An efficient algorithm for finding the general solution of a matrix equation

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1 Introduction & Motivation

Consider the following problem.

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y + z \\ 2x - y - z \end{pmatrix} \text{ for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

Find 2 different linear transformations S_1 and S_2 such that $(T \circ S_1)$ and $(T \circ S_2)$ are both the identity operator on $\mathbb{R}^2$.

To solve the problem, we first deduce that the standard matrix $\boldsymbol{A}$ of T is $\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \end{pmatrix}$. We then try to find distinct matrices $\boldsymbol{B_1}, \boldsymbol{B_2}$ such that $\boldsymbol{AB_1} = \boldsymbol{AB_2} = \boldsymbol{I}$. To do so, we can find the general solution to the matrix equation $\boldsymbol{AX} = \boldsymbol{I}$.

The most straightforward way to do this would be to assign each entry in $\boldsymbol{X}$ to an unknown variable, expand the matrix $\boldsymbol{AX}$, compare the entries of $\boldsymbol{AX}$ to those of $\boldsymbol{B}$, and perform Gaussian Elimination on the resulting linear system. However, this method is computationally inefficient, especially for larger matrices, as the resulting linear system would have $m \cdot n$ equations and $p \cdot n$ variables, so performing Gaussian Elimination would require a time complexity of $O(mp^2n^3)$. The following describes a more computationally efficient way to obtain a general solution, without requiring any advanced linear algebra concepts.

2 Theory of the algorithm

Theorem 1 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times p$ and $m \times n$ matrices respectively. Let $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ be the reduced row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{B})$.

1. The matrix equation $\boldsymbol{AX} = \boldsymbol{B}$ is inconsistent if and only if $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ has a pivot entry on the right-hand side.
2. If $\boldsymbol{AX} = \boldsymbol{B}$ is consistent and every column on the left-hand side of $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ is a pivot column, then $\boldsymbol{AX} = \boldsymbol{B}$ has unique solution $\boldsymbol{X_0}$, where $\boldsymbol{X_0}$ is the matrix obtained by taking the first n rows of $\boldsymbol{B_0}$.
3. If $\boldsymbol{AX} = \boldsymbol{B}$ is consistent and there is a non-pivot column on the left-hand side of $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$, then the nullspace of $\boldsymbol{A_0}$ is not the zero space. Let $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be a basis for the nullspace of $\boldsymbol{A_0}$. Let $\boldsymbol{X_0}$ be the $n \times p$ matrix such that for $i = 1, 2, ..., n$, if the i-th column of $\boldsymbol{A_0}$ is the j-th pivot column, then the i-th row of $\boldsymbol{X_0}$ is the j-th row of $\boldsymbol{B_0}$, and if the i-th column of $\boldsymbol{A_0}$ is not a pivot column, then the i-th row of $\boldsymbol{X_0}$ is $\boldsymbol{0}$. Then $\boldsymbol{AX} = \boldsymbol{B}$ has solution set
$$\left\{\boldsymbol{X_0} + \begin{pmatrix}\boldsymbol{v_1} & \boldsymbol{v_2} & ... & \boldsymbol{v_k}\end{pmatrix} \boldsymbol{T} \mid \boldsymbol{T} \text{ is a } k \times n \text{ matrix.}\right\}$$

Before proving the theorem, we need to establish some simple lemmas. Notice that these are extremely similar to corresponding theorems for systems of linear equations.

Lemma 1 If $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ is the reduced row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{B})$, then the matrix equations $\boldsymbol{AX} = \boldsymbol{B}$ and $\boldsymbol{A_0X} = \boldsymbol{B_0}$ have the same solution set.

Proof Since $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ is the reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{B})$, $(\boldsymbol{A} \mid \boldsymbol{B})$ and $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ are row equivalent. Hence, there exist elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, ..., \boldsymbol{E_n}$ such that $(\boldsymbol{A_0} \mid \boldsymbol{B_0}) = \boldsymbol{E_n}...\boldsymbol{E_2}\boldsymbol{E_1}(\boldsymbol{A} \mid \boldsymbol{B})$. Let $\boldsymbol{E} = \boldsymbol{E_n}...\boldsymbol{E_2}\boldsymbol{E_1}$. Then $(\boldsymbol{A_0} \mid \boldsymbol{B_0}) = \boldsymbol{E}(\boldsymbol{A_0} \mid \boldsymbol{B_0}) = (\boldsymbol{EA_0} \mid \boldsymbol{EB})$, which implies $\boldsymbol{A_0} = \boldsymbol{EA}$ and $\boldsymbol{B_0} = \boldsymbol{EB}$.

