STAT 330: Lecture 5

2024 Spring

May 28, 2024

Last Lecture

- MGF: Definition $M(t) = E(e^{tX})$
- MGF: Properties
  1. Find mgf after location-scale transformation.
  2. Use mgf to obtain moments.
  3. mgf & distribution one-to-one.
- Techniques for Integration/Summation
  1. change variable.
  2. "make" a pdf/pmf out of the integrand/summand.

Chapter 1.

Chapter 2.

This Lecture

1. Joint cdf
2. Joint pmf
3. Joint pdf

Joint Distributions

Purpose: We want to characterize the behaviors of two (or multiple) random variables simultaneously.

Joint CDF:

Definition X & Y are two rvs.

The joint cdf is given by.

$$F(x,y) = Pr(X \le x, Y \le y) = Pr(\{X \le x\} \cap \{Y \le y\})$$

Suppose we have $X_1, \cdots, X_n$, then their joint cdf is

$$F(x_1, \cdots, x_n) = Pr(X_1 \le x_1, \cdots, X_n \le x_n)$$

$\boxed{\text{In this chapter, we focus on the bivariate case}}$

Properties of Joint CDF

1) Fix x, $F(x,y)$ is a non-decreasing function of y

Fix y, $F(x,y)$ is a non-decreasing function of x

2). $\lim_{x \to -\infty} F(x,y) = 0 = \lim_{y \to -\infty} F(x,y)$

3) $\lim_{x,y \to \infty} F(x,y) = 1.$

4). $F(x,y) \Rightarrow \underline{F_X(x) \& F_Y(y)}$ marginal cdfs.

$F_X(x) = Pr(X \le x)$

$= \lim_{y \to \infty} F(x,y)$

$F_Y(y) = Pr(Y \le y)$

$= \lim_{x \to \infty} F(x,y)$

$F_X(x), F_Y(y) \not\Rightarrow F(x,y)$

Joint Discrete Random Variables

Definition: Joint pmf

X & Y are both discrete rvs.

Joint pmf is defined as. $f(x,y) = Pr(X=x, Y=y)$, $x, y \in \mathbb{R}$.

Support $A = \{(x,y) \mid f(x,y) > 0\}$

Properties:

(1) $f(x,y) \geq 0$ for $x, y \in \mathbb{R}$.

(2) $\sum_{(x,y) \in A} f(x,y) = 1$

(3) $Pr\{(X,Y) \in B\} = \sum_{(x,y) \in B} f(x,y)$

e.g. $Pr(X \leq Y)$, $B = \{(x,y) \mid x \leq y\}$

$= \sum_{(x,y) \in B} f(x,y)$

Marginal pmf:

$f(x,y)$ is a joint pmf.

Sample space.

marginal pmf of X : $f_X(x) = Pr(X=x) = Pr(X=x, Y<\infty)$

$= \sum_{y \in \mathbb{R}} f(x,y)$

marginal pmf of Y : $f_Y(y) = Pr(Y=y) = Pr(Y=y, X<\infty)$

$= \sum_{x \in \mathbb{R}} f(x,y)$

Example: X & Y are discrete rvs.

Joint pmf: $f(x,y) = \underbrace{kq^2 p^{x+y}}_{p^x \cdot p^y}$, $x, y \in \{0, 1, \dots\}$
$p \in (0,1)$
$q = 1-p$

① Find value of k.

$$\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} f(x,y) = 1 \iff k\Big(\sum_{x=0}^{\infty} p^x\Big)\Big(\sum_{y=0}^{\infty} p^y\Big) q^2 = 1$$

$\frac{1}{1-p}$ geometric series.

$$\Rightarrow k \cdot \frac{1}{1-p} \cdot \frac{1}{1-p} \cdot q^2 = 1 \Rightarrow k = 1$$

②. Find the marginal pmfs of X and Y.

$$f_X(x) = \sum_{y=0}^{\infty} f(x,y) = \sum_{y=0}^{\infty} q^2 p^x p^y = q^2 p^x \sum_{y=0}^{\infty} p^y = q^2 \frac{1}{1-p} \cdot p^x = q p^x$$

