University of Waterloo

STAT330, Spring 2024

Tutorial 3

Last Updated:

June 20, 2024

Problem 1. Suppose that the conditional distribution of Y given $X = x$ is Poisson with mean $E(Y|X = x) = x$ and that the p.d.f of X is $f_X(x) = e^{-x}$ if $x \geq 0$ and 0 elsewhere.

(a) Find $E(Y)$.

(b) Find $Var(Y)$.

Solution. (a) We use the law of total expectation:

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}(\mathbb{E}[Y|X]) \\
&= \int_0^\infty \mathbb{E}[Y|X = x] f_X(x) \, \mathrm{d}x \\
&= \int_0^\infty x e^{-x} \, \mathrm{d}x \\
&= 1
\end{aligned}$$

(b) We use the law of total variance:

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathbb{E}(\mathrm{Var}(Y|X)) + \mathrm{Var}(\mathbb{E}(Y|X)) \\
&= \int_0^\infty \mathrm{Var}(\underbrace{Y|X = x}_{\sim poi(x)}) f_X(x) \, \mathrm{d}x + \mathrm{Var}(X) \\
&= \int_0^\infty x e^{-x} + 1 \\
&= 2
\end{aligned}$$

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Problem 2. Let X_1 and X_2 be independent normal random variables, $X_i \sim N(0,1)$, and let $Y_1 = aX_1 + bX_2$ and $Y_2 = cX_1 + dX_2$

(a) Show that Y_1 and Y_2 are bivariate normal.

(b) Specify the mean vector and covariance matrix in the joint distribution of Y_1 and Y_2

(c) Calculate directly, the mean, variance and covariance of Y_1, Y_2.

Solution. (a) We check the joint mgf.

$$
\begin{aligned}
M_{Y_1,Y_2}(t_1,t_2) &= \mathbb{E}[\exp(t_1Y_1 + t_2Y_2)] \\
&= \mathbb{E}[\exp(t_1(aX_1 + bX_2) + t_2(cX_1 + dX_2))] \\
&\cancel{= \mathbb{E}[\exp(t_1(aX_1 + bX_2) + t_2(cX_1 + dX_2))]} \\
&= \mathbb{E}[\exp(t_1a + t_2c)X_1 + (t_1b + t_2d)X_2] \\
&= M_{X_1}(t_1a + t_2c) \cdot M_{X_2}(t_1b + t_2d) \\
&= \exp(\frac{1}{2}(t_1^2a^2 + 2t_1t_2ac + t_2^2c^2)) \cdot \exp(\frac{1}{2}(t_1^2b^2 + 2t_1t_2bd + t_2^2d^2)) \\
&= \exp(\frac{1}{2}(t_1^2(a^2+b^2) + 2t_1t_2(ac+bd) + t_2^2(c^2+d^2)) \\
&= \exp(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t})
\end{aligned}
$$

$M_{X_1,X_2}(S_1,S_2) = M_{X_1}(S_1) \cdot M_{X_2}(S_2)$

$S_1 = t_1a + t_2c$

$S_2 = t_1b + t_2d$

Where Σ is $\begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$.

(b) The mean is $\mathbf{0}$ and the covariance matrix is Σ.

(c) Same answer as (b).

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\begin{aligned}
&\mathrm{Cov}(Y_1, Y_1) \\
&= \mathrm{Cov}(aX_1 + bX_2, aX_1 + bX_2) \\
&= a^2\,\mathrm{cov}(X_1, X_1) + b^2\,\mathrm{cov}(X_2, X_2) \\
&= a^2 + b^2.
\end{aligned}
$$

Problem 3. Let X have p.d.f. $f_X(x) = \frac{2}{9}(x+1)$ for $-1 \le x \le 2$. Find the p.d.f. of $Y = X^2$.

Solution. Let $g(x) = x^2$. Observe that for $y \le 1$, there are two pre-images and for $y > 1$ there is only one in the support. The distorting factor is:

$$|g'(x)| = g'(\sqrt{y}) = 2\sqrt{y}$$

For $y \in [0, 1]$:

$$f_Y(y) = \frac{f_X(-\sqrt{y})}{2\sqrt{y}} + \frac{f_X(\sqrt{y})}{2\sqrt{y}} = \frac{2}{9\sqrt{y}}$$

For $y \in (1, 4]$:

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} = \frac{2 \cdot (1+\sqrt{y})}{9 \cdot 2 \cdot \sqrt{y}} = \frac{1}{9} + \frac{1}{9\sqrt{y}}$$

The p.d.f. of Y is

$$f_Y(y) = \begin{cases} 2/(9\sqrt{y}) & 0 \le y \le 1 \\ 1/9 + 1/(9\sqrt{y}) & 1 < y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

