Lecture 17

2024 Spring

July 11, 2024

Last Lecture

- Markov Inequality $Pr(|X| \geq a) \leq \dfrac{E(|X|^k)}{a^k}$, $a > 0$, $k = 1, 2, \ldots$
- Weak Law of Large Number $\bar{X}_n \xrightarrow{P} E(X)$
- Central Limit Theorem

with conditions

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Norm(0,1)$$

limit of mgf is the mgf of Norm(0,1)

cdf cdf

This Lecture

- (Cont') Central Limit Theorem
- Continuous Mapping Theorem & Slutsky's Theorem & Delta Method

How to combine different convergences

(Cont') Central Limit Theorem

Examples: ① . $X_1 \cdots X_n \overset{iid}{\sim} \chi^2_1$, $Y_n = \sum_{i=1}^{n} X_i$

what is the limiting distn of

$$\frac{\frac{Y_n - n}{\sqrt{2n}}}{}$$

$X \sim \chi^2_1$. $E(X) = 1$. $Var(X) = \frac{1}{2} \cdot 2^2 = 2$

$X \sim Gamma(\frac{1}{2}, 2)$

$E(X) = \alpha\beta$,

$Var(X) = \alpha\beta^2$

$$\frac{Y_n - n}{\sqrt{2n}} = \frac{n(\overline{X}_n - 1)}{\sqrt{2n}} = \frac{\sqrt{n}(\overline{X}_n - 1)}{\sqrt{2}}$$

$$= \frac{\sqrt{n}(\overline{X}_n - E(X))}{\sqrt{Var(X)}} \xrightarrow{d} Norm(0,1)$$

② . $X_1 \cdots X_n \overset{iid}{\sim} Poi(\mu)$. $Y_n = \sum_{i=1}^{n} X_i$

Find the limiting distribution of

$$\frac{\frac{Y_n - n\mu}{\sqrt{n\mu}}}{}$$

$E(X) = Var(X) = \mu$

$$\frac{Y_n - n\mu}{\sqrt{n\mu}} = \frac{n(\overline{X}_n - \mu)}{\sqrt{n\mu}} = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sqrt{\mu}} = \frac{\sqrt{n}(\overline{X}_n - E(X))}{\sqrt{Var(X)}} \xrightarrow{d} Norm(0,1)$$

Continuous Mapping Theorem (CMT) The CMT answers the question: what is the asymptotic behavior of a sequence of random variables after transformation?

Theorem: Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function.

1. If $X_n \xrightarrow{d} X$, then $g(X_n) \xrightarrow{d} g(X)$.
2. If $X_n \xrightarrow{p} X$, then $g(X_n) \xrightarrow{p} g(X)$.

The CMT also holds for general $g : \mathbb{R}^d \to \mathbb{R}^b$ (more general)

Example 1: Define $S_n = \sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)}$, where $X_1, \ldots, X_n$ is an iid sample from $\text{Norm}(\mu, \sigma^2)$. Find the c such that $S_n \xrightarrow{p} c$.

$S_n^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$

$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$ (see previous notes)

$\frac{S_n^2}{\sigma^2} \sim \frac{1}{n-1}\chi^2_{n-1}$ (scaled χ^2_{n-1})

$\frac{1}{n-1}\chi^2_{n-1} \xrightarrow{p} 1$ $\left(\frac{1}{n-1}\chi^2_{n-1} \sim \frac{1}{n-1}\sum_{i=1}^{n-1} Z_i, \ Z_i \sim \chi^2_1\right)$

$\xrightarrow{p} E(Z_1) = 1$

$\frac{S_n^2}{\sigma^2} \xrightarrow{p} 1$

$S_n^2 \xrightarrow{p} \sigma^2$ (By CMT)

$S_n \xrightarrow{p} \sigma$ (By CMT, $g(x) = \sqrt{x}$)

Slutsky's Theorem (Implied by the CMT) If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} b$, then

r.v. const.

