University of Waterloo

STAT330, Spring 2024

# Tutorial 2

Last Updated:

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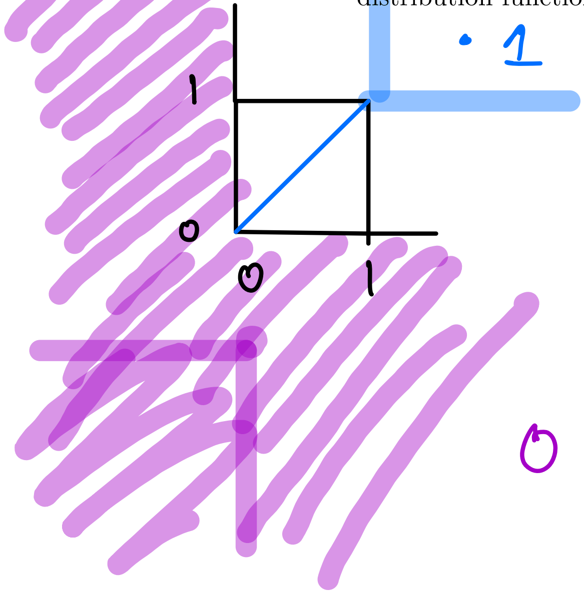
**Problem 1.** Let $X = Y$ having uniform $[0,1]$ distribution. Find the joint distribution function.




$$F(x,y) = \begin{cases} 0 & x \le 0, \text{ or } y \le 0 \\ 1 & x \ge 1, \text{ and } y \ge 1 \\ \min(x,y) & \text{otherwise.} \end{cases}$$

$$F(x,y) = P(X \le x \text{ and } X \le y)$$

$$= P(X \le \min\{x,y\})$$

$$= \min\{x,y\}.$$

**Problem 2.** Let $C$ be the discrete uniform random variable on two points $\{0, 1\}$ independent from $X$, which has continuous uniform $[0, 1]$ distribution. Let $Z = X$ if $C = 0$, and $Z = 0.5$ if $C = 1$. Find the distribution function of $Z$. (Extra: Find the joint distribution function for $Z$ and $C$. )

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 & x \ge 1 \\ \frac{1}{2}x & x \in [0, 0.5) \\ \frac{1}{2}x + \frac{1}{2} & x \in [0.5, 1] \end{cases}$$

if $x \geq 0.5$

$\frac{1}{2}$ $C = 1 \xrightarrow{1} Z = 0.5 \le x$ $\quad \frac{1}{2} + \frac{1}{2}x$

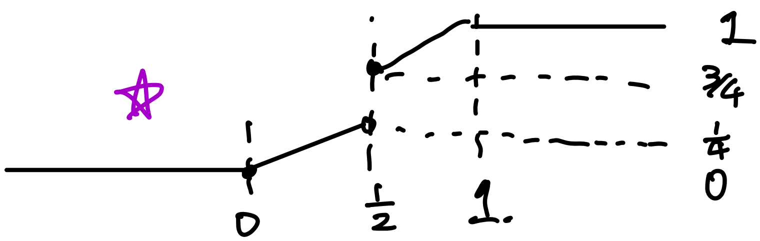
$\frac{1}{2}$ $C = 0 \xrightarrow[X \le x]{x} Z = X \le x$

