STAT 330: Lecture 4

2024 Spring

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$\begin{cases} \Gamma(1) = 1 \\ \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \end{cases} \Rightarrow \Gamma(n) = (n-1)!$, $n \in \mathbb{Z}^+$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Last Lecture

- Continuous random variable: Gamma function and some commonly used continuous distributions.
- Expectation, variance, and moment.

$E(|X|) < \infty$ $\quad Var(X) = E(X^2) - \{E(X)\}^2$ $\quad E(X^k)$ $\quad E[\{X - E(X)\}^k]$ $\quad E\{X(X-1)\cdots(X-k+1)\}$

This Lecture

- Finish what is left in the previous lecture.
- Moment generating function

Example: Gamma Distribution kth Moment $E(X^k)$

$X \sim Gam(\alpha, \beta)$.

$$E(X^k) = \int_0^\infty x^k \cdot \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx = \int_0^\infty \frac{x^{k+\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\cdot\beta^\alpha} dx.$$

Method 1. change variable. $y = x/\beta$, $x = \beta y$, $dx = \beta dy$

$$E(X^k) = \frac{\beta^k}{\Gamma(\alpha)} \cdot \int_0^{+\infty} y^{k+\alpha-1} e^{-y} dy = \frac{\beta^k}{\Gamma(\alpha)} \cdot \Gamma(k+\alpha)$$

Method 2:

$$E(X^k) = \int_0^\infty \frac{x^{k+\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\cdot\beta^\alpha} dx.$$

Let $\alpha' = k+\alpha$

Find pdf of $Gam(\alpha', \beta)$

$$= \left\{ \int_0^\infty \underbrace{\frac{x^{\alpha'-1} e^{-x/\beta}}{\Gamma(\alpha')\beta^{\alpha'}}}_{\text{pdf of } Gam(\alpha',\beta)} dx \right\} \cdot \frac{\Gamma(\alpha')\cdot\beta^{\alpha'}}{\Gamma(\alpha)\cdot\beta^\alpha} = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)}\cdot\beta^k$$

Let $k=1$. $E(X) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\cdot\beta = \alpha\beta$

Let $k=2$. $E(X^2) = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\cdot\beta^2 = (\alpha+1)\alpha\beta^2$

$$Var(X) = E(X^2) - \{E(X)\}^2$$
$$= \alpha\beta^2$$

Moment Generating Function

Definition Suppose X is r.v.

$M(t) = E(e^{tX})$ is called the moment-generating function.

if $M(t)$ exists on $t \in (-h, h)$ for some $h > 0$.

※ The support of $M(t)$ has to contain 0.

Example Find out the mgf of $X \sim \mathrm{Gam}(\alpha, \beta)$

$$M(t) = E(e^{tX}) = \int_0^{+\infty} e^{tx} \cdot \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\, \beta^{\alpha}} \cdot dx = \int_0^{+\infty} \frac{x^{\alpha-1} e^{-x(\frac{1}{\beta}-t)}}{\Gamma(\alpha) \cdot \beta^{\alpha}} \cdot dx$$

Method 1: Change variable. $y = x(\frac{1}{\beta} - t)$, $x = \frac{y}{\frac{1}{\beta}-t}$, $dx = \frac{dy}{\frac{1}{\beta}-t}$.

$$M(t) = \int_0^{+\infty} \frac{y^{\alpha-1}}{(\frac{1}{\beta}-t)^{\alpha-1}} \cdot e^{-y} \cdot \frac{1}{\frac{1}{\beta}-t} \cdot \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha}}\, dy$$

$$= \frac{\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} \cdot \frac{1}{(\frac{1}{\beta}-t)^{\alpha}} \cdot \frac{1}{\beta^{\alpha}} = (1-\beta t)^{-\alpha}$$

Method 2:
$$\int_0^{+\infty} \frac{x^{\alpha-1} e^{-x(\frac{1}{\beta}-t)}}{\Gamma(\alpha) \cdot \beta^{\alpha}} \cdot dx \qquad \text{let } \beta' = \frac{1}{\frac{1}{\beta}-t}$$

$$= \left\{ \int_0^{+\infty} \underbrace{\frac{x^{\alpha-1} e^{-x/\beta'}}{\Gamma(\alpha) \cdot (\beta')^{\alpha}}}_{\text{pdf of } \mathrm{Gam}(\alpha, \beta')} dx \right\} \cdot \frac{(\beta')^{\alpha}}{\beta^{\alpha}} = \left(\frac{\beta'}{\beta}\right)^{\alpha}$$

$$= (1-\beta t)^{-\alpha}$$

e.g. $\alpha = \frac{1}{2}$. $M(t) = \frac{1}{\sqrt{1-\beta t}} \Rightarrow 1 - \beta t > 0 \iff t < \frac{1}{\beta}$

Support of $M(t)$ $(-\infty, \frac{1}{\beta})$ contains 0.

