Solution

Q1. (a) For given function $f(x) = \ln(1+x)$, we have its n-th order derivative as

$$f^{(n)}(x) = (-1)^n \frac{n!}{(1+x)^n}, \quad n = 1,2,3,\cdots$$

Then the Taylor polynomial of degree n about the point $x_0 = 0$ is

$$P_n(x) = f(x_0) + \sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i = f(x_0) + \sum_{i=1}^{n} \frac{(-1)^i \frac{i!}{(1+x_0)^i},}{i!}(x-x_0)^i = \sum_{i=1}^{n} (-1)^i x^i$$

(b) With $x_0 = 0$, the remainder can be written as

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} = \frac{(-1)^{n+1} \frac{(n+1)!}{(1+\xi)^{n+1}}}{(n+1)!}(x-x_0)^{n+1} = (-1)^{n+1} \frac{x^{n+1}}{(1+\xi)^{n+1}}$$

where $\xi \in [x, x_0]([x_0, x])$.

Then $\ln(1.1) = \ln(1+0.1) = f(0.1)$, the remainder is

$$R_{n+1}(0.1) = (-1)^{n+1} \frac{0.1^{n+1}}{(1+\xi)^{n+1}} \quad \text{where} \quad \xi \in [0, 0.1]$$

Hence we have

$$|R_{n+1}(0.1)| = \left| (-1)^{n+1} \frac{0.1^{n+1}}{(1+\xi)^{n+1}} \right| = \frac{0.1^{n+1}}{(1+\xi)^{n+1}} \le \frac{0.1^{n+1}}{(1+0)^{n+1}} = 0.1^{n+1} = 10^{-(n+1)}$$

To make the error within 10^{-5}, n is at least 4.

Q2. (a) The Taylor series of $\sin(x)$ about the point $x_0 = 0$ is

$$\sin(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$$

Then the Taylor series of $f(x)=\sin(\pi x/2)$ is

$$\sum_{i=0}^{\infty}(-1)^i\frac{\left(\frac{\pi x}{2}\right)^{2i+1}}{(2i+1)!}=\sum_{i=0}^{\infty}(-1)^i\left(\frac{\pi}{2}\right)^{2i+1}\frac{x^{2i+1}}{(2i+1)!}$$

(b) For a Taylor polynomial of degree n, the remainder is

$$R_{n+1}(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}=\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}, \quad \xi\in[0,x]$$

If n is odd, then $n+1$ is even, $f^{(n+1)}(\xi)=(-1)^{\frac{n+1}{2}}\left(\frac{\pi}{2}\right)^{n+1}\sin\left(\frac{\pi\xi}{2}\right)$, then

$$R_{n+1}(x)=\frac{(-1)^{\frac{n+1}{2}}\left(\frac{\pi}{2}\right)^{n+1}\sin\left(\frac{\pi\xi}{2}\right)}{(n+1)!}x^{n+1},\quad \xi\in[0,x]$$

If n is even, then $n+1$ is odd, $f^{(n+1)}(\xi)=(-1)^{\frac{n}{2}}\left(\frac{\pi}{2}\right)^{n+1}\cos\left(\frac{\pi\xi}{2}\right)$, then

$$R_{n+1}(x)=\frac{(-1)^{\frac{n}{2}}\left(\frac{\pi}{2}\right)^{n+1}\cos\left(\frac{\pi\xi}{2}\right)}{(n+1)!}x^{n+1},\quad \xi\in[0,x]$$

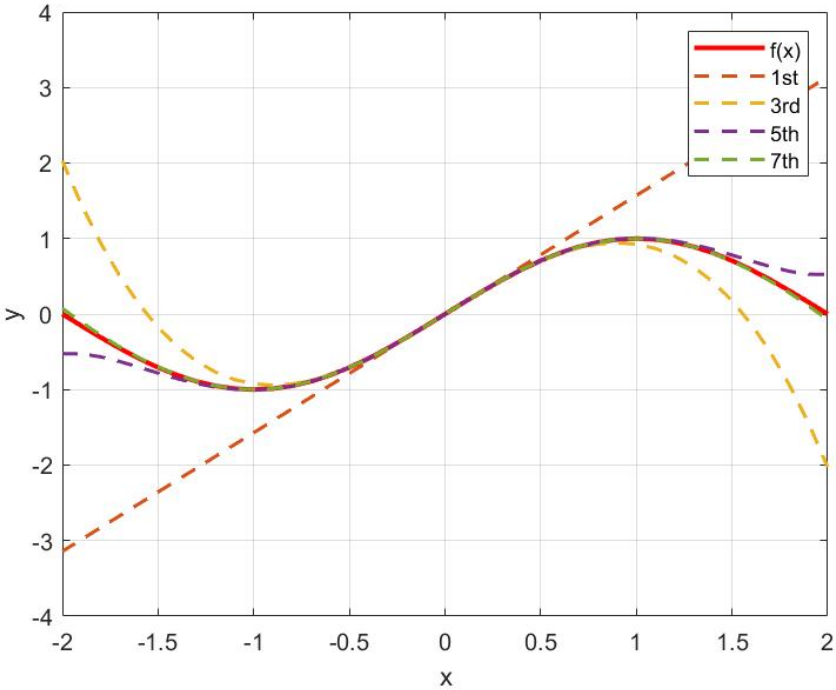
(c) For all $x\in[-1,1]$, we have $\xi\in[-1,1]$, then $\sin\left(\frac{\pi\xi}{2}\right)\in[-1,1]$ and $\cos\left(\frac{\pi\xi}{2}\right)\in[0,1]$, we have $\left|\sin\left(\frac{\pi\xi}{2}\right)\right|\le 1$ and $\left|\cos\left(\frac{\pi\xi}{2}\right)\right|\le 1$.

