University of Waterloo

STAT330, Spring 2024

Tutorial 1

Last Updated:

May 21, 2024

In this tutorial, we examine the uniform distribution $U[0,1]$. Its cumulative distribution function is

$$F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & x \in (0,1) \\ 1 & x \geq 1 \end{cases}$$

[Explanation]

[Solutions]

[Added after]

Problem 1. Find the probability density function $f_U(x)$.

$f(x) = F'(x)$

if $x < 0$: $f(x) = \frac{d}{dx} 0 = 0$

if $x \in (0,1)$: $f(x) = \frac{d}{dx}(x) = 1$

if $x > 0$: $f(x) = \frac{d}{dx} 1 = 0$

$f(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$

F not differentiable at 0, 1

$f(0) = f(1) = 0$ can be arbitrarily defined.

Problem 2. Find the expectation of U.

$E(U) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$

$= \int_{-\infty}^{0} x f(x)\, dx + \int_{0}^{1} x f(x)\, dx + \int_{1}^{\infty} x f(x)\, dx.$

$= \int_{-\infty}^{0} x \cdot 0\, dx + \int_{0}^{1} x \cdot 1\, dx + \int_{1}^{\infty} x \cdot 0\, dx$

$= \int_0^1 x \cdot 1 \, dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}$

One way of computing the variance is $\mathrm{Var}(X) = E[(X-\mu)^2]$ where μ is the expectation.

Problem 3. Compute the variance of U.

$E[(U-\frac{1}{2})^2] = \int_0^1 (x-\frac{1}{2})^2 f(x)\, dx$

$= \int_0^1 (x-\frac{1}{2})^2 \cdot 1 \, dx.$

$= \frac{1}{3}(x-\frac{1}{2})^3 \Big|_0^1$

$= \frac{1}{3}\left((\frac{1}{2})^3 + (\frac{1}{2})^3\right) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

The moment generating function for a random variable X is defined to be $M_X(t) = E[e^{tX}]$, whenever the expectation is finite. Recall that

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx$$

when X has density f.

Problem 4. Compute the moment generating function of U.

if $t \neq 0$ $M_t(U) = E[e^{t \cdot U}] = \int_0^1 e^{tx} \cdot 1 \, dx.$

$= \frac{1}{t} e^{tx} \Big|_0^1 = \frac{1}{t} e^t - \frac{1}{t} \cdot 1$

$= \frac{e^t - 1}{t}$

$t = 0$: $\int_0^1 e^0 \cdot 1 \, dx$

$= \int_0^1 1 \, dx = 1$

$M_t(U) = \begin{cases} \frac{e^t - 1}{t} & t \neq 0 \\ 1 & t = 0. \end{cases}$

The moment generating function gets its name because it "generates the moments". Write

$$e^{tX} = 1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots$$

Since expectation is linear, we have

$$M_X(t) = E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}\boxed{E[X^2]} + \frac{t^3}{3!}E[X^3] + \dots$$

$\star$

$M_X'(0) = E[X]$

$M_X''(0) = E[X^2]$

Not suitable for this problem

5. $M_t(U) = \begin{cases} \frac{e^t-1}{t} & t \neq 0 \\ 1 & t = 0. \end{cases}$

$$\frac{e^t-1}{t} = \left(1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}\cdots -1\right)/t = \frac{t^0}{1!}+\frac{t^1}{2!}+\frac{t^2}{3!}\cdots$$

Problem 5. Compute the second moment of U using the method above, and compute again the variance using the formula $\mathrm{Var}(X) = E[X^2] - E[X]^2$.

$E[U^2] = \frac{1}{3}$ by matching coefficients. ☆ $\quad = 1+\frac{t^1}{1!}\cdot\frac{1}{2}+\frac{t^2}{2!}\boxed{\frac{1}{3}}$ ☆

$$\mathrm{Var}[U] = E[U^2] - E[U]^2$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$E[U^2] = \int_0^1 x^2\, dx = \frac{1}{3}x^2\Big|_0^1 = \frac{1}{3}.$$

We wish to establish an easy version of *distributional transform* of a random variable X,which constructs a random variable from $[0,1]$ to $\mathbb{R}$ with the same distribution as X. Intuitively, this sorts the values of X in increasing order without changing probabilities.
However there are some technical difficulties we need to address first.

