University of Waterloo

Department of Statistics and Actuarial Science

STAT330 : Homework 4 Solutions

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[Notes for Graders]
[Extra Explanations]
[Problems and Solutions]

Problem 1. Consider an iid sample $\{X_1, \ldots, X_n\}$ from the logistic distribution; that is,

$$F(x) = \{1 + \exp(-x)\}^{-1}, \quad x \in (-\infty, \infty).$$

Letting $X_{(n)}$ be $X_{(n)} = \max\{X_1, \ldots, X_n\}$, $a_n = \log(n)$, and $b_n = n/(1+n)$, find the limiting distribution of

$$Y_n = \frac{X_{(n)} - a_n}{b_n}.$$

Solution. $\mathbb{P}(X_{(n)} \leq x) = \mathbb{P}(X_1 \leq x \ldots X_n \leq x) = [F(x)]^n$. We have 7pts

$$\begin{aligned}
\mathbb{P}(Y_n \leq x) &= \mathbb{P}(\frac{X_{(n)} - a_n}{b_n} \leq x) \\
&= \mathbb{P}(X_{(n)} \leq a_n + b_n x) \\
&= \left\{1 + \exp(-\log(n) - \frac{n}{n+1}x\right\}^{-n} \\
&= \left\{1 + \frac{1}{n}\exp(-\frac{n}{n+1}x)\right\}^{-n}
\end{aligned}$$

Fix x, let $c_n = \exp(-nx/(n+1))$. Since $n/(n+1) \to 1$, for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $n > N \implies \exp(-x) \leq c_n \leq \exp(-(1-\epsilon)x)$. Hence taking $n \to \infty$, we have

$$\lim_{n\to\infty}\left\{1 + \frac{1}{n}\exp(-x)\right\}^{-n} \leq \lim_{n\to\infty}\left\{1 + \frac{1}{n}\exp(-\frac{n}{n+1}x)\right\}^{-n} \leq \lim_{n\to\infty}\left\{1 + \frac{1}{n}\exp(-(1-\epsilon)x)\right\}^{-n}$$

Evaluating the limits on the sides:

$$\exp(-\exp(-x)) \leq \lim_{n\to\infty}\left\{1 + \frac{1}{n}\exp(-\frac{n}{n+1}x)\right\}^{-n} \leq \exp(-\exp(-(1-\epsilon)x))$$

This holds for all $\epsilon > 0$ so by the squeeze theorem, it converges to $F_Y(x) = \exp(-\exp(-x))$, $x \in (-\infty, \infty)$. ♠ 3pts

Problem 2. Consider an iid sample $\{X_1, \dots, X_n\}$ from the exponential distribution; that is,

$$F(x) = 1 - \exp(-\lambda x), \quad x \in (0, \infty).$$

Letting $X_{(n)}$ be $X_{(n)} = \max\{X_1, \dots, X_n\}$, $a_n = \log(n)/\lambda$, and $b_n = n/(n+1)$, find the limiting distribution of

$$Y_n = \frac{X_{(n)} - a_n}{b_n}.$$

Solution. $\mathbb{P}(X_{(n)} \leq x) = \mathbb{P}(X_1 \leq x \dots X_n \leq x) = [F(x)]^n$. We have 7pts

$$\begin{aligned}
\mathbb{P}(Y_n \leq x) &= \mathbb{P}(\frac{X_{(n)} - a_n}{b_n} \leq x) \\
&= \mathbb{P}(X_{(n)} \leq a_n + b_n x) \\
&= \left\{1 - \exp(-\lambda \left(\frac{\log(n)}{\lambda} + \frac{n}{n+1}x\right))\right\}^n \\
&= \left\{1 - \frac{1}{n}\exp(\frac{-\lambda n}{n+1}x)\right\}^n
\end{aligned}$$

By a similar argument, this converges to $F_Y(x) = \exp(-\exp(-\lambda x))$, $x \in (0, \infty)$. ♠ 3pts

Problem 3. Stirling's Formula is used to approximate factorials.

$$n! \approx \frac{n^n}{e^n}\sqrt{2\pi n}, \quad n \to \infty.$$

You will prove the Stirling's Formula in this question.