$f_Y(x) = q p^y$ $\qquad X, Y \sim Geo(q)$

③ Find $Pr(X \le Y)$

$B = \{(x,y) \mid x \le y\}$

$$Pr(X \le Y) = \sum_{(x,y) \in B} f(x,y) = \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} q^2 p^x p^y = \sum_{x=0}^{\infty} p^x q^2 \sum_{y=x}^{\infty} p^y$$

$$= \sum_{x=0}^{\infty} p^x q^2 \cdot p^x \sum_{y=0}^{\infty} p^y = \sum_{x=0}^{\infty} q p^{2x} = q \sum_{x=0}^{\infty} p^{2x}$$

$$= q \cdot \frac{1}{1-p^2} = \frac{1}{1+p}$$

$p^x + p^{x+1} + p^{x+2} + \cdots$

$p^x(\underline{p^0 + p + \cdots})$

$p^x \cdot \frac{1}{1-p}$

Joint Continuous Random Variables

Definition: Suppose X, Y are continuous rvs.
If their joint cdf can be written as

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\,ds\,dt$$

for some $f(\cdot,\cdot)$. Then $f(x,y)$ is the joint pdf.

$$f(x,y) = \begin{cases} \dfrac{\partial^2 F(x,y)}{\partial x \partial y} & \text{if the derivative exists.} \\ 0 & \text{otherwise.} \end{cases}$$

Support. $A = \{(x,y) \mid f(x,y) > 0\}$

Properties:

(1) $f(x,y) \geq 0$

(2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

(3) $Pr((X,Y) \in B) = \iint_{(x,y)\in B} f(x,y)\,dx\,dy$.

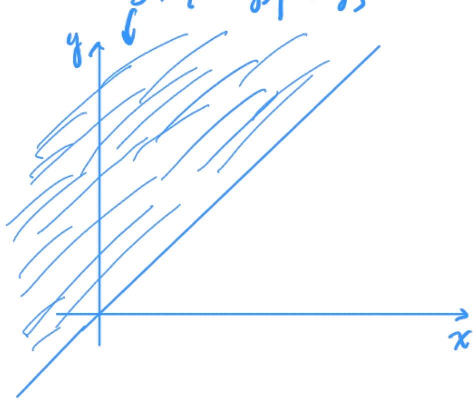
(4). From joint to marginal.

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$$

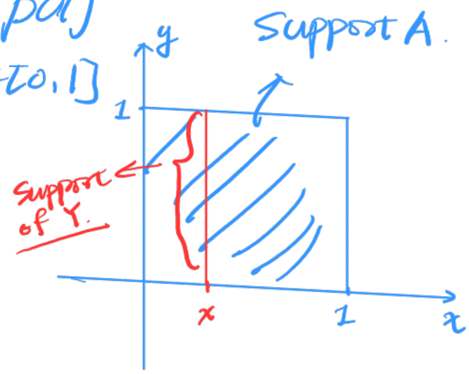
E.g. $Pr(X \leq Y)$

$B: \{(x,y) \mid x \leq y\}$



Example 1: Suppose X, Y with joint pdf

$$f(x,y)=\begin{cases} x+y, & x\in[0,1],\ y\in[0,1] \\ 0, & \text{otherwise.}\end{cases}$$



① Verify that $f(x,y)$ is a joint pdf.

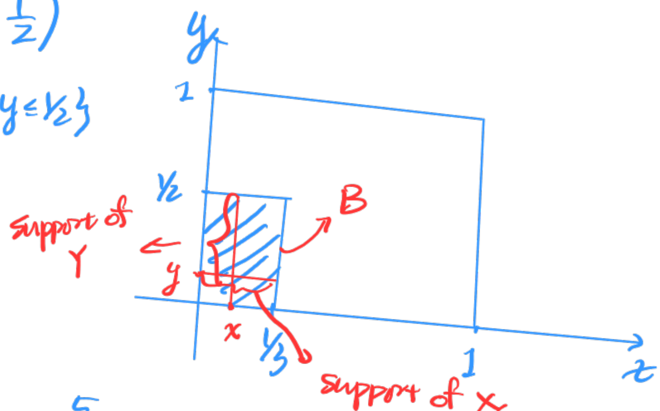
$f(x,y)\ge 0$, $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy$

