University of Waterloo

Department of Statistics and Actuarial Science

STAT330: Homework 1 Solutions

Last Updated:

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[Notes for Graders]
[Extra Explanations]
[Problems and Solutions]

Problem 1. [8pts]

Let's suppose that 5% of men and 0.25% of women are color-blind. A person is chosen randomly from the population and is color-blind. What is the probability that the person is male (assuming that males and females are in equal numbers in the population.) [*hint: conditional probability*]

Solution. We first calculate relevant quantities:

- The percentage of color blind men is $50\% \times 5\% = 2.5\%$ [2pts]
- The percentage of color blind women is $50\% \times 0.25\% = 0.125\%$
- The percentage of color blind people is $2.5\% + 0.125\% = 2.625\%$ [2pts]

Using the formula for conditional probability, we have:

$$\Pr(\text{male}|\text{color blind}) = \frac{\Pr(\text{male} \cap \text{color blind})}{\Pr(\text{color blind})} = \frac{2.5\%}{2.625\%} = \frac{20}{21} \approx 95.238\%$$

♠ [4pts]

Problem 2 (6pts)**.**

Suppose we have two events A and B and they satisfy $\Pr(A) > 0$ and $\Pr(B) > 0$. Prove that

- If A and B are mutually exclusive (i.e., $A \cap B = \emptyset$), they cannot be independent.
- If A and B are independent, they cannot be mutually exclusive.

Solution. Suppose $A \cap B = \emptyset$, we have [3pts]

$$\Pr(A \cap B) = \Pr(\emptyset) = 0$$

but

$$\Pr(A) \cdot \Pr(B) > 0,$$

so

$$\Pr(A \cap B) \neq \Pr(A) \cdot \Pr(B)$$

and hence they cannot be independent.
Next, suppose A and B are independent. We have [3pts]

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B) > 0.$$

So $A \cap B$ can't possibly be empty, hence they are not mutually exclusive.
Alternatively, one can show that the two statements are logically equivalent by

$$(F \to \neg G) \equiv (G \to \neg F)$$

and only show one direction.
The keywords "Logically equivalent" or "Contrapositive" would suffice. ♠

Problem 3 (6pts)**.**
Consider the following function $f(x)$.

$$f(x) = \begin{cases} \exp(-x) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Is $f(x)$ a pdf? If no, show why. If yes, find its corresponding cdf.

Solution. Yes it is. f is non-negative because the exponential function is non-negative. Next we show that f integrates to 1: [2pts]

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_{0}^{\infty} e^{-x}\ \mathrm{d}x = -e^{-x}\Big|_0^\infty = 0 - (-1) = 1.$$

If $y \leq 0$: [1pt]

$$F(y) = \int_{-\infty}^{y} f(x)\ \mathrm{d}x = \int_{-\infty}^{y} 0\ \mathrm{d}x = 0.$$

If $y > 0$: [3pts]

$$F(y) = \int_{-\infty}^{y} f(x)\ \mathrm{d}x = \int_{0}^{y} e^{-x}\ \mathrm{d}x = -e^{-x}\Big|_0^y = 1 - e^{-y}.$$

So

$$F(y) = \begin{cases} 0 & y \leq 0 \\ 1 - e^{-y} & y > 0 \end{cases}$$

♠

Problem 4 (8pts)**.** For a **discrete** random variable X, letting $F(\cdot)$ be its cdf and $f(\cdot)$ be its pmf. Express the following probabilities using $F(\cdot)$ and $f(\cdot)$.

- $\Pr(a < X \leq b)$
- $\Pr(a \leq X \leq b)$
- $\Pr(a \leq X < b)$
- $\Pr(a < X < b)$

Solution.

$$\begin{aligned}
\Pr(a < X \le b) =& F(b) - F(a)\\
\Pr(a \le X \le b) =& F(b) - F(a) + f(a)\\
\Pr(a \le X < b) =& F(b) - F(a) + f(a) - f(b)\\
\Pr(a < X < b) =& F(b) - F(a) - f(b)
\end{aligned}$$

♠ [2pts] each

Problem 5 (6pts, Logistic distribution)**.**
For the following cdf, find out the pdf.

$$F(x) = \frac{1}{1 + \exp(-x)}, \quad x \in \mathbb{R}.$$

Solution. The pdf is the derivative of the cdf.

$$\begin{aligned}
f(x) &= F'(x) && [1pt]\\
&= \frac{\mathrm{d}}{\mathrm{d}x}(1 + e^{-x})^{-1}\\
&= -(1 + e^{-x})^{-2}(-e^{-x}) && \text{(Chain Rule)}[2pts]\\
&= \frac{(e^{-x})}{(1 + e^{-x})^2} && [2pts]\\
&= \frac{1}{(e^{x/2} + e^{-x/2})^2} && \text{(Alternative answer)}
\end{aligned}$$

The support is $x \in \mathbb{R}$. ♠ [1pts]

Problem 6 (6pts)**.**
Suppose a random variable X has the following pdf.

$$f(x) = \begin{cases} 2x & x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Find out the cdf.

