READING

9

Time-Series Analysis

by Richard A. DeFusco, PhD, CFA, Dennis W. McLeavey, CFA, Jerald E. Pinto, PhD, CFA, and David E. Runkle, PhD, CFA

Richard A. DeFusco, PhD, CFA, is at the University of Nebraska-Lincoln (USA). Dennis W. McLeavey, CFA, is at the University of Rhode Island (USA). Jerald E. Pinto, PhD, CFA, is at CFA Institute (USA). David E. Runkle, PhD, CFA, is at Trilogy Global Advisors (USA).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** calculate and evaluate the predicted trend value for a time series, modeled as either a linear trend or a log-linear trend, given the estimated trend coefficients;
☐	**b.** describe factors that determine whether a linear or a log-linear trend should be used with a particular time series and evaluate limitations of trend models;
☐	**c.** explain the requirement for a time series to be covariance stationary and describe the significance of a series that is not stationary;
☐	**d.** describe the structure of an autoregressive (AR) model of order *p* and calculate one- and two-period-ahead forecasts given the estimated coefficients;
☐	**e.** explain how autocorrelations of the residuals can be used to test whether the autoregressive model fits the time series;
☐	**f.** explain mean reversion and calculate a mean-reverting level;
☐	**g.** contrast in-sample and out-of-sample forecasts and compare the forecasting accuracy of different time-series models based on the root mean squared error criterion;
☐	**h.** explain the instability of coefficients of time-series models;
☐	**i.** describe characteristics of random walk processes and contrast them to covariance stationary processes;
☐	**j.** describe implications of unit roots for time-series analysis, explain when unit roots are likely to occur and how to test for them, and demonstrate how a time series with a unit root can be transformed so it can be analyzed with an AR model;

(continued)

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
□	**k.** describe the steps of the unit root test for nonstationarity and explain the relation of the test to autoregressive time-series models;
□	**l.** explain how to test and correct for seasonality in a time-series model and calculate and interpret a forecasted value using an AR model with a seasonal lag;
□	**m.** explain autoregressive conditional heteroskedasticity (ARCH) and describe how ARCH models can be applied to predict the variance of a time series;
□	**n.** explain how time-series variables should be analyzed for nonstationarity and/or cointegration before use in a linear regression;
□	**o.** determine an appropriate time-series model to analyze a given investment problem and justify that choice.

1 INTRODUCTION TO TIME-SERIES ANALYSIS

As financial analysts, we often use time-series data to make investment decisions. A **time series** is a set of observations on a variable's outcomes in different time periods: the quarterly sales for a particular company during the past five years, for example, or the daily returns on a traded security. In this reading, we explore the two chief uses of time-series models: to explain the past and to predict the future of a time series. We also discuss how to estimate time-series models, and we examine how a model describing a particular time series can change over time. The following two examples illustrate the kinds of questions we might want to ask about time series.

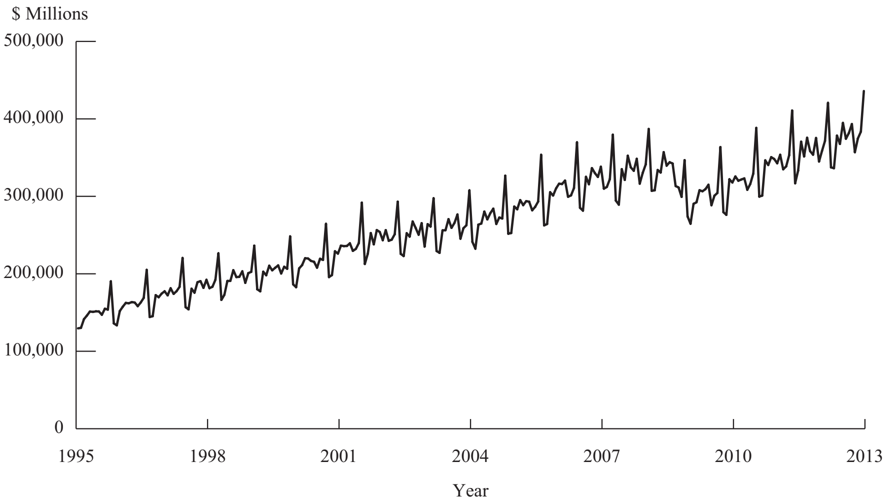
Suppose it is the beginning of 2014 and we are managing a US-based investment portfolio that includes Swiss stocks. Because the value of this portfolio would decrease if the Swiss franc depreciates with respect to the dollar, and vice-versa, holding all else constant, we are considering whether to hedge the portfolio's exposure to changes in the value of the franc. To help us in making this decision, we decide to model the time series of the franc/dollar exchange rate. Figure 1 shows monthly data on the franc/dollar exchange rate. (The data are monthly averages of daily exchange rates.) Has the exchange rate been more stable since 1987 than it was in previous years? Has the exchange rate shown a long-term trend? How can we best use past exchange rates to predict future exchange rates?

Figure 1 Swiss Franc/US Dollar Exchange Rate, Monthly Average of Daily Data

Source: Board of Governors of the Federal Reserve System.

As another example, suppose it is the beginning of 2014. We cover retail stores for a sell-side firm and want to predict retail sales for the coming year. Figure 2 shows monthly data on US retail sales. The data are not seasonally adjusted, hence the spikes around the holiday season at the turn of each year. Because the reported sales in the stores' financial statements are not seasonally adjusted, we model seasonally unadjusted retail sales. How can we model the trend in retail sales? How can we adjust for the extreme seasonality reflected in the peaks and troughs occurring at regular intervals? How can we best use past retail sales to predict future retail sales?

Figure 2 Monthly US Retail Sales



Source: US Department of Commerce, Census Bureau.

Some fundamental questions arise in time-series analysis: How do we model trends? How do we predict the future value of a time series based on its past values? How do we model seasonality? How do we choose among time-series models? And how do we model changes in the variance of time series over time? We address each of these issues in this reading.

The reading[1] is organized as follows. Section 2 describes typical challenges in applying the linear regression model to time series data. Section 3 presents linear and log-linear trend models, which describe, respectively, the value and the natural log of the value of a time series as a linear function of time. Section 4 presents autoregressive time series models—which explain the current value of a time series in terms of one or more lagged values of the series. Such models are among the most commonly used in investments, and the section addresses many related concepts and issues. Section 5 addresses time series that are random walks. Because such time series are not covariance stationary, they cannot be modeled using autoregressive models unless they can be transformed into stationary series. The section explores appropriate transformations and tests of stationarity. Section 6 addresses moving-average time-series models, and Section 7 discusses the problem of seasonality in time series and how to address it. Section 8 covers autoregressive moving-average models, a more complex alternative to autoregressive models. Section 9 addresses modeling changing variance of the error term in a time series. Section 10 examines the consequences of regression of one time series on another when one or both time series may not be covariance stationary.

2 CHALLENGES OF WORKING WITH TIME SERIES

Throughout the reading, our objective will be to apply linear regression to a given time series. Unfortunately, in working with time series we often find that the assumptions of the linear regression model are not satisfied. To apply time-series analysis, we need to assure ourselves that the linear regression model assumptions are met. When those assumptions are not satisfied, in many cases we can transform the time series, or specify the regression model differently, so that the assumptions of the linear regression model are met.

We can illustrate assumption difficulties in the context of a common time-series model, an autoregressive model. Informally, an autoregressive model is one in which the independent variable is a lagged (that is, past) value of the dependent variable, such as the model $x_t = b_0 + b_1x_{t-1} + \varepsilon_t$.[2] Specific problems that we often encounter in dealing with time series include the following:

- The residual errors are correlated instead of being uncorrelated. In the calculated regression, the difference between x_t and $b_0 + b_1x_{t-1}$ is called the residual error (ε_t). The linear regression assumes that this error term is not correlated across observations. The violation of that assumption is frequently more critical in terms of its consequences in the case of time-series models involving past values of the time series as independent variables than for other models (such as cross-sectional) in which the dependent and independent variables are distinct. As we discussed in the reading on multiple regression, in a regression in which the dependent and independent variables are distinct, serial correlation of the errors in this model does not affect the consistency of our estimates of intercept or slope coefficients. By contrast, in an autoregressive

1 Examples in this reading were updated in 2014 by Professor Sanjiv Sabherwal of the University of Texas, Arlington.

2 We could also write the equation as $y_t = b_0 + b_1y_{t-1} + \varepsilon_t$.

time-series regression such as $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$, serial correlation in the error term causes estimates of the intercept (b_0) and slope coefficient (b_1) to be inconsistent.