(a) Suppose $X_1, \dots, X_n$ is an iid sample from the exponential distribution with mean 1. Prove that

$$\frac{\bar{X}_n - 1}{1/\sqrt{n}} \xrightarrow{d} \mathrm{Norm}(0,1).$$

(b) Show that

$$\frac{\sqrt{n}}{\Gamma(n)}(x\sqrt{n}+n)^{n-1}\exp\{-(x\sqrt{n}+n)\} \approx \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

(c) Prove the Stirling's Formula.

Solution. (a) Since each X_i has mean 1 and variance 1, this follows directly from CLT. 3pts

(b) The right hand side is the normal density. So we are hoping to show that the left side would be the density of $Y_n = \sqrt{n}(\bar{X}_n - 1)$. We first calculate F_{Y_n}. 4pts

$$\begin{aligned}
\mathbb{P}(\sqrt{n}(\bar{X}_n - 1) \le x) &= \mathbb{P}(\bar{X}_n \le \frac{x}{\sqrt{n}} + 1) \\
&= \mathbb{P}(X_1 + \cdots + X_n \le \sqrt{n}x + n) \\
&= \mathbb{P}(\mathrm{Gamma}(n,1) \le \sqrt{n}x + n) \\
&= \int_{-\infty}^{\sqrt{n}x+n} \frac{1}{\Gamma(n)} t^{n-1}e^{-t}\, \mathrm{d}t
\end{aligned}$$

Taking the derivative with respect to x, the density is 3pts

(use the chain rule for $g(x) = \sqrt{n}x + n$)

$$f_{Y_n}(x) = \frac{\sqrt{n}}{\Gamma(n)}(\sqrt{n}x+n)^{n-1}\exp\{-(\sqrt{n}x+n)\}$$

The justification of $Y_n \xrightarrow{d} Y$ implying $f_{Y_n} \to f_Y$ seems quite non-trivial. We are skipping this.

(c) Set x to 0 for the (approximate) equation in (b). We get 3pts

$$\frac{\sqrt{n}}{\Gamma(n)} n^{n-1}e^{-n} \approx \frac{1}{\sqrt{2\pi}}$$

Note that $\Gamma(n)$ is $(n-1)!$ and so $n! = n \cdot \Gamma(n)$. We have

$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$

♠

Problem 4. $X \sim \text{Poi}(\mu)$, the conditional distribution of Y given $X = x$ is Chi-square with degrees of freedom $2x$: χ^2_{2x}.

(a) Find $E(Y)$ and $\text{Var}(Y)$.

(b) Find the limiting distribution of

$$\frac{Y - E(Y)}{\sqrt{\text{Var}(Y)}}$$

as $\mu \to \infty$.

Solution. (a) We use the law of total expectation and the law of total variance. Recall that χ^2_k has mean k and variance $2k$. 2pts

$$\mathbb{E}(Y) = \mathbb{E}\left\{\mathbb{E}(Y|X)\right\} = \mathbb{E}(2X) = 2\mu$$

and 2pts

$$\begin{aligned}\text{Var}(Y) &= \text{Var}(\mathbb{E}(Y|X)) + \mathbb{E}\left\{\text{Var}(Y|X)\right\} \\ &= \text{Var}(2X) + \mathbb{E}(4X) \\ &= 4\mu + 4\mu = 8\mu\end{aligned}$$

(b) We use the MGF method. Let Z denote $\{Y - \mathbb{E}(Y)\}/\sqrt{\text{Var}(Y)}$ 6pts

$$\begin{aligned}M_Z(t) &= \mathbb{E}\exp(tZ) \\ &= \mathbb{E}\left\{\exp(t\frac{Y-2\mu}{\sqrt{8\mu}})\right\} \\ &= \exp(-\frac{2\mu t}{\sqrt{8\mu}}) \cdot \mathbb{E}\left\{\exp(t\frac{Y}{\sqrt{8\mu}})\right\} \\ &= \exp(-\frac{\sqrt{2\mu}t}{2}) \cdot M_Y(\frac{t}{\sqrt{8\mu}})\end{aligned}$$

We need to calculate the MGF of Y.