$F(x) = P(Z \le x)$

if $x < 0.5$

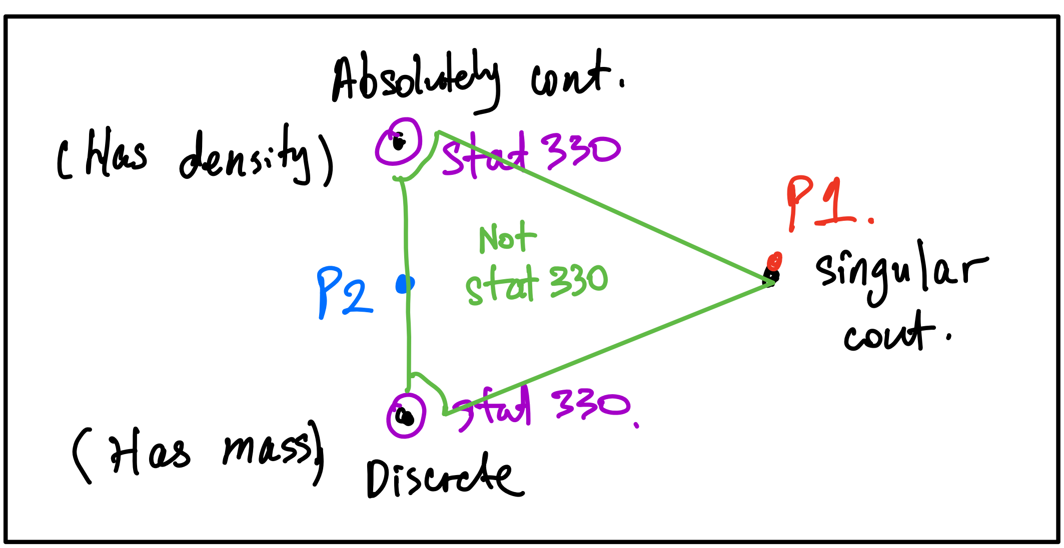
$\frac{1}{2}$ $C = 1 \xrightarrow{0} 0.5 = Z \le x < 0.5$

$\frac{1}{2}$ $C = 0 \xrightarrow[X \le x]{x} Z = X \le x$ $\quad 0 + \frac{1}{2}x$



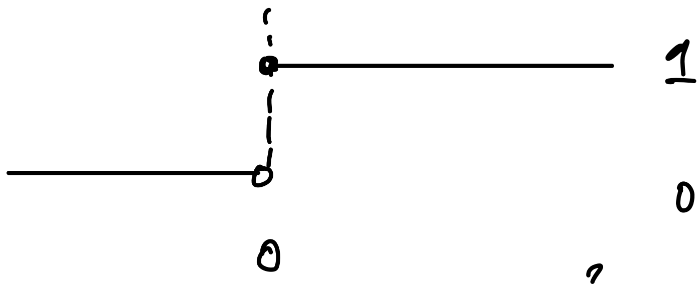

$F = \frac{1}{2}F(\text{mass at } 0.5)$
$+ \frac{1}{2}F(\text{unif } [0,1])$

Double expectation Formula is really "discuss cases"



**Problem 3.** Let $X$ be the random variable with 100% chance taking the value 0.

- What is the cdf of $X$?
- Is $X$ independent with $X$?




$$P(X\in A)\cdot P(X\in B) \overset{?}{=} P(X\in A \text{ and } X\in B)$$

$$P(X\in A) = \begin{cases} 1 & \text{if } 0\in A \\ 0 & \text{if } 0\notin A. \end{cases}$$

$$\min\big(P(X\in A),\ P(X\in B)\big)$$

So $X$ is independent with $X$.

**Problem 4.** (Rock Paper Scissors) Suppose Player 1 and Player 2 are playing rock (R), Paper (P) and Scissors (S). against each other. They each independently randomize a choice with 1/3 chance. Let $X$ and $Y$ be player 1 and player 2's choice respectively. To make $X$ and $Y$ random variables, we can map "R", "P", "S" to $\{-1, 0, 1\}$ but let's keep things intuitive. Let $Z$ be player 1's payoff: $-1$ for a loss, 0 for a draw and 1 for a win.

1. • find the joint p.m.f of $X$ and $Y$.
2. • find the joint p.m.f of $X$ and $Z$.
3. • find the joint p.m.f of $Y$ and $Z$.
4. • Show that $X$ and $Z$ are independent.
5. • Show that $Y$ and $Z$ are independent.
6. • Is $(X, Y, Z)$ independent? Explain.

$$\boxed{\begin{aligned} &P(X\in A \text{ and } Y\in B \text{ and } Z\in C) \\ &= P(X\in A)\cdot P(X\in B)\cdot P(Z\in C). \end{aligned}}$$

① $X$ vs. $Y$

| X \ Y | R | P | S. |
|---|---|---|---|
| R | 1/9 | 1/9 | 1/9 |
| P | " | ⋮ | " |
| S | 1/9 | 1/9 | 1/9 |

② $X$ vs. $Z$

| X \ Z | -1 | 0 | 1. |
|---|---|---|---|
| R | 1/9 | " | " |
| P | " | ⋱ | " |
| S | " | " | 1/9 |

④ X, Z independent

③ ⑤ Similar.

