University of Waterloo

Department of Statistics and Actuarial Science

STAT330 : Homework 3 Solutions

Teaching Assisstant: Muqiao (James) Huang

August 4, 2024

[Notes for Graders]
[Extra Explanations]
[Problems and Solutions]

Problem 1. Suppose $(X_1, \ldots, X_k)$ has a multinomial distribution $\text{Mult}(n, p_1, \ldots, p_k)$.

(a) [6pts] Verify the moment generating function for $(X_1, \ldots, X_k)$ given in the lecture note.

(b) [6pts] Verify that $X_i | X_j + X_i = t \sim \text{Bin}\left(t, \frac{p_i}{p_i + p_j}\right)$.

(c) [6pts] Find the conditional distribution of $(X_1, \ldots, X_t)$ given $(X_{t+1}, \ldots, X_k)$, where $1 < t < k$. For the convenience of notation, you can denote $s = x_{t+1} + \cdots + x_k$ in your derivations.

Solution. Let $X_{k+1} = n - (X_1 + \cdots + X_k)$ and $p_{k+1} = 1 - (p_1 + \cdots + p_k)$.

(a) We calculate the mgf directly.

$$\begin{aligned}
M_{X_1, \cdots, X_{k+1}}(t_1, \cdots, t_{k+1}) &= \mathbb{E}[\exp(\mathbf{t}^T \mathbf{X})] \\
&= \sum_{x_1 + \cdots x_{k+1} = n} \exp(\mathbf{t}^T \mathbf{x}) \binom{n}{x_1, \cdots, x_{k+1}} p_1^{x_1} \cdots p_{k+1}^{x_{k+1}} \\
&= \sum_{x_1 + \cdots x_{k+1} = n} \binom{n}{x_1, \cdots, x_{k+1}} e^{t_1 x_1} p_1^{x_1} \cdots e^{t_{k+1} x_{k+1}} p_{k+1}^{x_{k+1}} \\
&= \sum_{x_1 + \cdots x_{k+1} = n} \binom{n}{x_1, \cdots, x_{k+1}} (e^{t_1} p_1)^{x_1} \cdots (e^{t_{k+1}} p_{k+1})^{x_{k+1}} \\
&= (e^{t_1} p_1 + \cdots e^{t_{k+1}} p_{k+1})^n
\end{aligned}$$

To find the marginal of $X_1, \cdots, X_k$, simply set $t_{k+1} = 0$ and we get $(e^{t_1} p_1 + \cdots e^{t_k} p_k + p_{k+1})^n$

(b) First calculate the distribution of $X_i + X_j$.

$$\begin{aligned}
M_{X_i + X_j}(t) &= \mathbb{E}[\exp((X_i + X_j)t)] \\
&= \mathbb{E}[\exp(0 \cdot X_1 + \cdots + t \cdot X_i + \cdots + t \cdot X_j + \ldots 0 \cdot + X_{k+1}] \\
&= M_{X_1, \ldots, M_{k+1}}(t_1 = 0, \ldots, t_i = t, \ldots, t_j = t, \ldots t_{k+1} = 0) \\
&= (e^t (p_i + p_j) + (1 - p_i - p_j))^n
\end{aligned}$$

So $X_i + X_j$ is distributed as $\text{Bin}(n, p_i + p_j)$.

Next we find the marginal distribution of X_i, X_j:

$$\begin{aligned} M_{X_i,X_j}(t_i,t_j) &= M_{X_1,\ldots,X_{k+1}}(t_1=0,\ldots,t_i,\ldots,t_j,\ldots,t_{k+1}=0) \\ &= (e^{t_i}p_i + e^{t_j}p_j + (1-p_i-p_j))^n \end{aligned}$$

