STAT 330: Lecture 12

2024 Spring

June 20, 2024

Last Lecture

- Function of random variables: The CDF method. $\begin{cases} \text{1. get the cdf.} \\ \text{2. take the derivative} \end{cases}$

This Lecture

- Function of random variables: The CDF method and the one-to-one transformation method.

Review

order stistics.

$X_{(1)} = \min\{X_1, \cdots, X_n\}$, $X_{(n)} = \max\{X_1, \cdots, X_n\}$

CDF of $X_{(n)}$

$G(y) = Pr(X_{(n)} \le y) = Pr(X_1 \le y, \cdots, X_n \le y)$

CDF of $X_{(1)}$

$G(y) = Pr(X_{(1)} \le y) = 1 - Pr(X_{(1)} > y)$

$= 1 - Pr(X_1 > y, \cdots, X_n > y)$

CDF Method

Examples ① If $Z \sim Norm(0,1)$. find the pdf of $Y = Z^2$

Step 1 : $G_Y(y) = Pr(Y \le y) = Pr(Z^2 \le y)$, $y \ge 0$.

$= Pr(-\sqrt{y} \le Z \le \sqrt{y})$

$= F_Z(\sqrt{y}) - F_Z(-\sqrt{y})$

$$\frac{d\,h(g(x))}{dx} = h'(g(x))g'(x)$$

Step 2 : $g_Y(y) = G'_Y(y) = F'_Z(\sqrt{y})\frac{1}{2}y^{-\frac{1}{2}} - F'_Z(-\sqrt{y})\cdot(-1)\cdot\frac{1}{2}y^{-\frac{1}{2}}$

$= f_Z(\sqrt{y})\frac{1}{2}y^{-\frac{1}{2}} + f_Z(-\sqrt{y})\cdot\frac{1}{2}y^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2\pi}}\exp(-\frac{y}{2})\cdot y^{-\frac{1}{2}} \sim Gamma(\frac{1}{2}, 2)$

we call it χ^2_1

pdf of χ^2_n : $\frac{1}{2^{\frac{n}{2}}\cdot\Gamma(\frac{n}{2})}\cdot x^{\frac{n}{2}-1}e^{-x/2} \sim Gamma(\frac{n}{2}, 2)$

②. $f(x) = \frac{\theta}{x^{\theta+1}}$, $x \ge 1$, $\theta > 0$. Find pdf of $Y = \log X$.

support of Y, $y \in [0, \infty)$.

$G_Y(y) = Pr(Y \le y) = Pr(\log X \le y) = Pr(X \le e^y)$

$= \int_1^{e^y} \frac{\theta}{x^{\theta+1}}\,dx = -x^{-\theta}\Big|_1^{e^y} = 1 - e^{-y\theta}$. $y > 0$

$g_Y(y) = G'_Y(y) = \theta e^{-y\theta}$. $y > 0$.

One-to-one Univariate Transformation Method

X is rv. $Y = h(X)$, $h(\cdot)$ is a one-to-one function.

If $h(x)$ is one-to-one, then the pdf of Y is

$$g_Y(y) = f(x)\left|\frac{dx}{dy}\right| \quad , \quad y = h(x), \quad x = h^{-1}(y)$$

$f(x)$: $f(h^{-1}(y))$ is a function of y.

$|\cdot|$: absolute value.

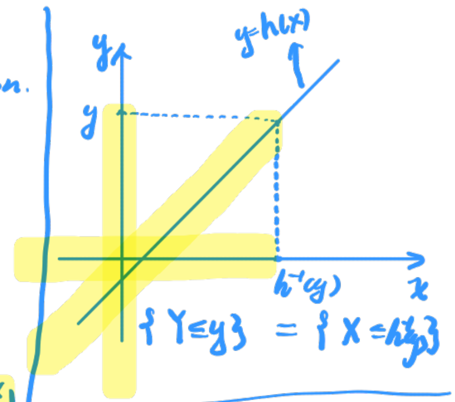
$\frac{dx}{dy}$: $\frac{d\,h^{-1}(y)}{dy}$

proof: $G_Y(y) = Pr(Y \le y) = Pr(h(X) \le y)$

1). if $h(\cdot)$ is an increasing function.

$$Pr(h(X) \le y) = Pr(X \le h^{-1}(y))$$
$$= F_X(h^{-1}(y))$$

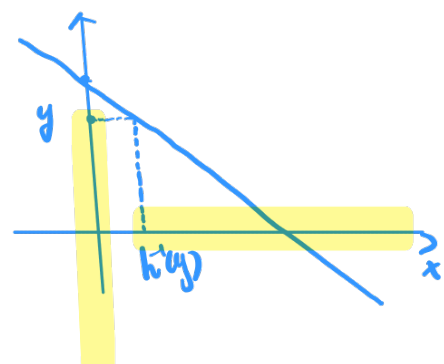
$$g_Y(y) = \frac{d F_X(h^{-1}(y))}{dy} = F_X'(h^{-1}(y))\,\frac{d\,h^{-1}(y)}{dy}$$
$$= f(h^{-1}(y))\,\frac{d\,h^{-1}(y)}{dy}$$
$$= f(x)\,\frac{dx}{dy} = f(x)\left|\frac{dx}{dy}\right|$$

