Q1.

(a) Firstly we have

$$A^T A = \begin{bmatrix} 2 & 1 & -2 & 4 \\ 1 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -2 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -2 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 4 \\ 1 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 & -2 & 8 \\ 4 & 5 & 2 & 4 \\ -2 & 2 & 8 & -8 \\ 8 & 4 & -8 & 16 \end{bmatrix}$$

By solving $\det\left(\lambda I - A^T A\right) = 0$, we obtain eigenvalues $\lambda_1 = 25$ and $\lambda_2 = 9$, then by solving $\left(\lambda I - A^T A\right) v = 0$, we obtain corresponding eigenvectors $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

Also, $\lambda_1 = 25$ and $\lambda_2 = 9$ are also eigenvalues of AA^T which can be verified by $\det\left(\lambda I - AA^T\right) = 0$, then by solving $\left(\lambda I - AA^T\right) u = 0$, we obtain corresponding eigenvectors $u_1 = \begin{bmatrix} 2/5 & 1/5 & -2/5 & 4/5 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 1/3 & 2/3 & 2/3 & 0 \end{bmatrix}^T$.

Let $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$, $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, then we have the singular value decomposition of A as

$$A = U\Sigma V^T = \begin{bmatrix} 2/5 & 1/3 \\ 1/5 & 2/3 \\ -2/5 & 2/3 \\ 4/5 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) From part (a), we have singular value decomposition of A as $A = U\Sigma V^T$, then we can easily obtain the singular value decomposition of A^T as

$$A^T = \left(U\Sigma V^T\right)^T = V\Sigma^T U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/5 & 1/5 & -2/5 & 4/5 \\ 1/3 & 2/3 & 2/3 & 0 \end{bmatrix}$$

Q2.

By the Shur decomposition, matrix A can be written as

$$A = UDU^H$$

where U is a unitary matrix and D is an upper-triangle matrix. Since matrix A is normal, we have $AA^H = A^H A$, that is

$$\left(UDU^H\right)\left(UDU^H\right)^H = \left(UDU^H\right)^H \left(UDU^H\right)$$

$$DD^H = D^H D$$

Then matrix D must be a diagonal matrix whose diagonal elements are the eigenvalues of matrix A, then

$$A^H A = \left(UDU^H\right)^H \left(UDU^H\right) = UD^H DU^H = U\left(D^H D\right)U^H$$

where $D^H D$ is also a diagonal matrix whose diagonal elements are square of the diagonal elements of D, that is, eigenvalues of $A^H A$ are square of eigenvalues of A.

According to the definition of singular value, matrix A's singular values are arithmetic square roots of $A^H A$'s eigenvalues, thus for A's eigenvalues λ_i, it has corresponding singular value $\sigma_i = \sqrt{\lambda_i^2} = |\lambda_i|$.