8. Constrained Optimization and Second Order Conditions

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A Lagrange problem can be written in the standard form as

$$\max / \min f(\mathbf{x}) \quad \text{when} \begin{cases} g_1(\mathbf{x}) = a_1 \\ g_2(\mathbf{x}) = a_2 \\ \vdots \\ g_m(\mathbf{x}) = a_m \end{cases}$$

where $f, g_1, \ldots, g_m$ are functions in n variables and $a_1, \ldots, a_m$ are given numbers. It can be solved in the following way:

1. Write down all Lagrange conditions (FOC + C) and solve them, to obtain a list of candidates for max/min. Compute the value $f(\mathbf{x})$ of each candidate on the list.
2. Check if there are any admissible points where NDCQ fails. If there are any such points, these are also candidates for max/min, and we compute $f(\mathbf{x})$ for these points as well.
3. There are no other candidates for max/min than the ones found above, so by comparing values we can find the best candidate for max/min (if there are any candidates).
4. If there is a best candidate for max/min, we try to determine if this point is in fact the max/min using either the Second order condition or the Extreme value theorem.

Proposition 8.1 (Second Order Condition). *Let $(\mathbf{x}^*; \lambda^*)$ be a candidate for max or min that satisfies the Lagrange conditions, and consider the function*

$$\mathbf{x} \mapsto \mathscr{L}(\mathbf{x}, \lambda^*)$$

If this function is concave as a function in $\mathbf{x}$, then $\mathbf{x}^*$ is a max, and if it is convex as a function in $\mathbf{x}$ then $\mathbf{x}^*$ is a min.

If we use the Second order condition, it is not necessary to check NDCQ. If we use the Extreme value theorem, we must check that the set D of points satisfying all constraints is bounded; if this is the case, there is a max and a min and therefore the best candidate for max/min is in fact max/min.

A Kuhn-Tucker problem can be written in standard form as

$$\max f(\mathbf{x}) \quad \text{when} \begin{cases} g_1(\mathbf{x}) \leq a_1 \\ g_2(\mathbf{x}) \leq a_2 \\ \vdots \\ g_m(\mathbf{x}) \leq a_m \end{cases}$$

It can be solved in the following way:

1. Write down all Kuhn-Tucker conditions (FOC + C + CSC) and solve them, to obtain a list of candidates for max. Compute the value $f(\mathbf{x})$ of each candidate on the list.
2. Check if there are any admissible points where NDCQ fails. If there are any such points, these are also candidates for max, and we compute $f(\mathbf{x})$ for these points as well.
3. There are no other candidates for max, so by comparing values we can find the best candidate for max (if there are any candidates).
4. If there is a best candidate for max, we try to determine if this point is in fact the max using either the Second order condition or the Extreme value theorem.

Proposition 8.2 (Second Order Condition). *Let $(\mathbf{x}^*; \lambda^*)$ be a candidate for max that satisfies the Kuhn-Tucker conditions, and consider the function*

$$\mathbf{x} \mapsto \mathscr{L}(\mathbf{x}, \lambda^*)$$

If this function is concave as a function in $\mathbf{x}$, then $\mathbf{x}^*$ is a max.

If we use the Second order condition, it is not necessary to check NDCQ. If we use the Extreme value theorem, we must check that the set D of points satisfying all constraints is bounded; if this is the case, there is a max and therefore the best candidate for max is in fact max.