Solution. $F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t$. [1pt]
For $x \le 0$: [1pt]

$$F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t = \int_{-\infty}^{x} 0\ \mathrm{d}t = 0$$

For $x \in (0, 1)$: [3pts]

$$F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t = \int_{-\infty}^{0} 0\ \mathrm{d}t + \int_{0}^{x} 2t\ \mathrm{d}t = x^2$$

For $x \ge 1$: [1pt]

$$F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t = \int_{-\infty}^{0} 0\ \mathrm{d}t + \int_{0}^{1} 2t\ \mathrm{d}t + \int_{1}^{x} 0\ \mathrm{d}t = 1$$

So

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^2 & x \in (0, 1) \\ 1 & x \ge 1 \end{cases}$$

♠

Problem 7 (10pts)**.**

The following are pdf; find out the value of c.

(a) $f(x) = c\sin(x), x \in (0, \pi/2)$

(b) $f(x) = ce^{-|x|}, x \in \mathbb{R}$

Solution. To be a valid pdf, f must integrate to 1.

(a)

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_0^{\pi/2} c\sin x\ \mathrm{d}x = -c\cos x\Big|_0^{\pi/2} = c$$

So we must have $c = 1$. [4pts]

(b)

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_{-\infty}^{0} ce^{x}\ \mathrm{d}x + \int_0^{\infty} ce^{-x}\ \mathrm{d}x = ce^{x}\Big|_{-\infty}^{0} - ce^{-x}\Big|_0^{\infty} = 2c$$

So we must have $c = 1/2$. [6pts]

♠

Problem 8 (18pts)**.**

Compute the expectation $E(X)$ and the variance $\mathrm{Var}(X)$ for each of the following probability distributions.

(a) $f_X(x) = ax^{a-1}$, $0 < x < 1$, $a > 0$

(b) $f_X(x) = 1/n$, $x = 1, 2, \ldots, n > 0$

(c) $f_X(x) = \frac{3}{2}(x-1)^2$, $0 < x < 2$

Solution. (a) $f(x) = ax^{a-1}$, $0 < x < 1$, $a > 0$ [6pts]

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)\ \mathrm{d}x = \int_0^1 ax^{a}\ \mathrm{d}x = \frac{a}{a+1}x^{a+1}\Big|_0^1 = \frac{a}{a+1} \\
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\ \mathrm{d}x = \int_0^1 ax^{a+1}\ \mathrm{d}x = \frac{a}{a+2}x^{a+2}\Big|_0^1 = \frac{a}{a+2} \\
\mathrm{Var}(X) &= E(X^2) - E(X)^2 = \frac{a}{a+2} - (\frac{a}{a+1})^2 = \frac{a}{(a+1)^2(a+2)}
\end{aligned}$$

(b) $f(x) = 1/n$, $x = 1, 2, \ldots, n > 0$ [6 pts]

$$\begin{aligned}
E(X) &= \sum_{x=1}^{n} xf(x) = \sum_{x=1}^{n} x\frac{1}{n} = \frac{n+1}{2} \\
E(X^2) &= \sum_{x=1}^{n} x^2 f(x) = \sum_{x=1}^{n} x^2\frac{1}{n} = \frac{(n+1)(2n+1)}{6} \\
\mathrm{Var}(X) &= E(X^2) - E(X)^2 = \frac{(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 = \frac{1}{12}(n^2-1)
\end{aligned}$$

(c) $f(x) = \frac{3}{2}(x-1)^2$, $0 < x < 2$ [6pts]

$$E(X) = \int_{-\infty}^{\infty} xf(x)\ \mathrm{d}x = \int_0^2 \frac{3}{2}x(x-1)^2\ \mathrm{d}x = \frac{3}{2}(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2})|_0^2 = 1$$
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\ \mathrm{d}x = \int_0^2 \frac{3}{2}x^2(x-1)^2\ \mathrm{d}x = \frac{3}{2}(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3})|_0^2 = \frac{8}{5}$$
$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{8}{5} - 1^2 = \frac{3}{5}$$

♠

Problem 9 (32pts)**.**

Find the moment-generating function corresponding to

(a) $f(x) = 1/c$, $0 < x < c$

(b) $f(x) = 2x/c^2$, $0 < x < c$

(c) $f(x) = \dfrac{1}{2\beta} \exp\left\{-\dfrac{|x-\alpha|}{\beta}\right\}$, where $x, \alpha \in \mathbb{R}$, $\beta > 0$