- The mean and/or variance of the time series changes over time. Regression results are invalid if we estimate an autoregressive model for a time series with mean and/or variance that changes over time.

Before we try to use time series for forecasting, we may need to transform the time-series model so that it is well specified for linear regression. With this objective in mind, you will observe that time-series analysis is relatively straightforward and logical.

TREND MODELS

3

Estimating a trend in a time series and using that trend to predict future values of the time series is the simplest method of forecasting. For example, we saw in Figure 2 that monthly US retail sales show a long-term pattern of upward movement—that is, a **trend**. In this section, we examine two types of trends—linear trends and log-linear trends—and discuss how to choose between them.

3.1 Linear Trend Models

The simplest type of trend is a **linear trend**, one in which the dependent variable changes at a constant rate with time. If a time series, y_t, has a linear trend, then we can model the series using the following regression equation:

$$y_t = b_0 + b_1 t + \varepsilon_t,\ t = 1, 2, \ldots, T \qquad (1)$$

where

y_t = the value of the time series at time t (value of the dependent variable)
b_0 = the y-intercept term
b_1 = the slope coefficient
t = time, the independent or explanatory variable
ε_t = a random-error term

In Equation 1, the trend line, $b_0 + b_1 t$, predicts the value of the time series at time t (where t takes on a value of 1 in the first period of the sample and increases by 1 in each subsequent period). Because the coefficient b_1 is the slope of the trend line, we refer to b_1 as the trend coefficient. We can estimate the two coefficients, b_0 and b_1, using ordinary least squares, denoting the estimated coefficients as $\hat{b}_0$ and $\hat{b}_1$.[3]

Now we demonstrate how to use these estimates to predict the value of the time series in a particular period. Recall that t takes on a value of 1 in Period 1. Therefore, the predicted or fitted value of y_t in Period 1 is $\hat{y}_1 = \hat{b}_0 + \hat{b}_1(1)$. Similarly, in a subsequent period, say the sixth period, the fitted value is $\hat{y}_6 = \hat{b}_0 + \hat{b}_1(6)$. Now suppose that we want to predict the value of the time series for a period outside the sample, say period $T + 1$. The predicted value of y_t for period $T + 1$ is $\hat{y}_{T+1} = \hat{b}_0 + \hat{b}_1(T + 1)$. For example, if $\hat{b}_0$ is 5.1 and $\hat{b}_1$ is 2, then at $t = 5$ the predicted value of y_5 is 15.1 and

3 Recall that ordinary least squares is an estimation method based on the criterion of minimizing the sum of a regression's squared residuals.

at $t = 6$ the predicted value of y_6 is 17.1. Note that each consecutive observation in this time series increases by $\hat{b}_1 = 2$ irrespective of the level of the series in the previous period.

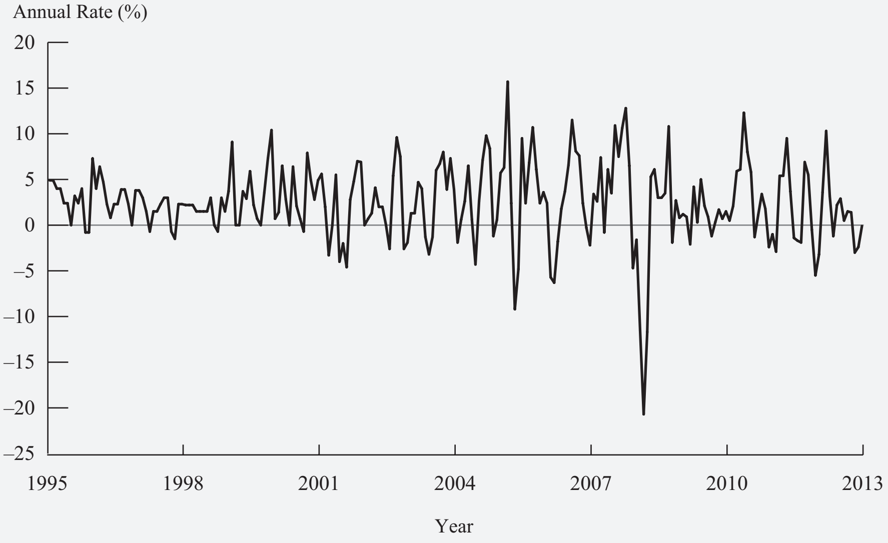
EXAMPLE 1

The Trend in the US Consumer Price Index

It is January 2014. As a fixed income analyst in the trust department of a bank, Lisette Miller is concerned about the future level of inflation and how it might affect portfolio value. Therefore, she wants to predict future inflation rates. For this purpose, she first needs to estimate the linear trend in inflation. To do so, she uses the monthly US Consumer Price Index (CPI) inflation data, expressed as an annual percentage rate,[4] shown in Figure 3. The data include 228 months from January 1995 through December 2013, and the model to be estimated is $y_t = b_0 + b_1 t + \varepsilon_t$, $t = 1, 2, \ldots, 228$. Table 1 shows the results of estimating this equation. With 228 observations and two parameters, this model has 226 degrees of freedom. At the 0.05 significance level, the critical value for a t-statistic is 1.97. The intercept $\left(\hat{b}_0 = 2.8853\right)$ is statistically significant because the value of the t-statistic for the coefficient is well above the critical value. The trend coefficient is negative $\left(\hat{b}_1 = -0.0038\right)$, suggesting a declining trend in inflation during the sample time period. However, the trend is not statistically significant because the absolute value of the t-statistic for the coefficient is well below the critical value. The estimated regression equation can be written as

$$y_t = 2.8853 - 0.0038t$$

Figure 3 Monthly CPI Inflation, Not Seasonally Adjusted



Source: Bureau of Labor Statistics.

4 In these data, 1 percent is represented as 1.0.

Table 1 Estimating a Linear Trend in Inflation Monthly Observations, January 1995–December 2013

Regression Statistics	
R-squared	0.0033
Standard error	4.3297
Observations	228
Durbin–Watson	1.09

	Coefficient	Standard Error	t-Statistic
Intercept	2.8853	0.5754	5.0144
t (Trend)	−0.0038	0.0044	−0.8636

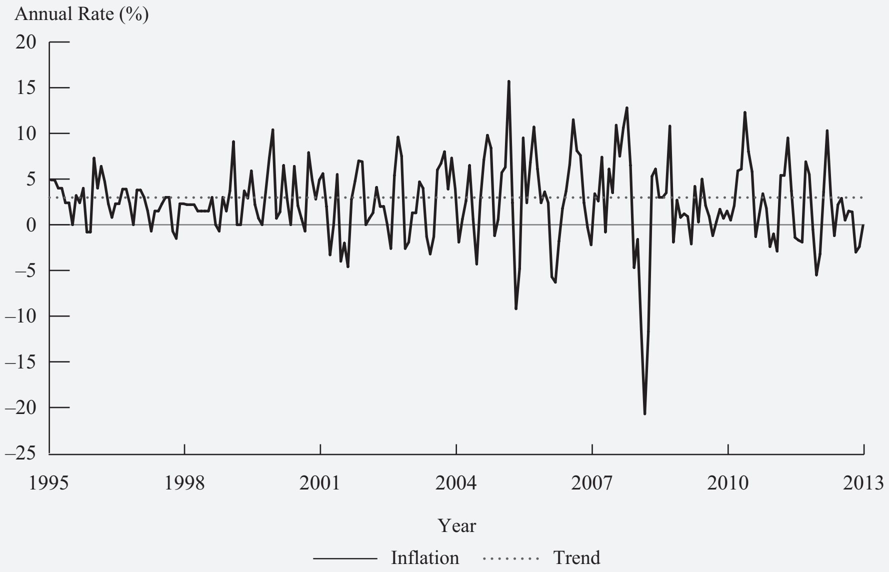
Source: US Bureau of Labor Statistics.

