model for "party"

$$\Pr\{Y \leq 1\} = F(\alpha_1 + \beta x) \text{ and } \Pr\{Y \leq 2\} = F(\alpha_2 + \beta x)$$

- The function F could be the CDF of
  - A logistic distribution (Proportional Odds Model)
  - A normal distribution (Ordered Probit Model)
  - An extreme value distribution (Proportional Hazards Model)

Probit and complementary log-log links

```
nmod.probit<-vglm(party ~ age + education + income,
family=cumulative(parallel=TRUE,link=probit),rnes96)
summary(nmod.probit)

nmod.cloglog<-vglm(party ~ age + education + income,
family=cumulative(parallel=TRUE,link=cloglog),rnes96)
summary(nmod.cloglog)
```

```
vglm(formula = party ~ age + education + income, family = cumulative(parallel = TRUE,
    link = probit), data = rnes96)



Coefficients:
               Estimate Std. Error z value Pr(>|z|)
(Intercept):1  0.306698   0.271604   1.129 0.258809
(Intercept):2  0.976247   0.272625   3.581 0.000342 ***
age           -0.001896   0.002305  -0.823 0.410731
education     -0.006770   0.017717  -0.382 0.702398
income        -0.008059   0.001331  -6.055  1.4e-09 ***
---

Names of linear predictors: probit(P[Y<=1]), probit(P[Y<=2])

Residual deviance: 1994.127 on 1883 degrees of freedom

Log-likelihood: -997.0636 on 1883 degrees of freedom


Exponentiated coefficients:
      age education    income
0.9981054 0.9932533 0.9919731
```

```
vglm(formula = party ~ age + education + income, family = cumulative(parallel = TRUE,
    link = cloglog), data = rnes96)


Coefficients:
               Estimate Std. Error z value Pr(>|z|)
(Intercept):1 -0.049780   0.296676  -0.168   0.8667
(Intercept):2  0.693885   0.296105   2.343   0.0191 *
age           -0.002080   0.002462  -0.845   0.3980
education     -0.010662   0.019396  -0.550   0.5825
income        -0.008054   0.001511  -5.331 9.77e-08 ***
---

Names of linear predictors: cloglog(P[Y<=1]), cloglog(P[Y<=2])

Residual deviance: 2002.988 on 1883 degrees of freedom

Log-likelihood: -1001.494 on 1883 degrees of freedom


Exponentiated coefficients:
      age education    income
0.9979217 0.9893947 0.9919786
```

Modeling multinomial data

Multinomial data without an order

- We have modeled "party" as an ordered multinomial response.
  - Democrat
  - Independent
  - Republican

- What if we treat "party" as a multinomial variable without an order?
- How can we build a model using "education", "age", and "income" as explanatory variables?

Review of logistic regression models for a binary response

- Suppose the binary response (two categories!)

$$Y \sim \text{Bernoulli}(p)$$

- We will make two additional assumptions on top of this assumption:
  1. Y relies on X only through its mean, e.g $Y \sim \text{Bernoulli}(p(X))$
  2. The logit transformation of the parameter p

$$\text{logit}(p) = \log(\frac{p}{1-p}) = \beta_1 + \beta_2 X$$

- How can we extend this modeling idea to multinomial data?

Modeling multinomial data

- Suppose Y follows a multinomial distribution

$$Y = A \text{ with prob } p_1,\ Y = B \text{ with } p_2,\ Y = C \text{ with } p_3$$

- We will make two additional assumptions on top of this assumption:
  1. Y relies on X only through the three probabilities $p_1(X), p_2(X), p_3(X)$
  2. The log odds

$$\log(\frac{p_2}{p_1}) = \alpha_2 + \beta_2 X \quad \text{and} \quad \log(\frac{p_3}{p_1}) = \alpha_3 + \beta_3 X$$

Notes

- A constraint: $p_1 + p_2 + p_3 = 1$

- We use p_1 as the baseline probability. It does not matter which category we use as the baseline.