Suppose $\boldsymbol{X_0}$ is a solution to $\boldsymbol{AX} = \boldsymbol{B}$. Then $\boldsymbol{A_0X_0} = \boldsymbol{EAX_0} = \boldsymbol{EB} = \boldsymbol{B_0}$. Hence $\boldsymbol{X_0}$ is also a solution to $\boldsymbol{A_0X} = \boldsymbol{B_0}$.

Suppose $\boldsymbol{X_0}$ is a solution to $\boldsymbol{A_0X} = \boldsymbol{B_0}$. Since $\boldsymbol{E}$ is a product of elementary matrices, $\boldsymbol{E}$ is invertible. Hence $\boldsymbol{AX_0} = \boldsymbol{E}^{-1}\boldsymbol{EAX_0} = \boldsymbol{E}^{-1}\boldsymbol{A_0X_0} = \boldsymbol{E}^{-1}\boldsymbol{B_0} = \boldsymbol{E}^{-1}\boldsymbol{EB} = \boldsymbol{B}$. Hence $\boldsymbol{X_0}$ is also a solution to $\boldsymbol{AX} = \boldsymbol{B}$.

Hence the matrix equations $\boldsymbol{AX} = \boldsymbol{B}$ and $\boldsymbol{A_0X} = \boldsymbol{B_0}$ have the same solution set.

Lemma 2 If $\boldsymbol{X_0}$ is a particular solution to the matrix equation $\boldsymbol{AX} = \boldsymbol{B}$, and S is the solution set of the matrix equation $\boldsymbol{AX} = \boldsymbol{0}$, then the solution set of $\boldsymbol{AX} = \boldsymbol{B}$ is $\{\boldsymbol{X_0} + \boldsymbol{Y} \mid \boldsymbol{Y} \in S\}$.

Proof Let $\boldsymbol{X_1}$ be a solution of $\boldsymbol{AX} = \boldsymbol{B}$. Then $\boldsymbol{AX_1} = \boldsymbol{B}$, hence $\boldsymbol{A}(\boldsymbol{X_1} - \boldsymbol{X_0}) = \boldsymbol{AX_1} - \boldsymbol{AX_0} = \boldsymbol{0} - \boldsymbol{0} = \boldsymbol{0}$. Then $\boldsymbol{X_1} = \boldsymbol{X_0} + (\boldsymbol{X_1} - \boldsymbol{X_0})$, where $\boldsymbol{A}(\boldsymbol{X_1} - \boldsymbol{X_0}) \in S$, hence $\boldsymbol{X_1} \in \{\boldsymbol{X_0} + \boldsymbol{Y} \mid \boldsymbol{Y} \in S\}$. Let $\boldsymbol{X_2} \in \{\boldsymbol{X_0} + \boldsymbol{Y} \mid \boldsymbol{Y} \in S\}$. Then $\boldsymbol{X_2} = \boldsymbol{X_0} + \boldsymbol{Y_0}$ for some $\boldsymbol{Y_0} \in S$. Hence $\boldsymbol{AX_2} = \boldsymbol{A}(\boldsymbol{X_0} + \boldsymbol{Y_0}) = \boldsymbol{AX_0} + \boldsymbol{AY_0} = \boldsymbol{B} + \boldsymbol{0} = \boldsymbol{B}$. Hence $\boldsymbol{X_2}$ is a solution of $\boldsymbol{AX} = \boldsymbol{B}$.