$$\begin{aligned}M_Y(t) &= \mathbb{E}\exp(tY) \\ &= \mathbb{E}\left\{\mathbb{E}(\exp(tY)|X)\right\} \\ &= \sum_{x=0}^{\infty} e^{-\mu}\frac{\mu^x}{x!}(1-2t)^{-x} \\ &= e^{-\mu}\sum_{x=0}^{\infty}\frac{(\mu/(1-2t))^x}{x!} \\ &= \exp(-\mu + \frac{\mu}{1-2t}) \\ &= \exp(\mu \cdot \frac{2t}{1-2t})\end{aligned}$$

Substituting, we get

$$\begin{aligned}M_Z(t) &= \exp(-\frac{\sqrt{2\mu}t}{2}) \cdot M_Y(\frac{t}{\sqrt{8\mu}}) \\ &= \exp(-\frac{\sqrt{2\mu}t}{2}) \cdot \exp\left\{\mu \cdot \frac{2t/\sqrt{8\mu}}{1-2t/\sqrt{8\mu}}\right\} \\ &= \exp\left(t \cdot \left\{\frac{\sqrt{2\mu}/2}{1-t/\sqrt{2\mu}} - \frac{\sqrt{2\mu}}{2}\right\}\right) \\ &= \exp\left(t \cdot \left\{\frac{\sqrt{2\mu}/2 - \sqrt{2\mu}/2 + t/2}{1-t/\sqrt{2\mu}}\right\}\right)\end{aligned}$$

As $\mu \to \infty$, we get $\exp(t^2/2)$ so $Z \to_D N(0, 1)$.

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Problem 5. Suppose $X_1, \dots, X_n$ is an iid sample from Bern(p). Typically, the parameter of interest is p.

(a) Find the Method-of-moment estimator and the Maximum Likelihood (ML) estimator for p.

(b) Another popular parameter is called the *odds*, defined as $p/(1-p)$, find the ML estimator of the odds.

(c) Find the limiting distribution of $\sqrt{n}(\widehat{p}-p)$, where $\widehat{p}$ is the ML estimator for p.

(d) Find the limiting distribution of $\sqrt{n}(\widehat{\lambda}-p/(1-p))$, where $\widehat{\lambda}$ is the ML estimator for the odds.

Solution. (a) Let's first do method-of-moment. 2pts

- Step 1. $\mu_1 = \mathbb{E}(X_1) = p$
- Step 2. $p = \mu_1$
- Step 3. $\widehat{p} = \bar{X}_n$

Next let's do MLE. The likelihood function is 3pts

$$L(p) = \prod (p^{x_i}(1-p)^{1-x_i}) = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

The log-likelihood function is

$$l(p) = (\sum x_i)\log(p) + (n - \sum x_i)\log(1-p)$$

Taking the derivative:

$$l'(p) = (\sum x_i)\frac{1}{p} - (n - \sum x_i)\frac{1}{1-p} = (\sum x_i)\frac{1}{p(1-p)} - \frac{n}{1-p}$$

Setting this to 0, we get

$$(\sum x_i)\frac{1}{p} = n$$

and hence $\widehat{p} = \bar{X}_n$

(b) Let $\lambda = p/(1-p)$, by the invariance property, $\widehat{\lambda} = \bar{X}_n/(1-\bar{X}_n)$. 2pts

(c) By the CLT, 3pts

$$\sqrt{n}\frac{\bar{X}_n - p}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1)$$

So that

$$\sqrt{n}(\bar{X}_n - p) \xrightarrow{d} N(0, p(1-p))$$

(d) Use the Delta method with $g(x) = x/(1-x)$ we have 3pts

$$\sqrt{n}(g(\bar{X}_n) - g(p)) \xrightarrow{d} g'(p)N(0, p(1-p))$$

Simplifying, we get

$$\sqrt{n}(\widehat{\lambda} - \frac{p}{1-p}) \xrightarrow{d} \frac{1}{(1-p)^2}N(0, p(1-p)) \sim N(0, \frac{p}{(1-p)^3})$$

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Problem 6. Suppose X_n is a sequence of random variables that satisfies

$$\sqrt{n}(X_n - \theta) \xrightarrow{d} \text{Norm}(0, \sigma^2).$$

For a given function $g(\cdot)$ that $g'(\theta) = 0$ and the second derivative exists and $g''(\theta) \neq 0$. Find the limiting distribution of

$$n\left\{g(X_n) - g(\theta)\right\}.$$

[*Hint: Use the Taylor expansion*]

