STAT 330: Lecture 11

2024 Spring

June 18, 2024

In the past few lectures Multinomial Distribution/Bivariate Normal Distribution

This Lecture

- More on Bivariate Normal Distribution.
- Function of random variables.

Review of BVN

$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim BVN(\mu, \Sigma)$,

mean vector $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ $\quad \mu_1 = E(X_1)$, $\mu_2 = E(X_2)$

variance-covariance matrix.

$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ $\quad \sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$, $\rho = \rho(X_1, X_2)$

① mgf.

② marginal distb. $X_1 \sim Norm(\mu_1, \sigma_1^2)$, $X_2 \sim Norm(\mu_2, \sigma_2^2)$

③ conditional distribution:

X_1, X_2 if $X_1 \perp X_2$

$E(X_1 X_2) = E(X_1) \cdot E(X_2)$

$\Rightarrow Cov(X_1, X_2) = 0$

$\Rightarrow \rho(X_1, X_2) = 0$

More the Properties of BVN

5. $X_1 \perp X_2 \Leftrightarrow \rho = 0 \Leftrightarrow Cov(X_1, X_2) = 0$

" $\Rightarrow$ " is obvious.

" $\Leftarrow$ " if $\rho = 0$, then $X_1 \perp X_2$.

Method 1:

$X_1 \perp X_2 \Leftrightarrow f(x_1, x_2) = f(x_1) \cdot f(x_2)$

when $\rho = 0$.

$f(x_1, x_2) =$

$\frac{1}{\sqrt{2\pi}\sigma_1} \cdot \exp\left\{-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right\} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma_2} \cdot \exp\left\{-\frac{1}{2} \cdot \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}$

$Norm(\mu_1, \sigma_1^2)$ — X_1

$Norm(\mu_2, \sigma_2^2)$ — X_2

$f(x_1, x_2) = f(x_1) f(x_2)$.

$\Rightarrow X_1 \perp X_2$.

Method 2

$X_1 \perp X_2 \Leftrightarrow M_{(X_1, X_2)}(t_1, t_2) = M_{X_1}(t_1) \cdot M_{X_2}(t_2)$

when $\rho = 0$

$M_{(X_1, X_2)} = \exp\left\{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}t_1^2\sigma_1^2 + \frac{1}{2}t_2^2\sigma_2^2\right\}$

$= \exp\left\{t_1\mu_1 + \frac{1}{2}t_1^2\sigma_1^2\right\} \cdot \exp\left\{t_2\mu_2 + \frac{1}{2}t_2^2\sigma_2^2\right\}$

$= M_{X_1}(t_1) M_{X_2}(t_2)$

$\Rightarrow X_1 \perp X_2$

Method 3.

$X_1 \perp X_2 \Leftrightarrow f(x_1 | x_2) = f(x_1)$

$X_2 | X_1 = x_1 \sim Norm\left(\mu_2 + \rho\sigma_2 \frac{(x_1-\mu_1)}{\sigma_1}, \sigma_2^2(1-\rho^2)\right)$

when $\rho = 0$.

$X_2 | X_1 = x_1 \sim Norm(\mu_2, \sigma_2^2)$

$X_2 \sim Norm(\mu_2, \sigma_2^2)$

$\Rightarrow X_1 \perp X_2$.

6.

linear transformation.

Let $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. $c^T X = [c_1, c_2]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = c_1 X_1 + c_2 X_2 \in \mathbb{R}$

$E(c^T X) = c_1 E(X_1) + c_2 E(X_2) = c^T \mu = [c_1, c_2]\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = c_1\mu_1 + c_2\mu_2$.

$var(c^T X) = c^T Var(X) c = c^T \Sigma c$

$c^T X \sim Norm(c^T\mu, c^T \Sigma c)$

Let $A_{2\times 2}$, $b_{2\times 1}$. $AX+b$?

$AX+b \sim BVN(A\mu+b, A\Sigma A^T)$

Mean vector: $E(AX+b) = AE(X)+b$
$= A\mu+b$

covariance matrix:

$Var(AX+b) = Var(AX)$
$= A\,Var(X)A^T$
$= A\Sigma A^T$

$A = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$AX+b = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$= \begin{bmatrix} a_1^T X + b_1 \\ a_2^T X + b_2 \end{bmatrix} \in \mathbb{R}^2$

7. $Z \sim Norm(0,1)$, find out the dist'n of Z^2?

$$M_{Z^2}(t) = E(e^{Z^2 t}) = \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{z^2}{2}) dz$$

try yourself

Method 1: make a normal pdf.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\{-\frac{z^2}{2}(1-2t)\} dz$$

Method 2: Change variable.

$$= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{z^2}{2}(1-2t)\} dz.$$

Let $y = \frac{z^2}{2}(1-2t)$, $z = \sqrt{\frac{2y}{1-2t}}$, $dz = \sqrt{\frac{2}{1-2t}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{y}} dy$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{1-2t}} \underbrace{\int_{0}^{\infty} \exp(-y) \cdot y^{-1/2} dy}_{=\Gamma(\frac{1}{2}) = \sqrt{\pi}}$$

$$= (1-2t)^{-1/2}$$

Gamma(α, β). mgf. $(1-\beta t)^{-\alpha}$

$Z^2 \sim$ Gamma$(\frac{1}{2}, 2)$

We call Gamma$(\frac{1}{2}, 2)$ as χ^2_1 "Chi-square".

8. $Z_1, Z_2 \sim Norm(0,1)$. $Z_1 \perp Z_2$.