Proof of Theorem 1.1 (First Part) Here, we prove that if $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ has a pivot entry on the right-hand side, then the matrix equation $\boldsymbol{AX} = \boldsymbol{B}$ is inconsistent. Suppose for a contradiction that $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ has a pivot entry on the right-hand side, but $\boldsymbol{X_0}$ is a solution to the matrix equation $\boldsymbol{AX} = \boldsymbol{B}$. By Lemma 1, $\boldsymbol{X_0}$ is a solution to $\boldsymbol{A_0X} = \boldsymbol{B_0}$, i.e. $\boldsymbol{A_0X_0} = \boldsymbol{B_0}$. Let the (i, j)-entry of $\boldsymbol{B_0}$ be the pivot entry. Then the (i, j)-entry of $\boldsymbol{B_0}$ is the leading entry of the i-th row of $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$. Hence, the i-th row of $\boldsymbol{A_0}$ is a zero row, i.e. $[\boldsymbol{A_0}]_{ik} = 0$ for $k = 1, 2, ..., p$. Then by the formula of matrix multiplication, we have

$$[\boldsymbol{B_0}]_{ij} = \sum_{k=1}^{p} [\boldsymbol{A_0}]_{ik}[\boldsymbol{X_0}]_{kj} = \sum_{k=1}^{p} 0 \cdot [\boldsymbol{X_0}]_{kj} = 0$$

This contradicts the fact that $[\boldsymbol{B_0}]_{ij}$ is a pivot entry. Hence we conclude that if $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ has a pivot entry on the right-hand side, then the matrix equation $\boldsymbol{AX} = \boldsymbol{B}$ does not have a solution.

To show the converse of this statement, that if $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ does not have a pivot entry on the right-hand side, then the matrix equation $\boldsymbol{AX} = \boldsymbol{B}$ is consistent, we will need to establish the results of Theorem 1.2 and 1.3 first.

Proof of Theorem 1.2 Suppose $\boldsymbol{AX} = \boldsymbol{B}$ is consistent and every column on the left-hand side of $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ is a pivot column. Since $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ is in reduced row-echelon form, this implies that $\boldsymbol{A_0}$ is in the form $\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{pmatrix}$. By the contrapositive of Theorem 1.1 (First Part), since $\boldsymbol{AX} = \boldsymbol{B}$ is consistent, $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ does not have a pivot entry on the right-hand side. Hence $\boldsymbol{B_0}$ is of the form $\begin{pmatrix} \boldsymbol{X_0} \\ \boldsymbol{0} \end{pmatrix}$, where $\boldsymbol{X_0}$ is an $n \times n$ matrix. Hence $\boldsymbol{A_0X_0} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{pmatrix} \boldsymbol{X_0} = \begin{pmatrix} \boldsymbol{IX_0} \\ \boldsymbol{0X_0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X_0} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{B_0}$. Hence $\boldsymbol{X_0}$ is a solution to $\boldsymbol{A_0X} = \boldsymbol{B_0}$. Let $\boldsymbol{X_1}$ be a solution to $\boldsymbol{A_0X} = \boldsymbol{0}$. Then $\boldsymbol{0} = \boldsymbol{A_0X_1} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{pmatrix} \boldsymbol{X_1} = \begin{pmatrix} \boldsymbol{IX_1} \\ \boldsymbol{0X_1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X_1} \\ \boldsymbol{0} \end{pmatrix}$. Hence $\boldsymbol{X_1} = \boldsymbol{0}$. Hence $\boldsymbol{A_0X} = \boldsymbol{0}$ only has the zero solution. By Lemma 2, every solution of $\boldsymbol{A_0X} = \boldsymbol{B_0}$ is of the form $\boldsymbol{X_0} + \boldsymbol{Y}$, where $\boldsymbol{Y}$ is a solution of $\boldsymbol{A_0X} = \boldsymbol{0}$. Hence $\boldsymbol{X_0}$ is the unique solution to $\boldsymbol{A_0X} = \boldsymbol{B_0}$. By Lemma 1, $\boldsymbol{X_0}$ is the unique solution to $\boldsymbol{AX} = \boldsymbol{B}$,

Proof of Theorem 1.3 Suppose $\boldsymbol{AX} = \boldsymbol{B}$ is consistent and there is a non-pivot column on the left-hand side of $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$.

