12.Difference Equation (need notes)

2018年10月25日 23:40

$y' = AY$:

$Y = C_1 V_1 e^{\lambda_1 t} + \cdots + C_n V_n e^{\lambda_n t}$

A *difference equation* is an equation relating the general term y_t in the sequence $\{y_t : t = 0, 1, 2, \ldots\}$ with one or more of the term preceeding y_t. In general, it has the form

$$y_t = F(t, y_{t-1}, y_{t-2}, \ldots, y_{t-d})$$

for some positive integer $d \geq 1$ called the *order* of the difference equation. A difference equation is also called a *recurrence relation*.

Notice that we write y_t instead of $y(t)$ for sequences (that is, when y is a function of t, where t only takes a discrete set of values $t = 0, 1, 2, \ldots$ instead of a continuous set of values). Also notice that any difference equation can be transformed to such a form that y_t is expressed in terms of $y_{t-1}, \ldots, y_{t-d}$. For instance, the difference equation $y_{t+1} - 2y_t = t$ can be tranformed to $y_t - 2y_{t-1} = t - 1$ be replacing t with $t - 1$.

A first order difference equation is a difference equation that relates two consequtive terms in the sequence. It can be written as

$$y_t = F(t, y_{t-1})$$

The general solution will depend on one parameter, and we therefore need one initial condition to determine a unique solution. A second order difference equation has the form

$$y_t = F(t, y_{t-1}, y_{t-2})$$

The general solution will depend on two parameters, and we therefore need two initial condition to determine a unique solution.

1 A *first order linear* difference equation (with constant coefficients) is a first order difference equation that has the form

$$\boxed{y_{t+1} + ay_t = f_t}$$

where a is a given number and f_t is an expression in t. It is called homogenous if $f_t = 0$, in which case it has solution

$$y_{t+1} + ay_t = 0 \quad \Leftrightarrow \quad y_t = C(-a)^t$$

Otherwise, it is called inhomogeneous, and has solution

$$y_t = y_t^h + y_t^p = C(-a)^t + y_t^p$$

where y_t^p is a particular solution of $y_{t+1} + ay_t = f_t$.

2 A *second order linear* difference equation (with constant coefficients) has the form

$$\boxed{y_{t+2} + ay_{t+1} + by_t = f_t}$$

where a, b are given numbers and f_t is an expression in t. It is called homogenous if $f_t = 0$. In this case, we consider the characteristic equation $r^2 + ar + b = 0$, and notice that r is a root in this equation if and only if r^t is a solution of the difference equation. There are three cases to consider:

1. If $a^2 - 4b > 0$, there are two distinct roots $r_1 \neq r_2$, and the general solution is

$$y = C_1 r_1^t + C_2 r_2^t$$

2. If $a^2 - 4b = 0$, there is one double root $r = -a/2$, and the general solution is

$$y = C_1 r^t + C_2 t r^t = (C_1 + C_2 t) r^t$$

3. If $a^2 - 4b < 0$, there are no (real) roots (but two complex roots). The general solution is

$$y = (\sqrt{b})^t (C_1 \cos(\theta t) + C_2 \sin(\theta t))$$

where θ is a number such that $\cos(\theta) = -\frac{a}{2\sqrt{b}}$.

If f_t is non-zero, the difference equation is called inhomogeneous, and it has solution

$$y_t = y_t^h + y_t^p$$

where y_t^h is the general solution of the homogeneous equation $y_{t+2} + ay_{t+1} + by_t = 0$, and y_t^p is a particular solution of $y_{t+2} + ay_{t+1} + by_t = f_t$.

To find a particular solution y_t^p of a linear difference equation, we may consider f_t, f_{t+1}, f_{t+2} and guess a solution $y_t = y_t^p$ such that

- The guess for y contains parameter that we may later adjust
- The guess for y has the same form as f_t, f_{t+1}, f_{t+2}

When a guess for y_t is given, we compute y_{t+1}, y_{t+2} and insert it in the difference equation to see if it is a solution for any choices of the parameters. In that case, we have found a particular solution y^p. If the initial guess for y does not work, we may try to multiply it by t (and thereafter by t^2, t^3, etc if necessary).

A differential or difference equation is *globally asymptotically stable* if its general solution y (with $y = y(t)$ for a differential equation or $y = y_t$ for a difference equation) satisfy

$$\bar{y} = \lim_{t \to \infty} y$$

exists (that is, $\bar{y}$ it is finite) and is independent of the undetermined coefficients. The interpretation of globally asymptotically stable equations is that there is a long term equilibrium $\bar{y}$ that is independent of the initial conditions.