So $(X_i, X_j) \sim$ multinomial$(n; p_i, p_j)$

We are ready to find the conditional probability $\mathbb{P}(X_i = x | X_i + X_j = t)$ for $0 \le x \le t$:

$$\begin{aligned} &\mathbb{P}(X = x | X_i + X_j = t) \\ &= \frac{\mathbb{P}(X_i = x, X_i + X_j = t)}{\mathbb{P}(X_i + X_j = t)} \\ &= \frac{\mathbb{P}(X_i = x, X_j = t - x)}{\mathbb{P}(X_i + X_j = t)} \\ &= \left\{ \binom{n}{x, t-x, n-t} {p_i}^x (p_j)^{t-x} (1-p_i-p_j)^{n-t} \right\} \div \left\{ \binom{n}{t} (p_i + p_j)^t (1-p_i-p_j)^{n-t} \right\} \\ &= \frac{n!}{x!(t-x)!(n-t)!} \div \frac{n!}{t!(n-t)!} \times \frac{{p_i}^x p_j^{t-x} (1-p_i-p_j)^{n-t}}{(p_i+p_j)^t (1-p_i-p_j)^{n-t}} \\ &= \frac{t!}{x!(t-x)!} \times \frac{p_i^x}{(p_i+p_j)^x} \frac{p_j^{t-x}}{(p_i+p_j)^{t-x}} \\ &= \binom{t}{x} (\frac{p_i}{p_i+p_j})^x (\frac{p_j}{p_i+p_j})^{t-x} \end{aligned}$$

Hence the conditional probability for $X_i = x$ is precisely the mass of Bin$(t, p_i/(p_i + p_j))$ with x successes.

(c) Similar to previous part, denote $q = p_{t+1} + \cdots + p_k$.

$$(X_1, \ldots, X_t, X_{k+1}) \sim \text{Multinomial}(n, p_1, \ldots, p_t, p_{k+1})$$

and

$$T := (X_1 + \cdots + X_t + X_{k+1}) \sim \text{Bin}(n, p_1 + \cdots + p_t + p_{k+1}).$$

The event $X_{t+1} + \cdots + X_k = s$ is the same as $T = n - s$. So the conditional probability is

$$\begin{aligned} &\mathbb{P}(X_1 = x_1, \ldots, X_t = x_t, X_{k+1} = x_{k+1} | T = n - s) \\ &= \left\{ \binom{n}{x_1, \ldots, x_t, x_{k+1}, s} p_1^{x_1} \cdots p_t^{x_t} p_{k+1}^{x_{k+1}} q^s \right\} \div \left\{ \binom{n}{n-s} (1-q)^{n-s} q^s \right\} \\ &= \frac{n!}{x_1! \ldots x_t! x_{k+1}! s!} \div \frac{n!}{(n-s)!s!} \frac{p_1^{x_1} \ldots p_t^{x_t} p_{k+1}^{x_{k+1}} q^s}{(1-q)^{x_1} \ldots (1-q)^{x_t} (1-q)^{x_{k+1}} q^s} \\ &= \binom{n-s}{x_1, \ldots, x_t, x_{k+1}} (\frac{p_1}{1-q})^{x_1} \cdots (\frac{p_t}{1-q})^{x_t} (\frac{p_{k+1}}{1-q})^{x_{k+1}} \end{aligned}$$

Which is precisely the probability mass function of

$$\text{Multinomial}(n - s; \frac{p_1}{1-q}, \ldots, \frac{p_t}{1-q})$$

♠

Problem 2. Three coins are tossed at the same time for n times.

(a) [4pts] Find the joint distribution of X, the number of times no heads appear; Y, the number of times one head appears; and Z, the number of times two heads appear.

(b) [4pts] Find the conditional density of X and Z given Y.

Solution. (a) Suppose the coins are fair and independent. Each time 3 coins are tossed, there is a 1/8 chance of no heads, 3/8 chance of 1 head and 3/8 chance of 2 heads. Hence

$$(X, Y, Z) \sim \text{Multinomial}(n, \frac{1}{8}, \frac{3}{8}, \frac{3}{8})$$

(b) We easily see that

$$(X, Z, Y) \sim \text{Multinomial}(n, \frac{1}{8}, \frac{3}{8}, \frac{3}{8})$$

Apply (c) of Problem 1, we have

$$(X, Z|Y = y) \sim \text{Multinomial}(n - y, \frac{1}{5}, \frac{3}{5})$$

♠

Problem 3. Suppose we have two random variables $Z_1, Z_2 \sim \mathrm{Norm}(0,1)$ and $Z_1 \in \mathbb{R}$ is independent of $Z_2 \in \mathbb{R}$. Find the distribution of the following random variables (or vectors).