First, we find a matrix $\boldsymbol{X_0}$ and show that it is a particular solution of $\boldsymbol{A_0X} = \boldsymbol{B_0}$. Let k be the number of pivot columns of $\boldsymbol{A_0}$. Then $\boldsymbol{A_0}$ has $n-k$ zero rows. By the contrapositive of Theorem 1.1 (First Part), since $\boldsymbol{AX} = \boldsymbol{B}$ is consistent, $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ does not have a pivot entry on the right-hand side. Hence $\boldsymbol{B_0}$ must also have $n-k$ zero rows. Let $\boldsymbol{A_0} = \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & ... & \boldsymbol{a_p} \end{pmatrix}$, $\boldsymbol{B_0} = \begin{pmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \vdots \\ \boldsymbol{b_k} \\ \boldsymbol{0} \end{pmatrix}$, where $\boldsymbol{a_1}, \boldsymbol{a_2}, ..., \boldsymbol{a_p}$ are column vectors, $\boldsymbol{b_1}, \boldsymbol{b_2}, ..., \boldsymbol{b_k}$ are row vectors and $\boldsymbol{0}$ is a $(n-k) \times m$ zero matrix. Define a $p \times n$ matrix $\boldsymbol{X_0} = \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \vdots \\ \boldsymbol{x_p} \end{pmatrix}$ such that, for $i = 1, 2, ..., p$,

$$\boldsymbol{x_i} = \begin{cases} \boldsymbol{b_j} & \text{if } \boldsymbol{a_i} \text{ is the } j\text{-th pivot column of } \boldsymbol{A_0} \\ \boldsymbol{0} & \text{if } \boldsymbol{a_i} \text{ is not a pivot column of } \boldsymbol{A_0} \end{cases}$$

Note that since $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ is in reduced row-echelon form, the j-th pivot column of $\boldsymbol{A_0}$ is $\boldsymbol{e_j}$. Hence, for $i = 1, 2, ..., n$, if $\boldsymbol{a_i}$ is the j-th pivot column of $\boldsymbol{A_0}$, then $\boldsymbol{a_i x_i} = \boldsymbol{e_j b_j}$, and if $\boldsymbol{a_i}$ is not a pivot column of $\boldsymbol{A_0}$, then $\boldsymbol{a_i x_i} = \boldsymbol{a_i 0} = \boldsymbol{0}$. Hence, we have

$$\begin{aligned} \boldsymbol{A_0X_0} &= \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & ... & \boldsymbol{a_p} \end{pmatrix} \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \vdots \\ \boldsymbol{x_p} \end{pmatrix} \\ &= \boldsymbol{a_1x_1} + \boldsymbol{a_2x_2} + ... + \boldsymbol{a_px_p} \\ &= \boldsymbol{e_1x_1} + \boldsymbol{e_2x_2} + ... + \boldsymbol{e_kb_k} \\ &= \begin{pmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \vdots \\ \boldsymbol{b_k} \\ \boldsymbol{0} \end{pmatrix} \\ &= \boldsymbol{B_0} \end{aligned}$$

Hence $\boldsymbol{X_0}$ is a solution to $\boldsymbol{A_0X} = \boldsymbol{B_0}$.

Next, let $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be a basis for the nullspace of $\boldsymbol{A_0}$, and let $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & ... & \boldsymbol{v_k} \end{pmatrix}$. We show that $\{\boldsymbol{VT} \mid \boldsymbol{T} \text{ is a } k \times n \text{ matrix.}\}$ is the solution set of $\boldsymbol{A_0X} = \boldsymbol{0}$. If $\boldsymbol{X_0} = \boldsymbol{VT}$

for some $k \times n$ matrix $\boldsymbol{T}$, then we have

$$\begin{aligned}
\boldsymbol{A_0 X_0} &= \boldsymbol{A_0 V T} \\
&= \boldsymbol{A_0} \begin{pmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \dots & \boldsymbol{v_k} \end{pmatrix} \boldsymbol{T} \\
&= \begin{pmatrix} \boldsymbol{A v_1} & \boldsymbol{A v_2} & \dots & \boldsymbol{A v_k} \end{pmatrix} \boldsymbol{T} \\
&= \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \end{pmatrix} \boldsymbol{T} \\
&= \boldsymbol{0}
\end{aligned}$$

Hence $\boldsymbol{X_0}$ is a solution of $\boldsymbol{A_0 X} = \boldsymbol{0}$.

