University of Waterloo

STAT330, Spring 2024

Tutorial 5

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Problem 1. Consider a two-dimensional array of numbers for x, y positive integers

$$A_{x,y} = \sum_{t=0}^{x} \frac{y^t}{t!} e^{-y}$$

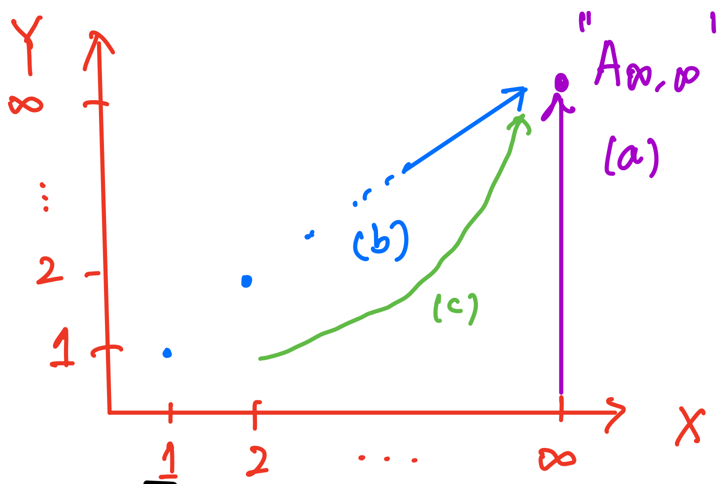
Compute the following limits.

(a) $\lim_{y\to\infty} \lim_{x\to\infty} A_{x,y}$

(b) $\lim_{s\to\infty} A_{s,s}$

(c) $\lim_{s\to\infty} A_{s+\lfloor k\sqrt{s}\rfloor,s}$

where $k \geq 0$. Hint: CLT



(a) $\lim_{y\to\infty} \left[\lim_{x\to\infty} \sum_{t=0}^{x} \frac{y^t}{t!} e^{-y} \right]$

$= \lim_{y\to\infty} e^{y} \cdot e^{-y} = \lim_{y\to\infty} 1 = 1$

(b) $\sum_{t=0}^{x} \frac{y^t}{t!} e^{-y} = P(X_y \leq x)$ where $X_y \sim Poi(y)$

$Z_1, Z_2 \ldots \overset{iid}{\sim} Poi(1)$.

$X_n \overset{d}{=} \sum_{i=1}^{n} Z_i$

CLT: $\frac{\sqrt{n}(\bar{Z}_n - \mu)}{\sigma} \xrightarrow{D} N(0,1)$

$A_{s,s} = P(X_s - s \leq 0)$

$= P\left(\frac{X_s - s}{\sqrt{s}} \leq \frac{0}{\sqrt{s}}\right)$

$$= \frac{1}{2}$$

(c) $A_{S+\lfloor K\sqrt{S}\rfloor, S} = P\left(X_S \le S + \lfloor K\sqrt{S}\rfloor\right)$

$$= P\Big(\underbrace{\frac{X_S - S}{\sqrt{S}}}_{\to_D N(0,1)} \le \underbrace{\frac{\lfloor K\sqrt{S}\rfloor}{\sqrt{S}}}_{\to K.} \Big)$$

$$\underset{S\to\infty}{\to} \Phi(K).$$

$$\frac{(X_n - n\cdot\mu)}{\sqrt{n}\,\sigma} \to_D N(0,1)$$

$$\|$$

$$\frac{(X_n - n)}{\sqrt{n}} \to_D N(0,1)$$

So anything in $[\frac{1}{2}, 1]$ are possible...

Problem 2. Let $X_n \sim \text{BIN}(n, \theta)$. Find the limiting distribution of

(a) $U_n = \underbrace{\frac{X_n}{n}}_{\to_p \theta}\left(1 - \underbrace{\frac{X_n}{n}}_{\to_p \theta}\right)$

(b) $W_n = \sqrt{n}\left(\frac{X_n}{n} - \theta\right)$

(c) $Z_n = \frac{W_n}{\sqrt{U_n}}$

(a) $BIN(n, \theta) \sim$ iid sum of $Ber(\theta)$.

