University of Waterloo

Department of Statistics and Actuarial Science

STAT330 : Homework 2 Solutions

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[Notes for Graders]
[Extra Explanations]
[Problems and Solutions]

Problem 1. [8pts] We define a function

$$F(x, y) = \begin{cases} 0, & x + y < -1 \\ 1, & x + y \geq -1. \end{cases}$$

Is $F(x, y)$ a valid joint cdf? If it is, show why. If not, give a counterexample.

Solution. We had four properties of joint cdfs from the lecture, but those are not enough to characterize joint cdfs. That is to say, it's possible that some F satisfies all four properties and is still not a valid cdf. The extra property you need to characterize joint cdfs is that "every rectangle must have probability value inside the range $[0, 1]$". For this problem, showing any set having a probability value outside of $[0, 1]$ will indicate that F is not a valid joint cdf.
Let A be the set $\{X \leq 0, Y \leq -1\}$ and B be the set $\{X \leq -1, Y \leq 0\}$. Then $A \cap B$ is the set $\{X \leq -1, Y \leq -1\}$. We calculate the probability of $A \cup B$.

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 1 + 1 - 0 = 2$$

Therefore F is not a valid joint cdf.
To be precise, $A \subseteq \Omega$ is the set

$$\{\omega : X(\omega) \leq 0 \ \text{ and } \ Y(\omega) \leq -1\}$$

♠

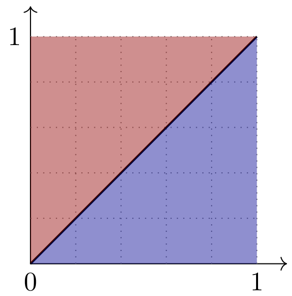
Any explicit example of a set with probability value outsie $[0, 1]$ is a correct solution.
Give 5/8 if students verified the 4 properties from lecture.

Problem 2. [10pts] Let $f_1(x, y)$ be an joint pdf

$$f_1(x, y) = \begin{cases} kx & \text{for } 0 \le y \le x \le 1, \\ ky & \text{for } 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find out the value of k and the marginal distributions of X and Y.

Solution. The red region has density ky and the blue region has density kx.



We add the blue and red regions separately:

$$\begin{aligned} f_X(x) &= \int_0^x kx \, \mathrm{d}y + \int_x^1 ky \, \mathrm{d}y \\ &= kx^2 + \frac{k}{2}[1 - x^2] \\ &= \frac{k}{2}[x^2 + 1] \end{aligned}$$

for $x \in (0, 1)$ and $f_X(x) = 0$ outside $(0, 1)$.
The marginal density of f must integrate to 1, so:

$$\begin{aligned} 1 &= \int_{\infty}^{\infty} f_X(x) \, \mathrm{d}x \\ &= \int_0^1 \frac{k}{2}[x^2 + 1] \, \mathrm{d}x \\ &= \frac{k}{2}[(\frac{1}{3}x^3 + x)\big|_0^1 \\ &= \frac{k}{2}(\frac{4}{3}) \\ &= k\frac{2}{3} \end{aligned}$$

Therefore we must have $k = 3/2$, and the marginal pdf of X is $f_X(x) = (0.75x^2 + 0.75)$ for $x \in (0, 1)$.
For the cumulative distribution of X, $F_X(x) = 0$ for $x \le 0$ and $F_X(x) = 1$ for $x \ge 1$.
For $x \in (0, 1)$:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^x f_X(t) \, \mathrm{d}t \\ &= \int_0^x 0.75t^2 + 0.75 \, \mathrm{d}t \\ &= 0.25t^3 + 0.75t\big|_0^x \\ &= 0.25x^3 + 0.75x \end{aligned}$$

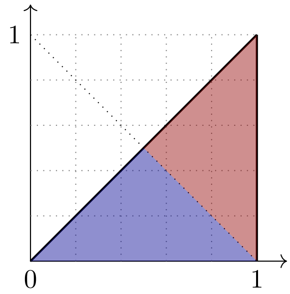
By symmetry, that is $f(x, y) = f(y, x)$, we have $F_Y(x) = F_X(x)$.
I would accept any of f_X and F_X as the "distribution" of X.
Support must be indicated (3pts), but what happens outside the support can be omitted.

