Chapter 1 Random variables

Overview: probability model, random variables, expeution
moment-generating function.

1.1. Introduction

1.1.1. Probability model.

A random experiment: eg. toss a coin/die, time for commuting between Waterloo & Toronto

discrete outcome — Continuous outcome.

(1) $\boxed{\text{Sample space}}$: all posible outcomes of a random experiment.

eg. If the random experiment is to toss one coin: $S_1 = \{H, T\}$

If ... ... two coins: $S_2 = \{(H,H), (H,T), (T,T), (T,H)\}$

If .. .. is the commuting time: $S_3 = \mathbb{R}^+$

(2) $\boxed{\text{Event}}$: A subset of the sample space.

For $S_1 = \{H, T\}$, eg. we define event A as the heads $A = \{H\} \subset S_1$

For $S_2 = \{(H,H), (H,T), (T,T), (T,H)\}$, eg. we define event A as the first and second tosses have different outcomes.

$A = \{(H,T), (T,H)\} \subset S_2$

For $S_3 = \mathbb{R}^+$, $A = \{t ; t \geq 2 \text{ hours}\} \subset S_3$

(3) $\boxed{\text{Probability function}}$ $Pr(\cdot)$ is a function of event and

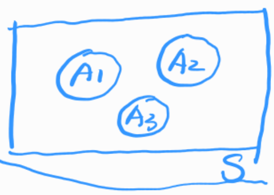
i) $0 \leq Pr(A) \leq 1$, $\forall A \subset S$

ii) $Pr(S) = 1$

iii) If $A_1, A_2, A_3 \cdots$ are pairwise mutally exclusive

then, $Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(A_i)$

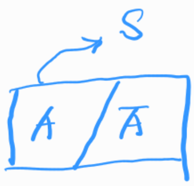
for $\forall i \neq j$
$A_i \cap A_j = \Phi$



eg. For $S_2 = \{(H,H), (H,T), (T,T), (T,H)\}$

Define a probability function as $Pr(A) = \dfrac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } S}$ ...... (*)

You can check that $Pr(\cdot)$ is a probability function.



Properties of probability functions

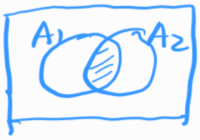
i) $Pr(\Phi) = 0$

ii) Let $\bar{A}$ be the complementary event of A (i.e. $\bar{A} \cup A = S$ & $\bar{A} \cap A = \Phi$)
then $Pr(A) + Pr(\bar{A}) = 1$

iii) if A_1 and A_2 are mutually exclusive. then

$$Pr(A_1 \cup A_2) = P(A_1) + P(A_2)$$

iv) In general, $Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2) - Pr(A_1 \cap A_2)$



v) If $A_1 \subset A_2 \Rightarrow Pr(A_2) \geq Pr(A_1)$

Sample space + events + probability function = probability model.

↳ for a random experiment.

Conditional probability

For two events A & B, $Pr(B) > 0$, $Pr(A) \geq 0$

The conditional probability of A given B is defined as

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Independence of two events

A & B are independent if $Pr(A \cap B) = Pr(A)\,Pr(B)$

consider S_2 and the probability func. in $(*)$

A= " The first toss is head" , B= " The second toss is head"

$= \{(H,H), (H,T)\}$ $\quad$ $= \{(H,H), (T,H)\}$

$Pr(A) = 1/2$. $\quad Pr(B) = 1/2$.

$A \cap B$ = " Both tosses are head" $= \{(H,H)\}$

$Pr(A \cap B) = 1/4 = Pr(A)\,Pr(B)$

1.1.2. Random variable.

We use uppercase letters to denote random variables. X, Y.

1) A Random variable is a function. $X: S \to \mathbb{R}$

The support is the sample space

The range is the real line.

For $S_2 = \{(H,H), (H,T), (T,H), (T,T)\}$. we define a random variable X as the number of heads in the two tosses.

$$X((H,H)) = 2, \quad X((H,T)) = 1, \quad X((T,H)) = 1, \quad X((T,T)) = 0$$

$\{X \leq x\}$ is an event.

If $x=2$. $\{X \leq x\} = S_2$; if $x=1$, $\{X \leq 1\} = \{(H,T), (T,H), (T,T)\}$

we assign probabilities to events.

$$Pr(\{X \leq x\}) = Pr(X \leq x) = \begin{cases} 1. & \text{if } x=2 \\ 3/4 & \text{if } x=1 \\ 1/4 & \text{if } x=0 \end{cases}$$

Cumulative distribution function of a random variable

cdf. denoted by $F(\cdot)$

$F(x) \triangleq Pr(X \le x)$, $x \in \mathbb{R}$.

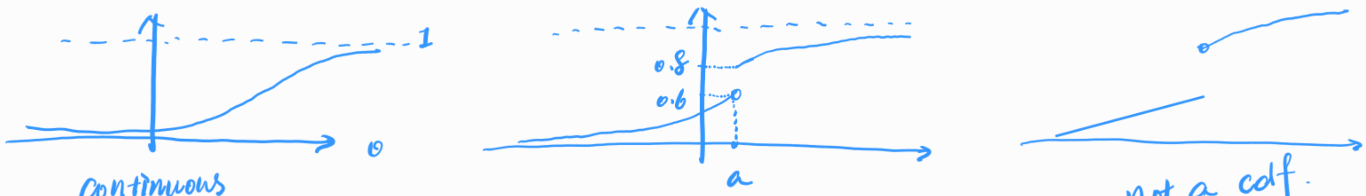
defined as.

(i) $F(x)$ is a non-decreasing function $F(x_1) \le F(x_2)$ if $x_1 \le x_2$

(ii) $\lim\limits_{x \to -\infty} F(x) = 0$ & $\lim\limits_{x \to +\infty} F(x) = 1$

(iii) $F(x)$ is a right continuous function of x

$\lim\limits_{x \to a^+} F(x) = F(a)$



continuous

$F(a) = 0.8$

not a cdf.

(iv) $Pr(a < X \le b) = Pr(X \le b) - Pr(X \le a)$

$= F(b) - F(a)$

(v). $Pr(X = x) = Pr(\text{Jump at } x) = \underline{F(x)} - \underline{\lim\limits_{t \to x^-} F(t)}$

limit from right

limit from left.

what if there is not jump? $Pr(X = x) = ?$