Solution to Quiz 11

Q1.

(a) Since $A(x)=\frac{1+x}{(1+2x)(1-3x)}=\frac{1}{5}\frac{1}{1+2x}+\frac{4}{5}\frac{1}{1-3x}$ is given, we have

$$A'(x)=\frac{-2}{5}\frac{1}{(1+2x)^2}+\frac{12}{5}\frac{1}{(1-3x)^2}$$

$$A''(x)=\frac{8}{5}\frac{1}{(1+2x)^3}+\frac{72}{5}\frac{1}{(1-3x)^3}$$

$$A'''(x)=\frac{-48}{5}\frac{1}{(1+2x)^4}+\frac{648}{5}\frac{1}{(1-3x)^4}$$

it follows by Taylor's Theorem that

$$a_0=A(0)=\frac{1}{5}\frac{1}{1+2\times 0}+\frac{4}{5}\frac{1}{1-3\times 0}=1$$

$$a_1=A'(0)=\frac{-2}{5}\frac{1}{(1+2\times 0)^2}+\frac{12}{5}\frac{1}{(1-3\times 0)^2}=2$$

$$a_2=\frac{A''(0)}{2!}=\frac{\frac{8}{5}\frac{1}{(1+2\times 0)^3}+\frac{72}{5}\frac{1}{(1-3\times 0)^3}}{2!}=8$$

$$a_3=\frac{A'''(0)}{3!}=\frac{\frac{-48}{5}\frac{1}{(1+2\times 0)^4}+\frac{648}{5}\frac{1}{(1-3\times 0)^4}}{3!}=20$$

According to the recursion equation $a_{n+2}=c_1a_{n+1}+c_2a_n$, we have

$$\begin{cases} a_2=c_1a_1+c_2a_0 \\ a_3=c_1a_2+c_2a_1 \end{cases}$$

That is

$$\begin{cases} 8 = 2c_1 + c_2 \\ 20 = 8c_1 + 2c_2 \end{cases}$$

By solving this set of equations, we finally obtain $c_1 = 1$ and $c_2 = 6$, thus the recursion equation is $a_{n+2} = a_{n+1} + 6a_n$, and we have initial conditions $a_0 = 1$ and $a_1 = 2$.

(b) We have

$$A(x) = \frac{1+x}{(1+2x)(1-3x)} = \frac{1}{5}\frac{1}{1+2x} + \frac{4}{5}\frac{1}{1-3x}$$

$$= \sum_{n\geq 0}\frac{1}{5}(-2x)^n + \sum_{n\geq 0}\frac{4}{5}(3x)^n = \sum_{n\geq 0}\frac{(-2)^n + 4\times 3^n}{5}x^n$$

Then we know that

$$a_n = \frac{(-2)^n + 4\times 3^n}{5}$$

Q2.

(a) By observing the first 10 terms in the sequence, we can deduce that the recurrence equation is

$$g_{n+2} = g_n + g_{n+1}, \quad n = 0,1,2,3,4,\cdots$$

(b) With generating function $G(x) = \sum_n g_n x^n$, we have

$$\sum_n g_{n+1}x^{n+1} = \sum_n g_n x^n - g_0 x^0 = G(x) - 4$$

$$\sum_n g_{n+2}x^{n+2} = \sum_n g_n x^n - g_0 x^0 - g_1 x^1 = G(x) - 4 - x$$

Also, using the recurrence we deduced in part (a), we have

$$\sum_n g_{n+2}x^{n+2} = \sum_n (g_n + g_{n+1})x^{n+2} = x^2\sum_n g_n x^n + x\sum_n g_{n+1}x^{n+1}$$

That is

$$G(x)-4-x=x^2\left[G(x)\right]+x\left[G(x)-4\right]$$

$$\left(x^2+x-1\right)G(x)=3x-4$$

We finally obtain the close form generating function as

$$G(x)=\frac{3x-4}{x^2+x-1}$$