- It holds that

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} X \\ b \end{pmatrix}$$

- If $g(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous at (x, b) for any $x \in \mathbb{R}$, then

$$g(X_n, Y_n) \xrightarrow{d} g(X, b).$$

Proof: By CMT.

- In particular,

1. If $g(x,y) = x+y$. $X_n + Y_n \xrightarrow{d} X+b$
2. If $g(x,y) = xy$. $X_n Y_n \xrightarrow{d} bX$
3. If $g(x,y) = \frac{x}{y}$, $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{b}$, if $b \neq 0$.

Examples: ① $X_n \xrightarrow{p} a \Rightarrow X_n^2 \xrightarrow{p} a^2$

if $a \geq 0$, $X_n \geq 0$. $\sqrt{X_n} \xrightarrow{p} \sqrt{a}$

② If $X_n \xrightarrow{d} \text{Norm}(0,1)$ $\quad Z \sim \text{Norm}(0,1)$

$2X_n \xrightarrow{d} 2Z \sim \text{Norm}(0,4)$

$2X_n + 1 \xrightarrow{d} 2Z+1 \sim \text{Norm}(1,4)$

$X_n^2 \xrightarrow{d} Z^2 \sim \chi_1^2$

③. $X_n \xrightarrow{d} Z \sim \text{Norm}(0,1)$. $Y_n \xrightarrow{p} b$ $(b \neq 0)$

$X_n + Y_n \xrightarrow{d} b + Z \sim \text{Norm}(b,1)$

$X_n Y_n \xrightarrow{d} bZ \sim \text{Norm}(0, b^2)$

$X_n / Y_n \xrightarrow{d} \frac{Z}{b} \sim \text{Norm}(0, \frac{1}{b^2})$

Examples: ① $X_1 \cdots X_n \overset{iid}{\sim} Poi(\mu)$. Find the limiting dist'n of $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sqrt{\bar{X}_n}$ & $U_n = \sqrt{n}(\bar{X}_n - \mu)$.

Sol: $Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu}} \cdot \frac{\sqrt{\mu}}{\sqrt{\bar{X}_n}}$

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu}} \xrightarrow{d} Norm(0,1)$

$\bar{X}_n \xrightarrow{P} \mu$, $\sqrt{\bar{X}_n} \xrightarrow{P} \sqrt{\mu}$, $\frac{\sqrt{\mu}}{\sqrt{\bar{X}_n}} \xrightarrow{P} 1$

$Z_n \xrightarrow{d} Norm(0,1)$ By the Slutsky's theorem,

$U_n = \sqrt{n}(\bar{X}_n - \mu) = \underbrace{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu}}}_{\xrightarrow{d} Norm(0,1)} \sqrt{\mu}$, $\sqrt{\mu} \xrightarrow{P} \sqrt{\mu}$

$\xrightarrow{d} \sqrt{\mu} \cdot Z$, $Z \sim Norm(0,1)$

$\sim Norm(0, \mu)$

② $X_1 \cdots X_n \overset{iid}{\sim} Unif(0,1)$.

$U_1 = X_{(1)}$ $U_n = X_{(n)}$.

$n(1 - U_n) \xrightarrow{d} Exp(1)$

$U_1 \xrightarrow{P} 0$, $U_n \xrightarrow{P} 1$.

1). $e^{U_n} \xrightarrow{P} e^1 = e$, $g(x) = e^x$

2). $\sin(U_1) \xrightarrow{P} \sin(0) = 0$, $g(x) = \sin(x)$.

3). Find the limiting distribution of $e^{-n(1-U_n)}$

$e^{-n(1-U_n)} \xrightarrow{d} e^{-X}$, $X \sim Exp(1)$ By CMT. $g(x) = e^{-x}$

Examples: (cont'). $Y = e^{-X}$, $X \sim Exp(1)$.

using one-to-one transformation.

pdf $h(y) = f(x) \cdot \left|\frac{dx}{dy}\right| = \exp(-x) \cdot \left|\frac{dx}{dy}\right|$.