Problem 6. Let X be any random variable, and F_X be its cumulative distribution function. Show that $F_X : \mathbb{R} \to [0,1]$ is never bijective.

suppose F_X is not surjective: It is not bijective.

else, suppose F_X is surjective: $\exists\, c \in \mathbb{R} : F_X(c) = 1$.

But F_X is increasing. $F_X(c+\frac{1}{2}) = 1$.

F_X is not injective hence not bijective.

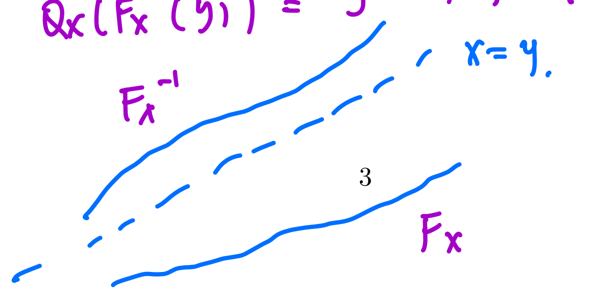
One way to fix this issue is to define $F_X : \mathbb{R} \cup \{\pm\infty\} = \bar{\mathbb{R}} \to [0,1]$ instead.
We assume that F_X is bijective and let $Q_X : [0,1] \to \bar{\mathbb{R}}$ be its inverse. Note that Q_X must be strictly increasing.

quantile.

Inverse means:

$$F_X \circ Q_X = id[0,1] \qquad F_X(Q_X(x)) = x \quad \forall x \in [0,1]$$

$$Q_X \circ F_X = id(\bar{\mathbb{R}}) \qquad Q_X(F_X(y)) = y \quad \forall y \in \bar{\mathbb{R}}$$



$$\Omega \xrightarrow{U} [0,1] \xrightarrow{Q_X} \overline{\mathbb{R}}$$

Random Variable.

Problem 7. Show that $Q_X(U)$ has the same distribution as X. That is, they have the same cumulative distribution function.

Let G be the cdf of $Q_X(U)$

$$\begin{aligned}
G(x) &= \Pr(Q_X(U) \le x) \\
&= \Pr(Q_X^{-1} Q_X(U) \le Q_X^{-1}(x)) \quad (Q_X^{-1} \text{ strictly increasing}) \\
&= \Pr(U \le \underbrace{F_X(x)}_{\in [0,1]}) \\
&= F_X(x).
\end{aligned}$$

Problem 8. Let X, Y be normal random variables with mean 0 and variances $\sigma_X^2 = 1$ and $\sigma_Y^2 = 2$. Let $X' = Q_X(U)$ and $Y' = Q_Y(U)$. Find the conditional probabilities $P(X' \ge 1 | Y' \ge 1)$ and $P(Y' \ge 1 | X' \ge 1)$.

Solution: Let Φ be the cdf of $N(0,1)$

$$\begin{aligned}
P(Y \ge 1) &= P(Y \le -1) = P(Y/\sqrt{2} \le -1/\sqrt{2}) \\
&= \Phi(-1/\sqrt{2}) \\
P(X \ge 1) &= P(X \le -1) = \Phi(-1) < \Phi(-1/\sqrt{2})
\end{aligned}$$

so $F_Y(1) < F_X(1)$

$$\begin{aligned}
& X' \ge 1 \\
\Leftrightarrow\ & Q_X(U) \ge 1 \\
\Leftrightarrow\ & U \ge F_X(1) \\
\Rightarrow\ & U \ge F_Y(1) \\
\Leftrightarrow\ & Q_Y(U) \ge 1 \\
\Leftrightarrow\ & Y' \ge 1.
\end{aligned}$$

so $X' \ge 1 \Rightarrow Y' \ge 1$.

$\{X' \ge 1 \cap Y' \ge 1\} = \{X' \ge 1\}$.

so $P(X' \ge 1 \mid Y' \ge 1)$

$$= \frac{P\{X' \ge 1 \cap Y' \ge 1\}}{P\{Y' \ge 1\}} = \frac{P\{X' \ge 1\}}{P(Y' \ge 1)}$$

By Q7

$$= \frac{P\{X \ge 1\}}{P(Y \ge 1)} = \frac{\Phi(-1)}{\Phi(-1/\sqrt{2})} = \frac{0.159}{0.240} \approx \frac{2}{3}$$

$$P(Y' \ge 1 \mid X' \ge 1) = \frac{P(X' \ge 1)}{P(X' \ge 1)} = 1.$$