$$=\int_0^1\int_0^1 (x+y)\,dx\,dy=\int_0^1\left\{\left(yx+\frac{x^2}{2}\right)\Big|_0^1\right\}dy$$

$$=\int_0^1\left(y+\frac{1}{2}\right)dy=1$$

② $\Pr(X\le 1/3,\ Y\le \frac{1}{2})$

$B=\{(x,y)\mid x\le 1/3,\ y\le 1/2\}$



$\Pr(X\le 1/3, Y\le 1/2)$

$$=\iint_{(x,y)\in B} f(x,y)\,dx\,dy$$

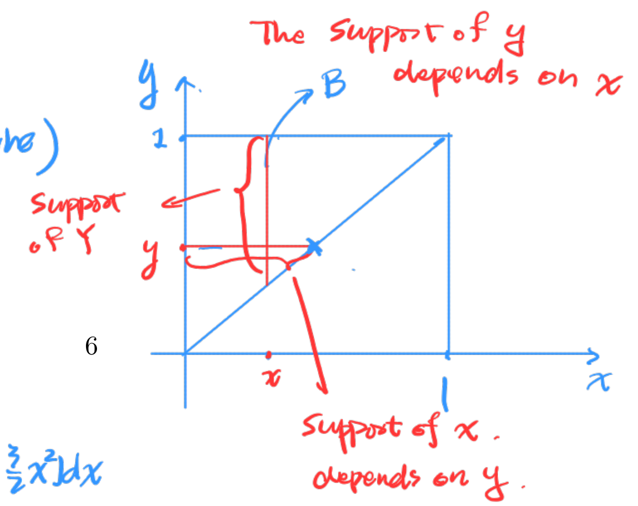
$$=\int_0^{1/3}\int_0^{1/2}(x+y)\,dy\,dx=\frac{5}{72}$$

OR

$$=\int_0^{1/2}\int_0^{1/3}(x+y)\,dx\,dy=\frac{5}{72}.$$

③ $\Pr(X\le Y)=1/2$ (Conjecture)

$B=\{(x,y)\mid x\le y\}$



$$\Pr(X\le Y)=\int_0^1\int_0^y (x+y)\,dx\,dy=\int_0^1 \frac{3}{2}y^2\,dy=1/2$$

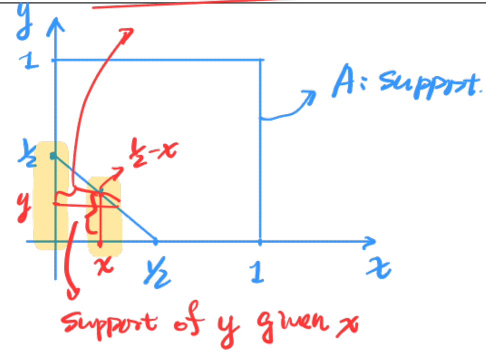
$$\Pr(X\le Y)=\int_0^1\int_x^1 (x+y)\,dy\,dx=\int_0^1\left(x+\frac{1}{2}-\frac{3}{2}x^2\right)dx=\frac{1}{2}$$

Support of x given y. $(0, \frac{1}{2}-y)$

Cont'

④. $Pr(X+Y \le 1/2)$.

$B = \{(x,y) \mid x+y \le 1/2\}$



$Pr(X+Y \le \frac{1}{2})$

$$= \iint_{(x,y)\in B} f(x,y)\,dx\,dy = \int_0^{1/2}\int_0^{1/2-x} (x+y)\,dy\,dx = \int_0^{1/2} \left(\frac{1}{8} - \frac{1}{2}x^2\right)dx$$

$$= \frac{1}{24}$$

OR.

$$= \int_0^{1/2}\int_0^{1/2-y} (x+y)\,dx\,dy = \int_0^{1/2} \left\{ \frac{1}{2}\left(\frac{1}{2}-y\right)^2 + \left(\frac{1}{2}-y\right)y \right\} dy$$

