Solution to Quiz 7

Q1. For the left side, we know $\begin{bmatrix} n \\ n-1 \end{bmatrix}$ denotes the number of permutations on $[n]$ with $n-1$ cycles.

Consider $(1)(2)(3)\cdots(n)$, it is the only permutation on $[n]$ with n cycles, to build permutations with $n-1$ cycles, we need to combine any two chosen cycles in $(1)(2)(3)\cdots(n)$ into one cycle, hence the number is equal to the number of ways to pick two elements from n given elements, which is $\binom{n}{2}$. Therefore we have

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$$

Q2. The other 8 elements include (1)(2)(3)(4), (142)(3), (1)(234), (1)(243), (2)(134), (2)(143), (4)(123) and (4)(132).

Q3. For a permutation whose cube is the identity, it must contain cycles of size 1 or 3.

Now we consider the element $n+1$:

If element $n+1$ forms a single cycle $(n+1)$ and we have $g^3(n+1) = n+1$, with the first n elements also forming permutation whose cube is the identity, that is T_n, we combine the cycle $(n+1)$ with the cycles in T_n to build a $n+1$-permutation which obviously satisfies the condition. The number of this kind of cases is obviously t_n.

If element $n+1$ is in a 3-size cycle, we need to find the other 2 elements from the first n elements, the number of ways is $\binom{n}{2} = \frac{n(n-1)}{2}$, within the cycle, we can arrange the three elements in $2! = 2$ ways, outside the cycle, the remaining $n-2$ elements can for a permutation in T_{n-2} so that we can combine them into a $n+1$-permutation whose cube is the identity. The number of this kind of cases is $\frac{n(n-1)}{2}2t_{n-2}=n(n-1)t_{n-2}$

Thus we have $t_{n+1}=t_n+n(n-1)t_{n-2}$.