Solution to Quiz 10

Q1.

(a) Using the recursion, we have

$$a_2 = 3a_1 - a_0 = 3\times 4 - 1 = 11$$
$$a_3 = 3a_2 - a_1 = 3\times 11 - 4 = 29$$
$$a_4 = 3a_3 - a_2 = 3\times 29 - 11 = 76$$

(b) With generating function as $A(x) = \sum_{n\ge 0} a_n x^n$, we have

$$\sum_{n\ge 0} a_{n+1} x^{n+1} = A(x) - a_0 x^0 = A(x) - 1$$

$$\sum_{n\ge 0} a_{n+2} x^{n+2} = A(x) - a_0 x^0 - a_1 x^1 = A(x) - 1 - 4x$$

Using the recursion, we have

$$\begin{aligned}\sum_{n\ge 0} a_{n+2} x^{n+2} &= \sum_{n\ge 0} \left(3a_{n+1} - a_n\right) x^{n+2} = 3x\sum_{n\ge 0} a_{n+1} x^{n+1} - x^2 \sum_{n\ge 0} a_n x^n \\ &= 3x\left[A(x) - 1\right] - x^2 A(x) = \left(3x - x^2\right) A(x) - 3x\end{aligned}$$

That is

$$A(x) - 1 - 4x = \left(3x - x^2\right) A(x) - 3x$$

$$\left(x^2 - 3x + 1\right) A(x) = x + 1$$

Then we finally have

$$A(x) = \frac{x+1}{x^2 - 3x + 1}$$

Q2.

With generating function as $B(x) = \sum_{n\ge 0} b_n x^n$, we have

$$\sum_{n\ge 0} b_{n+1} x^{n+1} = B(x) - b_0 x^0 = B(x) - 1$$

$$\sum_{n\ge 0} b_{n+2} x^{n+2} = B(x) - b_0 x^0 - b_1 x^1 = B(x) - 1 - 5x$$

$$\sum_{n\geq 0} b_{n+3}x^{n+3} = B(x) - b_0x^0 - b_1x^1 - b_2x^2 = B(x) - 1 - 5x - 12x^2$$

Using the recursion, we have

$$\begin{aligned}\sum_{n\geq 0} b_{n+3}x^{n+3} &= \sum_{n\geq 0}\left(3b_{n+2} - b_{n+1} + 4b_n\right)x^{n+3} \\ &= 3x\sum_{n\geq 0} b_{n+2}x^{n+2} - x^2\sum_{n\geq 0} b_{n+1}x^{n+1} + 4x^3\sum_{n\geq 0} b_n x^n\end{aligned}$$

That is

$$B(x) - 1 - 5x - 12x^2 = 3x\left[B(x) - 1 - 5x\right] - x^2\left[B(x) - 1\right] + 4x^3 B(x)$$

$$\left(1 - 3x + x^2 - 4x^3\right)B(x) = 1 + 2x - 2x^2$$

Hence we finally obtain

$$B(x) = \frac{1 + 2x - 2x^2}{1 - 3x + x^2 - 4x^3}$$