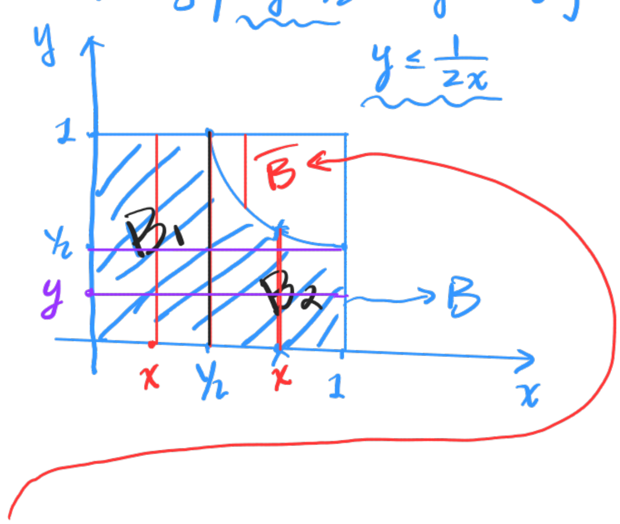
$$= \frac{1}{24}$$

Choosing $\iint$ dxdy or $\iint$ dydx doesn't seem to make a big difference in the previous examples. But in some cases, one is preferred over the other.

⑤. $Pr(XY \le 1/2)$

$B = \{(x,y) \mid xy \le 1/2,\ x,y \in [0,1]\}$

$y \le \frac{1}{2x}$



Method 1: $\displaystyle\iint_{(x,y)\in B} (x+y)\,dx\,dy =$

$B = B_1 \cup B_2$. $B_1 = \{(x,y) \mid x \in [0, 1/2],\ y \in [0,1]\}$

$B_2 = \{(x,y) \mid x \in [1/2, 1],\ y \in [0, \frac{1}{2x}]\}$

$$= \iint_{(x,y)\in B_1} (x+y)\,dx\,dy + \iint_{(x,y)\in B_2} (x+y)\,dx\,dy$$

$$= \int_0^{1/2}\int_0^1 (x+y)\,dy\,dx + \int_{1/2}^1 \int_0^{\frac{1}{2x}} (x+y)\,dy\,dx$$

$$= 3/4$$

Method 2.

$Pr(XY \le 1/2) = 1 - \underline{Pr(XY \ge 1/2)}$

$\bar{B} = \{(x,y) \mid xy \ge 1/2,\ x,y \in [0,1]\}$

$= \{(x,y) \mid x \in [1/2, 1],\ y \in [\frac{1}{2x}, 1]\}$

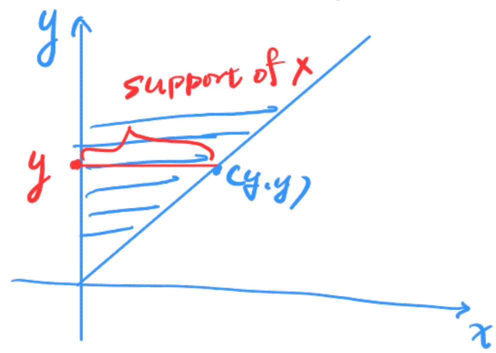
$$Pr(XY \ge \tfrac{1}{2}) = \iint_{(x,y)\in \bar{B}} (x+y)\,dx\,dy = \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2x}}^{1} (x+y)\,dy\,dx = 1/4$$

Example 2:

$$f(x,y) = \begin{cases} k e^{-x-y} & 0<x<y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

①. Find k.

$$1 = \iint_{(x,y)\in A} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{y} k e^{-x-y}\,dx\,dy \Rightarrow k=2$$