⑥ $P(R, R, -1.) = 0.$

$P(X=R)\cdot P(Y=R)\cdot P(Z=-1) = \frac{1}{27}.$

Not independent.

**Problem 5.** Let $X$ be a standard normal distribution and $Y = X^2$.

- Compute the covariance of $X$ and $Y$.
- Show that $X$ and $Y$ are not independent.

$$\mathrm{cov}(X, Y)$$

$$= \mathrm{cov}(X, X^2)$$

$$= \underbrace{E[X^3]}_{0} - \underbrace{E[X]}_{0} E[X^2]$$

$$= 0.$$

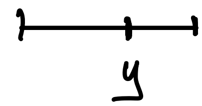
$$P(X^2 \le 1) = P(-1 \le X \le 1)$$

$$= P(-1 \le X \le 1 \text{ and } X^2 \le 1)$$

$$\ne P(X^2 \le 1) \cdot P(-1 \le X \le 1)$$

**Problem 6.** A stick of length of 1 is split at a point $X$ that is uniformly distributed over $(0,1)$. Determine the expected length of the piece that contains the point $y$, $0 \le y \le 1$.

Let $L$ be the length of the piece that contains $y$.




$$E[L] = E\big[E[L \mid X]\big]$$

$$= \int_0^1 E[L \mid X=x] \cdot f_X(x)\,dx$$

$$= \int_0^y E[L \mid X=x] \cdot f_X(x)\,dx + \int_y^1 E[L \mid X=x] \cdot f_X(x)\,dx$$

$$= \int_0^y (1-x)\,1\,dx + \int_y^1 x \cdot 1 \cdot dx.$$

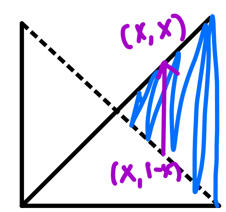
$$= x - \tfrac{1}{2}x^2 \Big|_0^y + \tfrac{1}{2}x^2 \Big|_y^1$$

$$= y - \tfrac{1}{2}y^2 + \tfrac{1}{2} - \tfrac{1}{2}y^2.$$

$$= \tfrac{1}{2} + y - y^2$$

**Problem 7.** Let us choose at random a point from the interval (0, 1) and let the random variable $X$ be equal to the number which corresponds to that point. Then choose a point at random from the interval $(0, x)$, where $x$ is the experimental value of $X$; and let the random variable $Y$ be equal to the number which corresponds to this point.

1. Find the marginal p.d.f of $X$ and the conditional p.d.f of $Y$ given $X = x$.
2. Compute $P(X + Y \geq 1)$.
3. Find the marginal p.d.f of Y and the conditional p.d.f of $X$ given $Y = y$.
4. Find the conditional expectation $E(X|Y = y)$.

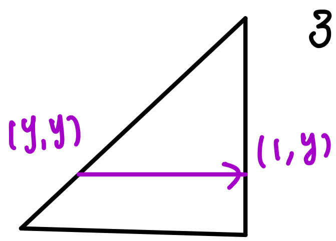



1. $X \sim U(0,1)$, $Y|X=x \sim U(0,x)$

$\searrow f(y|x) = \frac{1}{x}$, $y \in (0,x)$

2. $$\int_{\frac{1}{2}}^{1} \int_{1-x}^{x} \frac{1}{x} \, dy \, dx$$

$$= 1 - \ln 2$$




3. $$f_Y(y) = \int_y^1 \frac{1}{x} \, dx = -\ln y, \quad y \in (0,1)$$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{-x \ln y}, \quad y \in (0,1), \; x \in (y,1)$$

4. $$E[X|Y=y] = \int_y^1 x \cdot f(x|y) \, dx$$

$$= \frac{y-1}{\ln y}, \quad y \in (0,1)$$

# Slides Q7:

Let $X, Y$ be uncorrelated. $Z = -X$. $\mathrm{Var}(X) \neq 0$.

