STAT 330: Lecture 13

2024 Spring

June 25, 2024

Last Lecture

- One-to-one transformation method

This Lecture

- Continue on one-to-one transformation method
- MGF method

Review X, Y r.v.s. $\begin{cases} U = h_1(X, Y) \\ V = h_2(X, Y) \end{cases}$ find pdf of (U, V).

If the transformation is one-to-one.

Jacobian matrix : $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}$

Jacobian (determinant) : $\det(J)$

$g(u, v) = f(x, y)\,|\det(J)| \longrightarrow$ absolute value.

Examples: one-to-one bivariate transformation

Exaxmple 1 Suppose X and Y are continuous random variables with joint pdf $p(x, y) = \exp(-x - y)$, $x \in (0, \infty)$, $y \in (0, \infty)$. Letting $U = X + Y$ and $V = X$, find the joint pdf of U and V and the marginal pdf of U.

Step 1: Find the inverse of the transformation.

$$\begin{cases} u = X+Y \\ V = X \end{cases}$$ solve (X, Y) from the equations

$$\Rightarrow \begin{cases} X = V \\ Y = U - V \end{cases}$$

Step 2: Find the support for (U, V)

$v \in (0, +\infty)$ $\quad u = V + Y$, $y \in (0, +\infty)$

given v, $u \in (v, +\infty)$

Support $0 < v < u < \infty$

Step 3: $\begin{cases} X = V \\ Y = U - V \end{cases}$ $\quad J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$\det(J) = 0 \cdot (-1) - 1 \cdot 1 = -1$

$g(u, v) = f(x, y) \cdot |\det(J)| = f(x, y)$, replace x, y with u, v

$= f(v, u - v)$

$= \exp(-v - (u - v)) = \exp(-u)$, $0 < v < u < \infty$

$g(u) = \int_0^u g(u, v) dv = \int_0^u \exp(-u) dv = u \exp(-u)$.

Example 2. $X \perp Y \sim \text{Norm}(0, 1)$ $\quad U = X + Y$, $V = X - Y$. find the joint pdf of (U, V)

Step 1. $\begin{cases} U = X+Y \\ V = X-Y \end{cases} \Rightarrow \begin{cases} X = \frac{U+V}{2} \\ Y = \frac{U-V}{2} \end{cases}$

Step 2: $U = X+Y \in (-\infty, +\infty)$

given u, $v \in (-\infty, +\infty)$

Step 3: $J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, $\det(J) = \frac{1}{2} \cdot (-\frac{1}{2}) - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2}$

$g(u, v) = f(x, y) |\det(J)|$

$= \frac{1}{2} f(\frac{u+v}{2}, \frac{u-v}{2})$

$= \frac{1}{\sqrt{2\pi \cdot 2}} \exp(-\frac{u^2}{2 \cdot 2}) \cdot \frac{1}{\sqrt{2\pi \cdot 2}} \exp(-\frac{v^2}{2 \cdot 2})$, $u, v \in (-\infty, +\infty)$

$U \perp V$.
$U \sim Norm(0, 2)$.
$V \sim Norm(0, 2)$

Practice: course note 4.2.5 / 4.2.6

Method ②

$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$. $\begin{bmatrix} X \\ Y \end{bmatrix} \sim BVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$\begin{bmatrix} U \\ V \end{bmatrix} \sim BVN\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)$

MGF Technique

1. Find the mgf of a random variable.
2. Use the uniqueness of mgf to find the distribution of the new random variable.

<u>A Useful Result</u>: Suppose $X_1, \ldots, X_n$ are independent, then $T = \sum_{i=1}^{n} X_i$ has mgf

$$M_T(t) = E\left\{ \exp\left(t \sum_{i=1}^{n} X_i\right) \right\} = \prod_{i=1}^{n} E\{\exp(tX_i)\}$$
$$= \prod_{i=1}^{n} M_{X_i}(t)$$

Furthermore, if $X_1, \ldots, X_n$ are i.i.d., then $T = \sum_{i=1}^{n} X_i$ has mgf.

$$M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = [M_X(t)]^n$$. X has same dist'n as X_i.

Normal Distribution

1. If $X \sim \text{Norm}(\mu, \sigma^2)$, then $aX + b \sim \text{Norm}(a\mu + b, a^2\sigma^2)$.

Proof: $M_{aX+b}(t) = E\left[\exp\{(aX+b)t\}\right] = \exp(bt) \cdot \underbrace{E\{\exp(atX)\}}_{M_X(at)}$

$= \exp(bt) \cdot \exp\{a\mu t + \frac{1}{2}\sigma^2 a^2 t^2\}$

$= \exp\{(b+a\mu)t + \frac{1}{2}\sigma^2 a^2 t^2\}$

$\sim \text{Norm}(b+a\mu, a^2\sigma^2)$

$E(aX+b) = a\mu + b$

$Var(aX+b) = a^2 Var(X) = a^2\sigma^2$

2. If $X_i \sim \text{Norm}(\mu_i, \sigma_i^2)$ $i = 1, \ldots, n$ and $X_1, \ldots, X_n$ are independent. Then $\sum_{i=1}^n a_i X_i \sim \text{Norm}(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2))$.