- The probability

$$p_j = \frac{\exp \eta_j}{1+\sum_{j=2}^{J} \exp \eta_j}$$

$$\eta_1 = 0 \text{ and } \eta_j = \alpha_j + \beta_j x$$

```
library("faraway")
data(nes96)

### Process the data
party<-nes96$PID
levels(party)<-c("Democrat", "Democrat", "Independent",
"Independent", "Independent", "Republican", "Republican")
inca<-
c(1.5,4,6,8,9.5,10.5,11.5,12.5,13.5,14.5,16,18.5,21,23.5,27.5,32
.5,37.5,42.5,47.5,55,67.5,82.5,97.5,115)
income<-inca[unclass(nes96$income)]
rnes96<-data.frame(party, income, education=nes96$educ,
age=nes96$age)
edu.years<-c(9,10.5,12,13,14,16,18)
education<-edu.years[unclass(nes96$educ)]
rnes96$education<-education
```

```
install.packages("nnet")
library("nnet")
mmod<-multinom(party~age+education+income,rnes96)
summary(mmod)
```

How to interpret the coefficients?

```
> summary(mmod)
Call:
multinom(formula = party ~ age + education + income, data = rnes96)

Coefficients:
            (Intercept)          age     education      income
Independent   -1.177840 0.000238586 -0.0007666165 0.01614117
Republican    -1.332179 0.004328175  0.0132647908 0.01747883

Std. Errors:
            (Intercept)          age  education      income
Independent   0.6095759 0.005181846 0.03986776 0.003089633
Republican    0.5622177 0.004736439 0.03660254 0.002875076

Residual Deviance: 1984.366
AIC: 2000.366
```

Model selection and comparison

```
### Model selection based on AIC
mmodi<-step(mmod)

### Model comparison based on the significance
deviance(mmodi)-deviance(mmod)

mmod$edf-mmodi$edf

pchisq(deviance(mmodi)-deviance(mmod),mmod$edf-mmodi$edf,lower=F)
```

```
> mmodi<-step(mmod)
Start:  AIC=2000.37
party ~ age + education + income
 Df      AIC
- education  6 1996.539
- age        6 1997.325
<none>       8 2000.366
- income     6 2042.822
```

```
Step:  AIC=1996.54
party ~ age + income
 Df      AIC
- age      4 1993.424
<none>     6 1996.539
- income   4 2048.850
```

```
Step:  AIC=1993.42
party ~ income

Df      AIC
<none>     4 1993.424
- income   2 2045.272
```

Model comparison

```
### Difference between deviances
> deviance(mmodi)-deviance(mmod)
[1] 1.058172

### Difference between degrees of freedom
> mmod$edf-mmodi$edf
[1] 4

### Chi^2 test
> pchisq(deviance(mmodi)-deviance(mmod),mmod$edf-
mmodi$edf,lower=F)
[1] 0.9008507
```

