11.Second Order Differential Equations

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11.1 Main concepts

A *second order differential equation* may contain y'', y', y and t. The general solution of a second order differential equation will depend on *two* free variables. An *initial value problem* consists of the differential equation and two initial conditions, and has a unique solution, called the *particular solution*.

A second order differential equation is *linear* (with constant coefficients) if it can be written in the form

$$\boxed{y'' + ay' + by = f(t)}$$

linear

for some numbers a, b and an expression $f(t)$ in t. It is *homogeneous* if it can be written $y'' + ay' + by = 0$ (the case where $f = 0$), and *inhomogeneous* otherwise.

Proposition 11.1 (Superposition principle). *If y_1 is a solution of the differential equation $y'' + ay' + by = f_1(t)$ and y_2 is a solution of $y'' + ay' + by = f_2(t)$, then the linear combination $c_1y_1 + c_2y_2$ is a solution of $y'' + ay' + by = c_1f_1(t) + c_2f_2(t)$ for any given numbers c_1, c_2.*

1) **Homogeneous case.** To solve the differential equation $\boxed{y'' + ay' + by = 0}$, we find the roots of the characteristic equation $r^2 + ar + b = 0$, given by

$$\boxed{r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}}$$

since $y = e^{rt}$ is a solution if and only if r is a root. There are three subcases to consider:

1. If $a^2 - 4b > 0$, there are two distinct roots $r_1 \neq r_2$, and the general solution is
$$\boxed{y = C_1e^{r_1t} + C_2e^{r_2t}}$$
2. If $a^2 - 4b = 0$, there is one double root $r = -a/2$, and the general solution is
$$\boxed{y = C_1e^{rt} + C_2te^{rt} = (C_1 + C_2t)e^{rt}}$$
3. If $a^2 - 4b < 0$, there are no (real) roots (but two complex roots $r = \alpha \pm \beta\sqrt{-1}$). The general solution is
$$\boxed{y = e^{\alpha t}(C_1\cos(\beta t) + C_2\sin(\beta t))}$$
where $\alpha = -a/2$ and $\beta = \sqrt{4b - a^2}/2$.

2) **Inhomogeneous case.** By the superposition principle, the differential equation $y'' + ay' + by = f(t)$ has general solution of the form $y = y_h + y_p$, where y_h is the general solution of the corresponding *homogeneous* equation $y'' + ay' + by = 0$, and y_p is a *particular* solution of $y'' + ay' + by = f(t)$.

To find a particular solution y_p of $y'' + ay' + by = f(t)$, we may consider $f(t), f'(t), f''(t)$ and guess a solution $y = y_p$ such that

- The guess for y contains parameter that we may later adjust
- The guess for y has the same form as $f(t), f'(t), f''(t)$

When a guess for y is given, we compute y', y'' and insert it in $y'' + ay' + by = f(t)$ to see if it is a solution for any choices of the parameters. In that case, we have found a particular solution y_p. If the initial guess for y does not work, we may try to multiply it by t (and thereafter by t^2, t^3, etc if necessary).

Superposition principle for first order linear differential equations. We may also use the superposition principle to solve the first order linear differential equation $y' + ay = b(t)$, where a is a number and $b(t)$ is an expression in t. This is an alternative to the method using integrating factor used in Lecture 10. We obtain the solution $y = y_h + y_p$, and $y_h = Ce^{-at}$ since the characteristic equation in this case is $r + a = 0$ with root $r = -a$. The general solution is therefore

$$\boxed{y = Ce^{-at} + y_p}$$

where y_p is a particular solution of $\boxed{y' + ay = b(t)}$.