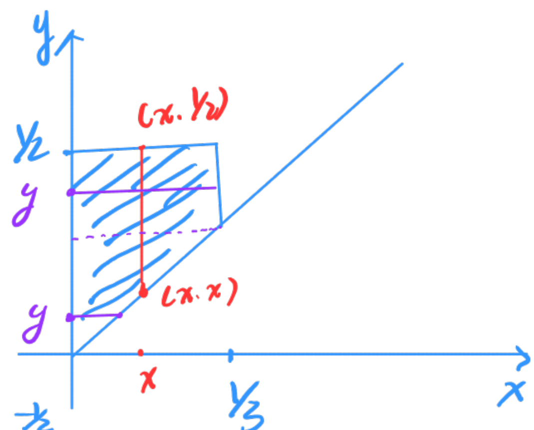


②. $Pr(X \le 1/3,\ Y \le 1/2)$

$B = \{(x,y) \mid (x,y)\in A,\ x \le 1/3,\ y \le 1/2\}$

$$\int_0^{1/3}\int_x^{1/2} 2e^{-x-y}\,dy\,dx$$

$$= \int_0^{1/3} 2e^{-x}\left\{\int_x^{1/2} e^{-y}\,dy\right\} dx = 1 - e^{-\frac{2}{3}} + 2e^{-5/6} - 2e^{-1/2}$$



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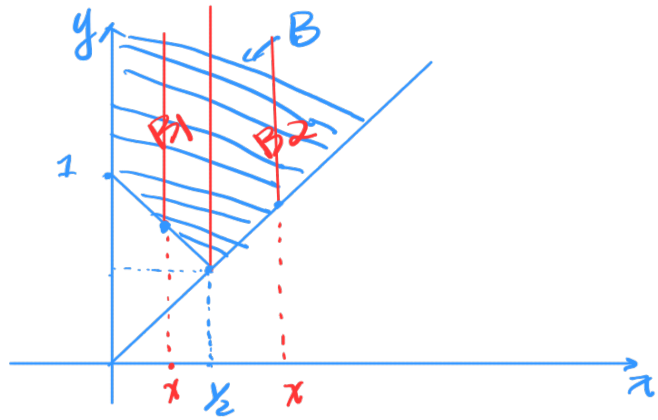
③. $\Pr(X+Y \geq 1)$

$B = \{(x,y) \mid (x,y) \in A,\ \underline{x+y \geq 1}\}$

$\boxed{y \geq 1-x}$



Method 1: fix x first.

$B = B_1 \cup B_2$.

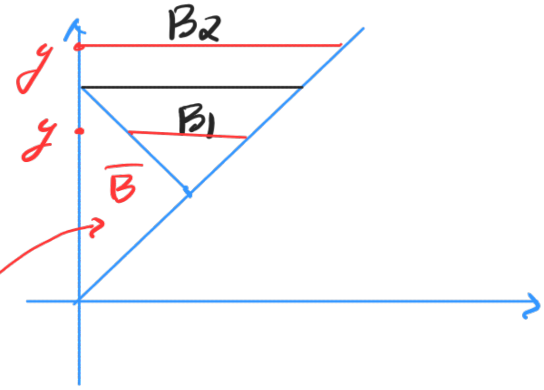
$B_1 = \{(x,y) \mid (x,y) \in A,\ x \in [0, \tfrac{1}{2}],\ y \in [1-x, \infty)\}$

$B_2 = \{(x,y) \mid (x,y) \in A,\ x \in [\tfrac{1}{2}, \infty),\ y \in [x, \infty)\}$

$$\iint_{(x,y) \in B} 2e^{-x-y}\,dy\,dx = \iint_{(x,y) \in B_1} (\cdots)\,dy\,dx + \iint_{(x,y) \in B_2} (\cdots)\,dy\,dx = 2e^{-1}$$

Method 2: fix y first.

skipped.

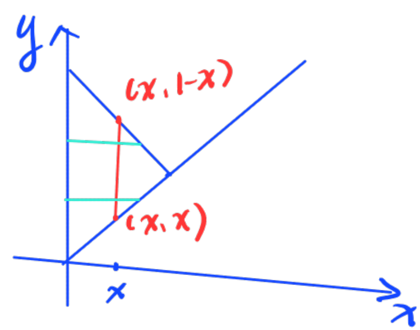


Method 3:

$\Pr(X+Y \geq 1) = 1 - \underline{\Pr(X+Y \leq 1)}$

$\overline{B}$

Fix x.



$\overline{B} = \{(x,y) \mid x \in [0, \tfrac{1}{2}],\ y \in [x, 1-x]\}$

$$\Pr(X+Y \leq 1) = \int_0^{1/2} \int_x^{1-x} (\cdots)\,dy\,dx$$

$$= 2e^{-1}$$