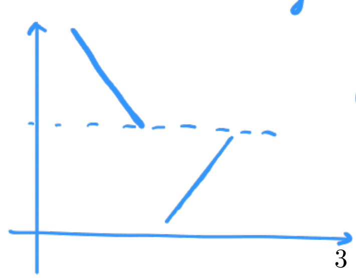


2). if $h(\cdot)$ is a decreasing function.

$$Pr(Y \le y) = Pr(h(X) \le y)$$
$$= Pr(X \ge h^{-1}(y))$$
$$= 1 - Pr(X < h^{-1}(y))$$
$$= 1 - F_X(h^{-1}(y))$$

$$g_Y(y) = 0 - F_X'(h^{-1}(y))\,\frac{d\,h^{-1}(y)}{dy}$$
$$= f(x)\left(-\frac{dx}{dy}\right) = f(x)\left|\frac{dx}{dy}\right|$$





one-to-one function

= piecewise monotone functions with no overlap.

Examples

① Suppose X is rv. its pdf is $f(x)$. its cdf is $F(x)$.
find out the distribution of $F(X)$. is a random variable.

Method 1

$Y = F(X)$. support of Y, $y \in [0,1]$, $X = F^{-1}(Y)$

pdf of Y:

$$g_Y(y) = f(x) \cdot \left|\frac{dx}{dy}\right|.$$

$$\frac{dx}{dy} = \frac{d\,F^{-1}(y)}{dy} = \frac{1}{F'(F^{-1}(y))} = \frac{1}{f(x)}$$

For an invertible function h

$$(h^{-1}(a))' = \frac{1}{h'(h^{-1}(a))}$$

$\Rightarrow g_Y(y) = 1$, Y has uniform $(0,1)$ regardless of the distribution as long as X is continuous.

Method 2

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\frac{dF(x)}{dx}} = \frac{1}{f(x)}$$

$y = h(x)$, h is inverible

$$1 = \frac{dy}{dy} = \frac{dh(x)}{dy} = \frac{dh(x)}{dx} \cdot \frac{dh^{-1}(y)}{dy} = \frac{dy}{dx} \cdot \frac{dx}{dy}. \Rightarrow \frac{dx}{dy} = \frac{1}{dy/dx}$$

Example. $X \sim \text{Unif}(0,1)$. Find the pdf of $Y = -\log X$. $x > 0$

support of Y: $y \in [0, +\infty)$, $y = -\log x$, $x = \exp(-y)$

$$g_Y(y) = f(x) \cdot \left|\frac{dx}{dy}\right| = 1 \cdot \left|\frac{d\exp(-y)}{dy}\right| = \exp(-y).$$ Method 1

$$= 1 \cdot \left|\frac{1}{dy/dx}\right| = \left|\frac{1}{-\frac{d\log x}{dx}}\right| = x = \exp(-y)$$ Method 2

$y \in [0, +\infty)$

One-to-one Bivariate Transformation Method

X, Y joint pdf $f(x,y)$, $\begin{cases} U = h_1(X,Y) \\ V = h_2(X,Y) \end{cases}$

The goal is to find the joint pdf of (U,V).

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① $h_1(x,y)$, $h_2(x,y)$ are one-to-one.

$(x,y) \to (u,v)$, $(u,v) \to (x,y)$ → unique.

The two functions h_1 h_2, are one-to-one if there exists another two unique functions s.t.

$$\begin{cases} x = g_1(u,v) \\ y = g_2(u,v) \end{cases}$$

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② Jacobian matrix.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

We call its determinant as Jacobian. of $\begin{cases} u = h_1(x,y) \\ v = h_2(x,y) \end{cases}$ → or Jacobian determinant.

$\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$ → determinant notation

or

$\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Jacobian matrix for $\begin{cases} x = g_1(u,v) \\ y = g_2(u,v) \end{cases}$

$$\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right) = \left[\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)\right]^{-1}$$

in fact $\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}$

The inverse of a Jacobian matrix is the Jacobian of the inverse function.

The joint pdf for $\begin{cases} U = h_1(X,Y) \\ V = h_2(X,Y) \end{cases}$ is

$$g(u,v) = f(x,y) \cdot \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right|$$

→ absolute value. notation

determinant of Jacobian matrix.