STAT 330: Lecture 7

2024 Spring

June 4, 2024

Last Lecture Independence $f(x,y)=f(x)f(y)$

1. Definition: Connection with Independence of Events/Equivalence to the product of marginal pdf/pmf
2. Function of Independence Random Variables $X \perp Y \Rightarrow g(X) \perp h(X)$
3. Factorization Theorem $\begin{cases} f(x,y) = c\, g(x)\cdot h(y) \\ A = A_X \times A_Y \end{cases} \Leftrightarrow X \perp Y$

This Lecture Joint Expectation, Variance, Covariance; Correlation Coefficient

Joint Expectation We extend the definition of expectation from univariate to bivariate (and multivariate) cases.

Definition: Suppose $h(x, y)$ is a real function. Then

$\in \mathbb{R}$.

$$E[h(X,Y)] = \begin{cases} \sum_x \sum_y h(x,y) f(x,y) & X \,\&\, Y \text{ are joint discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) f(x,y) dx dy & X \,\&\, Y \text{ are joint continuous} \end{cases}$$

if the expectation exists.

e.g. $E(XY) = \begin{cases} \sum_x \sum_y xy f(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y) dx dy & \text{continuous} \end{cases}$

$E(X) = \begin{cases} \sum_x \sum_y x f(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y) dx dy & \text{continuous} \end{cases}$

OR.

$E(X) = \begin{cases} \sum_x x f(x), & \text{discrete} \\ \int_{-\infty}^{\infty} x f(x) dx, & \text{continuous} \end{cases}$ First get $f(x)$ from $f(x,y)$

Properties:

1) linearity: exchange the order of summation & expectation.

$E[a g(X,Y) + b h(X,Y)] = a E[g(X,Y)] + b E[h(X,Y)]$

$E[\sum_{i=1}^n a_i X_i] = \sum_{i=1}^n a_i E(X_i)$

2) $E(XY) \neq E(X)E(Y)$ in general.

But $E(XY) = E(X)E(Y)$ if $X \perp Y$

proof. $E(XY) = \iint xy f(x,y) dx dy$

$= \iint xy f(x) f(y) dx dy$

$= \{\int x f(x) dx\}\{\int y f(y) dy\} = E(X)E(Y)$

$E\{g(X)h(Y)\} \stackrel{?}{=} E\{g(X)\}E\{h(Y)\}$ if $X \perp Y$.

If $X_1 \cdots X_n$ are independent

$E(\prod_{i=1}^n X_i) = \prod_{i=1}^n E(X_i)$

<u>*Covariance*</u>: Covariance of X & Y is defined as

$$\mathrm{Cov}(X, Y) = E\{(X - E(X))(Y - E(Y))\}$$
$$= E(XY) - E(X)E(Y)$$

If $X \perp Y$, $E(XY) = E(X)E(Y) \Rightarrow \mathrm{Cov}(X, Y) = 0$

$$\mathrm{Var}(X) = \mathrm{Cov}(X, X)$$

<u>*Variance Formulas*</u>:

1) $\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y)$

2) $\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) + \sum_{i \neq j} a_i a_j \mathrm{Cov}(X_i, X_j)$

$$= \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) + 2 \sum_{i<j} a_i a_j \mathrm{Cov}(X_i, X_j)$$

NOT a proof

$$(aX + bY)^2 = a^2 \underbrace{X^2}_{\mathrm{Var}(X)} + b^2 \underbrace{Y^2}_{\mathrm{Var}(Y)} + 2ab \underbrace{XY}_{\mathrm{Cov}(X, Y)}$$

$$\left(\sum_{i=1}^{n} a_i X_i\right)^2 = a_1^2 \underbrace{X_1^2}_{\mathrm{Var}(X_1)} + a_2^2 \underbrace{X_2^2}_{\mathrm{Var}(X_2)} + \cdots + a_n^2 X_n^2 + \sum_{i \neq j} a_i a_j \underbrace{X_i X_j}_{\mathrm{Cov}(X_i, X_j)}$$

3). If $X_1 \cdots X_n$ are independent.

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$$

Examples: Suppose we have joint pmf for (X, Y)

$$f(x,y) = \frac{\theta^{x+y} e^{-2\theta}}{x!\, y!}, \quad x, y \in \{0, 1, 2, \cdots\}$$

Find $\mathrm{Var}(2X+3Y)$

$$f(x,y) = e^{-2\theta} \cdot \frac{\theta^x}{x!} \cdot \frac{\theta^y}{y!} \Rightarrow X \perp Y \text{ By factorization theorem.}$$

$$\mathrm{Var}(2X+3Y) = 4\,\mathrm{Var}(X) + 9\,\mathrm{Var}(Y)$$

$X, Y \sim \mathrm{Poi}(\theta)$. $\mathrm{Var}(X) = \mathrm{Var}(Y) = \theta$

$\Rightarrow \mathrm{Var}(2X+3Y) = 13\theta$.

X & Y joint pdf. $f(x,y) = \begin{cases} x+y & , 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

Find out $\mathrm{Var}(X+Y)$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$$

$\mathrm{Var}(X) = \mathrm{Var}(Y)$ due to symmetry.