$Y_n \overset{iid}{\sim} Ber(\theta)$. $X_n = \sum_{i=1}^n Y_i$.

$$\frac{X_n}{n} = \frac{\sum_{i=1}^n Y_i}{n} \xrightarrow[\sigma^2(Y_i) = \theta \cdot (1-\theta)]{WLLN}_p \theta$$

Slutsky's thm: $A_n \to_p a$ $B_n \to_D B$ g continuous.

$$g(A_n, B_n) \to_D g(a, B).$$

$$g(x, y) = x \cdot (1 - y)$$

$$A_n \leftarrow \frac{X_n}{n} \quad B_n \leftarrow \frac{X_n}{n}$$

$$a = \theta \quad B = \theta.$$

$$U_n \to_D \theta(1-\theta) \Rightarrow U_n \to_p \theta(1-\theta).$$

(b) $W_n = \sqrt{n}\left(\frac{X_n}{n} - \theta\right)$

$$\frac{\sqrt{n}\left(\frac{X_n}{n} - \theta\right)}{\underbrace{\sqrt{\theta(1-\theta)}}} \to_D Z \sim N(0,1)$$

$$\Rightarrow \sqrt{n}\left(\overset{Y_n}{\frac{X_n}{n}} - \theta\right) \to_D \underbrace{\sqrt{\theta(1-\theta)}}_{X_n} \cdot Z \sim N(0, \theta(1-\theta)).$$

$\left(g(x,y) = x \cdot y \right)$ The other continuity thms are special cases of slutsky's thm

(c) $Z_n = \frac{W_n}{\sqrt{U_n}}$ $\begin{matrix} \to N(0, \theta(1-\theta)) \\ \hline \to \sqrt{\theta(1-\theta)} \end{matrix}$ $\sim N(0,1)$

$$\left.\begin{aligned} X_n &= U_n \\ Y_n &= W_n \\ g(x,y) &= \frac{y}{\sqrt{x}}. \end{aligned}\right\} \text{Apply slutsky's thm.}$$

$\mu = 2\theta$

$\sigma^2 = 2\theta^2$.

Problem 3. Suppose $X_1, \ldots, X_n$ is a random sample from Gamma$(2, \theta)$. Find the limiting distribution of

(a) $\bar{X}_n = \frac{1}{n} \sum X_i$

(a) By WLLN, $\bar{X}_n \to_p 2\theta$

(b) $V_n = \sqrt{n}(\bar{X}_n - \underset{\mu}{2\theta})$

(b) $\dfrac{\sqrt{n}(\bar{x}_n - 2\theta)}{\sigma} \xrightarrow{CLT}_p N \sim N(0,1)$.

$V_n \to_D \sigma \cdot N. \sim N(0, \sigma^2) = N(0, 2\theta^2)$

(c) $Z_n = \dfrac{\sqrt{n}(\bar{X}_n - 2\theta)}{\bar{X}_n/\sqrt{2}}$

(d) $U_n = \sqrt{n}[\log(\bar{X}_n) - \log(2\theta)]$

(c) $\bar{x}_n \to_p 2\theta$. $\bar{x}_n/\sqrt{2} \to_p \sqrt{2}\theta$.

$\underset{\sim}{V_n} \to_D N(0, \sigma^2)$

$\bar{x}_n/\sqrt{2} \to_p \sigma$ $\quad \xrightarrow[\text{Slutsky's thm}]{g(x,y) = (\frac{y}{x})}$ $Z_n \to_D N(0,1)$

(d) Delta Method

if $n^b(X_n - a) \to_D X$. (b) $N(0, 2\theta^2)$

Then $n^b(g(X_n) - g(a)) \to_D g'(a) \cdot X$.

$b = \frac{1}{2}$, $g = \log$, $X_n = \bar{x}_n$, $a = 2\theta$. $\quad g'(a) = \frac{1}{2\theta}$

$U_n \to_D \dfrac{N(0, 2\theta^2)}{2\theta} \overset{D}{=} N(0, \frac{1}{2})$ By Delta Method.