(a) [4pts] $(Z_1 - Z_2)^2/2$

(b) [4pts] $\begin{bmatrix} Z_1 + Z_2 \\ Z_1 - Z_2 \end{bmatrix}$

(c) [4pts] $(Z_1 + Z_2)/|Z_1 - Z_2|$

Solution. (a) $Z_1 - Z_2 \sim \mathrm{N}(0, \sqrt{2}^2)$ so that $(Z_1 - Z_2)/\sqrt{2} \sim \mathrm{N}(0,1)$.

Hence its square is distributed as χ_1^2.

(b)

$$\begin{bmatrix} Z_1 + Z_2 \\ Z_1 - Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

By Tutorial 3 Problem 2, we have multivariate normal with $\mu = 0$ and

$$\Sigma = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

(c)

$$\frac{Z_1 + Z_2}{|Z_1 - Z_2|} = \frac{Z_1 + Z_2}{\sqrt{(Z_1 - Z_2)^2}} = \frac{(Z_1 + Z_2)/\sqrt{2}}{\sqrt{(Z_1 - Z_2)^2/2}}$$

- $(Z_1 + Z_2)/\sqrt{2} \sim \mathrm{N}(0,1)$.
- $(Z_1 - Z_2)^2/2 \sim \chi_1^2$ by (a).
- $(Z_1 + Z_2)/\sqrt{2} \perp (Z_1 - Z_2)^2/2$

$Z_1 + Z_2 \perp Z_1 - Z_2$ by (b) and the fact that for bivariate normal, uncorrelated implies independence. Then we use the fact that $X \perp Y \implies f(X) \perp g(Y)$ for any maps f, g.

The above 3 facts implies that we have a t_1 distribution.

Alternatively, one could directly calculate the pdf and get $f(t) = 1/\{\pi(t^2 + 1)\}$ and conclude that it is a standard Cauchy distribution. Which is in fact the same as a t_1 distribution.

♠

Problem 4. [8pts] Z_1 and Z_2 are the same as Q3; it can be shown that their product $Z_1 Z_2$ has the same distribution as the sum of two independent scaled Chi-square random variables. That is,

$$Z_1 Z_2 \overset{d}{=} aY_1 + bY_2,$$

where $Y_1, Y_2 \sim \chi_1^2$ and Y_1 is independent of Y_2. Suppose constants $a > 0$ and $b < 0$, find out the values of a and b. Here "$\overset{d}{=}$" means the two sides have the same distribution.

Solution.

$$\begin{aligned} Z_1 Z_2 &= \frac{1}{4}((Z_1 + Z_2)^2 - (Z_1 - Z_2)^2) \\ &= \frac{1}{2}\frac{(Z_1 + Z_2)^2}{2} - \frac{1}{2}\frac{(Z_1 - Z_2)^2}{2} \\ &= \frac{1}{2}\left(\frac{Z_1 + Z_2}{\sqrt{2}}\right)^2 - \frac{1}{2}\left(\frac{Z_1 - Z_2}{\sqrt{2}}\right)^2 \end{aligned}$$

$(Z_1 + Z_2)/\sqrt{2} \sim \mathrm{N}(0, 1)$ and $(Z_1 - Z_2)/\sqrt{2} \sim \mathrm{N}(0, 1)$ so we have $a = 1/2, b = -1/2$. Independence is the same as in problem 3 part (c). ♠

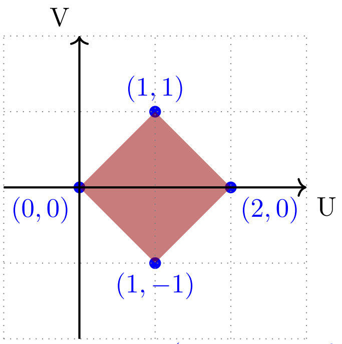
Problem 5. [8pts] Suppose X and Y are independent Uniform$(0,1)$. Define $U = X+Y$ and $V = X-Y$. Find the joint pdf of U and V. Find the marginal pdf of U and the marginal pdf of V.