(d) $\Pr(X = x) = \binom{r+x-1}{x} p^r (1-p)^x$, $x = 0, 1, \ldots$, $0 < p < 1$, and $r > 0$ is an integer

Solution. We use the following definition for moment generating function:

$$M_X(t) = E(e^{tX})$$

(a) $f(x) = 1/c$, $0 < x < c$ [6pts]

If $t \neq 0$:

$$\begin{aligned} E(e^{tX}) &= \int_0^c e^{tx}\frac{1}{c}\ \mathrm{d}x \\ &= \frac{1}{ct}e^{tx}|_0^c \\ &= \frac{1}{ct}(e^{ct} - 1) \end{aligned}$$

$M_X(0) = E(e^0) = 1$

(b) $f(x) = 2x/c^2$, $0 < x < c$ [6pts]

If $t \neq 0$:

$$\begin{aligned} E(e^{tX}) &= \int_0^c e^{tx}\frac{2x}{c^2}\ \mathrm{d}x \\ &= \frac{2}{c^2t^2}e^{tx}(tx-1)|_0^c \\ &= \frac{2}{c^2t^2}(e^{ct}(ct-1)+1) \end{aligned}$$

$M_X(0) = 1$

(c) $f(x) = \frac{1}{2\beta}e^{-|x-\alpha|/\beta}$, where $x, \alpha \in \mathbb{R}$, $\beta > 0$ [10pts]

$$
\begin{aligned}
&E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx}\frac{1}{2\beta}e^{-|x-\alpha|/\beta}\ \mathrm{d}x \\
&= \int_{-\infty}^{\alpha} e^{tx}\frac{1}{2\beta}e^{(x-\alpha)/\beta}\ \mathrm{d}x + \int_{\alpha}^{\infty} e^{tx}\frac{1}{2\beta}e^{-(x-\alpha)/\beta}\ \mathrm{d}x \\
&= \int_{-\infty}^{\alpha} \frac{e^{(-\alpha/\beta)}}{2\beta}e^{x(\beta t+1)/\beta}\ \mathrm{d}x + \int_{\alpha}^{\infty} \frac{e^{(\alpha/\beta)}}{2\beta}e^{x(\beta t-1)/\beta}\ \mathrm{d}x \\
&= \frac{e^{-\alpha/\beta}}{2(\beta t+1)}e^{x(\beta t+1)/\beta}|_{-\infty}^{\alpha} + \frac{e^{\alpha/\beta}}{2(\beta t-1)}e^{x(\beta t-1)/\beta}|_{\alpha}^{\infty} \\
&= \frac{e^{\alpha t}}{2(\beta t+1)} - \frac{e^{\alpha t}}{2(\beta t-1)} \quad (\text{This step needs } \beta\text{t}+1>0 \text{ and } \beta\text{t}-1<0) \\
&= \frac{e^{\alpha t}}{1-\beta^2 t^2} \quad \text{For } -\frac{1}{\beta} < \text{t} < \frac{1}{\beta}
\end{aligned}
$$

In order for the integral to converge, near $-\infty$, $x(\beta t+1)$ needs to be $-\infty$, so that $\beta t + 1$ needs to be positive. Similar for the other side.

(d) $f(x) = \binom{r+x-1}{x}p^r(1-p)^x$, $x = 0, 1, \ldots$ [10pts]

$$
\begin{aligned}
E(e^{tX}) &= \sum_{x=0}^{\infty} e^{tx}\binom{r+x-1}{x}p^r(1-p)^x \\
&= p^r\sum_{x=0}^{\infty}\binom{r+x-1}{x}[(1-p)e^t]^x \\
&= p^r\sum_{x=0}^{\infty}\binom{r+x-1}{x}y^x \qquad (1) \\
&= p^r\sum_{x=0}^{\infty}\binom{r+x-1}{x}y^x\frac{(1-y)^r}{(1-y)^r} \\
&= \frac{p^r}{(1-y)^r}\sum_{x=0}^{\infty}\binom{r+x-1}{x}y^x(1-y)^r \\
&= \frac{p^r}{(1-y)^r} \qquad (2) \\
&= \frac{p^r}{[1-(1-p)e^t]^r}
\end{aligned}
$$

(1) Let $y = (1-p)e^t$

(2) The summands resemble the pmf of negative-binomial $\text{NB}(r, 1-y)$, hence sums to 1.

This is valid when $y \in (0, 1)$, or equivalently $t < -\log(1-p)$.

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