STAT 330: Lecture 3

2024 Spring

May 16, 2024

Review of the Last Lecture

1. Discrete Random Variables
    - Definition
    - Pmf
    - Support
    - Properties
    - Commonly used discrete random variables.
2. Continuous Random Variables
    - Definition
    - Pdf
    - Support
    - Properties

Overview of This Lecture

1. Gamma Function
2. Some common continuous random variables
3. Expectation, Variance, Moment

1 Gamma Function

- Notation: $\Gamma(\alpha), \alpha > 0$
- Definition:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

- Properties:

① $\Gamma(1) = \int_0^\infty y^{1-1} e^{-y} dy = \int_0^{\infty} e^{-y} dy = -\{ e^{-y} \big|_0^{+\infty} \} = -(0-1) = 1$

② $\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1)$ [Integration by part]

$\Gamma(\alpha-1) = \int_0^\infty y^{\alpha-2} e^{-y} dy = \frac{1}{\alpha-1}\{ y^{\alpha-1} e^{-y} \big|_0^{+\infty} + \int_0^{+\infty} y^{\alpha-1} e^{-y} dy \} = \frac{1}{\alpha-1}\cdot\Gamma(\alpha)$

③ $n \in \mathbb{Z}^+$, $\Gamma(n) = (n-1)!$ [using ① & ②]

④ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

2 Some Continuous Random Variables

2.1 Gamma Distribution

Notation $X \sim \mathrm{Gam}(\alpha, \beta)$ and $\alpha > 0, \beta > 0$.

Support $(0, +\infty)$

pdf $f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\,\beta^\alpha}$, $x > 0$.

change variable

$y = \frac{x}{\beta} \Rightarrow x = \beta y$

$\Rightarrow dx = \beta dy$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\cdot\beta^\alpha} dx$$

$$= \{ \int_0^{+\infty} y^{\alpha-1} e^{-y} dy \} / \Gamma(\alpha) = 1$$

2.2 Weibull Distribution

Notation $X \sim Wei(\theta, \beta)$, $\theta, \beta > 0$

Support $(0, +\infty)$

pdf $f(x) = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(\frac{x}{\theta})^\beta}$, $x>0$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^{+\infty} \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(\frac{x}{\theta})^\beta} dx$$

$$= \int_0^{+\infty} e^{-y} dy = \Gamma(1) = 1$$

change variable.

$y = (\frac{x}{\theta})^\beta$.

$x = \theta y^{1/\beta}$

$dx = \theta \cdot y^{1/\beta - 1} \cdot \frac{1}{\beta} dy$

2.3 Normal Distribution

Notation $X \sim Norm(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$

Support $A = \mathbb{R}$.

pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$. $-\infty < x < \infty$

① Special case. $\mu = 0$, $\sigma^2 = 1$. Standard normal distribution.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\} dx = 2 \cdot \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\} dx$$

symmetric around 0

change variable. $t = \frac{x^2}{2} \Rightarrow x = \sqrt{2t}$, $dx = \sqrt{2} \cdot \frac{1}{2} \cdot t^{-\frac{1}{2}} dt$

$$= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt = 1$$

$= \Gamma(\frac{1}{2}) = \sqrt{\pi}$

② general case. μ, σ^2.

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\} dx$$

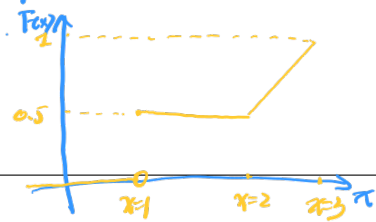
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{t^2}{2}\} dt = 1$$

Change variable

$t = \frac{x-\mu}{\sigma}$

$\begin{cases} x = \mu + \sigma t \\ dx = \sigma dt \end{cases}$

※: discrete vs. continuous is NOT a dichotomy. There are r.v.s neither discrete nor continuous.



3 Expectation (Mean)

Definition

Discrete: Suppose X is a discrete r.v. with support A, pmf: $f(x)$. Then.

The Expectation is defined as.

$$E(X) = \sum_{x \in A} x f(x), \quad \text{if } \sum_{x \in A} |x| f(x) < \infty$$

Continuous: Suppose X is a continuous r.v. pdf: $f(x)$, Then.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx. \quad \text{if. } \int_{-\infty}^{\infty} |x| f(x) dx < \infty$$

$$\boxed{\text{If } \sum_{x \in A} |x| f(x) = \infty \text{ or } \int_{-\infty}^{\infty} |x| f(x) dx = \infty \text{ Then, expectation does not exist.}}$$

Examples

$\boxed{\text{Discrete}}$ pmf. $f(x) = \frac{1}{x(x+1)}$. $A = \{1, 2, \dots\}$.

$$\sum_{x \in A} f(x) = \sum_{x \in A} \left(\frac{1}{x} - \frac{1}{x+1}\right) = 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}} + \cdots = 1$$

$$\sum_{x \in A} |x| f(x) = \sum_{x \in A} \frac{\cancel{x}}{\cancel{x}(x+1)} = \sum_{i=1}^{\infty} \frac{1}{i+1} = \infty$$

$\Rightarrow$ Expectation does NOT exist.