Solution. The Jacobian matrix of the map $(X,Y) \to (U,V)$ is: [2pts]

$$\begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So we have $|\det(J)| = 2$.
Next we will need to calculate the inverse transformation: [2pts]

$$(X,Y) = (\frac{1}{2}(U+V), \frac{1}{2}(U-V))$$



Hence we have [2pts]

$$f_{U,V}(u,v) = \frac{f_{X,Y}(x,y)}{2} = \frac{1}{2} f_{X,Y}(\frac{1}{2}(u+v), \frac{1}{2}(u-v)) = \frac{1}{2}$$

Provided the arguments of $f_{X,Y}$ are in the rectangle $[0,1] \times [0,1]$. To figure out this region, consider the map $f : (X,Y) \to (U,V)$. We have $f(0,0) = (0,0), f(0,1) = (1,-1), f(1,0) = (1,1)$ and $f(1,1) = (2,0)$. Since f is linear, it must map the rectangle enclosed by $(0,0), (0,1), (1,1), (1,0)$ to the region enclosed by $(0,0), (1,-1), (2,0)$ and $(1,1)$.
By inspection (We have uniform density, so we are calculating the length multipled by some factor.)

$$f_U(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \end{cases}$$

and [2pts]

$$f_V(x) = \begin{cases} 1-x, & 0 < x < 1 \\ 1+x, & -1 < x < 0 \end{cases}$$

In this solution, we calculated the determinant of the map $(X,Y) \to (U,V)$, and we needed to divide by $|\det J|$. In the following problems we calculate directly the determinant of the backward transformation and we will be multiplying $|\det J|$. ♠

Problem 6. [10pts] Suppose $X_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $X_2 \sim \text{Gamms}(\alpha_2, \beta)$ are independent

$$f_{X_i}(x) = \frac{1}{\Gamma(\alpha_i)\beta^{\alpha_i}} x^{\alpha_i - 1} \exp\left(-\frac{x}{\beta}\right), \quad x > 0.$$

Find the joint pdf of (Y_1, Y_2), which is given by

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_1/(X_1 + X_2) \end{cases}$$

Are Y_1 and Y_2 independent? What is the distribution of Y_1?

Solution. Playing around with the quantities, we find the inverse transformation is [2pts]

$$(X_1, X_2) = (Y_1 \cdot Y_2, Y_1 - Y_1 \cdot Y_2)$$

The Jacobian Matrix J of the map $(Y_1, Y_2) \to (X_1, X_2)$ is

$$\begin{bmatrix} Y_2 & Y_1 \\ 1 - Y_2 & -Y_1 \end{bmatrix}$$

The determinant $\det J$ is [2pts]

$$-Y_1 \cdot Y_2 - Y_1 + Y_1 \cdot Y_2 = -Y_1$$

Clearly Y_1 is positive, so $|\det J| = y_1$ at $Y_1 = y_1$. Hence we have [2pts]

$$f_{Y_1,Y_2}(y_1, y_2) = y_1 \cdot f_{X_1,X_2}(y_1 \cdot y_2, y_1 - y_1 \cdot y_2)$$