Solution. Taylor expand $g(X_n)$ around θ: 10pts

$$g(X_n) = g(\theta) + g'(\theta)(X_n - \theta) + \frac{g''(\theta)}{2}(X_n - \theta)^2 + \text{Remainder}$$

Since $g'(\theta) = 0$, multiplying n to both sides we get

$$n(g(X_n) - g(\theta)) = \frac{g''(\theta)}{2} n(X_n - \theta)^2 + \text{Remainder}$$

Now from the given, dividing by θ we get

$$\sqrt{n}\frac{X_n - \theta}{\sigma} \xrightarrow{d} \text{Norm}(0, 1)$$

So that by continuous mapping theorem with $(\cdot)^2$

$$n\frac{(X_n - \theta)^2}{\sigma^2} \xrightarrow{d} \chi_1^2$$

Hence, by continuous mapping theorem with the function being "multiplication by constant",

$$n(g(X_n) - g(\theta)) \approx \frac{g''(\theta)}{2} n(X_n - \theta)^2 \xrightarrow{d} \sigma^2 \frac{g''(\theta)}{2}\chi_1^2$$

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Problem 7. Suppose $X_1, \ldots, X_n$ is an iid sample from a distribution with $\mu = E(X)$ and the fourth moment of X exists. Find the limiting distribution of

$$\frac{\left(\bar{X}_n - \frac{\bar{X}_n^2}{\mu}\right)}{S_n/\sqrt{n}},$$

where S_n^2 is the sample variance.

Solution. Let μ and σ^2 be the mean and variance of X_i, we will first need to collect a few facts. 10pts

(1) $S_n \xrightarrow{p} \sigma$.

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{1}{n-1}\{(\sum_{i=1}^{n} X_i^2) - 2n\bar{X}_n^{\ 2} + n\bar{X}_n^{\ 2}\} = \frac{n}{n-1}((\bar{X^2})_n - \bar{X}_n^{\ 2})$$

Using the weak law of large numbers, which requires finite fourth moment of X, this converges in probability to $1 \cdot (\mathbb{E}[X^2] - \mathbb{E}[X]^2) = \sigma^2$.

The convergence of $(\bar{X}^2)_n$ requires finite fourth moment, the convergence of $\bar{X}_n^{\ 2}$ uses finite second moment (which is implied by finite 4th moment) and continuous mapping theorem. You can treat $n/(n-1)$ as a sequence of constant random variables and apply Slutsky's theorem.

(2) $\sqrt{n}(1/\bar{X}_n - 1/\mu) \xrightarrow{d} (1/\mu)^2\sigma N(0,1)$.

Using CLT, we have $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \sigma N(0,1)$. Apply Delta method with $g(x) = (1/x)$, with $g'(\mu) = 1/\mu^2$, we get the desired result.

Now we have

$$\begin{aligned}\frac{\bar{X}_n - \bar{X}_n^{\ 2}/\mu}{S_n/\sqrt{n}} &= \frac{\sqrt{n}(1/\bar{X}_n - 1/\mu)}{(1/\bar{X}_n)^2 S_n} \\ &= \frac{(1/\mu)^2}{(1/\bar{X}_n)^2} \cdot \frac{\sigma}{S_n} \cdot \frac{\sqrt{n}(1/\bar{X}_n - 1/\mu)}{(1/\mu)^2\sigma}\end{aligned}$$

Using continuous mapping theorem, the first term $\xrightarrow{p} 1$ because sample mean converges to real mean in probability, second term $\xrightarrow{p} 1$ by fact (1). The third term $\xrightarrow{d} N(0,1)$ by fact (2). Finally using Slutsky's theorem, we have

$$\frac{\bar{X}_n - \bar{X}_n^{\ 2}/\mu}{S_n/\sqrt{n}} \xrightarrow{d} N(0,1)$$

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Problem 8. Let the iid random sample $X_1, \ldots, X_n$ have a uniform density over $(\theta - 1/2, \theta + 1/2)$, where $\theta \in \mathbb{R}$. Find the ML estimator for θ.

