Solution

Q1.

(a) When $2n-1$ votes are counted, since candidate A never trails candidate B, the number is odd, then votes for A must be more than votes for B, and the corresponding number of ways is q_{2n-1}. Then, for $2n$ votes counted, since the first $2n-1$ are as described above, the $2n$-th vote can be either for A or for B, both situations can guarantee that A does not trails B, that is, $q_{2n}=2q_{2n-1}$, $n>0$.

When $2n$ votes are counted, since candidate A never trails candidate B, the number is even, then votes for A must be more than or equal to votes for B, and the corresponding number of ways is q_{2n}. Among them, the number of ways that A and B have the same amount of votes is c_n, then the number of ways that A has more votes than B is $q_{2n}-c_n$. Now for the $(2n+1)$-th vote, it can only be for A if A and B have the same amount of votes in the first $2n$ votes, and it can be either A or B if A has more votes than B in the first $2n$ votes, that is, $q_{2n+1}=2\left(q_{2n}-c_n\right)+c_n=2q_{2n}-c_n$, $n\geq 0$.

(b) We have

$$Q(x)=\sum\nolimits_n q_n x^n=\sum\nolimits_{n\ is\ odd} q_n x^n+\sum\nolimits_{n\ is\ even} q_n x^n$$

Using $q_{2n+1}=2q_{2n}-c_n$, we have

$$\begin{aligned}\sum\nolimits_{n\ is\ odd} q_n x^n &=\sum\nolimits_{n\ is\ odd}\left(2q_{n-1}-c_{\frac{n-1}{2}}\right)x^n=2x\sum\nolimits_{n\ is\ odd} q_{n-1}x^{n-1}-\sum\nolimits_{n\ is\ odd} c_{\frac{n-1}{2}}x^n\\ &=2x\sum\nolimits_{n\ \text{is even}} q_n x^n-\sum\nolimits_{n\ is\ odd} c_{\frac{n-1}{2}}x^n=2x\sum\nolimits_{n\ \text{is even}} q_n x^n-\sum\nolimits_n c_n x^{2n+1}\end{aligned}$$

Using $q_{2n}=2q_{2n-1}$, we have

$$\begin{aligned}\sum\nolimits_{n\ is\ even} q_n x^n &=1+\sum\nolimits_{n>0,\ n\ is\ even} q_n x^n=1+\sum\nolimits_{n>0,\ n\ is\ even} 2q_{n-1}x^n\\ &=1+2x\sum\nolimits_{n>0,\ n\ is\ even} q_{n-1}x^{n-1}=1+2x\sum\nolimits_{n\ \text{is odd}} q_n x^n\end{aligned}$$

Let $\sum\nolimits_{n\ is\ odd} q_n x^n=s$, $\sum\nolimits_{n\ is\ even} q_n x^n=t$, we have

$$\begin{cases} s = 2xt - \sum_n c_n x^{2n+1} \\ t = 1 + 2xs \end{cases}$$

By solving this set of equations, we obtain that

$$\begin{cases} s = \dfrac{2x - \sum_n c_n x^{2n+1}}{1-4x^2} \\ t = \dfrac{1 - 2x\sum_n c_n x^{2n+1}}{1-4x^2} \end{cases}$$

Using $C(x) = \sum_n c_n x^n = \dfrac{1-\sqrt{1-4x}}{2x}$, we can deduce that

$$\sum_n c_n x^{2n+1} = x\sum_n c_n \left(x^2\right)^n = x\frac{1-\sqrt{1-4x^2}}{2x^2} = \frac{1-\sqrt{1-4x^2}}{2x}$$

Hence we finally have

$$\begin{aligned} Q(x) &= \sum_{n \text{ is odd}} q_n x^n + \sum_{n \text{ is even}} q_n x^n = s + t = \frac{2x - \sum_n c_n x^{2n+1}}{1-4x^2} + \frac{1 - 2x\sum_n c_n x^{2n+1}}{1-4x^2} \\ &= \frac{2x - \dfrac{1-\sqrt{1-4x^2}}{2x}}{1-4x^2} + \frac{1 - 2x\dfrac{1-\sqrt{1-4x^2}}{2x}}{1-4x^2} \\ &= \frac{\dfrac{4x^2 - 1 + \sqrt{1-4x^2}}{2x}}{1-4x^2} + \frac{\sqrt{1-4x^2}}{1-4x^2} = -\frac{1}{2x} + \frac{\sqrt{1-4x^2}}{2x-4x^2} \end{aligned}$$