STAT 330: Lecture 9

2024 Spring

June 11, 2024

Last Lecture Conditional Distribution & Conditional Expectation

$h(x) = E(Y|X=x)$ is a function of x and x only.

$h(X) = E(Y|X)$ is a random variable.
function of X and X only

$var(Y|X) = E(Y^2|X) - \{E(Y|X)\}^2$ $E(Y) = E\{\underline{E(Y|X)}\}$

This Lecture Conditional ~~Distribution~~ Expectation and Joint MGF

(Cont') Conditional Expectation

Variance Decomposition Formula (Law of Total Variance)

$$\mathrm{Var}(Y) = E\{\mathrm{Var}(Y|X)\} + \mathrm{Var}\{E(Y|X)\}$$

proof: $\mathrm{Var}(Y|X) = E(Y^2|X) - \{E(Y|X)\}^2$

$E\{\mathrm{Var}(Y|X)\} = E\{E(Y^2|X)\} - E[\{E(Y|X)\}^2]$

$= E(Y^2) - E[\{E(Y|X)\}^2]$

$\mathrm{Var}\{E(Y|X)\} = E[\{E(Y|X)\}^2] - (E\{E(Y|X)\})^2$

$= E[\{E(Y|X)\}^2] - \{E(Y)\}^2$

$E\{\mathrm{Var}(Y|X)\} + \mathrm{Var}\{E(Y|X)\} = EY^2 - \{E(Y)\}^2$

$= \mathrm{Var}(Y)$

①. $\mathrm{Var}(Y) \geq E\{\mathrm{Var}(Y|X)\}$

e.g. Y is weight. X is height.

$\mathrm{Var}(Y)$: uncertainty in weight in the population.

$\mathrm{Var}(Y|X)$: uncertainty in weight after knowing height.

If we know the height, the uncertainty in weight will decrease.

$E\{\mathrm{Var}(Y|X)\}$: average uncertainty if we know height.

If $X \perp Y$. $\mathrm{Var}\{E(Y|X)\} = \mathrm{Var}\{E(Y)\} = 0$

$\mathrm{Var}(Y) = E\{\mathrm{Var}(Y|X)\}$

②. $\mathrm{Var}(Y) = E\{\mathrm{Var}(Y|X)\} + \mathrm{Var}\{E(Y|X)\}$

e.g. $X = \{$Canada, US, Mexico$\}$, $Y =$ weight.

$\mathrm{Var}(Y)$: variance of weight in North America.

$E\{\mathrm{Var}(Y|X)\}$: average within country variance.

$\mathrm{Var}(Y|X=CA)$ $\quad$ $\mathrm{Var}(Y|X=US)$ $\quad$ $\mathrm{Var}(Y|X=MX)$

average by taking expectation with respect to X

$\mathrm{Var}\{E(Y|X)\}$: $E(Y|X=CA)$ $\quad$ $E(Y|X=US)$ $\quad$ $E(Y|X=MX)$

variance of mean among different countries.

Example 1 $Y \sim Poi(\theta)$. $X|Y=y \sim Bin(y, p)$. Find $Var(X)$

Solution: $Var(X) = E\{Var(X|Y)\} + Var\{E(X|Y)\}$

$Var(X|Y) = Yp(1-p)$, $E(X|Y) = Yp$.

$E\{Var(X|Y)\} = p(1-p)E(Y)$, $Var\{E(X|Y)\} = Var(Yp)$

$= \theta p(1-p)$ $\qquad\qquad = p^2\theta$

$$\left.\begin{array}{l} Var(X) = \theta p(1-p) + p^2\theta = p\theta \\ \hline X \sim Poi(\theta p) . \ Var(X) = p\theta \end{array}\right\} \text{ Same result.}$$

Example 2 $X \sim Unif(0,1)$ & $Y|X=x \sim Bin(10, x)$

Find $E(Y)$, $Var(Y)$

$E(Y) = E\{E(Y|X)\} = E\{10X\} = 10E(X) = 5$.

$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}$

$= E\{10 \cdot X \cdot (1-X)\} + Var(10 \cdot X)$

$Var(X) = \frac{1}{12}$ $\qquad = 100 \cdot Var(X) + 10 \cdot \{E(X) - E(X^2)\}$

$E(X^2) = \frac{1}{12} + (\frac{1}{2})^2$ $\qquad = \frac{100}{12} + 10 \cdot \{\frac{1}{2} - \frac{1}{3}\} = 10$

$= \frac{1}{3}$

Example 3 $Y \sim Poi(\theta)$. $X|Y=y \sim Bin(y, p)$. Find mgf of X

$M(t) = E\{e^{tX}\} = E\{E(e^{tX}|Y)\}$

$E(e^{tX}|Y=y) = \sum_{x=0}^{y} \binom{y}{x} \cdot p^x (1-p)^{y-x} e^{tx} = \sum_{x=0}^{y} \binom{y}{x} (pe^t)^x (1-p)^{y-x}$

Let $p' = \dfrac{pe^t}{pe^t+1-p}$ $\quad 1-p' = \dfrac{1-p}{pe^t+1-p}$. $\quad = \underbrace{\sum_{x=0}^{y} \binom{y}{x} (p')^x (1-p')^{y-x}}_{\sim Bin(y, p')} \cdot \underline{(pe^t+1-p)^y}$

