Solution to Quiz 8

Q1.

(a) Using Sieve Formula, the number of tablecloths which had at least one stain is

$$\begin{gathered}|C \cup V \cup G \cup S| \\ =|C|+|V|+|G|+|S|-|VC|-|VG|-|SG|-|CS|-|CG|-|VS|+|VCS|+|VCG|+|VGS|+|CSG|-|CVGS| \\ =6+11+13+9-3-5-3-0-4-5+0+2+2+0-0=23\end{gathered}$$

Then the number of tablecloths that had no stains is 27-23=4.

(b) Using Sieve Formula, the number of tablecloths which has more than one stain is

$$\begin{aligned}
|VC \cup VG \cup SG \cup CS \cup CG \cup VS| &= |VC|+|VG|+|SG|+|CS|+|CG|+|VS| \\
&-|VC \cap VG|-|VC \cap SG|-|VC \cap CS|-|VC \cap CG|-|VC \cap VS| \\
&-|VG \cap SG|-|VG \cap CS|-|VG \cap CG|-|VG \cap VS| \\
&-|SG \cap CS|-|SG \cap CG|-|SG \cap VS| \\
&-|CS \cap CG|-|CS \cap VS| \\
&-|CG \cap VS| \\
&+\sum_{A_1, A_2, A_3, \in\{VC, VG, SG, CS, CG, VS\}}(-1)^3\left|A_1 \cap A_2 \cap A_3\right| \\
&+\sum_{A_1, A_2, A_3, A_4 \in\{VC, VG, SG, CS, CG, VS\}}(-1)^3\left|A_1 \cap A_2 \cap A_3 \cap A_4\right| \\
&+\sum_{A_1, A_2, A_3, A_4, A_5 \in\{VC, VG, SG, CS, CG, VS\}}(-1)^4\left|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5\right| \\
&-|VC \cap VG \cap SG \cap CS \cap CG \cap VS| \\
=|VC|+|VG|+|SG|+|CS|+|CG|+|VS|-2|VCS|-2|VCG|-2|VGS|-2|CSG|+4|CVGS| \\
=3+5+3+0+4+5-2\times 0-2\times 2-2\times 2-2\times 0+4\times 0=12
\end{aligned}$$

From part (a) we know that the number of tablecloths that had at least one stain is 23, then the number of tablecloths that has exactly one stain is 23-12=11.

Q2.

For $i \in[n-1]$, $j \in[n]$, assume that $\pi(i)=j$ and exactly $\pi(i)>\pi(i+1)$ while $\pi(k)<\pi(k+1)$ for $k \neq i$.

Obviously there are only $j-1$ elements smaller than j, that is $[j-1]$, we hope that $\pi(k)<j=\pi(i)$ when $k<i$. One of $[j-1]$ is equal to $\pi(i+1)$, thus we have at most $j-2$ elements before $\pi(i)$, k is at most $j-2$, then i is at most $j-1$, that is,

$i=1,2,\cdots,j-1$ and $j=2,3,\cdots,n$.

For a given j and a given i, to make a n-permutation whose size of drop sets is 1, we need to first select $i-1$ elements from $[j-1]$ and put them before index i with ascending order, the number of ways is $\binom{j-1}{i-1}$, then we put the remaining elements of $[n]$ after index i with ascending order, since the first selected $i-1$ elements from $[j-1]$ do not make all of $[j-1]$, thus $\pi(i+1)$ must be from $[j-1]$ so that $\pi(i)>\pi(i+1)$.

Therefore, the total number of n-permutations with drop sets of size 1 is equal to

$$\sum_{j=2}^{n}\sum_{i=1}^{j-1}\binom{j-1}{i-1}$$