University of Waterloo

STAT330, Spring 2024

Tutorial 4

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Problem 1. Suppose $X_1 \sim \text{Poisson}(\theta_1)$ and $X_2 \sim \text{Poisson}(\theta_2)$ are independent. Find the conditional distribution of X_1 given $S = X_1 + X_2$. $\sim Poi(\theta_1+\theta_2)$

$P(X_1 = x \mid S = s)$ $\quad 0 \le x \le s$ integers

$$= \frac{P(X_1 = x \text{ and } S = s)}{P(S=s)}$$

$$= \frac{P(X_1 = x \text{ and } X_2 = s-x)}{P(S=s)}$$

$$= \frac{P(X_1 = x) \cdot P(X_2 = s-x)}{P(S=s)}$$

$$= \frac{e^{\theta_1} \cdot \frac{\theta_1^{x}}{x!} \cdot e^{\theta_2} \cdot \frac{\theta_2^{(s-x)}}{(s-x)!}}{e^{\theta_1+\theta_2} \cdot \frac{(\theta_1+\theta_2)^{(s-x)+x}}{s!}}$$

$$= \binom{s}{x} \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{x} \cdot \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{s-x}.$$

$$X_1 \mid S \sim bin\left(S, \frac{\theta_1}{\theta_1+\theta_2}\right)$$

Proof: $M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}]$

$= M_{X_1,X_2}(t,t)$

$= M_{X_1}(t) \cdot M_{X_2}(t)$ (Independence)

$= \exp(\theta_1(e^t-1)) \cdot \exp(\theta_2(e^t-1))$

$= \exp((\theta_1+\theta_2)(e^t-1))$

is MGF of $Poi(\theta_1+\theta_2)$

Hard mode: This is the last problem. Don't look at (a) , (b) and solve (c) directly.

Problem 2. $X_1, \ldots, X_n$ are i.i.d following an exponential distribution with mean 1. Let $Y_1 = \min\{X_i\}$ and $Y_n = \max\{X_i\}$. Let $R = Y_n - Y_1$.

(a)

(b)

(c) Find the pdf of R. $(n-1)e^{-r}(1-e^{-r})^{n-2}$

Problem 3. $X_1, \ldots, X_n$ are i.i.d following an exponential distribution with mean 1. Let $Y_1 = \min\{X_i\}$ and $Y_n = \max\{X_i\}$.

(a) Find the joint pdf of (Y_1, Y_n).

(b) Letting $S = (Y_n + Y_1)$ and $R = Y_n - Y_1$, find the joint pdf of (R, S).

(c) Find the pdf of R.

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$$F_{Y_1, Y_n}(x, y) = P(Y_1 \le x \text{ and } Y_n \le y)$$

$$= \underbrace{P(Y_n \le y)}_{X_i \le y \ \forall i} - \underbrace{P(Y_1 > x \text{ and } Y_n \le y)}_{x < X_i \le y.}$$

$$= \prod_{i=1}^{n} F_{X_i}(y) - \prod_{i=1}^{n} \left(F_{X_i}(y) - F_{X_i}(x)\right)$$

$$= (1 - e^{-y})^n - \left[e^{-x} - e^{-y} \right]^n$$

$$f_{Y_1, Y_n}(x, y) = \frac{\partial}{\partial y} \frac{\partial}{\partial x} F_{Y_1, Y_n}(x, y)$$

$$= \frac{\partial}{\partial y} \left(n (e^{-x} - e^{-y})^{n-1} e^{-x} \right)$$

$$= n(n-1)(e^{-x} - e^{-y})^{n-2} e^{-x} e^{-y}$$

(b) $$(S, R) = (Y_1 + Y_n, -Y_1 + Y_n)$$

$$(Y_1, Y_n) = \left(\frac{S-R}{2}, \frac{S+R}{2} \right)$$

$$J = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \quad |\det J| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$$f_{S,R}(s, r) = \frac{1}{2} f_{Y_1, Y_n}(x, y)$$

$$= \frac{1}{2} f_{Y_1, Y_n}\left(\frac{s-r}{2}, \frac{s+r}{2}\right) \quad \text{on "support" } 2$$

$$= \frac{1}{2} n(n-1) \left(e^{-\frac{s-r}{2}} - e^{-\frac{s+r}{2}} \right)^{n-2} e^{-\frac{s-r}{2}} e^{-\frac{s+r}{2}} \ \star$$

$$= \frac{1}{2} n(n-1) \left(e^{-s/2} \right)^{n-2} \left(e^{\frac{r}{2}} - e^{-\frac{r}{2}} \right)^{n-2} e^{-s}$$

③ function vs formula.

$$f_{S,R}(s,r) = \begin{cases} \star & \text{when } x \in A \\ 0 & \text{when } x \in A^c \end{cases}$$

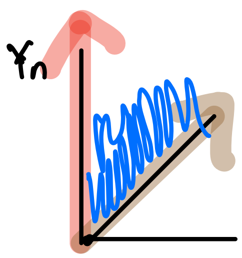
supp $f_{Y_1,Y_n} = B$ = $0 \le Y_1 \le Y_n$

$\downarrow T$ $\quad\downarrow$ $\quad\downarrow$

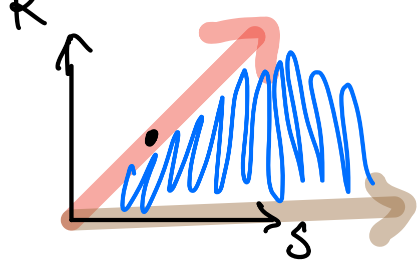
supp $f_{S,R} = A = T(B)$

$T(x,y) = (x+y, -x+y)$

0 1 $\quad$ (1, 1)



Y_1 $\overset{T}{\rightsquigarrow}$



$A = \{(s,r) : 0 \le r \le s\}$.

$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$\Leftrightarrow$ X, Y independent

$\Leftrightarrow$ · Formulas on support satisfy equation

· supp $f_{X,Y}$ = supp $f_X \times$ supp f_Y.

$T(B) = A$.

$\Updownarrow$

$T(B) \subseteq A \Leftrightarrow B \subseteq T^{-1}(A)$

$A \subseteq T(B)$

(c) $f_R(r) = \int_{-\infty}^{\infty} f_{S,R}(s,r)\, ds$.

$$= \int_A \star \; ds$$

$$= \int_r^{\infty} \frac{1}{2} n(n-1) \left(e^{-s/2} \right)^{n-2} \left(e^{\frac{r}{2}} - e^{-\frac{r}{2}} \right)^{n-2} e^{-s}\, ds$$

$$= \int_r^{\infty} \frac{1}{2} n(n-1) \left(e^{r/2} - e^{-r/2} \right)^{n-2} \left(e^{-s/2} \right)^{n-2} e^{-s}\, ds.$$

$$= \frac{1}{2} n(n-1)(e^{r/2} - e^{-r/2})^{n-2} \quad (-1)(e^{-s/2})^{n}\Big|_{r}^{\infty}.$$

$$= (n-1)e^{-r} \cdot (1-e^{-r})^{n-2}.$$