STAT 330: Lecture 2

2024 Spring

May 14, 2024

Review of the Last Lecture

- Probability Model
- Conditional Probability, Independence
- Random Variable & cdf

Overview of This Lecture

- Discrete Random Variables
- Continuous Random Variables

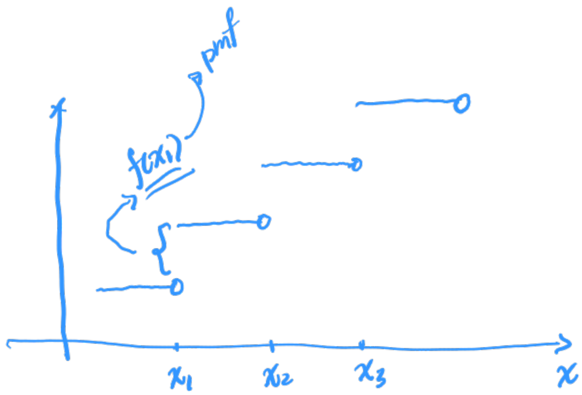
1 Discrete Random Variables rv.

Definition

If a rv X can only take finite or countable number of values, then ....
0/1 0, 1, 2, ...

Cdf

$F(x)$



right-continous step function.

Probability Mass Function pmf.

$$f(x) = \begin{cases} \Pr(X=x), & \text{if } X \text{ can take the value } x \\ 0 & \text{otherwise.} \end{cases}$$

Support of X

$A = \{x : f(x) > 0\}$
all possible values of X.

Properties of pmf.

(1) $f(x) \geq 0$

(2) $\sum_{x \in A} f(x) = 1$. summation over all possible values.
A: support of X

2 Important Discrete Random Variables

Bernoulli $X \sim Bern(p)$, $p \in (0,1)$

Motivation: If toss a coin, and prob. of heads is p. X = # of heads you get.

Support: $\{0, 1\}$

$f(0) = Pr(X=0) = 1-p$

$f(1) = Pr(X=1) = p$.

$$\begin{cases} f(0), f(1) \geq 0 \\ \sum_{x \in A} f(x) = f(0) + f(1) = p + (1-p) = 1. \end{cases}$$

Binomial $X \sim Bin(n, p)$

Motivation: n coins, independent. the prob. of heads is p. X = # of heads you get.

Support: $A = \{0, 1, \cdots, n\}$

pmf: $f(x) = Pr(X=x) = \underbrace{\binom{n}{x}}_{\boxed{\#\text{ of cases}}} \underbrace{p^x (1-p)^{n-x}}_{\boxed{\text{prob. of one case}}}$, $x \in A$.

$$\begin{cases} f(x) \geq 0, x \in A \\ \sum_{x \in A} f(x) = 1. \end{cases}$$

$X = \sum_{i=1}^{n} X_i \rightarrow$ a collection of Bern.

Geometric $X \sim Geo(p)$

Motivation: The # of tails you get before your first heads. p = prob. of heads.

Support: $A = \{0, 1, 2, \cdots\}$

pmf: $f(x) = Pr(X=x) = (1-p)^x p$, $x \in A$.

$f(x) \geq 0$, $\sum_{x \in A} f(x) = 1$.

e.g. $x=2$, × × O $(1-p)^2 p$

$x=0$ O p

$x=3$, × × × O $(1-p)^3 p$

Negative Binomial

$X \sim NegBin(r, p)$

Motivation: X is the # of tails you get, when you first reach r heads.

Support: $A = \{0, 1, 2, \cdots\}$

pmf: $f(x) = Pr(X=x) = \binom{x+r-1}{x}(1-p)^x p^{r-1} \cdot p$

$= \binom{x+r-1}{x} \cdot (1-p)^x \cdot p^r$.

$X = \sum_{i=1}^{r} X_i$. $X_i \sim Geo(p)$.

Negative-Binomial & Geo relation.

e.g. $X = 2$. $\boxed{r=2}$

× × O O

O × × O

× O × O

$\binom{3}{2}(1-p)^2 p \times p$

$= \binom{3}{2} \cdot (1-p)^2 p^2$

Poisson

$X \sim Poi(\mu)$.

Poisson dist. is the limit of Bin. when $n \to \infty$.