♠

Problem 3. Suppose (X, Y) has joint pdf $f(x, y) = k/x$ for $0 < y < x < 1$.

1. [4pts] Find the value of k.
2. [6pts] Calculate $\Pr(X + Y < 1)$.

Solution. The blue region is for part(2), and the two colored-regions are for part (1).



$$
\begin{aligned}
(1)\quad 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^1 \int_0^x k/x\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^1 k\ \mathrm{d}x = k
\end{aligned}
$$

So we must have $k = 1$.

(2) We integrate over the blue region.

$$
\begin{aligned}
\mathbb{P}(X + Y < 1) &= \int_0^{0.5}\int_0^x 1/x\ \mathrm{d}y\ \mathrm{d}x + \int_{0.5}^1 \int_0^{1-x} 1/x\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^{0.5} 1\ \mathrm{d}x + \int_{0.5}^1 (1/x)(1-x)\ \mathrm{d}x \\
&= x\Big|_1^{0.5} + \ln(x)\Big|_{0.5}^1 - x\Big|_{0.5}^1 \\
&= 0.5 + \ln(1) - \ln(0.5) - (1 - 0.5) \\
&= \ln(2)
\end{aligned}
$$

$\spadesuit$

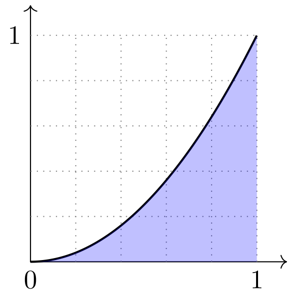
Problem 4. (a) [5pts] Find $\Pr(X > \sqrt{Y})$ if X and Y are jointly distributed with pdf

$$f(x, y) = x + y, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

(b) [5pts] Find $\Pr(X^2 < Y < X)$ if X and Y are jointly distributed with pdf

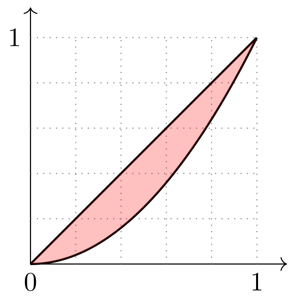
$$f(x, y) = 2x, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Solution. (a) The region of integration is $0 \le y \le x^2$, $0 \le x \le 1$.



$$\begin{aligned}
\mathbb{P}(X > \sqrt{Y}) &= \int_0^1 \int_0^{x^2} x + y \; \mathrm{d}y \; \mathrm{d}x \\
&= \int_0^1 (xy + \frac{1}{2}y^2)\Big|_0^{x^2} \; \mathrm{d}x \\
&= \int_0^1 x^3 + \frac{1}{2}x^4 \; \mathrm{d}x \\
&= \frac{1}{4}x^4 + \frac{1}{10}x^5\Big|_0^1 \\
&= 0.35 = 7/20
\end{aligned}$$

(b) The region of integration (red) is $x^2 < y < x$, $0 \le x \le 1$.



$$\begin{aligned}
\mathbb{P}(X^2 < Y < X) &= \int_0^1 \int_{x^2}^{x} 2x \; \mathrm{d}y \; \mathrm{d}x \\
&= \int_0^1 2xy\Big|_{x^2}^{x} \; \mathrm{d}x \\
&= \int_0^1 2x^2 - 2x^3 \; \mathrm{d}x \\
&= \frac{2}{3}x^3 - \frac{1}{2}x^4\Big|_0^1 \\
&= 1/6
\end{aligned}$$

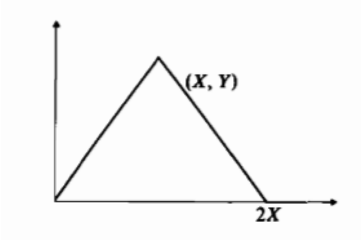
Give 4/5 if the answer is 0.15 for (b) ♠

Problem 5. [16pts] The random pair (X, Y) has the following joint pdf

$$f_{X,Y}(x, y) = 1 - \alpha(1 - 2x)(1 - 2y), x, y \in (0, 1)$$

where the parameter α satisfies $-1 \leq \alpha \leq 1$.