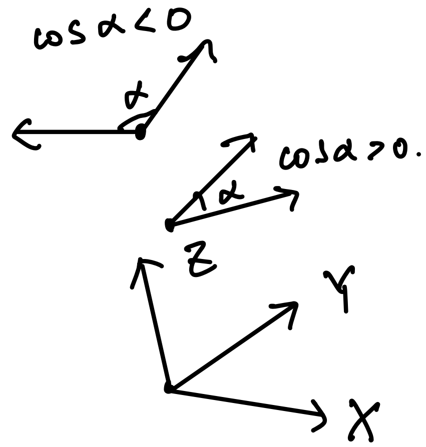
$$\rho(X, Y) = 0.$$

$$\rho(Y, Z) = 0.$$

$$\rho(X, Z) = \rho(X, -X) = -1 < 0$$

R.V with mean 0. $\quad \mathrm{cov} \sim \langle \cdot , \cdot \rangle$

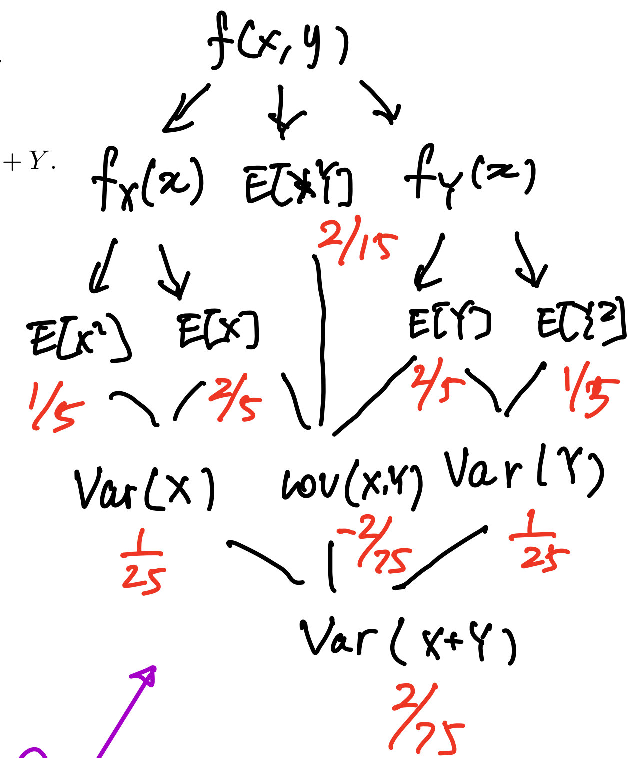
"dot product."




$$\rho \sim \cos(\angle \cdot)$$

**Problem 8.** Suppose $X$ and $Y$ are continuous random variables with joint p.d.f. $f(x,y) = 24xy$ if $0 < x, 0 < y$ and $x + y < 1$, and zero otherwise.

1. Find the covariance of $X$ and $Y$
2. Find the correlation coefficient of $X$ and $Y$
3. Find the correlation coefficient of $3X + 5$ and $X$.
4. Find the correlation coefficient of $3X + 5$ and $Y$.
5. Find the correlation coefficient of $3X + 5$ and $X + Y$. — $\sqrt{6}/6$




$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma(X)\,\sigma(Y)}$$

3. $\mathrm{cor}(3X+5, X) = \mathrm{cor}(X,X)$

4. $\mathrm{cor}(3X+5, Y) = \mathrm{cor}(X,Y)$

5. $\mathrm{cor}(3X+5, X+Y)$

$$= \mathrm{cor}(X, X+Y)$$

$$= \frac{\mathrm{var}(X) + \mathrm{cov}(X,Y)}{\sigma(X) + \sigma(X+Y)}$$

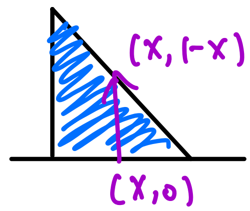
② substitute numbers.

① work through this mess

E[XY] shown next page.

other computations similar.

$$E[XY] = \int_0^1 \text{Vertical strips at } x. \; dx$$




$$= \int_0^1 \int_0^{1-x} xy \cdot f(x,y)\, dy\, dx.$$

$$= \int_0^1 \int_0^{1-x} 24 x^2 y^2 \, dy\, dx.$$

$$= \int_0^1 24x^2 \; \frac{1}{3} y^3 \Big|_0^{1-x} \; dx$$

$$= \int_0^1 8x^2 (1-x)^3 \, dx.$$

$$= \int_0^1 8x^2 - 24x^3 + 24x^4 - 8x^5 \, dx$$

$$= \frac{8}{3} - 6 + \frac{24}{5} - \frac{8}{6}$$

$$= \frac{4}{3} - 6 + \frac{24}{5} = \frac{20-90+72}{15} = \frac{2}{15}$$