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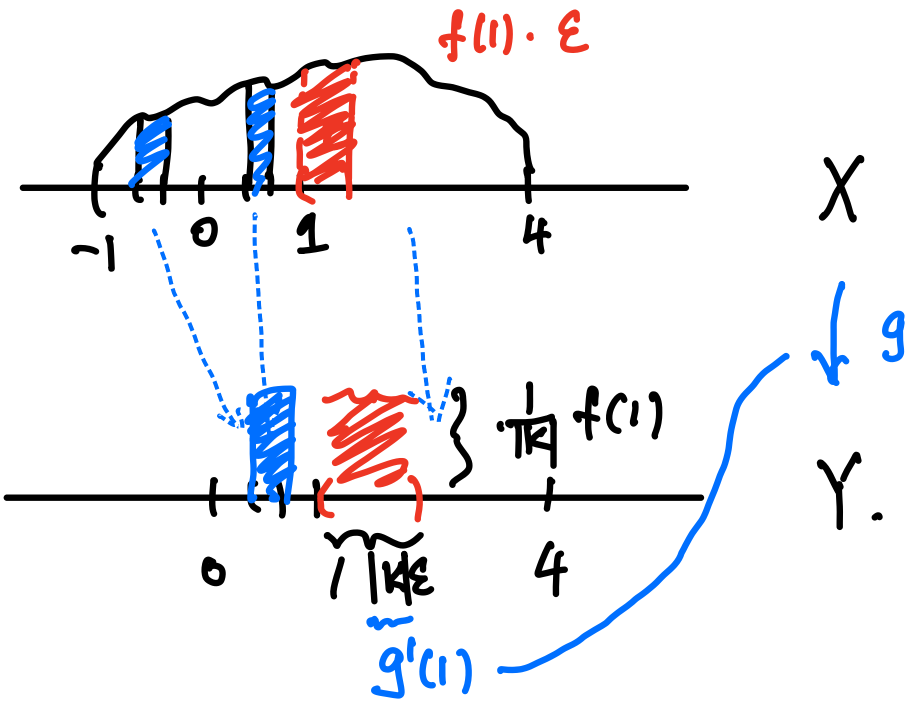


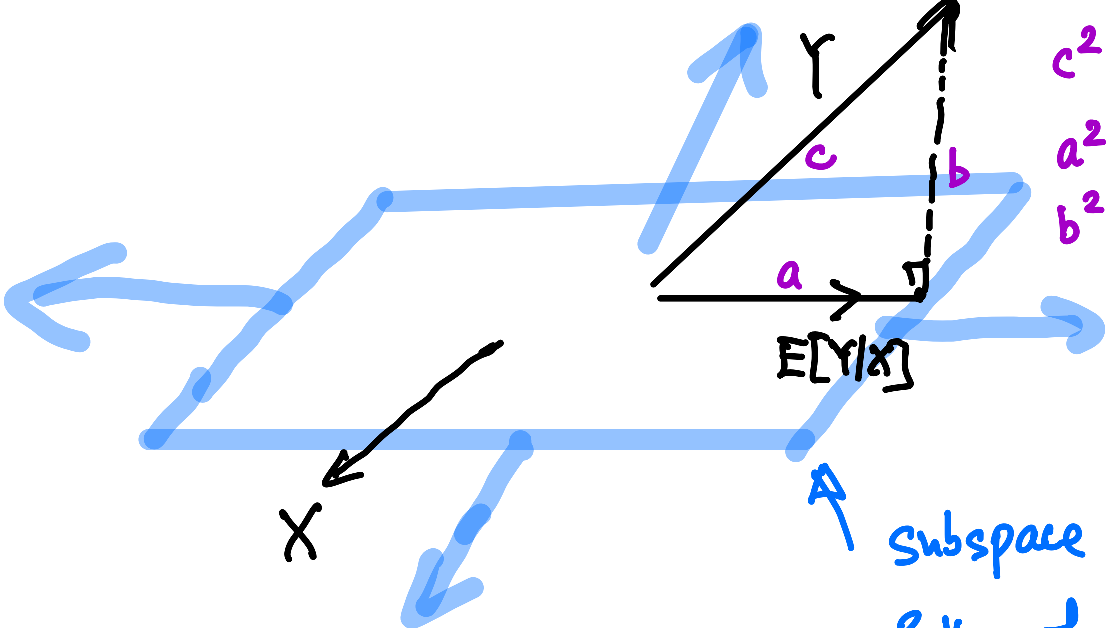
Expression	Type of object.
$Y \mid X = x$	Random variable in Y
$E^{Y}[Y \mid X = x]$	Number
$E[Y \mid X]$	Random variable in X, $\varphi(X)$ $\varphi : \mathbb{R} \to \mathbb{R}$
$Var(Y \mid X)$	R.V in X, $\varphi(X)$ for some φ.
$E^{X}[Var(Y \mid X)]$	number
$Var^{X}[E(Y \mid X)]$	number
$Y \mid X$	??

Useful fact:

① $E[\,E[Y \mid X]\,] = E[Y]$.

② $E[\varphi(X) \cdot Y \mid X] = \varphi(X) \cdot E[Y \mid X]$.

③ $Y^2 \mid X = x \quad = \quad (Y \mid X = x)^2$



$c^2 = \mathrm{Var}\, Y$

$a^2 = \mathrm{Var}(E[Y|X])$

$b^2 = E(\mathrm{Var}(Y|X))$

subspace of R.V. of the form $\varphi(X)$, for some $\varphi : \mathbb{R} \to \mathbb{R}$.

conjecture: $\mathrm{Cov}(Y, E[Y|X])$

$= \mathrm{Cov}(E[Y|X], E[Y|X])$