$Y = e^{-X}$, $x = -\log(y)$. $\frac{dx}{dy} = \frac{-1}{y}$

$\Rightarrow h(y) = \exp(-x) \left|\frac{-1}{y}\right| = y \cdot \frac{1}{y} = 1$. $y \in (0,1)$

$Y \sim unif(0,1)$

4). Find the limiting distribution of

$$\underbrace{(U_n+1)^2}_{\xrightarrow{P} (1+1)^2 = 4} \underbrace{\{n(1-U_n)\}}_{\xrightarrow{d} Exp(1)} \xrightarrow{d} 4X, \text{ where } X \sim Exp(1)$$

By Slutsky's theorem.

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(5). $X_1 \cdots X_n$ iid $Norm(\mu, \sigma^2)$.

$S_n^2 = \frac{\Sigma(X_i - \bar{X})^2}{n-1}$, $S_n \xrightarrow{P} \sigma$, $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim Norm(0,1)$

Find the limiting distribution of $\sqrt{n} \cdot \frac{\bar{X}-\mu}{S_n} \sim t_{n-1}$.

Sol. $$\frac{\sqrt{n}(\bar{X}-\mu)}{S_n} = \underbrace{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}_{\sim Norm(0,1)} \cdot \underbrace{\frac{\sigma}{S_n}}_{\xrightarrow{P} 1} \xrightarrow{d} Norm(0,1)$$ By Slutsky's theorem.

$\Rightarrow \xrightarrow{d} Norm(0,1)$ $\quad t_{n-1} \xrightarrow{d} Norm(0,1)$

Delta Method The Delta Method finds the limiting distribution of a distribution $g(Y_n)$ of a sequence of random variables Y_n, which has a normal limit distribution after centering and scaling.

Background: $\sqrt{n}(Y_n-\theta) \xrightarrow{d} \text{Norm}(0, \sigma^2)$

what about $g(Y_n)$?

Theorem: suppose $\sqrt{n}(Y_n-\theta) \xrightarrow{d} \text{Norm}(0.\sigma^2)$ and $g(\cdot)$ is differentiable at θ and $g'(\theta)\neq 0$

Then $\sqrt{n}(g(Y_n)-g(\theta)) \xrightarrow{d} \text{Norm}(0, \{g'(\theta)\}^2\sigma^2)$

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proof: $\sqrt{n}\cdot(g(Y_n)-g(\theta)) = \sqrt{n}(Y_n-\theta)\cdot \dfrac{g(Y_n)-g(\theta)}{Y_n-\theta}$

$$= \sqrt{n}(Y_n-\theta)\cdot\left\{g'(\theta) + \frac{g(Y_n)-g(\theta)}{Y_n-\theta} - g'(\theta)\right\}$$

$$= \underbrace{\sqrt{n}(Y_n-\theta)\cdot g'(\theta)}_{\xrightarrow{d} \text{Norm}(0,\{g'(\theta)\}^2\sigma^2)} + \underbrace{\sqrt{n}(Y_n-\theta)}_{\xrightarrow{d} \text{Norm}}\cdot \underbrace{\left\{\frac{g(Y_n)-g(\theta)}{Y_n-\theta} - g'(\theta)\right\}}_{u(Y_n) \xrightarrow{P} 0}$$

①. $Y_n \xrightarrow{P} \theta$.

$$Y_n-\theta = \underbrace{\frac{1}{\sqrt{n}}}_{\xrightarrow{P} 0}\cdot\underbrace{\sqrt{n}(Y_n-\theta)}_{\xrightarrow{d} \text{Norm}(0.\sigma^2)} \xrightarrow{P} 0, \quad Y_n \xrightarrow{P} \theta$$

②. Define. $u(y) = \begin{cases} 0, & y=\theta \\ \dfrac{g(y)-g(\theta)}{y-\theta} - g'(\theta), & y\neq\theta \end{cases}$ $u(y)$ is continuous.

By CMT. $u(Y_n) \xrightarrow{P} u(\theta) = 0$

$\sqrt{n}(g(Y_n)-g(\theta)) \xrightarrow{d} \text{Norm}(0, \{g'(\theta)\}^2\sigma^2)$