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$\mathrm{Cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$$

We need. $E(X^2), E(X), E(XY)$.

$$E(X) = E(Y) = \int_0^1 x \underline{f(x)}\, dx = \int_0^1 x \cdot (x + \tfrac{1}{2})\, dx = \frac{7}{12}$$

marginal of x

$$E(X^2) = E(Y^2) = \int_0^1 x^2 f(x)\, dx = \int_0^1 x^2 (x + \tfrac{1}{2})\, dx = \frac{5}{12}$$

$$E(XY) = \int_0^1 \int_0^1 xy(x+y)\, dx\, dy = \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy = \frac{1}{3}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$$

$$\mathrm{Var}(X+Y) = 2\left(E(X^2) - \{E(X)\}^2\right) + 2\,\mathrm{Cov}(X,Y)$$

$$= \frac{11}{144} \cdot 2 - \frac{2}{144} = \frac{5}{36}.$$

Correlation Coefficient

Definition: The correlation coefficient of X and Y is defined as

$$\rho(X,Y) = \frac{\mathrm{Cov(X, Y)}}{\sqrt{\mathrm{Var(X)}}\sqrt{\mathrm{Var(Y)}}}$$
$$= \frac{Cov(X,Y)}{sd(X)\cdot sd(Y)}$$

If $X \perp Y$, $\rho(X,Y)=0$

Properties: ① $-1 \le \rho(X,Y) \le 1$

$$\boxed{\begin{array}{l}\text{Cauchy Schwarz Inequality}: X, Y \text{ rvs} \\ \{E(XY)\}^2 \le E(X^2)E(Y^2)\end{array}}$$

$$[E\{X-E(X)\}\{Y-E(Y)\}]^2 \le E\{(X-E(X))^2\}\cdot E\{(Y-E(Y))^2\}$$
$$\{Cov(X,Y)\}^2 \le Var(X)\cdot Var(Y)$$
$$-\sqrt{Var(X)}\sqrt{Var(Y)} \le Cov(X,Y) \le \sqrt{Var(X)}\cdot\sqrt{Var(Y)}$$
$$\Leftrightarrow \quad \rho(X,Y) \in [-1, 1]$$

② Let $Y=aX+b$

$$Cov(X,Y) = Cov(X, aX+b) = Cov(X, aX) = a\,\underline{Cov(X,X)}$$
$$= a\,Var(X)$$

Example: $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \dfrac{a\,Var(X)}{\sqrt{Var(X)}\sqrt{a^2 Var(X)}} = \dfrac{a}{|a|} = \begin{cases} 1, & a>0 \\ -1, & a<0 \end{cases}$

Example. $f(x,y) = \begin{cases} x+y, & x\in[0,1],\ y\in[0,1] \\ 0, & \text{otherwise.} \end{cases}$

Find out $\rho(X,Y)$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} = \frac{-1/144}{\frac{11}{144}} = -\frac{1}{11}$$

Conditional Distribution

continuous/discrete.

Definition: Suppose X and Y are joint ~~discrete~~ with joint pdf/pmf $f(x, y)$. Then the conditional pdf/pmf of X given $Y = y$ is

random. non-random

$$f(x|y) = \frac{f(x, y)}{f(y)}, \quad \text{fix } Y=y$$

given that $f(y) > 0$. The conditional pdf/pmf of Y given $X = x$ is

$$f(y|x) = \frac{f(x, y)}{f(x)}, \quad \text{fix } X=x$$

given that $f(x) > 0$.

Properties: ① Conditional pdf/pmf are still valid pdf/pmf.

$$\begin{cases} f(x|y) \geq 0 \;\&\; \sum_x f(x|y) = 1 \\ f(y|x) \geq 0 \;\&\; \sum_y f(y|x) = 1 \end{cases} \quad \text{discrete}$$

$$\begin{cases} f(x|y) \geq 0 \;\&\; \int_{-\infty}^{\infty} f(x|y) = 1 \\ f(y|x) \geq 0 \;\&\; \int_{-\infty}^{\infty} f(y|x) = 1 \end{cases} \quad \text{continuous}$$

Example:

Example. 1.

$$f(x, y) = \begin{cases} 8xy & , \; 0<y<x<1 \\ 0 & , \; \text{otherwise.} \end{cases}$$

Find out 1) $f(x|y)$ 2) $f(y|x)$

First step: $f(x) = 4x^3, \; 0<x<1$

Find marginals $f(y) = 4y - 4y^3, \; 0<y<1.$

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y-4y^3} = \frac{2x}{1-y^2}, \quad x \in (y, 1)$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}, \quad y \in (0, x)$$

Example 2.

$$f(x,y) = \begin{cases} x+y & , x,y \in [0,1] \\ 0 & , \text{otherwise} \end{cases}$$

First, find out the marginals.

$$f(x) = x + \tfrac{1}{2}, \quad x \in [0,1]$$

$$f(y) = y + \tfrac{1}{2}, \quad y \in [0,1]$$

Second. take the ratio.

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{x+y}{y+\frac{1}{2}}, \quad x \in [0,1]$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{x+\frac{1}{2}}, \quad y \in [0,1]$$

Example 3. X, Y rvs. Joint discrete.

$$f(x,y) = q^2 \cdot p^{x+y}, \quad q = 1-p, \quad x,y \in \{0,1,2,\cdots\}$$

marginals. $f(x) = q p^x$, $f(y) = q p^y$

$$f(x,y) = q^2 \cdot p^x \cdot p^y. \Rightarrow X \perp Y. \text{ by Fatorization theorem.}$$

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{q^2 \cdot p^{x+y}}{q p^y} = q p^x = f(x)$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = q \cdot p^y = f(y)$$

marginal dist'n = conditional dist'n iif $X \perp Y$