Because the trend line slope is estimated to be −0.0038, Miller concludes that the linear trend model's best estimate is that the annualized rate of inflation declined at a rate of about 38 basis points per month during the sample time period. The decline is not statistically significantly different from zero.

In January 1995, the first month of the sample, the predicted value of inflation is $\hat{y}_1 = 2.8853 - 0.0038(1) = 2.8815$ percent. In December 2013, the 228th or last month of the sample, the predicted value of inflation is $\hat{y}_{228} = 2.8853 - 0.0038(228) = 2.0189$ percent. Note, though, that these predicted values are for in-sample periods. A comparison of these values with the actual values indicates how well Miller's model fits the data; however, a main purpose of the estimated model is to predict the level of inflation for out-of-sample periods. For example, for December 2014 (12 months after the end of the sample), $t = 228 + 12 = 240$, and the predicted level of inflation is $\hat{y}_{240} = 2.8853 - 0.0038(240) = 1.9733$ percent.

Figure 4 shows the inflation data along with the fitted trend. Consistent with the negative but small and statistically insignificant trend coefficient, the fitted trend line is slightly downward sloping. Note that inflation does not appear to be above or below the trend line for a long period of time. No persistent differences exist between the trend and actual inflation. The residuals (actual minus trend values) appear to be unpredictable and uncorrelated in time. Therefore, using a linear trend line to model inflation rates from 1995 through 2013 does not appear to violate the assumptions of the linear regression model. Note also that the R^2 in this model is quite low, indicating great uncertainty in the inflation forecasts from this model. In fact, the estimated model explains only 0.33 percent of the variation in monthly inflation. Although linear trend models have their uses, they are often inappropriate for economic data. Most economic time series reflect trends with changing slopes and/or intercepts over time. The linear trend model identifies the slope and intercept that provides the best linear fit for all past data. The model's deviation from the actual data can be greatest near the end of a data series which can compromise forecasting accuracy. Later in this reading, we will examine whether we can build a better model of inflation than a model that uses only a trend line.

Figure 4 Monthly CPI Inflation with Trend



Source: US Bureau of Labor Statistics.

3.2 Log-Linear Trend Models

Sometimes a linear trend does not correctly model the growth of a time series. In those cases, we often find that fitting a linear trend to a time series leads to persistent rather than uncorrelated errors. If the residuals from a linear trend model are persistent, we then need to employ an alternative model satisfying the conditions of linear regression. For financial time series, an important alternative to a linear trend is a log-linear trend. Log-linear trends work well in fitting time series that have exponential growth.

Exponential growth means constant growth at a particular rate. For example, annual growth at a constant rate of 5 percent is exponential growth. How does exponential growth work? Suppose we describe a time series by the following equation:

$$y_t = e^{b_0 + b_1 t},\ t = 1, 2, \ldots, T \tag{2}$$

Exponential growth is growth at a constant rate $\left(e^{b_1} - 1\right)$ with continuous compounding. For instance, consider values of the time series in two consecutive periods. In Period 1, the time series has the value $y_1 = e^{b_0 + b_1(1)}$, and in Period 2, it has the value $y_2 = e^{b_0 + b_1(2)}$. The resulting ratio of the values of the time series in the first two periods is $y_2/y_1 = \left(e^{b_0 + b_1(2)}\right)\Big/\left(e^{b_0 + b_1(1)}\right) = e^{b_1(1)}$. Generally, in any period t, the time series has the value $y_t = e^{b_0 + b_1(t)}$. In period $t + 1$, the time series has the value $y_{t+1} = e^{b_0 + b_1(t+1)}$. The ratio of the values in the periods $(t + 1)$ and t is $y_{t+1}/y_t = e^{b_0 + b_1(t+1)}\Big/e^{b_0 + b_1(t)} = e^{b_1(1)}$. Thus, the proportional rate of growth in the time series over two consecutive periods is always the same: $(y_{t+1} - y_t)/y_t = y_{t+1}/y_t - 1 = e^{b_1} - 1$.[5] Therefore, exponential growth is growth at a constant rate. Continuous compounding is a mathematical convenience that allows us to restate the equation in a form that is easy to estimate.

If we take the natural log of both sides of Equation 2, the result is the following equation:

$$\ln y_t = b_0 + b_1 t,\ t = 1, 2, \ldots, T$$

Therefore, if a time series grows at an exponential rate, we can model the natural log of that series using a linear trend.[6] Of course, no time series grows exactly at a constant rate. Consequently, if we want to use a **log-linear model**, we must estimate the following equation:

$$\ln y_t = b_0 + b_1 t + \varepsilon_t,\ t = 1, 2, \ldots, T \tag{3}$$

Note that this equation is linear in the coefficients b_0 and b_1. In contrast to a linear trend model, in which the predicted trend value of y_t is $\hat{b}_0 + \hat{b}_1 t$, the predicted trend value of y_t in a log-linear trend model is $e^{\hat{b}_0 + \hat{b}_1 t}$ because $e^{\ln y_t} = y_t$.

Examining Equation 3, we see that a log-linear model predicts that $\ln y_t$ will increase by b_1 from one time period to the next. The model predicts a constant growth rate in y_t of $e^{b_1} - 1$. For example, if $b_1 = 0.05$, then the predicted growth rate of y_t in each period is $e^{0.05} - 1 = 0.051271$ or 5.13 percent. In contrast, the linear trend model (Equation 1) predicts that y_t grows by a constant amount from one period to the next.

Example 2 illustrates the problem of nonrandom residuals in a linear trend model, and Example 3 shows a log-linear regression fit to the same data.

EXAMPLE 2

A Linear Trend Regression for Quarterly Sales at Starbucks

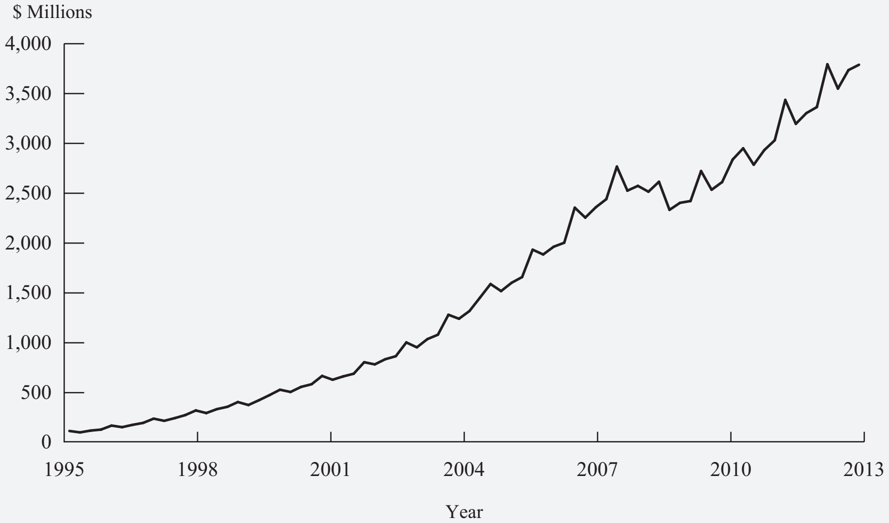
In October 2013, technology analyst Ray Benedict wants to use Equation 1 to fit the data on quarterly sales for Starbucks Corporation shown in Figure 5. Starbucks' fiscal year ends in September. Benedict uses 76 observations on Starbucks' sales from the first quarter of fiscal year 1995 (starting in October 1994) to the fourth quarter of fiscal year 2013 (ending in September 2013) to estimate the linear trend regression model $y_t = b_0 + b_1 t + \varepsilon_t$, $t = 1, 2, \ldots, 76$.[7] Table 2 shows the results of estimating this equation.

5 For example, if we use annual periods and $e^{b_1} = 1.04$ for a particular series, then that series grows by $1.04 - 1 = 0.04$, or 4 percent a year.

6 An exponential growth rate is a compound growth rate with continuous compounding.

7 In discussions of Starbucks' sales in this reading, year refers to Starbucks' fiscal year.

Figure 5 Starbucks Quarterly Sales by Fiscal Year



Source: Compustat.

