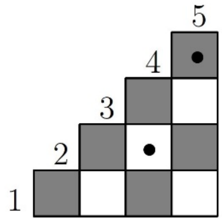
Solution to Exam 1

Q1.

(a) Here we use black dot to represent rooks, then 1/245/3 can be expressed as below.



(b) The number of rooks is $n-k$.

The size of triangle board is $n-1\times n-1$.

(c) Assume that the two rooks are named as r1 and r2, obviously r1 and r2 cannot be in the same row or the same column, r1 is always at the left side of r2.

As r2 is in the 4th column, number of positions of r1 is 3+5+6=14.

As r2 is in the 3rd column, number of positions of r1 is 1+2+3=6.

As r2 is in the 2nd column, number of positions of r1 is 1.

Hence the number of partitions in $B([5])$ under this restriction is 14+6+1=21.

The example of a two-rook partition than is forbidden under this restriction can be $125/3/4$.

Q2.

(a) The problem is equal to a weak composition of 15 into 3 parts, thus the number of ways is

$$\binom{15+3-1}{15}=\binom{17}{15}=\frac{17!}{15!(17-15)!}=136$$

(b) The number of ways that 15 oranges are chosen is 1, that is 0 apple + 15 oranges + 0 grapefruit.

The number of ways that 15 grapefruits are chosen is 1, that is 0 apple + 0 orange + 15 grapefruits.

The number of ways that 14 grapefruits are chosen is 2, that is 1 apple + 0 orange + 14 grapefruits, and 0 apple + 1 orange + 14 grapefruits.

The number of ways that 13 grapefruits are chosen is 3, that is 1 apple + 1 orange + 13 grapefruits, 2 apple + 0 orange + 13 grapefruits, and 0 apple + 2 oranges + 13 grapefruits

Then the number of different ways with only 14 oranges and 12 grapefruits in the store is 136-1-1-2-3=129.

Q3.

Using the method of proof by contradiction, we take an assumption that each two of these numbers are more than 10 units apart.

With ascending order, we denote the 5 numbers as x_1, x_2, x_3, x_4 and x_5, then equation $x_1+x_2+x_3+x_4+x_5=100$ is satisfied. According to the assumption above, we have

$$\begin{aligned}
&x_2 > x_1+10\\
&x_3 > x_2+10 > x_1+20\\
&x_4 > x_3+10 > x_2+20 > x_1+30\\
&x_5 > x_4+10 > x_3+20 > x_2+30 > x_1+40
\end{aligned}$$

Then we have

$$x_1+x_2+x_3+x_4+x_5 > x_1+(x_1+10)+(x_1+20)+(x_1+30)+(x_1+40)$$

$$100 > x_1+100$$

$$x_1 < 0$$

It contradicts that x_1 is a positive real number, indicating that our assumption is wrong, hence at least two of these numbers must be no more than 10 units apart, proof is finished.

Q4.

We have

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

Then

$$\begin{aligned}\sum_{k=0}^{m}\binom{n}{k}(-1)^k &= \binom{n}{0}(-1)^0+\sum_{k=1}^{m}\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right](-1)^k = 1+\sum_{k=1}^{m}\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right](-1)^k\\
&= \binom{n-1}{0}+\sum_{k=1}^{m}\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right](-1)^k\\
&= \binom{n-1}{0}-\left[\binom{n-1}{0}+\binom{n-1}{1}\right]+\left[\binom{n-1}{1}+\binom{n-1}{2}\right]-\left[\binom{n-1}{2}+\binom{n-1}{3}\right]+\\
&\quad \cdots+(-1)^m\left[\binom{n-1}{m-1}+\binom{n-1}{m}\right]\\
&= \left[\binom{n-1}{0}-\binom{n-1}{0}\right]+\left[\binom{n-1}{1}-\binom{n-1}{1}\right]+\left[\binom{n-1}{2}-\binom{n-1}{2}\right]+\\
&\quad \cdots+\left[(-1)^{m-1}\binom{n-1}{m-1}+(-1)^m\binom{n-1}{m-1}\right]+(-1)^m\binom{n-1}{m}\\
&= 0+0+0+\cdots+0+(-1)^m\binom{n-1}{m}\\
&= (-1)^m\binom{n-1}{m}\end{aligned}$$

Hence we have

$$\sum_{k=0}^{m}\binom{n}{k}(-1)^k=(-1)^m\binom{n-1}{m}$$

Q5.