$\Rightarrow M(t)$ is a legitimate mgf.

Example Find the mgf of the Poisson distribution.

$X \sim Poi(\theta)$, pmf. $f(x) = \frac{\theta^x e^{-\theta}}{x!}$, $x \in \{0, 1, \cdots\}$

$M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{\theta^x e^{-\theta}}{x!}$

$= e^{-\theta} \sum_{x=0}^{\infty} \boxed{\frac{(e^t\theta)^x}{x!}} = e^{-\theta} \sum_{x=0}^{\infty} \boxed{\frac{(e^t\theta)^x}{x!} \exp\{-e^t\theta\}} \exp\{e^t\theta\}$

$\sim Poi(e^t\theta)$

$= \exp\{(e^t - 1)\theta\}$, $t \in \mathbb{R}$.

$M(t)$ is a mgf

Properties of the MGF X, its location-scale transformation means.

$Y = aX + b$ (a: scale, b: location)

MGF after location-scale transformation

Suppose. X with mgf $M_X(t)$

What is the mgf of $Y = aX + b$?

Solution: $M_Y(t) = E\{e^{t(aX+b)}\} = E\{e^{atX + bt}\}$

$= e^{bt} E\{e^{atX}\} = e^{bt} M(at)$

$= M_X(at)$

Example: (1) $Z \sim Norm(0, 1)$, $M_Z(t)$.

$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{+\infty} e^{tz} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}) dz = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{z^2 + 2tz + t^2 - t^2}{2}\} dz$

$= \int_{-\infty}^{+\infty} \boxed{\frac{1}{\sqrt{2\pi}} \exp\{-\frac{(z-t)^2}{2}\}} \exp(\frac{t^2}{2}) dz = \exp(\frac{t^2}{2})$, $t \in \mathbb{R}$.

$\sim Norm(t, 1)$

(2) $X \sim Norm(\mu, \sigma^2)$ $X = \sigma Z + \mu$

$M_X(t) = e^{\mu t} M_Z(\sigma t) = e^{\mu t} \cdot e^{\sigma^2 t^2 / 2}$

From MGF to moment Find moments using the mgf.

Suppose. X with mgf $M_X(t)$, then.

$$E(X^k) = \underbrace{M_X^{(k)}(0)}$$

↳ Kth-derivative of $M_X(t)$ at $t=0$

Example 1: $Gam(\alpha, \beta)$, $M(t) = (1-\beta t)^{-\alpha}$, $t < 1/\beta$

$E(X) = \alpha\beta$, $M'(t) = (-\alpha)(1-\beta t)^{-\alpha-1}(-\beta) = \alpha\beta(1-\beta t)^{-(\alpha+1)}$

Set $t=0$, $M'(0) = \alpha\beta = E(X)$

$E(X^2) = \alpha(\alpha+1)\beta^2$, $M''(t) = \alpha\beta(\alpha+1)\cdot(1-\beta t)^{-(\alpha+2)}\cdot\beta$

Let $t=0$, $M''(0) = \alpha(\alpha+1)\beta^2 = E(X^2)$

Example 2: Poisson: $Poi(\theta)$, $M(t) = \exp\{\theta(e^t-1)\}$

$E(X) = \theta$, $M'(t) = \exp\{\theta(e^t-1)\}\,\theta e^t$, Let $t=0$.

$M'(0) = \theta = E(X)$

$E(X^2) = Var(X) + \{E(X)\}^2$, $M''(t) = \theta e^t e^{\theta(e^t-1)} + e^{\theta(e^t-1)}\theta^2(e^t)^2$

$= \theta + \theta^2$ $\qquad M''(0) = \theta + \theta^2 = E(X^2)$

MGF vs. Distribution Uniqueness of mgf

X and Y have the same distribution

$\Longleftrightarrow$ X and Y have the same mgf.

mgf and distribution : one to one.

Example: X has mgf $M_X(t) = e^{t^2/2}$ mgf of standard normal

(1) Find the mgf of $Y = 2X - 1$

$M_Y(t) = e^{-t} e^{2t^2}$

$X \sim Norm(0,1)$
$Y \sim Norm(-1, 4)$

(2) Find out $E(Y)$ & $Var(Y)$

$E(Y)$: $M_Y'(t) = e^{2t^2-t}(4t-1)$, $M_Y'(0) = -1 = E(Y)$

$E(Y^2)$: $M_Y''(t) = e^{2t^2-t}(4t-1)^2 + 4e^{2t^2-t}$, $M_Y''(0) = 4+1 = 5$

$Var(Y) = E(Y^2) - \{E(Y)\}^2 = 5 - 1 = 4$.

(3) What is the dist'n of Y?

Sol: $M_X(t)$ is the mgf of standard normal dist'n.

$\Rightarrow X \sim Norm(0,1)$

$Y = 2X - 1 \sim Norm(-1, 4)$