Table 2 Estimating a Linear Trend in Starbucks Sales

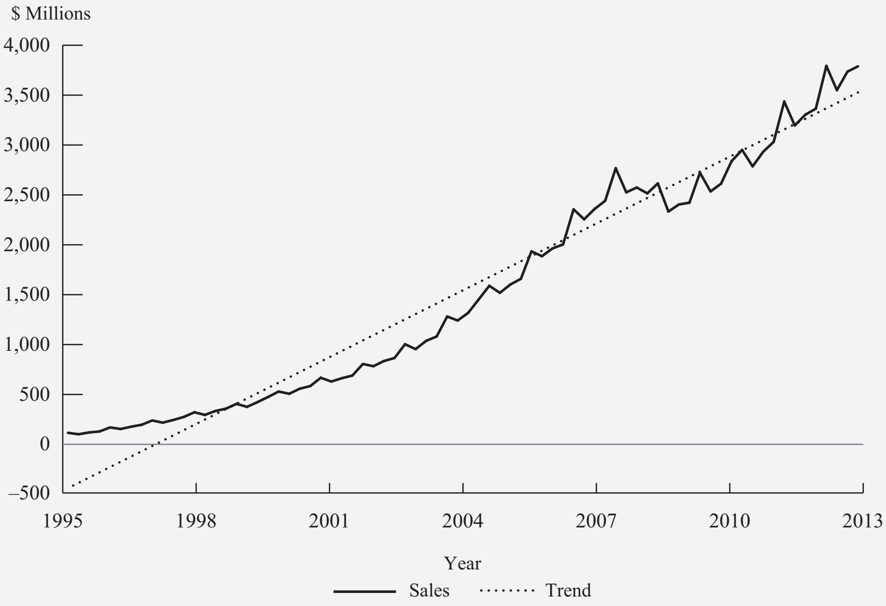
Regression Statistics	
R-squared	0.9595
Standard error	233.21
Observations	76
Durbin–Watson	0.32

	Coefficient	**Standard Error**	***t*-Statistic**
Intercept	−428.5380	54.0345	−7.9308
t (Trend)	51.0866	1.2194	41.8949

Source: Compustat.

At first glance, the results shown in Table 2 seem quite reasonable: Both the intercept and the trend coefficient are highly statistically significant. When Benedict plots the data on Starbucks' sales and the trend line, however, he sees a different picture. As Figure 6 shows, before 1998 the trend line is persistently below sales. Subsequently, until 2006, the trend line is persistently above sales and then varies somewhat thereafter.

Figure 6 Starbucks Quarterly Sales with Trend

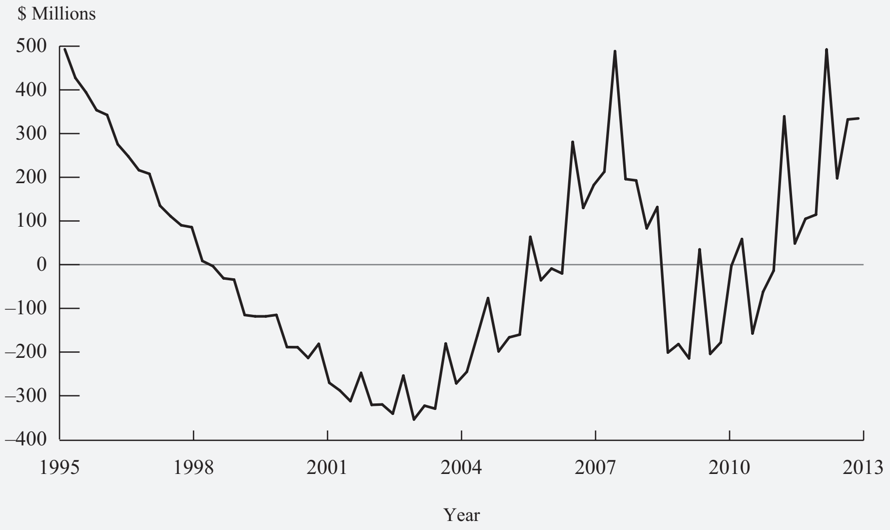


Source: Compustat.

Recall a key assumption underlying the regression model: that the regression errors are not correlated across observations. If a trend is persistently above or below the value of the time series, however, the residuals (the difference between the time series and the trend) are serially correlated. Figure 7 shows the residuals (the difference between sales and the trend) from estimating a linear trend model with the raw sales data. The figure shows that the residuals are persistent: they are consistently positive from 1995 to 1998, 2007 to 2009, and after 2012 and consistently negative from 1999 to 2006.

Because of this persistent serial correlation in the errors of the trend model, using a linear trend to fit sales at Starbucks would be inappropriate, even though the R^2 of the equation is high (0.96). The assumption of uncorrelated residual errors has been violated. Because the dependent and independent variables are not distinct, as in cross-sectional regressions, this assumption violation is serious and causes us to search for a better model.

Figure 7 Residual from Predicting Starbucks Sales with a Trend



Source: Compustat.

EXAMPLE 3

A Log-Linear Regression for Quarterly Sales at Starbucks

Having rejected a linear trend model in Example 2, technology analyst Benedict now tries a different model for the quarterly sales for Starbucks Corporation from the first quarter of 1995 to the fourth quarter of 2013. The curvature in the data plot shown in Figure 5 is a hint that an exponential curve may fit the data. Consequently, he estimates the following linear equation:

$$\ln y_t = b_0 + b_1 t + \varepsilon_t,\ t = 1, 2, \ldots, 76$$

This equation seems to fit the sales data well. As Table 3 shows, the R^2 for this equation is 0.95. An R^2 of 0.95 means that 95 percent of the variation in the natural log of Starbucks' sales is explained solely by a linear trend.

Table 3 Estimating a Linear Trend in Lognormal Starbucks Sales

Regression Statistics	
R-squared	0.9453
Standard error	0.2480
Observations	76
Durbin–Watson	0.12

Table 3 (Continued)

	Coefficient	Standard Error	*t*-Statistic
Intercept	5.1304	0.0575	89.2243
t (Trend)	0.0464	0.0013	35.6923

Source: Compustat.

Although both Equations 1 and 3 have a high R^2, Figure 8 shows how well a linear trend fits the natural log of Starbucks' sales (Equation 3). The natural logs of the sales data lie very close to the linear trend during the sample period, and log sales are not substantially above or below the trend for long periods of time. Thus, a log-linear trend model seems better suited for modeling Starbucks' sales than does a linear trend model.

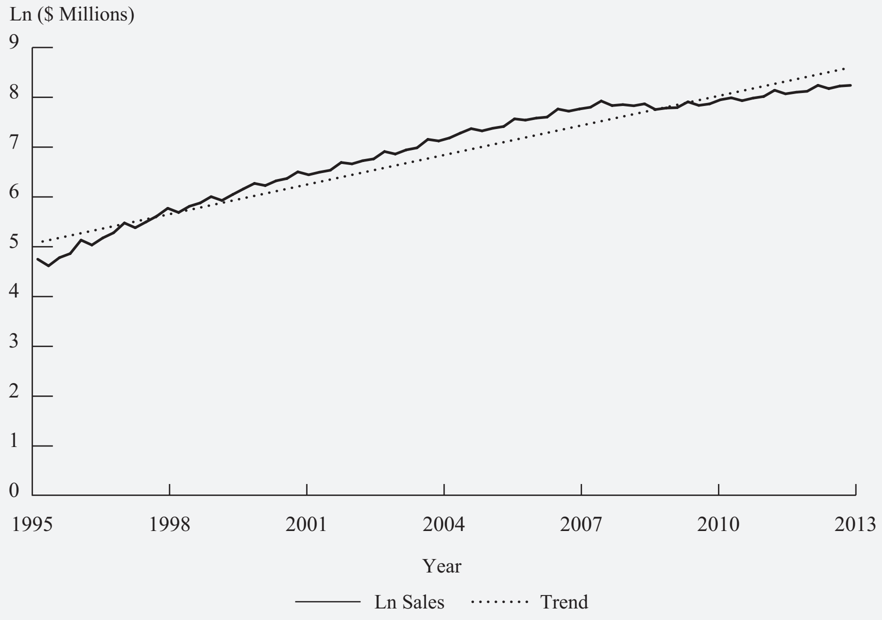
1 Benedict wants to use the results of estimating Equation 3 to predict Starbucks' sales in the future. What is the predicted value of Starbucks' sales for the first quarter of 2014?

