Solution to Math 481 – Quiz 5

Q1.

(a) In canonical form, σ can be written as

$$\sigma=\{\{1,7\},\{2,4,5\},\{3,8\},\{6\}\}$$

(b) In standard block form, δ can be written as

$$\delta=\{\{1,3,8\},\{2,5,6,7\},\{4\},\{9\},\{10\}\}$$

Q2.

From the textbook, we know that for all positive integers n and k, the number of compositions of n into k parts is $\binom{n-1}{k-1}$.

Then the number of compositions of n into an even number of parts can be calculated with addition.

If n is odd, the maximum even number of parts is $n-1$, then the number of compositions of n is $\sum_{k=1}^{\frac{n-1}{2}}\binom{n-1}{2k-1}$.

If n is even, the maximum even number of parts is n, then the number of compositions of n is $\sum_{k=1}^{\frac{n}{2}}\binom{n-1}{2k-1}$.

Q3.

(a) Assuming that the number of all partitions of $[n]$ is denoted by Bell number B_n, then the number of partitions of $[9]$ exclude blocks with the single element 9 is equal to B_9-B_8.

To calculate B_8 and B_9, we use the Bell triangle

1								
1	2							
2	3	5						
5	7	10	15					
15	20	27	37	52				
52	67	87	114	151	203			
203	255	322	409	523	674	877		
877	1080	1335	1657	2066	2589	3263	4140	
4140	5017	6097	7432	9089	11155	13744	17007	21147

From the triangle, we obtain that $B_8 = 4140$ and $B_9 = 21147$, thus the number of partitions of [9] exclude blocks with the single element 9 is $21147 - 4140 = 17007$.

(b) If the number of set partitions of [n] exclude blocks with the single element n is denoted by X_n, then

$$X_n = B_n - B_{n-1}$$

where B_n and B_{n-1} are Bell numbers.

The proof is quite simple, the number of set partitions of [n] is B_n, if we put singleton element n in a block and put other elements in other blocks, we only need to consider the number of partitions of [n-1] which is B_{n-1}, hence we have $X_n = B_n - B_{n-1}$ where Bell numbers satisfy that $B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i$ and $B_0 = B_1 = 1$.