(a) Prove or disprove: X and Y are independent if and only if X and Y are uncorrelated.



An isosceles triangle is formed as indicated in the sketch.

(b) If (X, Y) has the joint density given above, pick α to maximize the expected area of the triangle.

(c) What is the probability that the triangle falls within the unit square with corners at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$?

Solution. Let us first calculate relevant quantities. For $x \in (0, 1)$:

$$f_X(x) = \int_0^1 1 - \alpha(1 - 2x)(1 - 2y)\ \mathrm{d}y = y\big|_0^1 - \alpha(1 - 2x)\left\{y - y^2\right\}\big|_0^1 = 1$$

By symmetry, $f_Y(y) = 1$ Hence we have $\mathbb{E}[X] = \mathbb{E}[Y] = 1/2$ [3pts], each being a continuous uniform distribution on $[0, 1]$. Let's next calculate $\mathbb{E}(XY)$.

$$\begin{aligned}
\mathbb{E}(XY) &= \int_0^1 \int_0^1 xy(1 - \alpha(1 - 2x - 2y + 4xy))\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_0^1 \int_0^1 xy - \alpha xy + 2\alpha x^2 y + 2\alpha xy^2 - 4\alpha x^2 y^2\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_0^1 \left\{\frac{1}{2}x^2 y - \frac{\alpha}{2}x^2 y + \frac{2\alpha}{3}x^3 y + \alpha x^2 y^2 - \frac{4\alpha}{3}x^3 y^2\right\}\big|_0^1\ \mathrm{d}y \\
&= \int_0^1 \frac{1}{2}y - \frac{\alpha}{2}y + \frac{2\alpha}{3}y + \alpha y^2 - \frac{4\alpha}{3}y^2\ \mathrm{d}y \\
&= \frac{1}{4} - \alpha(\frac{1}{4} - \frac{1}{3} - \frac{1}{3} + \frac{4}{9}) \\
&= \frac{1}{4} - \alpha(\frac{1}{36}) \qquad \text{[6pts]}
\end{aligned}$$

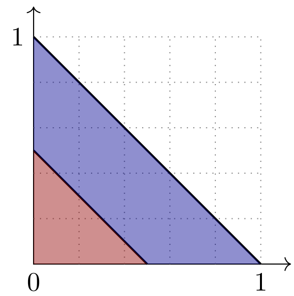
(a) [3pts] X, Y is uncorrelated
if and only if $\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0$
if and only if $\alpha = 0$ [This step is a good predictor of a correct solution]
if and only if $1 \cdot 1 = 1 - \alpha(1 - 2x)(1 - 2y)$ for all $(x, y) \in [0, 1] \times [0, 1]$
if and only if $f_X(x)f_y(y) = f_{X,Y}(x, y)$ for all x, y
if and only if X, Y are independent.
(b) [2pts] The expected area is $\mathbb{E}(XY)$, which is maximized at $\alpha = -1$.
(c) [2pts] The probability is $\Pr(0 \leq Y \leq 1 \text{ and } 0 \leq 2X \leq 1) = \Pr(0 \leq 2X \leq 1) = 0.5$.
The restriction on Y does nothing, as the support is between 0 and 1. We calculated that X follows a uniform [0,1] distribution!
If students solved the problem differently, parts a,b,c are out of 6,5,5 respectively. ♠

Problem 6. The joint pdf of X and Y is given by

$$f(x,y) = 3(x+y), \quad x,y \in (0,1), x+y \in (0,1)$$

1. [3pts] Find the marginal pdf of Y.
2. [4pts] Find $\Pr(X+Y<0.5)$
3. [4pts] Find $E(Y|X=x)$
4. [6pts] Find $\mathrm{Cov}(X,Y)$

Solution. The colored region indicates the support, and the red region indicates the region for part (2).