Solution to 1:

The estimated value $\hat{b}_0$ is 5.1304, and the estimated value $\hat{b}_1$ is 0.0464. Therefore, for the first quarter of 2014 (t = 77), the estimated model predicts that ln $\hat{y}_{77}$ = 5.1304 + 0.0464(77) = 8.7032 and that predicted sales are $\hat{y} = e^{\ln \hat{y}_{77}} = e^{8.7032}$ = \$6,022.15 million.[8]

8 Note that $\hat{b}_1$ = 0.0464 implies that the exponential growth rate per quarter in Starbucks' sales will be 4.75 percent ($e^{0.0464} - 1 = 0.0475$).

Figure 8 Natural Log of Starbucks Quarterly Sales



Source: Compustat.

2 How much different is the above forecast from the prediction of the linear trend model?

Solution to 2:

Table 2 showed that for the linear trend model, the estimated value of $\hat{b}_0$ is −428.5380 and the estimated value of $\hat{b}_1$ is 51.0866. Thus, if we predict Starbucks' sales for the first quarter of 2014 (t = 77) using the linear trend model, the forecast is $\hat{y}_{77}$ = −428.5380 + 51.0866(77) = $3,505.13 million. This forecast is far below the prediction made by the log-linear regression model. Later in this reading, we will examine whether we can build a better model of Starbucks' quarterly sales than a model that uses only a log-linear trend.

3.3 Trend Models and Testing for Correlated Errors

Both the linear trend model and the log-linear trend model are single-variable regression models. If they are to be correctly specified, the regression-model assumptions must be satisfied. In particular, the regression error for one period must be uncorrelated with the regression error for all other periods.[9] In Example 2 in the previous section, we could infer an obvious violation of that assumption from a visual inspection of a plot of residuals (Figure 7). The log-linear trend model of Example 3 appeared to fit the

9 Note that time-series observations, in contrast to cross-sectional observations, have a logical ordering: They must be processed in chronological order of the time periods involved. For example, we should not make a prediction of the inflation rate using a CPI series in which the order of the observations had been scrambled, because time patterns such as growth in the independent variables can negatively affect the statistical properties of the estimated regression coefficients.

data much better, but we still need to confirm that the uncorrelated errors assumption is satisfied. To address that question formally, we must carry out a Durbin–Watson test on the residuals.

In the reading on regression analysis, we showed how to test whether regression errors are serially correlated using the Durbin–Watson statistic. For example, if the trend models shown in Examples 1 and 3 really capture the time-series behavior of inflation and the log of Starbucks' sales, then the Durbin–Watson statistic for both of those models should not differ significantly from 2.0. Otherwise, the errors in the model are either positively or negatively serially correlated, and that correlation can be used to build a better forecasting model for those time series.

In Example 1, estimating a linear trend in the monthly CPI inflation yielded a Durbin–Watson statistic of 1.09. Is this result significantly different from 2.0? To find out, we need to test the null hypothesis of no positive serial correlation. For a sample with 228 observations and one independent variable, the critical value, d_l, for the Durbin–Watson test statistic at the 0.05 significance level is above 1.77. Because the value of the Durbin–Watson statistic (1.09) is below this critical value, we can reject the hypothesis of no positive serial correlation in the errors. We can conclude that a regression equation that uses a linear trend to model inflation has positive serial correlation in the errors.[10] We will need a different kind of regression model because this one violates the least-squares assumption of no serial correlation in the errors.

In Example 3, estimating a linear trend with the natural logarithm of sales for the Starbucks example yielded a Durbin–Watson statistic of 0.12. Suppose we wish to test the null hypothesis of no positive serial correlation. The critical value, d_l, is above 1.60 at the 0.05 significance level. The value of the Durbin–Watson statistic (0.12) is below this critical value, so we can reject the null hypothesis of no positive serial correlation in the errors. We can conclude that a regression equation that uses a trend to model the log of Starbucks' quarterly sales has positive serial correlation in the errors. So, for this series as well, we need to build a different kind of model.

Overall, we conclude that the trend models sometimes have the limitation that errors are serially correlated. Existence of serial correlation suggests that we can build better forecasting models for such time series than trend models.

4 AUTOREGRESSIVE (AR) TIME-SERIES MODELS

A key feature of the log-linear model's depiction of time series and a key feature of time series in general is that current-period values are related to previous-period values. For example, Starbucks' sales for the current period are related to its sales in the previous period. An **autoregressive model (AR)**, a time series regressed on its own past values, represents this relationship effectively. When we use this model, we can drop the normal notation of y as the dependent variable and x as the independent variable because we no longer have that distinction to make. Here we simply use x_t. For example, Equation 4 shows a first-order autoregression, AR(1), for the variable x_t:

$$x_t = b_0 + b_1 x_{t-1} + \varepsilon_t \tag{4}$$

10 Significantly small values of the Durbin–Watson statistic indicate positive serial correlation; significantly large values point to negative serial correlation. Here the DW statistic of 1.09 indicates positive serial correlation. For more information, see the readings on regression analysis.

Thus, in an AR(1) model, we use only the most recent past value of x_t to predict the current value of x_t. In general, a pth-order autoregression, AR(p), for the variable x_t is shown by

$$x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-2} + \ldots + b_p x_{t-p} + \varepsilon_t \quad (5)$$

In this equation, p past values of x_t are used to predict the current value of x_t. In the next section we discuss a key assumption of time-series models that include lagged values of the dependent variable as independent variables.

4.1 Covariance-Stationary Series

Note that the independent variable (x_{t-1}) in Equation 4 is a random variable. This fact may seem like a mathematical subtlety, but it is not. If we use ordinary least squares to estimate Equation 4 when we have a randomly distributed independent variable that is a lagged value of the dependent variable, our statistical inference may be invalid. To conduct valid statistical inference, we must make a key assumption in time-series analysis: We must assume that the time series we are modeling is **covariance stationary**.[11]

What does it mean for a time series to be covariance stationary? The basic idea is that a time series is covariance stationary if its properties, such as mean and variance, do not change over time. A covariance stationary series must satisfy three principal requirements.[12] First, the expected value of the time series must be constant and finite in all periods: $E(y_t) = \mu$ and $|\mu| < \infty$, $t = 1, 2, \ldots, T$. Second, the variance of the time series must be constant and finite in all periods. Third, the covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods. The second and third requirements can be summarized as follows:[13]

$$\text{Cov}(y_t, y_{t-s}) = \lambda_s,\ |\lambda_s| < \infty,\ t = 1, 2, \ldots, T;\ s = 0, \pm 1, \pm 2, \ldots, \pm T$$

where λ signifies a constant. What happens if a time series is not covariance stationary but we model it using Equation 4? The estimation results will have no economic meaning. For a non-covariance-stationary time series, estimating the regression in Equation 4 will yield spurious results. In particular, the estimate of b_1 will be biased, and any hypothesis tests will be invalid.

How can we tell if a time series is covariance stationary? We can often answer this question by looking at a plot of the time series. If the plot shows roughly the same mean and variance through time without any significant seasonality, then we may want to assume that the time series is covariance stationary.

Some of the time series we looked at in Figures 1 to 4 appear to be covariance stationary. For example, the inflation data shown in Figure 3 appear to have roughly the same mean and variance over the sample period. Many of the time series one encounters in business and investments, however, are not covariance stationary. For example, many time series appear to grow (or decline) steadily through time and so have a mean that is nonconstant, which implies that they are nonstationary. As an example, the time series of quarterly sales in Figure 5 clearly shows the mean

11 "Weakly stationary" is a synonym for covariance stationary. Note that the terms "stationary" or "stationarity" are often used to mean "covariance stationary" or "covariance stationarity," respectively. You may also encounter the more restrictive concept of "strictly" stationary, which has little practical application. For details, see Diebold (2008).

12 In the first requirement, we will use the absolute value to rule out the case in which the mean is negative without limit (minus infinity).

