STAT 330: Lecture 8

2024 Spring

June 6, 2024

Last Lecture

- Joint Expectation
- Variance & Covariance
- Correlation Coefficient
- Conditional Distribution

$E\{h(X,Y)\} = \iint h(x,y) f(x,y)\,dx\,dy.$

$Cov(X,Y) = E\{(X-E(X))(Y-E(Y))\}$

$Var(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{i<j} a_i a_j Cov(X_i, X_j)$

$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} \in [-1,1]$

$f(x|y) = \frac{f(x,y)}{f(y)}$. $f(y|x) = \frac{f(x,y)}{f(x)}$.

This Lecture Conditional Distribution & Conditional Expectation

e.g. $f(x,y) = q^2 p^{x+y}$. $x,y \in \{0,1,\cdots\}$

$X \perp Y$ $\quad f(x) = f(x|y)$

$f(y) = f(y|x)$

(Continued)

Conditional Distribution

Some Properties of Conditional Distribution:

1. $X \perp Y \iff f(x|y) = f(x) \iff f(y|x) = f(y)$

   we can use the conditional distribution to determine if X and Y are independent.

2. we can use it to obtain joint and marginal dist'ns.

   $f(x|y) = \dfrac{f(x,y)}{f(y)}$, $f(x,y) = f(x|y) \cdot f(y)$ ← recover the joint

   $f(x) = \int f(x,y)\,dy$

Example 1: $Y \sim Poi(\theta)$, $X|Y=y \sim Bin(y,p)$

Find the marginal of X.

① Find $f(x,y) = f(x|y) f(y) = \binom{y}{x} p^x (1-p)^{y-x} \dfrac{\theta^y}{y!} \cdot \exp(-\theta)$

② $f(x) = \sum_y f(x,y) = \sum_{y=x}^{\infty} \binom{y}{x} \cdot p^x (1-p)^{y-x} \dfrac{\theta^y}{y!} \exp(-\theta)$

$\binom{y}{x} = \dfrac{y!}{x!(y-x)!}$, $n = y-x$

$$f(x) = \sum_{n=0}^{\infty} \frac{\cancel{y!}}{x!\,n!} \cdot p^x (1-p)^n \frac{\theta^n \cdot \theta^x}{\cancel{y!}} \exp(-\theta)$$

$$= \sum_{n=0}^{\infty} \frac{(p\theta)^x}{x!} \cdot \frac{1}{n!} \{(1-p)\theta\}^n \exp(-\theta)$$

$$= \frac{(p\theta)^x}{x!} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \{(1-p)\theta\}^n \exp(-\theta)$$

$$= \frac{(p\theta)^x}{x!} \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} \cdot \{(1-p)\theta\}^n \cdot \exp\{-(1-p)\theta\}} \exp\{(1-p)\theta - \theta\}$$

$$= \frac{(p\theta)^x}{x!} \cdot \exp(-p\theta) \sim Poi(p\theta)$$

Example 2: Suppose Y. $f(y) = \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)}$. $y>0$. $Y \sim Gam(\alpha, 1)$

$f(x|y) = y e^{-xy}$, $x>0$. $y>0$. Find out marginal of X.

①. $f(x,y) = f(x|y) \cdot f(y) = y \cdot e^{-xy} \cdot \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)}$

$$= \frac{y^{\alpha} e^{-(x+1)y}}{\Gamma(\alpha)} \quad , x, y > 0$$

②. $f(x) = \int_{-\infty}^{+\infty} f(x,y)\, dy = \int_0^{+\infty} \frac{y^{\alpha} e^{-(x+1)y}}{\Gamma(\alpha)}\, dy$.

Method 1. Change variable.

Let $t = (x+1)y$, $y = \frac{t}{x+1}$, $dy = \frac{dt}{x+1}$

$$f(x) = \frac{1}{(x+1)^{\alpha+1}} \cdot \frac{1}{\Gamma(\alpha)} \cdot \underbrace{\int_0^{+\infty} t^{\alpha+1-1} e^{-t} dt}_{\Gamma(\alpha+1)}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{1}{(x+1)^{\alpha+1}} = \frac{\alpha}{(x+1)^{\alpha+1}}$$

Method 2. $\alpha' = \alpha+1$. $\beta = \frac{1}{1+x}$

$$f(x) = \underbrace{\boxed{\int_0^{+\infty} \frac{y^{\alpha'-1} \cdot \exp\{-\frac{y}{\beta}\}}{\Gamma(\alpha') \cdot \beta^{\alpha'}}}}_{\sim Gam(\alpha', \beta)} \cdot \frac{\Gamma(\alpha') \cdot \beta^{\alpha'}}{\Gamma(\alpha)} dy$$

$$= \frac{\Gamma(\alpha')}{\Gamma(\alpha)} \cdot \beta^{\alpha'} = \alpha \cdot \left(\frac{1}{x+1}\right)^{\alpha+1}$$

Conditional Expectation The conditional pmf/pdf is also a valid pmf/pdf. We can define the conditional expectation based on the conditional pmf/pdf.