(a) For $\left\{ {n+1 \atop k} \right\}_p$, it is equal to the number of all set partitions of $[n+1]$ with k blocks such that each block has no fewer than p elements, which can be deduced from only considering the first n elements.

If the first n elements, that is, $[n]$, can be divided into k blocks which all have no fewer than p elements, then $n+1$ can be added to any block of them so that we have set partitions of $[n+1]$ with k blocks whose sizes are not smaller than p, the number of this kind of cases is $k\left\{ {n \atop k} \right\}_p$.

If the first n elements, that is, $[n]$, can be divided into k blocks but not all blocks have no fewer than p elements, then $n+1$ will be key. Only if there is only one block with $p-1$ elements while the other blocks have no fewer than p elements, by adding $n+1$ to this one block can we have set partitions of $[n+1]$ with k blocks whose sizes are not smaller than p. The number of different ways of building this one block is $\binom{n}{p-1}$, the number of different ways to divide remaining $n-(p-1)=n-p+1$ elements into $k-1$ blocks with sizes not smaller than p is $\left\{ {n-p+1 \atop k-1} \right\}_p$, hence the number of this kind of cases is $\binom{n}{p-1}\left\{ {n-p+1 \atop k-1} \right\}_p$.

The above two kinds of cases can cover all possible ways of dividing $[n+1]$ into k blocks such that each block has no fewer than p elements, thus the final number is

$$\left\{ {n+1 \atop k} \right\}_p = k\left\{ {n \atop k} \right\}_p + \binom{n}{p-1}\left\{ {n-p+1 \atop k-1} \right\}_p$$

(b) If we hope to divided $[n]$ into set partitions than has no fewer than 2 elements, then the number of set partitions should be not greater than $n/2$, let K denotes the maximum integer than is not greater than $n/2$. Then we have

$$b(n)_2 = \left\{ {n \atop 1} \right\}_2 + \left\{ {n \atop 2} \right\}_2 + \cdots + \left\{ {n \atop K} \right\}_2 = \sum_{k\in N^+,\, k\le\frac{n}{2}} \left\{ {n \atop k} \right\}_2$$

(c) The table is as below.

	1	2	3	4	5	6	$b(n)_2$
2	1	0	0	0	0	0	1
3	1	0	0	0	0	0	1
4	1	3	0	0	0		4
5	1	10	0	0	0	0	11
6	1	25	15	0	0	0	41

7	1	56	105	0	0	0	162
8	1	119	490	105	0	0	715
9	1	246	1918	1260	0	0	3425
10	1	501	6825	9450	945	0	17722

Q6.

We have

$$3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} == \sum_{k=0}^{n} \binom{n}{k} 2^k$$

$$(-1)^n = (-2+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-2)^k 1^{n-k} == \sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^k$$

Then

$$3^n + (-1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k + \sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^k = \sum_{k=0}^{n} \binom{n}{k} \left[1+(-1)^k\right] 2^k$$

As k is odd, $1+(-1)^k = 1-1=0$, as k is even, $1+(-1)^k = 1+1=2$, hence

$$3^n + (-1)^n = \sum_{k=0}^{n} \binom{n}{k} \left[1+(-1)^k\right] 2^k = \sum_{k \text{ even}} \binom{n}{k} 2 \cdot 2^k = 2 \sum_{k \text{ even}} \binom{n}{k} 2^k$$

And we finally have

$$\sum_{k \text{ even}} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}$$