13 When s in this equation equals 0, then this equation imposes the condition that the variance of the time series is finite. This is so because the covariance of a random variable with itself is its variance: $\text{Cov}(y_t, y_t) = \text{Var}(y_t)$.

increasing as time passes. Thus Starbucks' quarterly sales are not covariance stationary.[14] Macroeconomic time series such as those relating to income and consumption are often strongly trending as well. A time series with seasonality (regular patterns of movement with the year) also has a nonconstant mean, as do other types of time series that we discuss later.[15]

Figure 2 showed that monthly retail sales (not seasonally adjusted) are also not covariance stationary. Sales in December are always much higher than sales in other months (these are the regular large peaks), and sales in January are always much lower (these are the regular large drops after the December peaks). On average, sales also increase over time, so the mean of sales is not constant.

Later in the reading, we will show that we can often transform a nonstationary time series into a stationary time series. But whether a stationary time series is original or transformed, a caution applies: Stationarity in the past does not guarantee stationarity in the future. There is always the possibility that a well-specified model will fail when the state of the world changes and yields a different underlying model that generates the time series.

4.2 Detecting Serially Correlated Errors in an Autoregressive Model

We can estimate an autoregressive model using ordinary least squares if the time series is covariance stationary and the errors are uncorrelated. Unfortunately, our previous test for serial correlation, the Durbin–Watson statistic, is invalid when the independent variables include past values of the dependent variable. Therefore, for most time-series models, we cannot use the Durbin–Watson statistic. Fortunately, we can use other tests to determine whether the errors in a time-series model are serially correlated. One such test reveals whether the autocorrelations of the error term are significantly different from 0. This test is a *t*-test involving a residual autocorrelation and the standard error of the residual autocorrelation. As background for the test, we next discuss autocorrelation in general before moving to residual autocorrelation.

The **autocorrelations** of a time series are the correlations of that series with its own past values. The order of the correlation is given by k where k represents the number of periods lagged. When $k = 1$, the autocorrelation shows the correlation of the variable in one period to its occurrence in the previous period. For example, the ***k*th order autocorrelation** (ρ_k) is

$$\rho_k = \frac{\mathrm{Cov}(x_t, x_{t-k})}{\sigma_x^2} = \frac{E\left[(x_t - \mu)(x_{t-k} - \mu)\right]}{\sigma_x^2}$$

where E stands for the expected value. Note that we have the relationship $\mathrm{Cov}(x_t, x_{t-k}) \le \sigma_x^2$ with equality holding when $k = 0$. This means that the absolute value of ρ_k is less than or equal to 1.

14 In general, any time series accurately described with a linear or log-linear trend model is not covariance stationary, although a transformation of the original series might be covariance stationary.

15 In particular, random walks are not covariance stationary.

Of course, we can never directly observe the autocorrelations, ρ_k. Instead, we must estimate them. Thus we replace the expected value of x_t, μ, with its estimated value, $\bar{x}$, to compute the estimated autocorrelations. The kth order estimated autocorrelation of the time series x_t, which we denote $\hat{\rho}_k$, is

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^{T}\left[\left(x_t - \bar{x}\right)\left(x_{t-k} - \bar{x}\right)\right]}{\sum_{t=1}^{T}\left(x_t - \bar{x}\right)^2}$$

Analogous to the definition of autocorrelations for a time series, we can define the autocorrelations of the error term for a time-series model as[16]

$$\begin{aligned}\rho_{\varepsilon,k} &= \frac{\text{Cov}\left(\varepsilon_t, \varepsilon_{t-k}\right)}{\sigma_\varepsilon^2} \\ &= \frac{E\left[\left(\varepsilon_t - 0\right)\left(\varepsilon_{t-k} - 0\right)\right]}{\sigma_\varepsilon^2} \\ &= \frac{E\left(\varepsilon_t \varepsilon_{t-k}\right)}{\sigma_\varepsilon^2}\end{aligned}$$

We assume that the expected value of the error term in a time-series model is 0.[17]

We can determine whether we are using the correct time-series model by testing whether the autocorrelations of the error term (**error autocorrelations**) differ significantly from 0. If they do, the model is not specified correctly. We estimate the error autocorrelation using the sample autocorrelations of the residuals (**residual autocorrelations**) and their sample variance.

A test of the null hypothesis that an error autocorrelation at a specified lag equals 0 is based on the residual autocorrelation for that lag and the standard error of the residual correlation, which is equal to $1/\sqrt{T}$, where T is the number of observations in the time series.[18] Thus, if we have 100 observations in a time series, the standard error for each of the estimated autocorrelations is 0.1. We can compute the t-test of the null hypothesis that the error correlation at a particular lag equals 0, by dividing the residual autocorrelation at that lag by its standard error $\left(1/\sqrt{T}\right)$.

How can we use information about the error autocorrelations to determine whether an autoregressive time-series model is correctly specified? We can use a simple three-step method. First, estimate a particular autoregressive model, say an AR(1) model. Second, compute the autocorrelations of the residuals from the model.[19] Third, test to see whether the residual autocorrelations differ significantly from 0. If significance tests show that the residual autocorrelations differ significantly from 0, the model is not correctly specified; we may need to modify it in ways that we will discuss shortly.[20] We now present an example to demonstrate how this three-step method works.

16 Whenever we refer to autocorrelation without qualification, we mean autocorrelation of the time series itself rather than autocorrelation of the error term or residuals.

17 This assumption is similar to the one made in the previous two readings about the expected value of the error term.

18 This calculation is derived in Diebold (2008).

19 We can compute these residual autocorrelations easily with most statistical software packages. In Microsoft Excel, for example, to compute the first-order residual autocorrelation, we compute the correlation of the residuals from observations 1 through T – 1 with the residuals from observations 2 through T.

20 Often, econometricians use additional tests for the significance of residual autocorrelations. For example, the Box–Pierce Q-statistic is frequently used to test the joint hypothesis that all autocorrelations of the residuals are equal to 0. For further discussion, see Diebold (2008).

EXAMPLE 4

Predicting Gross Margins for Intel Corporation

Analyst Melissa Jones decides to use a time-series model to predict Intel Corporation's gross margin [(Sales – Cost of goods sold)/Sales] using quarterly data from the second quarter of 1999 through the fourth quarter of 2013. She does not know the best model for gross margin but believes that the current-period value will be related to the previous-period value. She decides to start out with a first-order autoregressive model, AR(1): Gross margin$_t$ = $b_0 + b_1$(Gross margin$_{t-1}$) + ε_t. Her observations on the dependent variable are 2Q:1999 through 4Q:2013. Table 4 shows the results of estimating this AR(1) model, along with the autocorrelations of the residuals from that model.

Table 4 Autoregression: AR(1) Model Gross Margin of Intel Quarterly Observations, April 1999–December 2013

Regression Statistics	
R-squared	0.5429
Standard error	0.0337
Observations	59
Durbin–Watson	2.0987

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.1795	0.0635	2.8268
Gross margin$_{t-1}$	0.7449	0.0905	8.2309

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	−0.0495	0.1302	−0.3802
2	−0.0392	0.1302	−0.3011
3	0.0524	0.1302	0.4025
4	0.1450	0.1302	1.1137

Source: Compustat.

The first thing to note about Table 4 is that both the intercept ($\hat{b}_0$ = 0.1795) and the coefficient on the first lag ($\hat{b}_1$ = 0.7449) of the gross margin are highly significant in the regression equation.[21] The t-statistic for the intercept is about 2.8, whereas the t-statistic for the first lag of the gross margin is more than 8. With 59 observations and two parameters, this model has 57 degrees of freedom. At the 0.05 significance level, the critical value for a t-statistic is about 2.0. Therefore, Jones must reject the null hypotheses that the intercept is equal to 0 (b_0 = 0) and the coefficient on the first lag is equal to 0 (b_1 = 0) in favor of the alternative hypothesis that the coefficients, individually, are not equal to 0. But are these statistics valid? Although the Durbin–Watson statistic is presented in

21 The first lag of a time series is the value of the time series in the previous period.

Table 4, it cannot be used to test serial correlation when the independent variables include past values of the dependent variable. The correct approach is to test whether the residuals from this model are serially correlated.