(1) For $y \in (0,1)$, we have :

$$\begin{aligned} f_Y(y) &= \int_0^{1-y} f(x,y)\ \mathrm{d}x \\ &= \int_0^{1-y} 3x+3y\ \mathrm{d}x \\ &= \frac{3}{2}x^2 + 3xy\Big|_0^{1-y} \\ &= 1.5 - 3y + 1.5y^2 + 3y - 3y^2 \\ &= 1.5 - 1.5y^2 \end{aligned}$$

(2) We integrate f in the red region.

$$\begin{aligned} \mathbb{P}(X+Y<0.5) &= \int_0^{0.5}\int_0^{0.5-x} f(x,y)\ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^{0.5}\int_0^{0.5-x} 3x+3y\ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^{0.5} 3xy + 1.5y^2\Big|_0^{0.5-x}\ \mathrm{d}x \\ &= \int_0^{0.5} 3x(0.5-x) + 1.5(0.5-x)^2\ \mathrm{d}x \\ &= \int_0^{0.5} 0.375 - 1.5x^2\ \mathrm{d}x \\ &= 0.375 \cdot 0.5 - 0.5 \cdot (0.5^3) \\ &= 0.125 = 1/8 \end{aligned}$$

(3) By (1) and symmetry, $f_X(x) = 1.5 - 1.5x^2$. So for $y \in (0, 1-x)$,

$$f(y|x) = \frac{3(x+y)}{1.5(1-x^2)} = 2\frac{x+y}{1-x^2}$$

Computing the expectation:

$$\begin{aligned} \mathbb{E}(Y|X=x) &= \int_0^{1-x} yf(y|x)\ \mathrm{d}y \\ &= \frac{2}{1-x^2}\int_0^{1-x} xy+y^2\ \mathrm{d}y \\ &= \frac{2}{1-x^2}\left(\frac{1}{2}x(1-x)^2 + \frac{1}{3}(1-x)^3\right) \\ &= \frac{1}{3(1+x)}(3x - 3x^2 + 2 - 4x + 2x^2) \\ &= \frac{2-x-x^2}{3(1+x)} = \frac{2-3x+x^3}{3(1-x^2)} \end{aligned}$$

(4) To compute $\mathbb{E}(Y)$, We use double expectation formula:

$$
\begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}(\mathbb{E}(Y|X)) \\
&= \int_0^1 \frac{2-x-x^2}{3(1+x)} \frac{3}{2}(1-x^2) \, \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 2-x-x^2-2x+x^2+x^3 \, \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 2-3x+x^3 \, \mathrm{d}x \\
&= \frac{1}{2}(2-\frac{3}{2}+\frac{1}{4}) \\
&= \frac{3}{8} = 0.375
\end{aligned}
$$

By symmetry, $\mathbb{E}(X) = 3/8$. Next we compute $\mathbb{E}(XY)$.

$$
\begin{aligned}
\mathbb{E}(XY) &= \int_0^1 \int_0^{1-x} xyf(x,y) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} 3x^2y + 3xy^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \frac{3}{2}x^2y^2 + xy^3|_0^{1-x} \, \mathrm{d}x \\
&= \int_0^1 \frac{3}{2}x^2(1-x)^2 + x(1-x)^3 \, \mathrm{d}x \\
&= \int_0^1 \frac{3}{2}x^2 - 3x^3 + \frac{3}{2}x^4 + x - 3x^2 + 3x^3 - x^4 \, \mathrm{d}x \\
&= \int_0^1 x - 1.5x^2 + 0.5x^4 \, \mathrm{d}x \\
&= 0.5 - 0.5 + 0.1 = 0.1
\end{aligned}
$$

Hence $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0.1 - 0.375^2 = -13/320 = -0.040625$ ♠

Problem 7. [8pts] Consider two random variables X and Y and their expectations exist. Show that

$$\min_{g(x)} E\left\{Y - g(X)\right\}^2 = E\left\{Y - E(Y|X)\right\}^2,$$

where $g(x)$ can be all functions. (We say $E(Y|X)$ is the best predictor of Y conditional on X).