Find $Z_1^2 + Z_2^2$?

$$M_{Z_1^2+Z_2^2}(t) = E(e^{(Z_1^2+Z_2^2)t}) = E(e^{Z_1^2 t}) \cdot E(e^{Z_2^2 t})$$

$$= (1-2t)^{-1/2} (1-2t)^{-1/2}$$

$$= (1-2t)^{-1} \sim \text{Gamma}(1, 2) \quad \chi^2_2$$

$Z_1, \cdots Z_n \overset{iid}{\sim} Norm(0,1)$

Find $Z_1^2 + \cdots + Z_n^2 \sim$ Gamma$(\frac{n}{2}, 2)$ χ^2_n

9. $X \sim BVN(\mu, \Sigma)$. Find $(X-\mu)^T \Sigma^{-1} (X-\mu)$

NOT required $\leftarrow$ $\boxed{\text{Lemma I:}}$ If Σ is positive definite, $\exists A$, which is nonsingular. s.t. $\Sigma = AA^T$

Lemma 2: If $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim BVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$, $AZ+\mu \sim BVN(\mu, \Sigma)$

$E(AZ+\mu) = A\,E(Z) + \mu = \mu.$

$Var(AZ+\mu) = Var(AZ) = A\,Var(Z)\,A^T = AA^T = \Sigma$

By Lemmas 1&2, $\exists A$, s.t. $X = AZ+\mu$.

$$
\begin{aligned}
&(X-\mu)^T \Sigma^{-1} (X-\mu) \\
&= (AZ)^T (AA^T)^{-1} (AZ) \\
&= Z^T \cancel{A^T} \cancel{(A^T)^{-1}} \cancel{A^{-1}} \cancel{A} Z \\
&= Z^T Z = [z_1, z_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + z_2^2 \sim \chi^2_2
\end{aligned}
$$

Function of Random Variables

Given $X_1, \ldots, X_n$, which are (usually continuous) random variables, we want to determine the distribution of $Y = h(X_1, \ldots, X_n)$. $\in \mathbb{R}$.

We will discuss three methods:

- CDF method
- One-to-one transformation
- MGF method

CDF Method

Steps:

1. Find cdf of Y.
2. Find pdf of Y by
$$g(y) = G'(y)$$

Examples: $f(x,y) = 3y$. $0 < x \le y \le 1$. Find pdfs of $T = XY$, $S = Y/X$.

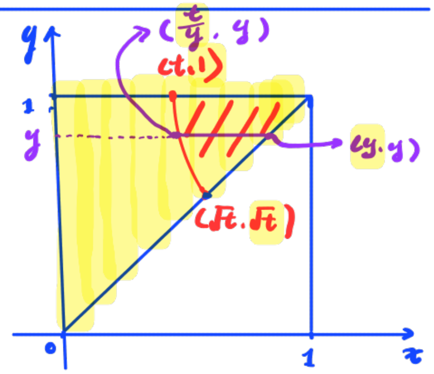
$G(t) = Pr(T \le t) = Pr(XY \le t)$

$= Pr(Y \le \frac{t}{X}) = 1 - Pr(Y > \frac{t}{X})$

$= 1 - \int_{\sqrt{t}}^{1} \int_{\frac{t}{y}}^{y} 3y \, dx \, dy$

$= 3t - 2t^{\frac{3}{2}}$, $t \in (0, 1]$

$g(t) = G'(t) = 3 - 3t^{\frac{1}{2}}$, $t \in (0, 1]$

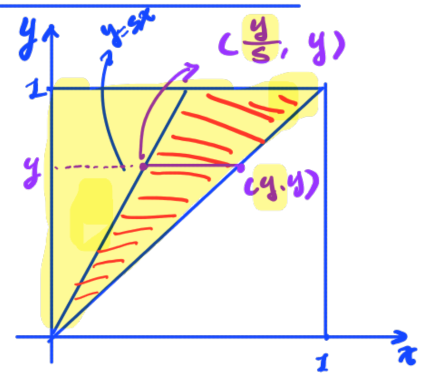


$S = Y/X$. $s \in [1, +\infty)$

$G(s) = Pr(S \le s) = Pr(Y \le sX)$, $s \ge 1$

$= \int_{0}^{1} \int_{\frac{y}{s}}^{y} 3y \, dx \, dy = 1 - \frac{1}{s}$, $s \ge 1$.

$g(s) = G'(s) = \frac{1}{s^2}$. for $s \ge 1$.



Example 2. (order statistic).

Suppose $X_1 \cdots X_n \overset{iid}{\sim} unif(0, \theta)$

$X_{(n)} = \max(X_1, \cdots, X_n)$. $X_{(1)} = \min(X_1 \cdots, X_n)$

Find pdfs of $X_{(n)}$. $X_{(1)}$

$$G_{X_{(n)}}(y) = Pr(X_{(n)} \le y) = Pr(X_1 \le y, \cdots, X_n \le y)$$
$$= \prod_{i=1}^{n} Pr(X_i \le y) = \left(\frac{y}{\theta}\right)^n, \quad y \in [0, \theta]$$
$$g_{X_{(n)}}(y) = G'_{X_{(n)}}(y) = \frac{n y^{n-1}}{\theta^n}, \quad y \in [0, \theta]$$

$$G_{X_{(1)}}(y) = Pr(X_{(1)} \le y) = 1 - Pr(X_{(1)} > y)$$
$$= 1 - Pr(X_1 > y, \cdots, X_n > y)$$
$$= 1 - \prod_{i=1}^{n} Pr(X_i > y) = 1 - \left(1 - \frac{y}{\theta}\right)^n, \quad y \in [0, \theta]$$
$$g_{X_{(1)}}(y) = \frac{n}{\theta} \cdot \left(1 - \frac{y}{\theta}\right)^{n-1}, \quad y \in [0, \theta]$$

Practice. $X_1 \cdots X_n \sim Exp(1)$. Find $X_{(n)}$. $X_{(1)}$