Definition:

The conditional expectation of $g(Y)$ given $X=x$ is defined as.

$$E[g(Y)|X=x] = \begin{cases} \sum_y g(y) \cdot f(y|x), & Y \text{ is discrete.} \\ \int_{-\infty}^{\infty} g(y) \cdot f(y|x) dy. & Y \text{ is continuous} \end{cases}$$

E.g. $E(Y|X=x)$

$Var(Y|X=x) = E(Y^2|X=x) - \{E(Y|X=x)\}^2$

$E(e^{tY}|X=x)$

Properties:

①. If $X \perp Y$.

$E\{g(Y)|X=x\} = E\{g(Y)\}$ b/c. $f(y|x) = f(y)$

$E\{h(X)|Y=y\} = E\{h(X)\}$ b/c $f(x|y) = f(x)$.

e.g. $g(Y)=Y$. $h(X)=X$

$E(Y|X=x) = E(Y)$

$E(X|Y=y) = E(X)$.

Properties: ② Substitution Rule.

suppose. $h(X,Y) \in \mathbb{R}$.

$$E\{h(X,Y)|X=x\} = E\{h(x,Y)|X=x\}$$

upper case X. lower case x

e.g. $E(X+Y|X=x) = E(x+Y|X=x) = x + E(Y|X=x)$

$E(XY|X=x) = E(xY|X=x) = x\,E(Y|X=x)$

③. Conditional Expectation has all properties of expectation (e.g. linearity)

Examples: $f(x,y) = \begin{cases} 8xy & , 0<y<x<1 \\ 0 & , \text{otherwise.} \end{cases}$

$f(x|y) = \frac{2x}{1-y^2}$, $x \in (y,1)$. Find out. $E(X|Y=y)$, $Var(X|Y=y)$

Need $E(X|Y=y)$ $E(X^2|Y=y)$

$$E(X|Y=y) = \int_y^1 x \cdot \frac{2x}{1-y^2}\,dx = \frac{2}{3}\cdot\frac{1-y^3}{1-y^2}, \quad y \in (0,1).$$

$$E(X^2|Y=y) = \int_y^1 x^2 \cdot \frac{2x}{1-y^2}\,dx = \frac{1}{2}\cdot(1+y^2), \quad y \in (0,1)$$

$$Var(X|Y=y) = E(X^2|Y=y) - \{E(X|Y=y)\}^2$$
$$= \frac{1}{2}\cdot(1+y^2) - \frac{4}{9}\left(\frac{1-y^3}{1-y^2}\right)^2.$$

Example 2. $Y \sim Poi(\theta)$. $X|Y=y \sim Bin(y,p)$.

Find $E(X|Y=y)$ $Var(X|Y=y)$.

$E(X|Y=y) = yp$.

$Var(X|Y=y) = y \cdot p(1-p)$

Double Expectation Theorem

$E(Y|X=x)$ and $E(Y|X)$:

$E(Y|X=x)$ is a function of x and x only.

$h(x) = E(Y|X=x)$

If we replace x with X, we have $h(X)$

$h(X)$ is a random variable.

We denote $h(X)$ as $E(Y|X)$

1. $E(Y|X)$ is a random variable.
2. $E(Y|X)$ is a function of r.v. X and X only.

Double Expectation Theorem:

$g(Y)$. $E\{g(Y)\} = E[\underbrace{E\{g(Y)|X\}}_{\text{a random variable func of } X \text{ and } X \text{ only.}}]$

proof. $E\{g(Y)\} = \int g(y) f(y) dy$

$= \int g(y) \cdot \int f(x,y) dx dy$

$= \int g(y) \cdot \int f(y|x) f(x) dx dy$

$= \int \{ \int g(y) f(y|x) dy \} f(x) dx$

By def. of cond. exp.

$= \int \boxed{E(g(Y)|X=x)} f(x) dx$

$= \int \boxed{h(x)} f(x) dx = E\{h(\boxed{X})\}$ random variable.

$= E(E\{g(Y)|X\})$

Example. $Y \sim Poi(\theta)$. $X|Y=y \sim Bin(y, p)$.

Find $E(X)$.

$E(X) = E\{E(X|Y)\} = E\{\underbrace{pY}_{\text{random variable.}}\} = pE(Y)$

$= \theta p$.