10.First Order Differential Equations

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10.1 Main concepts

A *differential equation* in the function $y = y(t)$ is an equation that contains the derivative $y' = y'(t)$, or higher order derivatives, such as $y'' = y''(t)$ and $y''' = y'''(t)$, or both. A *solution* is a function $y = y(t)$ that fits in the differential equation. There are usually many solutions. The general form of a solution is called the *general solution*.

A *first order differential equation* involves only y', y and t since the *order* of a differential equation is the highest order of the derivatives involved. In general, a first order differential equation has the form

$$y' = F(t, y)$$

where $F(t, y)$ is some expression in t and y. The general solution of a first order differential equation will depend on *one* free variable. An *initial value problem* consists of the differential equation and an initial condition, and has a unique solution, called the *particular solution*.

Separable differential equations. A first order differential equation is *separable* if it can be written in the form

$$y' = f(t) \cdot g(y)$$

for some expressions $f(t)$ in t and $g(y)$ in y. It can be solved using *separation of the variables*:

$$y' = f(t)g(y) \quad \Leftrightarrow \quad \frac{1}{g(y)} y' = f(t) \quad \Leftrightarrow \quad \int \frac{1}{g(y)} dy = \int f(t)\, dt$$

After the integrals are computed, one gets a solution in implicit form. To find an explicit form $y = y(t)$, one must solve for y.

Linear differential equations. A first order differential equation is *linear* if it can be written in the form

$$y' + a(t)y = b(t) \quad \Leftrightarrow \quad y' = b(t) - a(t)y$$

for some expressions $a(t), b(t)$ in t. It can be solved using *integrating factor*. The integrating factor is given by

$$u = e^{\int a(t)\, dt}$$

and multiplying with $u = u(t)$ in the differential equtation $y' + a(t)y = b(t)$ gives

$$uy' + a(t)uy = b(t)u \quad \Leftrightarrow \quad (uy)' = b(t)u \quad \Leftrightarrow \quad uy = \int b(t)u\, dt$$

The solution in explicit form is therefore given by

$$y = \frac{1}{u} \cdot \int b(t)u\, dt$$

If $a(t) = a$ and $b(t) = b$ are constants, then the solution is $y = b/a + Ce^{-at}$.

Exact differential equations. A first order differential equation is *exact* if it can be written in the form

$$p(t, y) \cdot y' + q(t, y) = 0$$

for some expressions $p(t, y), q(t, y)$ in t and y that satisfies the *exactness condition* $p'_t = q'_y$. This is the case if and only if there is a function $h = h(t, y)$ such that the equations

$$h'_y = p(t, y) \quad \text{and} \quad h'_t = q(t, y)$$

and then the solution of the differential equation is given by $h(t, y) = C$. To find an explicit form $y = y(t)$, one must solve for y.

exact: $p(t,y) \cdot y' + q(t,y) = 0$

exactness condition: $p'_t = q'_t$

have $h(t,y)$: $h'_y = p(t,y)$, $h'_t = q(t,y)$

$h(t,y) = C$

DE: differential equations

SDE: S~C ~ DE

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general solution ti

particular solution

$y' = f(t) \cdot g(y)$

$\Rightarrow \int \frac{1}{g(y)} dy = \int f(t) dt$

$y' + a(t)y = b(t)$

$\Rightarrow y' = b(t) - a(t)y$

$\boxed{u' = a(t)u}$

$u = e^{\int a(t) dt}$

$\Rightarrow uy' + a(t)uy = b(t)u \Rightarrow (uy)' = b(t)u$

$\Rightarrow uy = \int b(t) u dt$

$\Rightarrow y = \frac{1}{u} \int b(t) u dt$

if $a(t) = a$, $b(t) = b$

$\Rightarrow y = \frac{b}{a} - Ce^{-at}$