Solution to Worksheet (WS) 2

Question 1

(a) Using the definition of transfer matrix for both lens and free propagation, we have

$$\boldsymbol{M}_{a1} = \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{M}_{ba} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}, \ \boldsymbol{M}_{2b} = \begin{bmatrix} 1 & s_2 \\ 0 & 1 \end{bmatrix}$$

Hence the system transfer matrix is

$$\boldsymbol{M}_s = \boldsymbol{M}_{2b} \cdot \boldsymbol{M}_{ba} \cdot \boldsymbol{M}_{a1} = \begin{bmatrix} 1 & s_2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-s_2/f & s_1+s_2-s_1 s_2/f \\ -1/f & 1-s_1/f \end{bmatrix}.$$

(b) In transferring a ray from plane 1 to plane 2, we have

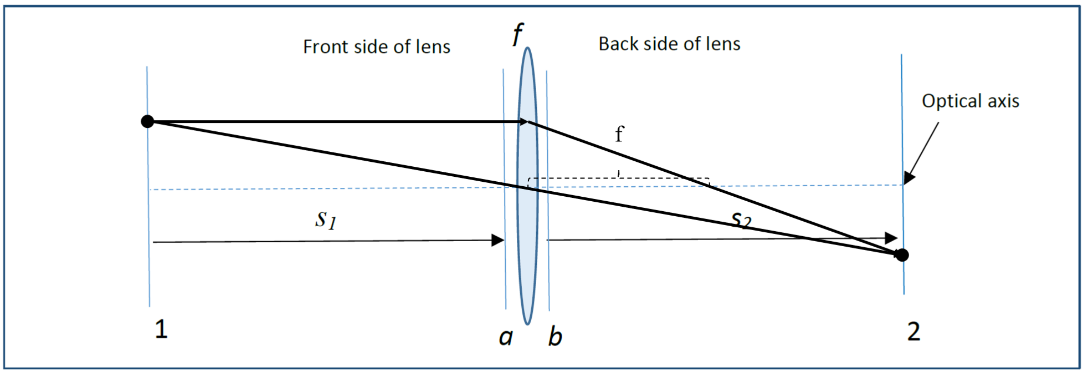
$$\begin{bmatrix} x \\ x' \end{bmatrix}_2 = \boldsymbol{M}_s \begin{bmatrix} x \\ x' \end{bmatrix}_1 = \begin{bmatrix} 1-s_2/f & s_1+s_2-s_1 s_2/f \\ -1/f & 1-s_1/f \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_1$$

By expanding the matrix equation, we have

$$\begin{cases} x_2 = (1-s_2/f)\,x_1 + (s_1+s_2-s_1 s_2/f)\,x_1' \\ x_2' = (-1/f)\,x_1 + (1-s_1/f)\,x_1' \end{cases}.$$

(c)

i) The graph is as below



ii) For a point on plane 1 with fixed x_1, all light with different directions should reach the same point on plane 2 with fixed x_2.

From part b, we have

$$x_2 = \left(1 - s_2/f\right)x_1 + \left(s_1 + s_2 - s_1 s_2/f\right)x_1'$$

Since the direction does not matter and $x_1' = n\alpha$, the coefficient of term x_1' should be 0, that is, $s_1 + s_2 - s_1 s_2/f = 0$, which is one element of the system matrix.

Hence, from $s_1 + s_2 - s_1 s_2/f = 0$, we can deduce that

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f},$$

now we replace s_1 by u and s_2 by v, so that we have

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f},$$

which is the thin lens equation.

(d) From part b, we have

$$\begin{cases} x_2 = \left(1 - s_2/f\right)x_1 + \left(s_1 + s_2 - s_1 s_2/f\right)x_1' \\ x_2' = \left(-1/f\right)x_1 + \left(1 - s_1/f\right)x_1' \end{cases}$$

In order to have

$$\begin{cases} x_2 = f\left(x_1'\right) \\ x_2' = f\left(x_1\right) \end{cases}$$

We need to set

$$\begin{cases} 1 - s_2/f = 0 \\ 1 - s_1/f = 0 \end{cases}$$

Hence we obtain the relation between s_1, s_2 and f as $s_1 = s_2 = f$.

Question 2

(a) From question 2, we easily have

$$\boldsymbol{M}_{a1}=\begin{bmatrix} 0 & f_1 \\ -1/f_1 & 1-s_1/f_1 \end{bmatrix} \text{ and } \boldsymbol{M}_{2a}=\begin{bmatrix} 1-s_2/f_2 & f_2 \\ -1/f_2 & 0 \end{bmatrix}.$$

Then the system transfer matrix is defined as

$$\begin{aligned}\boldsymbol{M}_s=\boldsymbol{M}_{2a}\cdot\boldsymbol{M}_{a1}&=\begin{bmatrix} 1-s_2/f_2 & f_2 \\ -1/f_2 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & f_1 \\ -1/f_1 & 1-s_1/f_1 \end{bmatrix} \\ &=\begin{bmatrix} -f_2/f_1 & f_1+f_2-s_2f_1/f_2-s_1f_2/f_1 \\ 0 & -f_1/f_2 \end{bmatrix}\end{aligned}.$$

(b) In transferring a ray from plane 1 to plane 2, we have

$$\begin{bmatrix} x \\ x' \end{bmatrix}_2=\boldsymbol{M}_s\begin{bmatrix} x \\ x' \end{bmatrix}_1=\begin{bmatrix} -f_2/f_1 & f_1+f_2-s_2f_1/f_2-s_1f_2/f_1 \\ 0 & -f_1/f_2 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}_1$$

By expanding the matrix equation, we have

$$\begin{cases} x_2=(-f_2/f_1)x_1+(f_1+f_2-s_2f_1/f_2-s_1f_2/f_1)x_1' \\ x_2'=(-f_1/f_2)x_1' \end{cases}.$$

(c)

i) For the case of a point light-source on the optical axis, we have $x_1=x_2=0$, hence the equations can be simplified as

$$\begin{cases} x_1=x_2=0 \\ x_2'=(-f_1/f_2)x_1' \end{cases}.$$

Since we have $x_2=(-f_2/f_1)x_1+(f_1+f_2-s_2f_1/f_2-s_1f_2/f_1)x_1'$, which turns to be

$$0=(f_1+f_2-s_2f_1/f_2-s_1f_2/f_1)x_1'.$$

x_1' is not fixed, hence we have $f_1+f_2-s_2f_1/f_2-s_1f_2/f_1=0$, then

$$s_2 = f_2 + \frac{f_2 \cdot f_2}{f_1} - s_1 \frac{f_2 \cdot f_2}{f_1 \cdot f_1}.$$

ii) Since we have $x_2 = \left(-f_2/f_1\right)x_1 + \left(f_1 + f_2 - s_2 f_1/f_2 - s_1 f_2/f_1\right)x_1'$ from part b and $f_1 + f_2 - s_2 f_1/f_2 - s_1 f_2/f_1 = 0$ from part c(i), we have $x_2 = \left(-f_2/f_1\right)x_1$, hence we can define that

$$\frac{x_2}{x_1} = -\frac{f_2}{f_1}.$$

iii) With $f_1 = f_2 = f$, the equations turn to be

$$\begin{cases} x_1 = x_2 = 0 \\ x_2' = -x_1' \end{cases}.$$