At the bottom of Table 4, the first four autocorrelations of the residual are displayed along with the standard error and the t-statistic for each of those autocorrelations.[22] The sample has 59 observations, so the standard error for each of the autocorrelations is $1/\sqrt{59} = 0.1302$. Table 4 shows that none of the first four autocorrelations has a t-statistic larger than 1.1137 in absolute value. Therefore, Jones can conclude that none of these autocorrelations differs significantly from 0. Consequently, she can assume that the residuals are not serially correlated and that the model is correctly specified, and she can validly use ordinary least squares to estimate the parameters and the parameters' standard errors in the autoregressive model.[23]

Now that Jones has concluded that this model is correctly specified, how can she use it to predict Intel's gross margin in the next period? The estimated equation is Gross margin$_t$ = 0.1795 + 0.7449(Gross margin$_{t-1}$) + ε_t. The expected value of the error term is 0 in any period. Therefore, this model predicts that gross margin in period t + 1 will be Gross margin$_{t+1}$ = 0.1795 + 0.7449(Gross margin$_t$). For example, if gross margin is 65 percent in this quarter (0.65), the model predicts that in the next quarter gross margin will increase to 0.1795 + 0.7449(0.65) = 0.6637 or 66.37 percent. On the other hand, if gross margin is currently 75 percent (0.75), the model predicts that in the next quarter, gross margin will fall to 0.1795 + 0.7449(0.75) = 0.7382 or 73.82 percent. As we show in the following section, the model predicts that gross margin will increase if it is below a certain level (70.36 percent) and decrease if it is above that level.

4.3 Mean Reversion

We say that a time series shows **mean reversion** if it tends to fall when its level is above its mean and rise when its level is below its mean. Much like the temperature in a room controlled by a thermostat, a mean-reverting time series tends to return to its long-term mean. How can we determine the value that the time series tends toward? If a time series is currently at its mean-reverting level, then the model predicts that the value of the time series will be the same in the next period. At its mean-reverting level, we have the relationship $x_{t+1} = x_t$. For an AR(1) model ($x_{t+1} = b_0 + b_1x_t$), the equality $x_{t+1} = x_t$ implies the level $x_t = b_0 + b_1x_t$, or that the mean-reverting level, x_t, is given by

$$x_t = \frac{b_0}{1 - b_1}$$

So the AR(1) model predicts that the time series will stay the same if its current value is $b_0/(1 - b_1)$, increase if its current value is below $b_0/(1 - b_1)$, and decrease if its current value is above $b_0/(1 - b_1)$.

In the case of gross margins for Intel, the mean-reverting level for the model shown in Table 4 is 0.1795/(1 − 0.7449) = 0.7036. If the current gross margin is above 0.7036, the model predicts that the gross margin will fall in the next period. If the current

22 For seasonally unadjusted data, analysts often compute the same number of autocorrelations as there are observations in a year (for example, four for quarterly data). The number of autocorrelations computed also often depends on sample size, as discussed in Diebold (2007).

23 Statisticians have many other tests for serial correlation of the residuals in a time-series model. For details, see Diebold (2008).

gross margin is below 0.7036, the model predicts that the gross margin will rise in the next period. As we will discuss later, all covariance-stationary time series have a finite mean-reverting level.

4.4 Multiperiod Forecasts and the Chain Rule of Forecasting

Often, financial analysts want to make forecasts for more than one period. For example, we might want to use a quarterly sales model to predict sales for a company for each of the next four quarters. To use a time-series model to make forecasts for more than one period, we must examine how to make multiperiod forecasts using an AR(1) model. The one-period-ahead forecast of x_t from an AR(1) model is as follows:

$$\hat{x}_{t+1} = \hat{b}_0 + \hat{b}_1 x_t \tag{6}$$

If we want to forecast x_{t+2} using an AR(1) model, our forecast will be based on

$$\hat{x}_{t+2} = \hat{b}_0 + \hat{b}_1 x_{t+1} \tag{7}$$

Unfortunately, we do not know x_{t+1} in period t, so we cannot use Equation 7 directly to make a two-period-ahead forecast. We can, however, use our forecast of x_{t+1} and the AR(1) model to make a prediction of x_{t+2}. The **chain rule of forecasting** is a process in which the next period's value, predicted by the forecasting equation, is substituted into the equation to give a predicted value two periods ahead. Using the chain rule of forecasting, we can substitute the predicted value of x_{t+1} into Equation 7 to get $\hat{x}_{t+2} = \hat{b}_0 + \hat{b}_1 \hat{x}_{t+1}$. We already know $\hat{x}_{t+1}$ from our one-period-ahead forecast in Equation 6. Now we have a simple way of predicting x_{t+2}.

Multiperiod forecasts are more uncertain than single-period forecasts because each forecast period has uncertainty. For example, in forecasting x_{t+2}, we first have the uncertainty associated with forecasting x_{t+1} using x_t, and then we have the uncertainty associated with forecasting x_{t+2} using the forecast of x_{t+1}. In general, the more periods a forecast has, the more uncertain it is.[24]

EXAMPLE 5

Multiperiod Prediction of Intel's Gross Margin

Suppose that at the beginning of 2014, we want to predict Intel's gross margin in two periods using the model shown in Table 4. Assume that Intel's gross margin in the current period is 65 percent. The one-period-ahead forecast of Intel's gross margin from this model is 0.6637 = 0.1795 + 0.7449(0.65). By substituting the one-period-ahead forecast, 0.6637, back into the regression equation, we can derive the following two-period-ahead forecast: 0.6739 = 0.1795 + 0.7449(0.6637). Therefore, if the current gross margin for Intel is 65 percent, the model predicts that Intel's gross margin in two quarters will be 67.39 percent.

24 If a forecasting model is well specified, the prediction errors from the model will not be serially correlated. If the prediction errors for each period are not serially correlated, then the variance of a multiperiod forecast will be higher than the variance of a single-period forecast.

EXAMPLE 6

Modeling US CPI Inflation

Analyst Lisette Miller has been directed to build a time-series model for monthly US inflation. Inflation and expectations about inflation, of course, have a significant effect on bond returns. For a 30-year period beginning with January 1984 and ending with December 2013, she selects as data the annualized monthly percentage change in the CPI. Which model should Miller use?

The process of model selection parallels that of Example 4 relating to Intel's gross margins. The first model Miller estimates is an AR(1) model, using the previous month's inflation rate as the independent variable: $\text{Inflation}_t = b_0 + b_1 \cdot \text{Inflation}_{t-1} + \varepsilon_t$, $t = 1, 2, \ldots, 359$. To estimate this model, she uses monthly CPI inflation data from January 1984 to December 2013 ($t = 1$ denotes February 1984). Table 5 shows the results of estimating this model.

Table 5 Monthly CPI Inflation at an Annual Rate: AR(1) Model Monthly Observations, February 1984–December 2013

Regression Statistics	
R-squared	0.2038
Standard error	3.4250
Observations	359
Durbin–Watson	1.8201

	Coefficient	**Standard Error**	***t*-Statistic**
Intercept	1.5703	0.2266	6.9298
Inflation_{t-1}	0.4510	0.0472	9.5551

Autocorrelations of the Residual

Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.0898	0.0528	1.7008
2	−0.1205	0.0528	−2.2822
3	−0.1571	0.0528	−2.9754
4	−0.0316	0.0528	−0.5985

Source: US Bureau of Labor Statistics.

As Table 5 shows, both the intercept ($\hat{b}_0 = 1.5703$) and the coefficient on the first lagged value of inflation ($\hat{b}_1 = 0.4510$) are highly statistically significant, with large t-statistics. With 359 observations and two parameters, this model has 357 degrees of freedom. The critical value for a t-statistic at the 0.05 significance level is about 1.97. Therefore, Miller can reject the individual null hypotheses that the intercept is equal to 0 ($b_0 = 0$) and the coefficient on the first lag is equal to 0 ($b_1 = 0$) in favor of the alternative hypothesis that the coefficients, individually, are not equal to 0.

Are these statistics valid? Miller will know when she tests whether the residuals from this model are serially correlated. With 359 observations in this sample, the standard error for each of the estimated autocorrelations is $1/\sqrt{359} = 0.0528$.

The critical value for the t-statistic is 1.97. Because both the second and the third estimated autocorrelation have t-statistics larger than 1.97 in absolute value, Miller concludes that the autocorrelations are significantly different from 0. This model is thus misspecified because the residuals are serially correlated.