By independence of $X_1 \perp X_2$, we have [4pts]

$$\begin{aligned}
&f_{Y_1,Y_2}(y_1, y_2) \\
&= y_1 \cdot \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} (y_1 \cdot y_2)^{\alpha_1 - 1} \exp\left(-\frac{y_1 \cdot y_2}{\beta}\right) \cdot \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} (y_1 - y_1 \cdot y_2)^{\alpha_2 - 1} \exp\left(-\frac{y_1 - y_1 \cdot y_2}{\beta}\right) \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1}\beta^{\alpha_2}} y_1 \cdot y_1^{(\alpha_1 - 1) + (\alpha_2 - 1)} \cdot \exp\left(-\frac{y_1}{\beta}\right) \cdot y_2^{\alpha_1 - 1} \cdot (1 - y_2)^{\alpha_2 - 1} \\
&= \frac{y_1^{\alpha_1 + \alpha_2 - 1} \exp\left(-\frac{y_1}{\beta}\right)}{\Gamma(\alpha_1 + \alpha_2)} \beta^{\alpha_1 + \alpha_2} \times y_2^{\alpha_1 - 1} (1 - y_2)^{\alpha_2 - 1} \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}
\end{aligned}$$

Where we multiplied $\Gamma(\alpha_1 + \alpha_2)$ on top and bottom. The first term is the pdf of $\text{Gamma}(\alpha_1 + \alpha_2, \beta)$ and the second term is the pdf of $\text{Beta}(\alpha_1, \alpha_2)$.
Since the support of Y_1, Y_2 is the rectangular region $[0, \infty] \times [0, 1]$, they are independent and $Y_1 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$. ♠

Problem 7. For independent $X_1, \dots, X_n$

(a) [4pts] If $X_i \sim \text{Bin}(n_i, p)$, find the distribution of $S = \sum_{i=1}^n X_i$.

(b) [4pts] If $X_i \sim \text{Gamma}(\alpha_i, \beta)$, find the distribution of $S = \sum_{i=1}^n X_i$.

(c) [4pts] If $X_i \sim \text{Poi}(\lambda_i)$, find the distribution of $S = \sum_{i=1}^n X_i$.

Prove using mgf. You can use the mgfs of the above distributions directly without proving them.

Solution. We use the fact that summing independent random variables correspond to multiplying their mgfs.

(a) Look up binomial mgf, we have

$$M_{X_i}(t) = (q + pe^t)^{n_i}$$

where $q = 1 - p$. Multiplying, we have

$$M_S(t) = (q + pe^t)^{n_1 + \cdots + n_n}$$

Hence $S \sim \text{Bin}(n_1 + \cdots + n_n, p)$

(b) Look up gamma mgf, we have

$$M_{X_i}(t) = (1 - \beta t)^{-\alpha_i}$$

Multiplying, we have

$$M_S(t) = (1 - \beta t)^{-(\alpha_1 + \cdots + \alpha_n)}$$

Hence $S \sim \text{Gamma}(\alpha_1 + \cdots + \alpha_n, \beta)$

(c) Look up poisson mgf, we have

$$M_{X_i}(t) = \exp\{\lambda_i e^{(t-1)}\}$$

Multiplying, we have

$$M_S(t) = \exp\{(\lambda_1 + \cdots + \lambda_n) e^{(t-1)}\}$$

Hence $S \sim \text{Poi}(\lambda_1 + \cdots + \lambda_n)$

If you want to be really careful, you could verify that the mgf exists on an open interval containing 0. ♠

Problem 8. Suppose X and Y are independent random variables with Uniform$(0,1)$ distribution. Let $U = XY$ and $V = X/Y$.

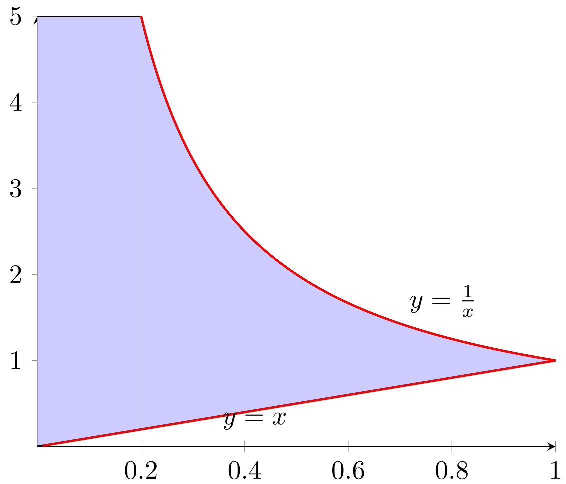
(a) [8pts] Find the joint pdf of (U, V).