Then we obtain

$$|R_{n+1}(x)|=\left|\frac{(-1)^{\frac{n+1}{2}}\left(\frac{\pi}{2}\right)^{n+1}\sin\left(\frac{\pi\xi}{2}\right)}{(n+1)!}x^{n+1}\right|\text{or}\left|\frac{(-1)^{\frac{n}{2}}\left(\frac{\pi}{2}\right)^{n+1}\cos\left(\frac{\pi\xi}{2}\right)}{(n+1)!}x^{n+1}\right|\le\frac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)!}$$

To guarantee accuracy of 10^{-6}, we need to guarantee $\dfrac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)!} \le 10^{-6}$, by calculation we obtain that n is at least 11.

(d) The graph is as below.



From the graph above, we can conclude that the higher degree of Taylor polynomial is, the more accurately it can approximate.

Q3. The progress of Gauss elimination is as below

$$[A|I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{r_2 \to r_2 - \frac{c}{a} r_1} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \dfrac{ad-cb}{a} & -\dfrac{c}{a} & 1 \end{array}\right] \xrightarrow{r_2 \to \frac{a}{ad-cb} r_2} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \dfrac{-c}{ad-cb} & \dfrac{a}{ad-cb} \end{array}\right]$$

$$\xrightarrow{r_1 \to \frac{1}{a} r_1} \left[\begin{array}{cc|cc} 1 & \dfrac{b}{a} & \dfrac{1}{a} & 0 \\ 0 & 1 & \dfrac{-c}{ad-cb} & \dfrac{a}{ad-cb} \end{array}\right] \xrightarrow{r_1 \to r_1 - \frac{b}{a} r_2} \left[\begin{array}{cc|cc} 1 & 0 & \dfrac{d}{ad-cb} & \dfrac{-b}{ad-cb} \\ 0 & 1 & \dfrac{-c}{ad-cb} & \dfrac{a}{ad-cb} \end{array}\right] = \left[I|A^{-1}\right]$$

That is, we have

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad-cb} & \dfrac{-b}{ad-cb} \\ \dfrac{-c}{ad-cb} & \dfrac{a}{ad-cb} \end{bmatrix}$$

It is obvious that the A^{-1} makes sense if and only if $ad-cb \neq 0$, that is, $\det(A) \neq 0$, hence we know that A^{-1} exists if and only if $\det(A) \neq 0$.

Q4. (a) We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -m_{2,1} & 1 & 0 & 0 \\ -m_{3,1} & 0 & 1 & 0 \\ -m_{4,1} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & 0 & 1 & 0 \\ m_{4,1} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & 0 & 1 & 0 \\ m_{4,1} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -m_{2,1} & 1 & 0 & 0 \\ -m_{3,1} & 0 & 1 & 0 \\ -m_{4,1} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

Since $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -m_{2,1} & 1 & 0 & 0 \\ -m_{3,1} & 0 & 1 & 0 \\ -m_{4,1} & 0 & 0 & 1 \end{pmatrix}$, we obtain $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & 0 & 1 & 0 \\ m_{4,1} & 0 & 0 & 1 \end{pmatrix}$.

(b) Since $\det(E_2) = 1 \neq 0$, E_2 is invertible, and we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & 0 & 1 \end{pmatrix} E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{3,2} & 1 & 0 \\ 0 & -m_{4,2} & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & 0 & 1 & 0 \\ m_{4,1} & 0 & 0 & 1 \end{pmatrix} = E_1^{-1}$$

Further, we have

$$E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & 0 & 1 \end{pmatrix} E_2 E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & 0 & 1 \end{pmatrix} I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & 0 & 1 \end{pmatrix}$$

(c) Since

$$P_1 P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

It is obvious that $P_1^{-1} = P_1$.

Q5. For given matrix A, we have

$$E_1 A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} = A_1$$

$$E_2 A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4/15 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} = A_2$$

$$E_3 A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -15/56 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{pmatrix} = U$$

That is $E_3 E_2 E_1 A = U$, then $A = E_1^1 E_2^1 E_3^1 U = LU$ where

$$L = E_1^1 E_2^1 E_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/16 & 1 \end{pmatrix}$$

Hence the LU factorization of A is

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/16 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{pmatrix}$$

To solve $Ax = b$, firstly we solve $Lc = b$, that is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/16 & 1 \end{pmatrix} c = \begin{pmatrix} 3 \\ -2 \\ 2 \\ -3 \end{pmatrix} \quad \Rightarrow \quad c = \begin{pmatrix} 3 \\ -11/4 \\ 41/15 \\ -209/56 \end{pmatrix}$$

Then we solve $Ux = c$, that is

$$\begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{pmatrix} x = \begin{pmatrix} 3 \\ -11/4 \\ 41/15 \\ -209/56 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Thus we finally obtain the solution.

Q6. Using the method of induction.

As $n = 1$, we have

$$\sum_{k=1}^{1} k^2 = 1 = \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6}$$

The equation holds.

As $n = p$ and p is an integer greater than 1, assuming the equation still holds, that is

$$\sum_{k=1}^{p} k^2 = \frac{p(p+1)(2p+1)}{6}$$

As $n = p + 1$, we have

$$\sum_{k=1}^{p+1} k^2 = \sum_{k=1}^{p} k^2 + (p+1)^2 = \frac{p(p+1)(2p+1)}{6} + (p+1)^2 = \frac{2p^3 + 3p^2 + p + 6p^2 + 12p + 6}{6}$$

$$= \frac{(p+1)(p+2)(2p+3)}{6} = \frac{(p+1)\left[(p+1)+1\right]\left[2(p+1)+1\right]}{6}$$

Obviously the equation still holds.

Hence we have proven that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.