Solution. The likelihood function is 10pts

$$L(\theta; x_1, \ldots, x_n) = \prod_{i=1}^{n} (f(x; \theta))$$

This is 1 if all of x_i satisfies $\theta - 1/2 \leq x_i \leq \theta + 1/2$ and 0 otherwise. Let y_1 and y_n be the smallest and largest observations respectively. Then any real number in the interval $[y_n - 1/2, y_1 + 1/2]$ is a MLE. For example $(y_1 + y_n)/2$. ♠

Problem 9. Let $X_1, \dots, X_n$ be iid Norm(μ, σ^2). The sample mean and sample variance are denoted by $\bar{X}_n$ and S_n^2, respectively.

(a) Find $E(\bar{X}_n)$ and $E(S_n^2)$.

(b) Find the MSE of S_n^2.

(c) For the ML estimator for σ^2 (available in the lecture notes), denoted by $\widehat{\sigma}_n^2$, find the MSE of $\widehat{\sigma}_n^2$.

(d) Does the sample variance S_n^2, as an estimator for σ^2, attain the Cramér-Rao Lower Bound?

Solution. (a) $\mathbb{E}(\bar{X}_n) = (\mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n))/n = \mu$. 5pts

$$
\begin{aligned}
\mathbb{E}(S_n^2) &= \mathbb{E}\Big(\frac{1}{n-1}\{\sum_{i=1}^n X_i^2 - 2X_i\bar{X}_n + \bar{X}_n{}^2\}\Big) \\
&= \frac{1}{n-1}\mathbb{E}\{(X_1^2 + \cdots + X_n^2) - \frac{1}{n}(X_1 + \cdots X_n)^2\} \\
&= \frac{1}{n-1}\mathbb{E}\{(1 - \frac{1}{n})\sum_{i=1}^n X_i^2 - \frac{1}{n}\sum_{i\neq j} X_i X_j\} \\
&= \frac{1}{n}\mathbb{E}\{(\sum_{i=1}^n X_i^2) - \frac{1}{(n-1)n}\sum_{i\neq j}\mathbb{E}(X_i X_j)\} \\
&= \mathbb{E}(X_1^2) - \mu^2 = \sigma^2
\end{aligned}
$$

(b) $(n-1)S_n^2/\sigma^2$ is distributed as χ^2_{n-1}, which has variance $2(n-1)$. So the variance of S_n^2 is $2\sigma^4/(n-1)$. This is precisely the MSE since by (a), the mean of S_n^2 is σ^2. 2pts

(c) From the course note, the MLE for σ^2 is 3pts

$$\frac{n-1}{n}S_n^2$$

The MSE can be decomposed as

$$\text{MSE} = \text{Var}\Big(\frac{n-1}{n}S_n^2\Big) + (\text{bias})^2 = \frac{2\sigma^4(n-1)}{n^2} + \frac{1}{n^2}\sigma^4 = \frac{\sigma^4(2n-1)}{n^2}$$

(d) The Cramér-Rao Lower Bound is $(\tau'(\theta))^2/J(\theta)$. In our case τ is the identity map so the lower bound is $1/J(\theta) = 1/(nJ_1(\theta))$. With $\theta = \sigma^2$: 4pts

$$
\begin{aligned}
J_1(\theta) &= \mathbb{E}(I(\theta)) = \mathbb{E}\Big(-\frac{\partial S(\theta)}{\partial \theta}\Big) = \mathbb{E}\Big(-\frac{\partial^2}{\partial \theta^2}l(\theta)\Big) \\
&= -\mathbb{E}\Big(\frac{\partial^2}{\partial \theta^2}\log f(X)\Big) \\
&= -\mathbb{E}\Big(\frac{\partial^2}{\partial \theta^2}\log\{(2\pi\theta)^{-1/2}\exp(-(1/2)(X-\mu)^2/\theta)\}\Big) \\
&= -\mathbb{E}\Big(\frac{\partial^2}{\partial \theta^2}\{-\frac{1}{2}\log(2\pi\theta) - \frac{1}{2}\frac{(X-\mu)^2}{\theta}\}\Big) \\
&= -\mathbb{E}\Big(\frac{1}{2\theta^2} - \frac{(X-\mu)^2}{\theta^3}\Big) \\
&= -\frac{1}{2\theta^2} + \frac{\sigma^2}{\theta^3} = \frac{1}{2\theta^2} = \frac{1}{2\sigma^4}
\end{aligned}
$$

Hence the bound $2\sigma^4/n$ is not attained, as $2\sigma^4/(n-1)$ is strictly larger.

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