(b) [4pts] Find the marginal pdf of U.

(c) [4pts] Does the first moment of V exist?

Solution. (a) We see that $UV = X^2$ and $U/V = Y^2$. So the inverse transformation is [2pts]

$$(X, Y) = (\sqrt{UV}, \sqrt{U/V})$$



It is non-trivial to figure out the region B. Try substituting the four corner points of A, you will get 3 points: $(0,0), (1,1), (0,\infty)$. Converting $0 \le X^2 = UV \le 1$ and $0 \le Y^2 = U/V \le 1$ to inequality relations between U and V gives the region. To rigorously verify that we are correct, denote $T(X,Y) = (U,V)$ and we will have to show that $T(A) \subseteq B$ and $B \subseteq T(A)$, the latter is equivalent to the easier to prove condition $T^{-1}(B) \subseteq A$ where T^{-1} denotes the inverse function.

The support $A = \text{supp}(X,Y) = [0,1] \times [0,1]$ is mapped to $B = \{(x,y) : x \in [0,1], y \in [x, 1/x]\}$. [2pts]
The Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{V/U} & \frac{1}{2}\sqrt{U/V} \\ \frac{1}{2}\sqrt{1/UV} & -\frac{1}{2V}\sqrt{U/V} \end{bmatrix}$$

The absolute value of the determinant is : [2pts]

$$|\det J| = |-\frac{1}{4V} - \frac{1}{4V}| = \frac{1}{2V}$$

By change of variable formula, we have on B: [2pts]

$$f_{U,V}(u,v) = \frac{1}{2v} f_{X,Y}(\sqrt{uv}, \sqrt{u/v}) = \frac{1}{2v}$$

(b) For $u \in (0,1)$:

$$f_U(u) = \int_u^{1/u} \frac{1}{2t}\, \mathrm{d}t = \frac{1}{2}\log\frac{1}{u} - \frac{1}{2}\log u = -\log u$$

(c) For $v \ge 1$, we have:

$$f_V(v) = \int_0^{1/v} \frac{1}{2v}\, \mathrm{d}t = \frac{1}{2v^2}$$

The expectation involves integrating $1/v$ from 1 to ∞ and more stuff which becomes irrelevant and is infinite.

♠

Problem 9. [8pts] Consider i.i.d. $X_1, \dots, X_n$ from Uniform$(0,1)$. Find the joint pdf of $(X_{(1)}, X_{(n)})$. It is recommended to use the cdf method (but it is no longer univariate).

Solution. Since $0 \le X_{(1)} \le X_{(n)} \le 1$, the support of f is $0 \le x \le y \le 1$. For such (x, y): [5pts]

$$\begin{aligned}
F_{X_{(1)},X_{(n)}}(x,y) &= \mathbb{P}(X_{(1)} \le x, X_{(n)} \le y) \\
&= \mathbb{P}(X_{(n)} \le y) - \mathbb{P}(X_{(1)} > x, X_{(n)} \le y) \\
&= [F(y)]^n - [F(y) - F(x)]^n \\
&= y^n - (y-x)^n
\end{aligned}$$

- $\{X_{(n)} \le y\}$ is the same event as all X_i are below y.
- $\{X_{(1)} > x, X_{(n)} \le y\}$ is the event that all X_i are between x and y.
- for continuous random variables we can ignore the difference between $<$ and $\le$.

Taking derivatives, we have, for $0 \le x \le y \le 1$: [3pts]

$$\begin{aligned}
f_{X_{(1)},X_{(n)}}(x,y) &= \frac{\partial}{\partial y}\frac{\partial}{\partial x}[y^n - (y-x)^n] \\
&= \frac{\partial}{\partial y}[n(y-x)^{n-1}] \\
&= n(n-1)(y-x)^{n-2}
\end{aligned}$$

♠