$\boxed{\text{Continuous.}}$ Cauchy distribution

pdf: $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ $x \in \mathbb{R}$.

$$\text{"}E(X)\text{"} = \int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{x}{1+x^2} dx = 0$$

$$\int_{-\infty}^{\infty} |x| \frac{1}{\pi} \cdot \frac{1}{1+x^2} dx = \frac{2}{\pi} \int_0^{\infty} \frac{x}{1+x^2} dx = \frac{2}{\pi} \cdot \log(1+x^2) \Big|_0^{+\infty} = \infty$$

Even though we can compute "E(x)", E(x) does NOT exist.

Examples

$X \sim Bern(p)$.

$E(X) = 1 \cdot p + 0 \cdot (1-p) = p$

$X \sim Bin(n, p)$

$E(X) = \sum_{i=0}^{n} \binom{n}{i} \cdot p^i (1-p)^{n-i} \cdot i$ (too complicated)

$X = \sum_{i=1}^{n} X_i$, $E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = np$

X: pdf $f(x) = \frac{\theta}{x^{\theta+1}}$, $x \geq 1$ & $\theta > 0$.

$\int_{-\infty}^{+\infty} |x| f(x) dx = \theta \int_1^{+\infty} \frac{1}{x^\theta} dx$

① $\theta = 1$, ∞

② $\theta < 1$, ∞

③ $\theta > 1$ $< \infty$

Expectation exists only when $\theta > 1$

$E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$= \theta \int_1^{+\infty} \frac{1}{x^\theta} dx$

$= \frac{\theta}{\theta - 1}$

Expectation of a Function of Random Variable

X is $\underset{\text{r.v.}}{\text{pmf } f(x)}$ $g(\cdot)$. what is $E\{g(X)\}$

Discrete. $E\{g(x)\} = \sum_{x \in A} g(x) f(x)$ if. $\sum_{x \in A} |g(x)| f(x) < \infty$

Continuous: $E\{g(x)\} = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$. if $\int_{-\infty}^{\infty} |g(x)| f(x) < \infty$

Linearity You can exchange the order of Expectation & Summation.

$E\{a g(x) + b h(x)\} = E\{a g(x)\} + E\{b h(x)\}$

$= a E\{g(x)\} + b E\{h(x)\}$

Variance Def: $Var(X) = E\{(X-\mu)^2\}$, where $\mu = E(X)$

$= E\{X^2 - 2\mu X + \mu^2\}$

$= E(X^2) - E(2\mu X) + \mu^2$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - \mu^2$

when X has $\mu = 0$. $Var(X) = E(X^2)$

Moments

$k \in \{1, 2, \cdots\}$

① Default : kth moment of X $E(X^k)$

② Central moment : kth central moment of X. $E\{(X-\mu)^k\}$

variance is 2nd central moment. $\mu = E(X)$ First moment.

③ kth factorial moment.

$E\{(X)_k\} = E\{X(X-1)\cdots(X-k+1)\}$

useful for finding the mean & variance of discrete r.v.s.

Examples

$X \sim Poi(\theta)$. $f(x) = \frac{\theta^x}{x!} \cdot e^{-\theta}$. $x \in \{0, 1, \cdots\}$.

$$E\{(X)_k\} = \sum_{x=0}^{\infty} x(x-1)\cdots(x-k+1) \frac{\theta^x}{x!} \cdot e^{-\theta}$$

$$= \sum_{x=k}^{\infty} x(x-1)\cdots(x-k+1) \frac{\theta^x}{x!} \cdot e^{-\theta}$$

$$= \sum_{x=k}^{\infty} \frac{\cancel{x(x-1)\cdots(x-k+1)}}{\cancel{x(x-1)\cdots(x-k+1)}(x-k)!} \cdot \theta^x e^{-\theta}$$

$$= \sum_{x=k}^{\infty} \frac{1}{(x-k)!} \cdot \theta^x e^{-\theta} \qquad n = x-k \Rightarrow x = n+k$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \theta^{n+k} e^{-\theta}$$

$$= \theta^k \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \theta^n e^{-\theta} = \theta^k$$

$k=1$. $E\{(X)_1\} = E(X) = \theta^1 = \theta$

$k=2$ $E\{(X)_2\} = E\{X(X-1)\}$

$= E(X^2) - E(X) = \theta^2$

$E(X^2) = \theta^2 + \theta$

$Var(X) = E(X^2) - \{E(X)\}^2 = \theta^2 + \theta - \theta^2 = \theta$

For Poisson. $Var(X) = E(X) = \theta$