Solution. For any (measurable) function $g : \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned}
\mathbb{E}[\{Y - g(X)\}^2] &= \mathbb{E}[\{Y - \mathbb{E}(Y|X) + \mathbb{E}(Y|X) - g(X)\}^2] \\
&= \mathbb{E}[\{Y - \mathbb{E}(Y|X)\}^2 + 2(Y - \mathbb{E}(Y|X))(\mathbb{E}(Y|X) - g(X)) + \{\mathbb{E}(Y|X) - g(X)\}^2] \\
&= \mathbb{E}[\{Y - \mathbb{E}(Y|X)\}^2 + \{\mathbb{E}(Y|X) - g(X)\}^2] \\
&= \mathbb{E}[\{Y - \mathbb{E}(Y|X)\}^2] + \mathbb{E}[\{\mathbb{E}(Y|X) - g(X)\}^2]
\end{aligned}$$

EDIT: To see why the middle term is 0, denote $\phi(X) = \mathbb{E}(Y|X) - g(X)$:

$$\begin{aligned}
&\mathbb{E}[(Y - \mathbb{E}(Y|X))(\mathbb{E}(Y|X) - g(X))] \\
&= \mathbb{E}[Y\phi(X)] - \mathbb{E}[\mathbb{E}(Y|X)\phi(X)] \\
&= \mathbb{E}[Y\phi(X)] - \mathbb{E}[\mathbb{E}(\phi(X)Y|X)] \\
&= \mathbb{E}[Y\phi(X)] - \mathbb{E}[Y\phi(X)] = 0
\end{aligned}$$

Where we can bring $\phi(X)$ inside by lemma 1 below.
By choosing $g(X) = E(Y|X)$, the second term becomes zero, and the objective function is minimized.
(1) Note that $\mathbb{E}[\mathbb{E}(\phi(X)Y|X)] = \mathbb{E}(\phi(X)Y)$ by double expectation formula. This important formula will be used extensively in problem 8.
(2) Fun fact: the optimal value $\mathbb{E}\left\{Y - \mathbb{E}(Y|X)\right\}^2$ equals $\mathbb{E}(\mathrm{Var}(Y|X))$, as will be implied by the next problem. Notice that I'm not claiming the two arguments of $\mathbb{E}(\cdot)$ are equal.

♠

Problem 8. Suppose we have two random variables X and Y, and their variances are finite (i.e., exist).

1. [6pts] Prove that Cov(X, Y) = $\text{Cov}(X, E(Y|X))$.
2. [3pts] Find out the correlation coefficient between X and $Y - E(Y|X)$.
3. [7pts] Find out the value of $\text{Var}\{Y - E(Y|X)\} - E\{\text{Var}(Y|X)\}$.

Solution. 1. Expand the right hand side using the definition of Cov:

$$\begin{aligned}\text{Cov}(X, \mathbb{E}(Y|X)) &= \mathbb{E}[X \cdot \mathbb{E}(Y|X)] - \mathbb{E}(X) \cdot \mathbb{E}[\mathbb{E}(Y|X)] \\ &= \mathbb{E}[\mathbb{E}(XY|X))] - \mathbb{E}(X) \cdot \mathbb{E}[\mathbb{E}(Y|X)] \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \\ &= \text{Cov}(X, Y)\end{aligned}$$

Going from the first line to the second line, we brought X inside the conditional expectation, which is justified by the following lemma:

Lemma 1. If $g : \mathbb{R} \to \mathbb{R}$ is any (measurable) function, $g(X)\mathbb{E}[Y|X] = \mathbb{E}[g(X)Y|X]$

We prove the lemma for the continuous case, by verifying the equality for each $X = x$.

$$g(x)\mathbb{E}[Y|X = x] = g(x)\int_{-\infty}^{\infty} y \cdot f(y|x) \text{ d}y = \int_{-\infty}^{\infty} g(x) \cdot y \cdot f(y|x) \text{ d}y = \mathbb{E}[g(X)Y|X = x]$$

Where the last step uses the substitution rule.