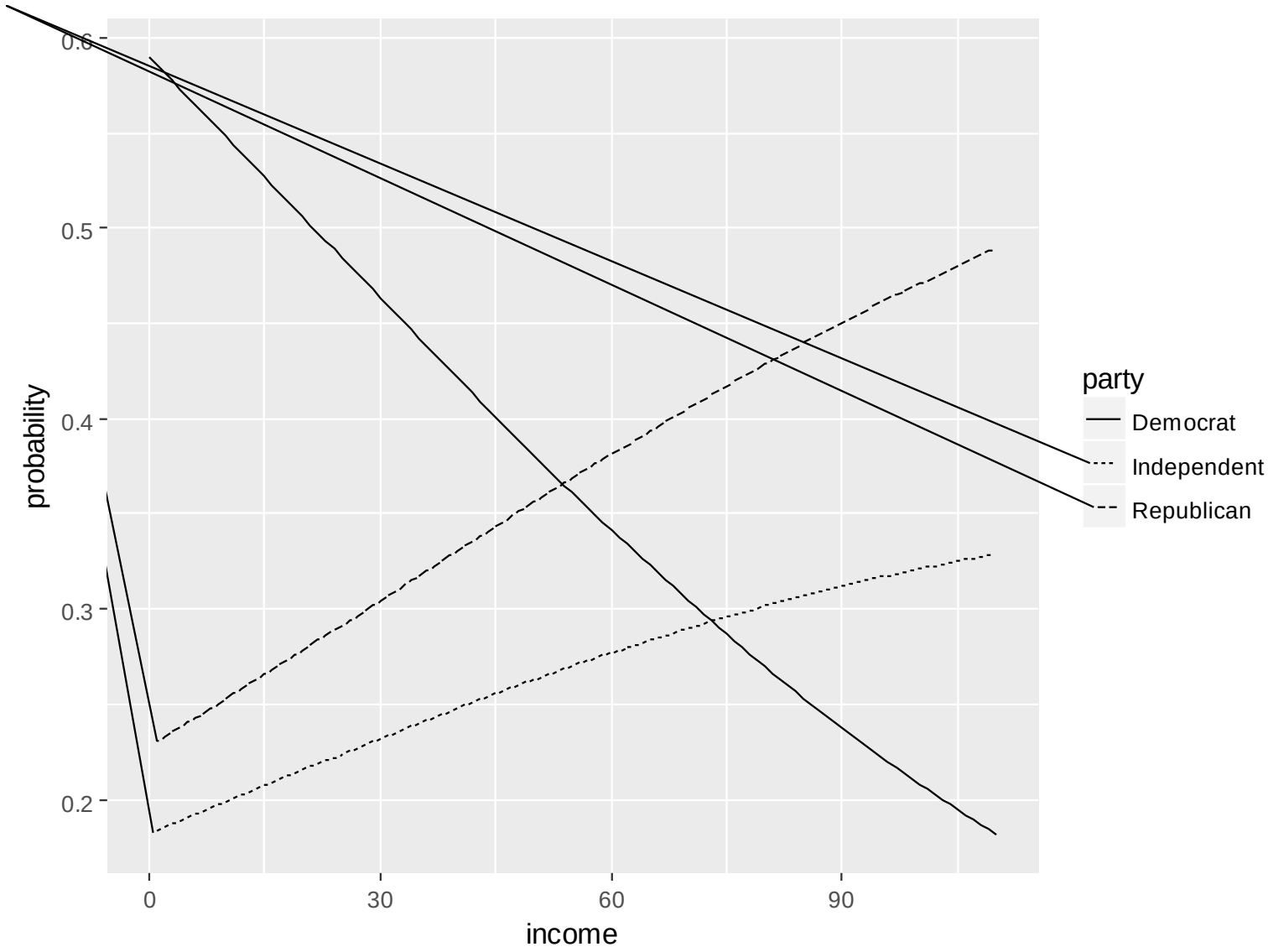
Prediction

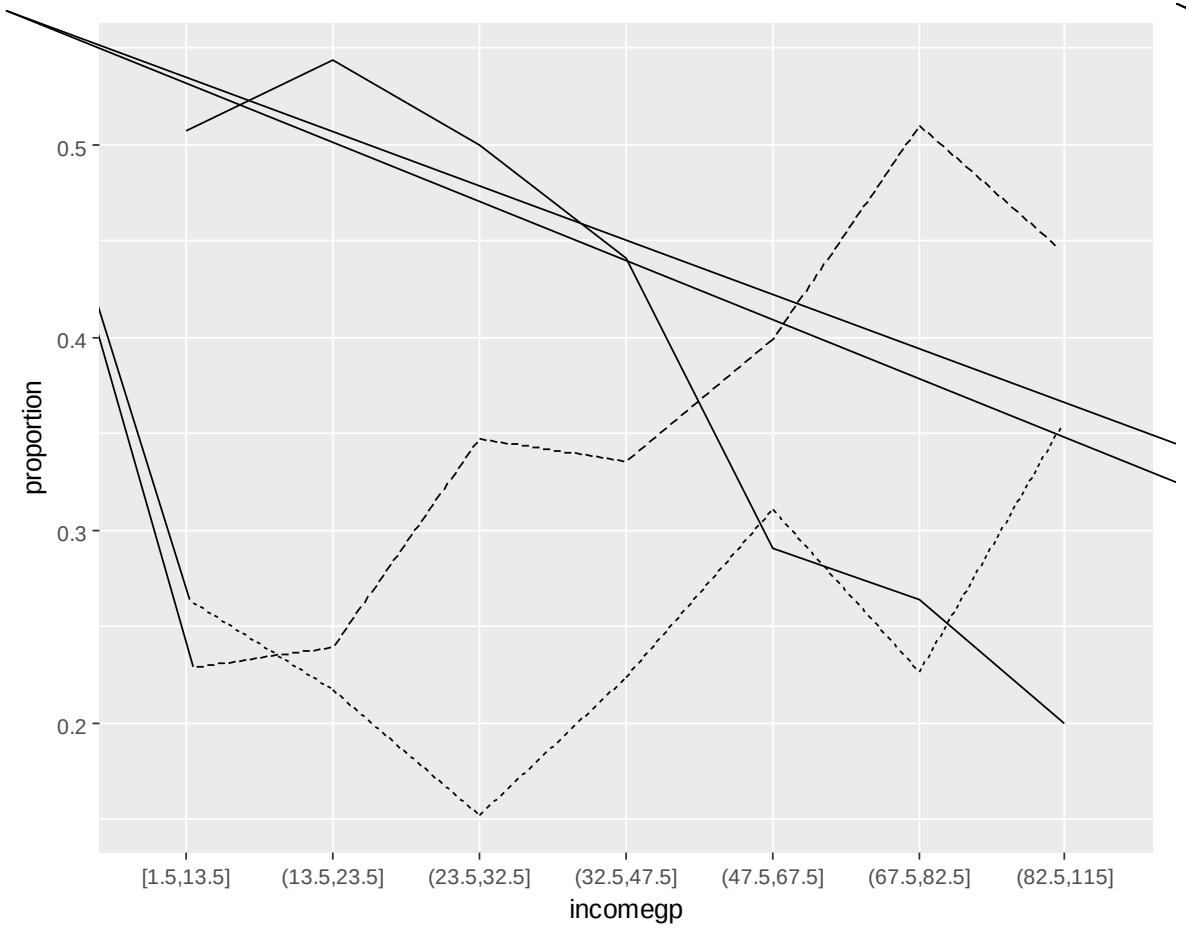
```
### Predict the outcome
predict(mmodi,data.frame(income=30))
predict(mmodi,data.frame(income=90))

### Give the probabilities of all possible outcomes
predict(mmodi,data.frame(income=30),type="probs")
predict(mmodi,data.frame(income=90),type="probs")
```

