Solution to Quiz 13

Q1. The Fibonacci number f_n is equal to the number of lists of 1's and 2's with sum as n.

Let s_k denotes the set of lists within which k items are 2's and the other $n-2k$ items are 1's, then we have $f_n = |s_0| + |s_1| + |s_2| + \cdots = \sum_{k\geq 0} |s_k|$.

In any list of s_k, the length of the list is $n-2k+k = n-k$, considering the order, we need to pick k positions from the $n-k$ positions to place 2, hence we have $|s_k| = \binom{n-k}{k}$.

Hence we have

$$f_n = |s_0| + |s_1| + |s_2| + \cdots = \sum_{k\geq 0} |s_k| = \sum_{k\geq 0} \binom{n-k}{k}$$

The proof is finished.

Q2. Let k be the number of days in the first part, since the first part contains 1 holiday, the second part contains 2 holidays and the third part contains 3 field trips, we have $1 \leq k \leq n-5$.

With the number of days in the first part fixed as k, the total number of days of the second part and the third part is $n-k$, then, let l be the number of days in the second part, similarly, we have $2 \leq l \leq n-k-3$, and the number of days of the third part is $n-k-l$.

For a fixed k as the number of days of the first part, there are $\binom{k}{1} = k$ ways to choose 1 holiday during the first part.

For a fixed l as the number of days of the second part, there are $\binom{l}{2}$ ways to choose 2 holidays during the second part.

For a fixed $n-k-l$ as the number of days of the third part, there are $\binom{n-k-l}{3}$ ways to choose 3 days for 3 field trips during the second part, since the order of field trips matters, the ways to arrange the 3 trips is $3!=6$, thus the total number of ways to arrange 3 field trips from $n-k-l$ days is $6\binom{n-k-l}{3}$.

By the product rule, summing over all values of k and l, we finally have

$$s_n = \sum_{k=1}^{n-5}\left\{k\sum_{l=2}^{n-k-3}\left[\binom{l}{2}\times 6\binom{n-k-l}{3}\right]\right\} = 6\sum_{k=1}^{n-5}\left\{k\sum_{l=2}^{n-k-3}\left[\binom{l}{2}\binom{n-k-l}{3}\right]\right\}$$