$f(x) = \underbrace{Pr(X=x)}_{\text{Binomial pmf}} = \binom{n}{x} p^x (1-p)^{n-x}$

When $n \to \infty$, $\mu = np > 0$

$$f(x) = \frac{n!}{(n-x)!\,x!} \cdot \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

$$= \frac{\mu^x}{x!} \cdot \boxed{\frac{n!}{(n-x)!}}\ \boxed{\frac{1}{n^x}}\ \underbrace{\boxed{\left(1-\frac{\mu}{n}\right)^n}}_{\to e^{-\mu}}\ \boxed{\left(1-\frac{\mu}{n}\right)^{-x}} \to 1 \longrightarrow \boxed{\frac{\mu^x}{x!} \cdot \exp(-\mu)} \rightarrow \text{Poisson pmf.}$$

$$\frac{n(n-1)\cdots(n-x+1)}{n^x} = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \to 1$$

Support: $A = \{0, 1, 2, \cdots\}$

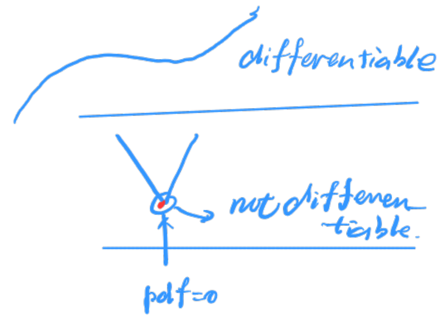
$$\begin{cases} f(x) \geq 0 \\ \sum_{x \in A} f(x) = 1. \end{cases}$$

3 Continuous Random Variable

Definition

If the possible values of X is an interval or real line.
then. X is a cont. r.v. e.g. $[0,1]$ $\mathbb{R}$.

Cdf ① $F(x)$: is a continuous function.
② $F(x)$ is almostly surely differentiable.
it is not differentiable for at most Countable # of points.



Probability Density Function pdf

$$f(x) = \begin{cases} F'(x), & \text{if } F(x) \text{ is differentiable at } x. \\ 0. & \text{otherwise.} \end{cases}$$

Support

$A = \{x : f(x) > 0\}$: possible value of x.

Properties

① $f(x) \geq 0$ ② $\int_{-\infty}^{+\infty} f(x)dx = 1$ $\left(\int_{-\infty}^{\infty} f(x)dx = F(x)\Big|_{-\infty}^{+\infty} = F(\infty) - F(-\infty) = 1 - 0 = 1 \right)$

Relation btw. $F(x)$. $f(x)$.

③ $F(x) = \int_{-\infty}^{x} f(t)\,dt$.

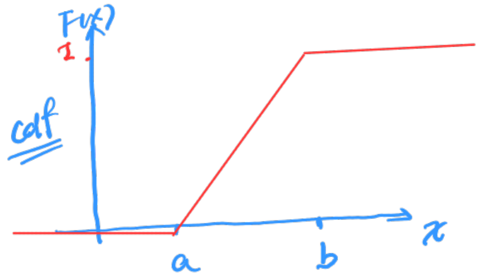
④ $f(x) = F'(x)$

⑤ $Pr(X=x) = 0$

⑥ $Pr(a < X \leq b) = Pr(a \leq X < b)$
$= Pr(a \leq X \leq b) = P(a < X < b)$
$= \int_a^b f(x)dx$.

Example

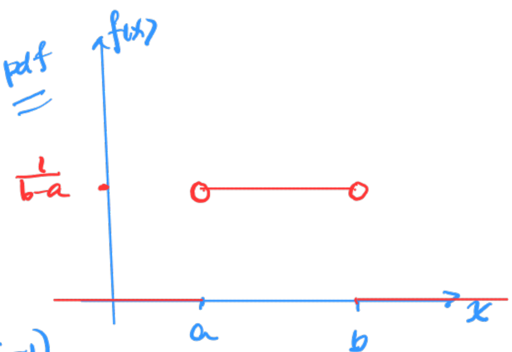
① Suppose $F(x) = \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a < x < b \\ 1 & x \ge b \end{cases}$, find the pdf.



$\underline{\underline{\text{cdf}}}$

$f(x) = \begin{cases} 0, & x \le a \text{ or } x \ge b. \\ \frac{1}{b-a} & a < x < b \end{cases}$

$\underline{\underline{\text{pdf}}}$



② $f(x) = \begin{cases} \frac{\theta}{x^{\theta+1}}, & x \ge 1 \\ 0, & x < 1 \end{cases}$

1). What is the value of θ?

$f(x) \ge 0 \Rightarrow \theta \ge 0$, if $\theta = 0$. $f(x)$ is not a pdf

$\Rightarrow \theta > 0$

$\int_{-\infty}^{\infty} f(x)dx = 1$ $\qquad \int_1^{+\infty} \frac{\theta}{x^{\theta+1}} dx = -x^{-\theta}\Big|_1^{+\infty} = (-0) - (-1) = 1$

↳ always true for any $\theta > 0$.

2). find $F(x)$?

$F(x) = \begin{cases} \int_1^x \frac{\theta}{x^{\theta+1}} dx = 1 - x^{-\theta}, & \text{for } x > 1 \\ 0 & \text{for } x \le 1 \end{cases}$

3). $\Pr(2 < x < 3)$?

$P(-2 < x < 3)$?

$\Pr(2 < x < 3) = F(3) - F(2)$

$\Pr(-2 < x < 3) = F(3) - F(-2)$