2. Since the covariance is linear, we have

$$\text{Cov}(X, Y - \mathbb{E}(Y|X)) = \text{Cov}(X, Y) - \text{Cov}(X, \mathbb{E}(Y|X)) = 0$$

by part 1. So the correlation coefficient is also 0.

3. By the law of total variance, we have $\mathbb{E}\{\text{Var}(Y|X)\} = \text{Var}(Y) - \text{Var}(\mathbb{E}(Y|X))$

$$\begin{aligned}&\text{Var}\{Y - \mathbb{E}(Y|X)\} - \mathbb{E}\{\text{Var}(Y|X)\} \\ &= \text{Var}(Y) - 2\text{Cov}(Y, \mathbb{E}(Y|X)) + \text{Var}(\mathbb{E}(Y|X)) - \text{Var}(Y) + \text{Var}(\mathbb{E}(Y|X)) \\ &= -2\text{Cov}(Y, \mathbb{E}(Y|X)) + 2\text{Cov}(\mathbb{E}(Y|X), \mathbb{E}(Y|X))\end{aligned}$$

Intuition: $\mathbb{E}[Y|X]$ is the "orthogonal projection" of Y onto the space of random variables of the form $g(X)$, so we expect the "dot product" above to be the same. So we try to show that the final answer is 0.

$$\begin{aligned}\text{Cov}(Y, \mathbb{E}(Y|X)) &= \mathbb{E}\{Y \cdot \mathbb{E}(Y|X) - \mathbb{E}(Y) \cdot \mathbb{E}(\mathbb{E}(Y|X))\} \\ &= \mathbb{E}\{Y \cdot \mathbb{E}(Y|X) - \mathbb{E}(Y) \cdot \mathbb{E}(Y)\} \\ \text{Cov}(\mathbb{E}(Y|X), \mathbb{E}(Y|X)) &= \mathbb{E}\{\mathbb{E}(Y|X) \cdot \mathbb{E}(Y|X) - \mathbb{E}[\mathbb{E}(Y|X)] \cdot \mathbb{E}[\mathbb{E}(Y|X)]\} \\ &= \mathbb{E}\{\mathbb{E}(Y|X) \cdot \mathbb{E}(Y|X) - \mathbb{E}(Y) \cdot \mathbb{E}(Y)\}\end{aligned}$$

Using lemma 1, $\mathbb{E}\{\mathbb{E}(Y|X) \cdot \mathbb{E}(Y|X)\} = \mathbb{E}\{\mathbb{E}[(\mathbb{E}(Y|X) \cdot Y)|X]\} = \mathbb{E}\{Y \cdot \mathbb{E}(Y|X)\}$. This shows that $\text{Cov}(Y, \mathbb{E}(Y|X)) = \text{Cov}(\mathbb{E}(Y|X), \mathbb{E}(Y|X))$ and the final answer is 0.

$\mathbb{E}(Y|X)$ is of the form $g(X)$, so we bring the first $\mathbb{E}(Y|X)$ inside the second $\mathbb{E}(Y|X)$.

Let me try that again:

$$\mathbb{E}\{\mathbb{E}(Y|X) \cdot \mathbb{E}(Y|X)\} = \mathbb{E}\{\mathbb{E}[(\mathbb{E}(Y|X) \cdot Y)|X]\} = \mathbb{E}\{Y \cdot \mathbb{E}(Y|X)\}$$

♠

Problem 9. [5pts] If the joint moment generating function of (X, Y) is given by

$$M_{X,Y}(t_1, t_2) = \exp\left\{\frac{1}{2}(t_1^2 + t_2^2)\right\}.$$

What is the distribution of Y?

Solution. Let's first find the marginal mgf for Y.

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \exp\left\{\frac{1}{2}(t_2^2)\right\}$$

We recognise that this is the standard normal mgf. So $Y \sim N(0, 1)$ by the uniqueness property of mgf. ♠