Solution

Q1. We divide the fruit collection into three groups, apple group, orange group and pear group respectively.

If the apple group has a content of 0, then the orange group's content can be 0~10 with remaining in the pear group, there are 11 cases.

If the apple group has a content of 1, then the orange group's content can be 0~9 with remaining in the pear group, there are 10 cases.

If the apple group has a content of 2, then the orange group's content can be 0~8 with remaining in the pear group, there are 9 cases.

……

If the apple group has a content of 10, then the orange group's content can only be 0 with remaining in the pear group (which is also 0), there is only 1 case.

Hence, the number of distinct collections is 11+10+9+8+…+3+2+1=66.

Q2. Using the combinatorial argument, we respectively interpret the two sides of the equation.

For the left-hand side

$$\binom{n}{k}\binom{k}{m}$$

We can interpret it as the number of distinct methods of choosing k items from n items and then choosing m items from the chosen k items.

For the right-hand side

$$\binom{n}{m}\binom{n-m}{k-m}$$

We can interpret it as the number of distinct methods of choosing m items from n items and then choosing k-m items from the remaining n-m items.

For the above two interpretations, it is easy to find that by performing the above two sets of steps separately we eventually get two groups chosen from the original n items, one group contains m items and the other group contains k-m items. Though the two sets of steps are different, their results are the same, hence the left-hand side must be equal to the right-hand side, we have proved

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Q3. With the identity

$$(t+1)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$$

By substitution $t = \frac{x}{y}$, we obtain

$$(t+1)^n = \sum_{k=0}^{n} \binom{n}{k} t^k \quad \Rightarrow \quad \left(\frac{x}{y}+1\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{y}\right)^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{-k}$$

Then we have

$$\begin{aligned}(x+y)^n &= \left[\left(\frac{x}{y}+1\right)y\right]^n = \left(\frac{x}{y}+1\right)^n y^n = \left[\sum_{k=0}^{n} \binom{n}{k} x^k y^{-k}\right] y^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{-k} y^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\end{aligned}$$

Hence we have proved

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$