On the other hand, suppose $\boldsymbol{X_0}$ is a solution of $\boldsymbol{A_0 X} = \boldsymbol{0}$. Let $\boldsymbol{X_0} = \begin{pmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} & \dots & \boldsymbol{x_p} \end{pmatrix}$, where $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_p}$ are column vectors. Then we have

$$\begin{aligned}
\boldsymbol{0} &= \boldsymbol{A_0 X_0} \\
&= \boldsymbol{A_0} \begin{pmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} & \dots & \boldsymbol{x_p} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{A_0 x_1} & \boldsymbol{A_0 x_2} & \dots & \boldsymbol{A_0 x_p} \end{pmatrix}
\end{aligned}$$

Hence, for $k = 1, 2, \dots, p$, $\boldsymbol{A_0 x_k} = 0$, hence $\boldsymbol{x_k}$ is in the nullspace of $\boldsymbol{A_0}$. Hence $\boldsymbol{x_k} \in \operatorname{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}\}$, so $\boldsymbol{x_k}$ is in the column space of V. Hence there exists $\boldsymbol{t_k}$ such that $\boldsymbol{x_k} = \boldsymbol{V t_k}$. Hence,

$$\begin{aligned}
\boldsymbol{X_0} &= \begin{pmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} & \dots & \boldsymbol{x_p} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{V t_1} & \boldsymbol{V t_2} & \dots & \boldsymbol{V t_p} \end{pmatrix} \\
&= \boldsymbol{V} \begin{pmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} & \dots & \boldsymbol{x_p} \end{pmatrix}
\end{aligned}$$

Hence $\boldsymbol{X_0} \in \{\boldsymbol{VT} \mid \boldsymbol{T} \text{ is a } k \times n \text{ matrix.}\}$.

Hence, we have shown that $\{\boldsymbol{VT} \mid \boldsymbol{T} \text{ is a } k \times n \text{ matrix.}\}$ is the solution set of $\boldsymbol{A_0 X} = \boldsymbol{0}$.

By Lemma 2, we conclude that $\boldsymbol{A_0 X} = \boldsymbol{B_0}$ has solution set

$$\{\boldsymbol{X_0} + \boldsymbol{VT} \mid \boldsymbol{T} \text{ is a } k \times n \text{ matrix.}\}$$

By Lemma 1, this is also the solution set of $\boldsymbol{AX} = \boldsymbol{B}$.

Proof of Theorem 1.1 (Second Part) We now prove that if $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ does not have a pivot entry on the right-hand side, then the matrix equation $\boldsymbol{AX} = \boldsymbol{B}$ is consistent. If every column of $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ is a pivot column, then Theorem 1.2 shows that $\boldsymbol{AX} = \boldsymbol{B}$ is consistent. On the other hand, if $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$ has a non-pivot column, then Theorem 1.3 shows that $\boldsymbol{AX} = \boldsymbol{B}$ is consistent.

3 Algorithmic Implementation

The algorithm can be implemented via the following Python code. The algorithm uses a custom Matrix class which is found in the Python file linalg.py.

```
def solve_matrix_equation(A, B):
    m, p, n = A.rows, A.cols, B.cols

    aug = A | B
    aug.rref()

    # if (A0 | B0) has a pivot entry on the RHS,
    # the matrix equation is inconsistent
    pivots = aug.get_pivots()
    for col in pivots:
        if col >= p:
            return ("Inconsistent", None)

    # check for non-pivot column in the LHS
    has_non_pivot = False
    for i in range(p):
        if i not in pivots:
            has_non_pivot = True

    # create the matrices A0, B0
    A0 = aug.slice(0, m, 0, p)
    B0 = aug.slice(0, m, p, p + n)

    # if every column on the LHS is a pivot column,
    # then the matrix equation has a unique solution
    if not has_non_pivot:
        return ("Unique Solution", B0.slice(0, p, 0, n))

    # if the LHS has a non pivot column,
    # then the matrix has infinitely many solutions
    pivot_indices = A0.get_pivot_indices()
    X0 = Matrix.from_formula(n, p,
        lambda i, j: 0 if pivot_indices[i] == -1 else B0[pivot_indices[i], j])
    V = A0.null()

    return ("Infinitely Many Solutions", X0, V)
```

4 Time Complexity Analysis

By performing Gaussian Elimination on the $m \times (n+p)$ augmented matrix $(\boldsymbol{A_0} \mid \boldsymbol{B_0})$, instead of an entirely different linear system, we reduce the time complexity of finding the general solution from $O(mp^2n^3)$, down to $O(m^2(p+n))$, which is faster by several orders of magnitude, especially for larger matrices.