$M_{\sum_{i=1}^n a_i X_i}(t) = \prod_{i=1}^n M_{a_i X_i}(t) = \prod_{i=1}^n E\{\exp(a_i t X_i)\}$

$= \prod_{i=1}^n M_{X_i}(a_i t)$

$= \prod_{i=1}^n \exp(a_i\mu_i t + \frac{1}{2} a_i^2 \sigma_i^2 t^2)$

$= \exp\{(\sum_{i=1}^n a_i\mu_i)t + \frac{1}{2}(\sum_{i=1}^n a_i^2\sigma_i^2)t^2\}$

$\sim \text{Norm}(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2)$.

$E(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n a_i \mu_i$

$Var(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n a_i^2\sigma_i^2$

3. If $X_1, \ldots, X_n \sim \text{Norm}(\mu, \sigma^2)$ and are iid. Then

$\sum_{i=1}^n X_i \sim \text{Norm}(n\mu, n\sigma^2)$

$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \sim \text{Norm}(\mu, \frac{\sigma^2}{n})$

Some Important Distributions

Chi-square Distribution

1. $Z \sim \text{Norm}(0,1)$, the distribution of Z^2.

$Z^2 \sim \text{Gamma}(\frac{1}{2}, 2)$
$\sim \chi_1^2$

Method 1: mgf.
Method 2: cdf method. } see previous notes.

$Z_1 \cdots Z_n \overset{iid}{\sim} \text{Norm}(0,1)$

$Z_1^2 + \cdots + Z_n^2 \sim \chi_n^2$

proof by mgf.
$M_{Z^2}(t) = (1-2t)^{-\frac{1}{2}}$
$M_{Z_1^2+\cdots+Z_n^2}(t) = [M_{Z^2}(t)]^n = (1-2t)^{-\frac{n}{2}}$
$\sim \text{Gamma}(\frac{n}{2}, 2)$
$\sim \chi_n^2$

2. If $Y_i \sim \chi^2_{k_i}$ and $Y_1, \ldots, Y_n$ are independent. The distribution of $\sum_{i=1}^n Y_i$.

Method 1. $Y_i = \sum_{j=1}^{k_i} Z_{ij}^2$, $Z_{ij} \sim \text{Norm}(0,1)$

$\sum_{i=1}^n Y_i = \sum_{i=1}^n \sum_{j=1}^{k_i} Z_{ij}^2 \sim \chi^2_{k_1+\cdots+k_n}$

Method 2: $M_{\sum_{i=1}^n Y_i}(t) = \prod_{i=1}^n M_{Y_i}(t) = \prod_{i=1}^n (1-2t)^{-\frac{k_i}{2}} = (1-2t)^{-(\sum_{i=1}^n k_i)/2}$

$\sim \text{Gamma}(\frac{\sum_{i=1}^n k_i}{2}, 2)$
$\sim \chi^2_{\sum_{i=1}^n k_i}$

3. if $X_1, \ldots, X_n \sim \text{Norm}(\mu, \sigma^2)$ are iid, then

$\frac{X_i - \mu}{\sigma} \sim \text{Norm}(0,1)$

$$\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$$

$\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_1^2$

t Distribution

Def: $\begin{cases} ① \ Z \sim \text{Norm}(0,1) \\ ② \ Y \sim \chi^2_n \\ ③ \ Z \perp Y \end{cases} \Rightarrow \dfrac{Z}{\sqrt{\frac{Y}{n}}} \sim$ t distribution with $df = n$ or t_n

F Distribution

Def: $\begin{cases} ① \ X \sim \chi^2_n \\ ② \ Y \sim \chi^2_m \\ ③ \ X \perp Y \end{cases} \Rightarrow \dfrac{X/n}{Y/m} \sim$ F distribution with numerator $df = n$ and denominator $df = m$. or $F_{n,m}$.

If $T \sim t_m$, what is the dist'n of T^2

$T^2 \sim F_{1,m}$.

$T = \dfrac{Z}{\sqrt{\frac{Y}{m}}}$. $Z \sim \text{Norm}(0,1)$, $Y \sim \chi^2_m$, $Z \perp Y$

$T^2 = \dfrac{Z^2/1}{Y/m}$. $\begin{cases} ① \ Z^2 \sim \chi^2_1 \\ ② \ Y \sim \chi^2_m \\ ③ \ Z \perp Y \Rightarrow Z^2 \perp Y \end{cases} \Rightarrow T^2 \sim F_{1,m}$