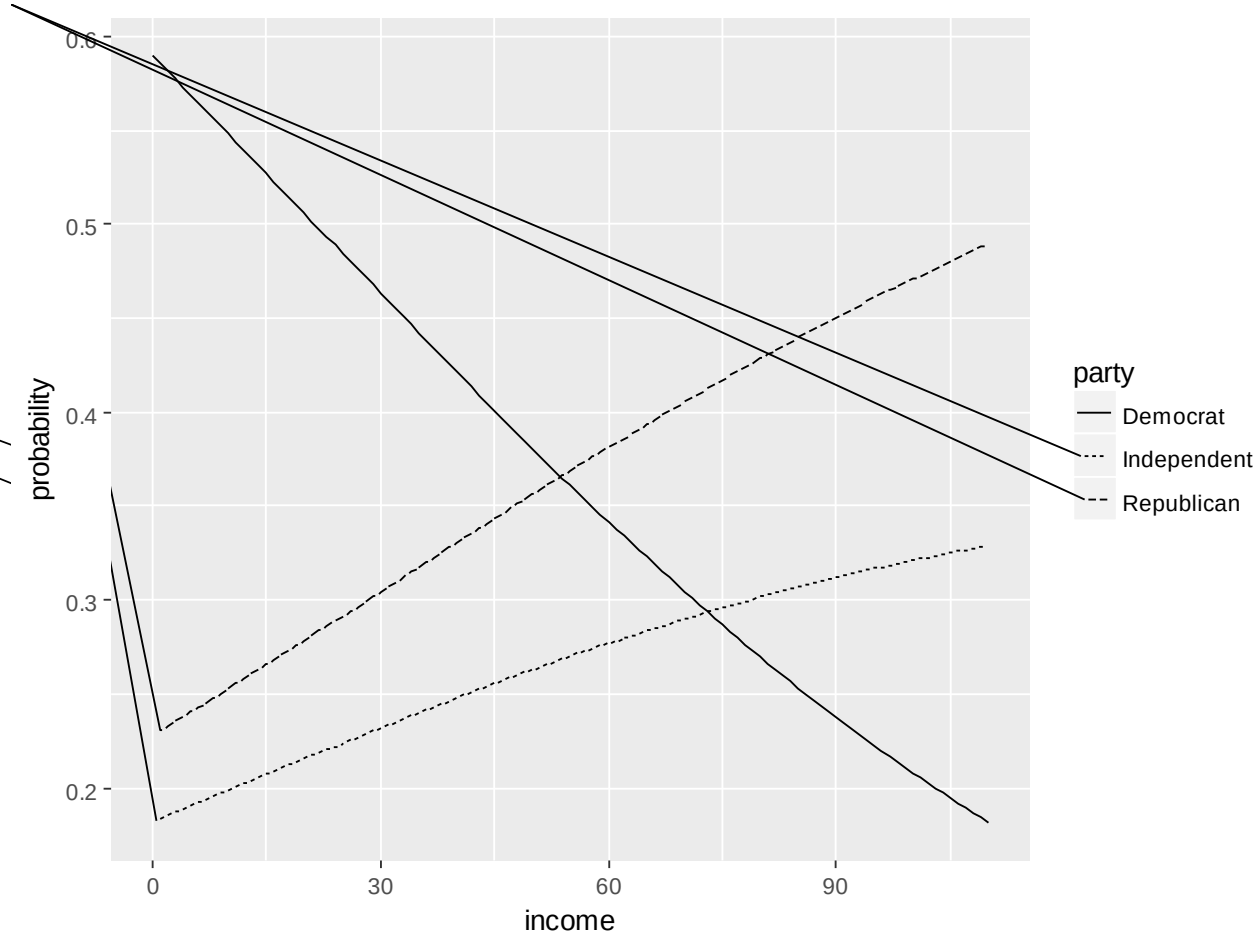
```
> predict(mmodi,data.frame(income=30))
[1] Democrat
Levels: Democrat Independent Republican
> predict(mmodi,data.frame(income=90))
[1] Republican
Levels: Democrat Independent Republican

> predict(mmodi,data.frame(income=30),type="probs")
   Democrat Independent  Republican
  0.4635683   0.2319741   0.3044576
> predict(mmodi,data.frame(income=90),type="probs")
   Democrat Independent  Republican
  0.2375755   0.3121136   0.4503109
```

Plot the fitted curve

```
inclevels<-0:110 ### Annual income 0-110K
preds<-
data.frame(income=inclevels,predict(mmodi,data.frame(i
ncome=inclevels),type="probs"))

install.packages("tidyr")
library("tidyr")
lpred<-gather(preds,party,probability,-income)
ggplot(lpred,aes(x=income,y=probability,group=party,li
netype=party))+geom_line()
```


probability
0.6
0.5
0.4
0.3
0.2
0
30
60
90
income
party
Democrat
Independent
Republican


0.5
0.4
0.3
0.2
proportion
[1.5,13.5]
(13.5,23.5]
(23.5,32.5]
(32.5,47.5]
(47.5,67.5]
(67.5,82.5]
(82.5,115]
incomegp


0.6
0.5
0.4
0.3
0.2
probability
0
30
60
90
income
party
Democrat
Independent
Republican

Prediction table

```
> xtabs(~predict(mmodi)+rnes96$party)
                rnes96$party
predict(mmodi)  Democrat Independent Republican
    Democrat         284         123        166
    Independent        0           0          0
    Republican        96         116        159
```