$= (pe^t+1-p)^y$

$M(t) = E\{(pe^t+1-p)^Y\}$

$= \sum_{y=0}^{\infty} (pe^t+1-p)^y \dfrac{\theta^y e^{-\theta}}{y!} = \sum_{y=0}^{\infty} \dfrac{\{\theta(pe^t+1-p)\}^y}{y!} \cdot e^{-\theta}$

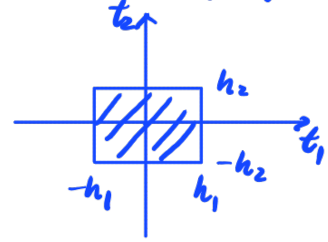
Let $\theta' = \theta \cdot (pe^t+1-p) = \underbrace{\sum_{y=0}^{\infty} \dfrac{(\theta')^y}{y!} e^{-\theta'}}_{\sim Poi(\theta')} e^{\theta'-\theta}$

$= e^{\theta'-\theta} = e^{p\theta(e^t-1)}$

Joint Moment Generating Function

Definition:

Bivariate: X, Y rvs. $M(t_1, t_2) = E[e^{t_1X + t_2Y}]$ is called the joint mgf for (X, Y) if $M(t_1, t_2)$ exists for $|t_1| < h_1$, $|t_2| < h_2$, for some $h_1 > 0$, $h_2 > 0$.



rectangle needs to contain $(0, 0)$

Multivariate: $X_1 \cdots X_n$ rvs.

$M(t_1, \cdots, t_n) = E[e^{t^T X}]$. where

$t^T = [t_1, \cdots, t_n]$. $t^T X = \sum_{i=1}^n t_i X_i$

$X^T = [X_1, \cdots, X_n]$

if $|t_i| < h_i$, for some $h_i > 0$, $i = 1, \cdots, n$.

Properties: ①. From joint mgf to marginal mgf.

Suppose. $M(t_1, t_2)$ for $|t_1| < h_1$, $|t_2| < h_2$. $h_1 > 0$, $h_2 > 0$.

$M_X(t_1) = M(t_1, t_2 = 0) = E(e^{t_1 X})$, $|t_1| < h_1$

$M_Y(t_2) = M(t_1 = 0, t_2) = E(e^{t_2 Y})$, $|t_2| < h_2$.

②. Independence

$X \perp Y \iff M(t_1, t_2) = M_X(t_1) \cdot M_Y(t_2)$

$X_1 \cdots X_n$ are independent, $\iff$.

$M(t_1, \cdots t_n) = \prod_{i=1}^n M_{X_i}(t_i)$

③. Uniqueness

joint mgf & joint dist'n are one-to-one.

Examples:

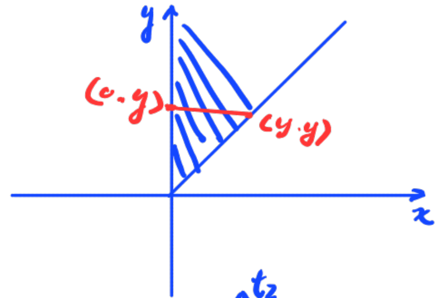
Example 1: $f(x,y) = e^{-y}$, $0<x<y$. Find joint and marginal mgfs.

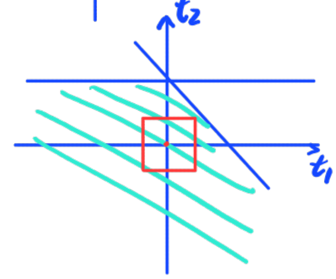
$$M(t_1,t_2) = E(e^{t_1X+t_2Y})$$

$$= \int_0^\infty \int_0^y e^{t_1x+t_2y} \cdot e^{-y} \, dx \, dy$$

$$= \int_0^\infty [e^{-(1-t_1-t_2)y} - e^{-(1-t_2)y}] \, dy$$

$$= \frac{1}{1-t_1-t_2} \cdot \frac{1}{1-t_2}, \quad t_2<1, \; t_1+t_2<1$$





$$M_X(t_1) = M(t_1, t_2=0) = \frac{1}{1-t_1}, \quad t_1 \neq 1$$

$$M_Y(t_2) = M(t_1=0, t_2) = \frac{1}{(1-t_2)^2}, \quad t_2<1$$

$$M_X(t_1) \cdot M_Y(t_2) \neq M(t_1, t_2) \Rightarrow X \not\perp Y$$

Example 2: Suppose $X \sim Poi(\theta_1)$, $Y \sim Poi(\theta_2)$, $X \perp Y$. Find out the dist'n of $X+Y$.

Solution: $M_{X+Y}(t) = E(e^{tX+tY}) = \{E(e^{tX})\}\{E(e^{tY})\}$

$$= \exp\{\theta_1(e^t-1)\} \exp\{\theta_2(e^t-1)\}$$

$$= \exp\{(\theta_1+\theta_2)(e^t-1)\} \sim Poi(\theta_1+\theta_2)$$