STAT 330: Lecture 6

2024 Spring

May 30, 2024

Last Lecture

1. Joint cdf
2. Joint pmf
3. Joint pdf

This Lecture Independence

Independence

Definition For any two random variables X and Y, we call them independent if

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B)$$

for any two sets A and B.

sets of real numbers.

Independence of rvs.

$\{X \in A\}$, $\{Y \in B\}$ are events.

Previously, independence b/w events.

A, B denote events.

$Pr(A \cap B) = Pr(A) \cdot Pr(B)$

Theorem ① Random variables X and Y are independent if and only if $\Leftrightarrow$

② $$f(x, y) = f_X(x)f_Y(y)$$

or $\Leftrightarrow$

③ $$F(x, y) = F_X(x)F_Y(y)$$

① = ② = ③

for any x and y.

f, f_X, f_Y : pdf / pmf.

F, F_X, F_Y : cdf.

Functions of Independent Random Variables If X and Y are independent, then $g(X)$ and $h(Y)$ are independent.

$Pr(g(X) \in A, h(Y) \in B)$

$= Pr(X \in g^{-1}(A), Y \in h^{-1}(B))$

$= Pr(X \in g^{-1}(A)) \, Pr(Y \in h^{-1}(B))$ b/c $X \perp Y$

$= Pr(g(X) \in A) \cdot Pr(h(Y) \in B)$

$\Rightarrow g(X) \perp h(Y)$ by definition.

If $X^2 \perp Y^2$, is $X \perp Y$?

Toss a coin. $\begin{cases} \text{Heads} & X=1, Y=-1 \\ \text{Tails} & X=-1, Y=1 \end{cases}$

X is not independent of Y

$pr(X^2 = 1) = 1$, $pr(Y^2 = 1) = 1$

$Pr(X^2 = 1, Y^2 = 1) = 1 = Pr(X^2 = 1) Pr(Y^2 = 1)$

$X^2 \perp Y^2$

$X^2 \mapsto X$ is not a function

Example

1. Consider the joint discrete random variable example

$$f(x,y) = q^2 p^{x+y}, \quad x, y = 0, 1, 2, \ldots$$

Are X and Y independent?

$f(x,y) = \underbrace{(qp^x)}_{\text{pmf of Geo}(q)} \cdot \underbrace{(qp^y)}_{\text{pmf of Geo}(q)} \quad \Rightarrow X \perp Y.$

2. Consider the previous example

$$f(x,y) = \begin{cases} x+y, & x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

Are X and Y independent?

① Find out the marginal pdfs of X & Y.

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} x + \frac{1}{2}, & x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$

$f_Y(y) = \begin{cases} y + \frac{1}{2}, & y \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$

② Check if $f(x,y) = f_X(x) f_Y(y)$.

$f(x,y) \neq f_X(x) f_Y(y)$. $\Leftrightarrow$ X and Y are not independent.

Factorization Theorem for Independence Suppose we have two random variables, X and Y. The support of X is denoted as A_X, the support of Y is denoted as A_Y, and the support of (X, Y) is denoted as A. Then the following two conditions together

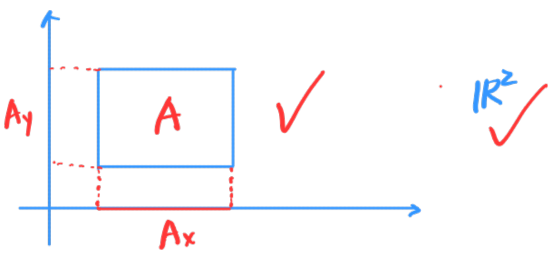
1. $f(x, y) = cg(x)h(y)$, where c is a constant.
2. $A = A_X \times A_Y$.

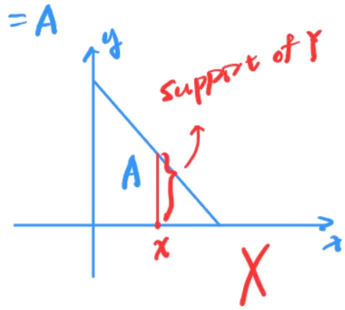
are the necessary and sufficient for X and Y to be independent.

$\Longleftrightarrow$

1. $g(x)$, $h(y)$ do not have to be pdfs/pmfs.
   Just need to be functions of x or y.

2. $A_x \times A_y = \{(x,y) \mid x \in A_x, y \in A_y\} = A$





Example 1 Given the joint pmf

$$f(x, y) = \frac{\theta^{x+y} e^{-2\theta}}{x!y!},$$

where $x, y = 0, 1, \ldots$. Are X and Y independent; find the marginal pmf of X and Y.

$f(x,y) = e^{-2\theta} \cdot \left(\frac{\theta^x}{x!}\right) \cdot \left(\frac{\theta^y}{y!}\right)$ satisfies condition I.

$A = \{(x,y) \mid x,y = 0,1,\cdots\} = \{x \mid x = 0,1,\cdots\} \times \{y \mid y = 0,1,2\cdots\}$
$= A_x \cdot A_y$ satisfies cond. 2.

$\Rightarrow X \perp Y$.

$f_X(x) = \frac{e^x}{x!} e^{-\theta}$. same for Y

Example 2 X and Y has a joint pdf

$$f(x,y) = \frac{3}{2}y(1-x^2), \quad x \in [-1,1], y \in [0,1]$$

Are they independent? Find out the marginal pdfs.

$f(x,y) = \frac{3}{2} \cdot y \cdot (1-x^2)$ Satisfies Cond. 1.

$A = A_X \times A_Y$

Satisfies cond 2.

$\Rightarrow X \perp Y$.

Method 1: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$, same for $f_Y(y)$

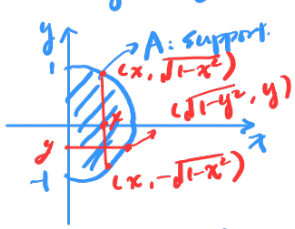
Method 2: $f_Y(y) = cy$, $\int f_Y(y)dy = 1$ $\Rightarrow c=2$

solve "c" from it

$f_X(x) = c(1-x^2)$, $\int f_X(x)dx = 1$ $c = 3/4$

Example 3 The joint pdf is $f(x,y) = \frac{2}{\pi}$, where $x \in [0, \sqrt{1-y^2}]$ and $y \in (-1,1)$. Are X and Y independent? Find the marginal pdfs for X and Y.

$f(x,y) = \frac{2}{\pi} \cdot 1 \cdot 1.$ Satisfies Cond. 1.



does not satisfies Cond. 2.

$\Rightarrow$ X and Y are not independent.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\pi}dy = \frac{4}{\pi}\sqrt{1-x^2}, \quad x \in [0,1]$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{\sqrt{1-y^2}} \frac{2}{\pi}dx = \frac{2}{\pi}\sqrt{1-y^2} \quad y \in [-1,1]$$