If the residuals in an autoregressive model are serially correlated, Miller can eliminate the correlation by estimating an autoregressive model with more lags of the dependent variable as explanatory variables. Table 6 shows the result of estimating a second time-series model, an AR(2) model using the same data as in the analysis shown in Table 5.[25] With 358 observations and three parameters, this model has 355 degrees of freedom. Because the degrees of freedom are almost the same as those for the estimates shown in Table 5, the critical value of the t-statistic at the 0.05 significance level also is almost the same (1.97). If she estimates the equation with two lags, $\text{Inflation}_t = b_0 + b_1 \text{ Inflation}_{t-1} + b_2 \text{ Inflation}_{t-2} + \varepsilon_t$, Miller finds that all three of the coefficients in the regression model (an intercept and the coefficients on two lags of the dependent variable) differ significantly from 0. The bottom portion of Table 6 shows that none of the first four autocorrelations of the residual has a t-statistic greater in absolute value than the critical value of 1.97. Therefore, Miller fails to reject the hypothesis that the individual autocorrelations of the residual equal 0. She concludes that this model is correctly specified because she finds no evidence of serial correlation in the residuals.

Table 6 Monthly CPI Inflation at an Annual Rate: AR(2) Model Monthly Observations, March 1984–December 2013

Regression Statistics	
R-squared	0.2349
Standard error	3.3637
Observations	358
Durbin–Watson	2.0273

	Coefficient	Standard Error	t-Statistic
Intercept	1.8953	0.2379	7.9668
Inflation_{t-1}	0.5406	0.0520	10.3962
Inflation_{t-2}	−0.2015	0.0520	−3.8750

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	t-Statistic
1	−0.0140	0.0529	−0.2647
2	0.0335	0.0529	0.6333

(continued)

25 Note that Table 6 shows only 358 observations in the regression because the extra lag of inflation requires the estimation sample to start one month later than the regression in Table 5. (With two lags, inflation for January and February 1984 must be known in order to estimate the equation starting in March 1984.)

Table 6 (Continued)

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
3	−0.0726	0.0529	−1.3724
4	−0.0056	0.0529	−0.1059

Source: US Bureau of Labor Statistics.

1 The analyst selected an AR(2) model because the residuals from the AR(1) model were serially correlated. Suppose that in a given month, inflation had been 4 percent at an annual rate in the previous month and 3 percent in the month before that. What would be the difference in the analyst forecast of the inflation for that month if she had used an AR(1) model instead of the AR(2) model?

Solution to 1:

The AR(1) model shown in Table 5 predicted that inflation in the next month would be 1.5703 + 0.4510(4) = 3.37 percent approximately, whereas the AR(2) model shown in Table 6 predicts that inflation in the next month will be 1.8953 + 0.5406(4) − 0.2015(3) = 3.45 percent approximately. If the analyst had used the incorrect AR(1) model, she would have predicted inflation to be 8 basis points lower (3.37 percent versus 3.45 percent) than using the AR(2) model. This incorrect forecast could have adversely affected the quality of her company's investment choices.

4.5 Comparing Forecast Model Performance

One way to compare the forecast performance of two models is to compare the variance of the forecast errors that the two models make. The model with the smaller forecast error variance will be the more accurate model, and it will also have the smaller standard error of the time-series regression. (This standard error usually is reported directly in the output for the time-series regression.)

In comparing forecast accuracy among models, we must distinguish between in-sample forecast errors and out-of-sample forecast errors. **In-sample forecast errors** are the residuals from a fitted time-series model. For example, when we estimated a linear trend with raw inflation data from January 1984 to December 2013, the in-sample forecast errors were the residuals from January 1984 to December 2013. If we use this model to predict inflation outside this period, the differences between actual and predicted inflation are **out-of-sample forecast errors**.

EXAMPLE 7

In-Sample Forecast Comparisons of US CPI Inflation

In Example 6, the analyst compared an AR(1) forecasting model of monthly US inflation with an AR(2) model of monthly US inflation and decided that the AR(2) model was preferable. Table 5 showed that the standard error from the AR(1) model of inflation is 3.4250, and Table 6 showed that the standard error from the AR(2) model is 3.3637. Therefore, the AR(2) model had a lower in-sample forecast error variance than the AR(1) model, which is consistent with our belief that the AR(2) model was preferable. Its standard error is 3.3637/3.4250 = 98.21 percent of the forecast error of the AR(1) model.

Often, we want to compare the forecasting accuracy of different models after the sample period for which they were estimated. We wish to compare the out-of-sample forecast accuracy of the models. Out-of-sample forecast accuracy is important because the future is always out of sample. Although professional forecasters distinguish between out-of-sample and in-sample forecasting performance, many articles that analysts read contain only in-sample forecast evaluations. Analysts should be aware that out-of-sample performance is critical for evaluating a forecasting model's real-world contribution.

Typically, we compare the out-of-sample forecasting performance of forecasting models by comparing their **root mean squared error (RMSE)**, which is the square root of the average squared error. The model with the smallest RMSE is judged most accurate. The following example illustrates the computation and use of RMSE in comparing forecasting models.

EXAMPLE 8

Out-of-Sample Forecast Comparisons of US CPI Inflation

Suppose we want to compare the forecasting accuracy of the AR(1) and AR(2) models of US inflation estimated over 1984 to 2013, using data on US inflation from January 2014 to September 2014.

Table 7 Out-of-Sample Forecast Error Comparisons: January 2014–September 2014 US CPI Inflation (Annualized)

Date	Infl(*t*)	Infl(*t*–1)	Infl(*t*–2)	AR(1) Error	Squared Error	AR(2) Error	Squared Error
2014							
January	4.5568	–0.1029	–2.4236	3.0329	9.1986	2.2288	4.9675
February	4.5289	4.5568	–0.1029	0.9034	0.8162	0.1494	0.0223
March	8.0077	4.5289	4.5568	4.3949	19.3154	4.5823	20.9978
April	4.0286	8.0077	4.5289	–1.1532	1.3298	–1.2831	1.6463
May	4.2726	4.0286	8.0077	0.8854	0.7839	1.8130	3.2869
June	2.2576	4.2726	4.0286	–1.2397	1.5368	–1.1357	1.2898
July	–0.4672	2.2576	4.2726	–3.0557	9.3373	–2.7221	7.4096
August	–1.9863	–0.4672	2.2576	–3.3459	11.1949	–3.1741	10.0750
September	0.9068	–1.9863	–0.4672	0.2324	0.0540	–0.0088	0.0001
				Average	5.9519	Average	5.5217
				RMSE	2.4396	RMSE	2.3498

Note: Any apparent discrepancies between error and squared error results are due to rounding.
Source: US Bureau of Labor Statistics.

For each month from January 2014 to September 2014, the first column of numbers in Table 7 shows the actual annualized inflation rate during the month. The second and third columns show the rate of inflation in the previous two months. The fourth column shows the out-of-sample errors (Actual – Forecast) from the AR(1) model shown in Table 5. The fifth column shows the squared errors from the AR(1) model. The sixth column shows the out-of-sample errors from the AR(2) model shown in Table 6. The final column shows the squared errors from the AR(2) model. The bottom of the table displays the average squared error and the RMSE. According to these measures, the AR(2) model was slightly more accurate than the AR(1) model in its out-of-sample forecasts of inflation from January 2014 to September 2014. The RMSE from the AR(2) model was only 2.3498/2.4396 = 96.32 percent as large as the RMSE from the AR(1) model. Therefore, the AR(2) model was more accurate both in-sample and out of sample. Of course, this was a small sample to use in evaluating out-of-sample forecasting performance. Sometimes, an analyst may have conflicting information about whether to choose an AR(1) or an AR(2) model. We must also consider regression coefficient stability. We will continue the comparison between these two models in the following section.

4.6 Instability of Regression Coefficients

One of the important issues an analyst faces in modeling a time series is the sample period to use. The estimates of regression coefficients of the time-series model can change substantially across different sample periods used for estimating the model. Often, the regression coefficient estimates of a time-series model estimated using an earlier sample period can be quite different from those of a model estimated using a later sample period. Similarly, the estimates can be different between models estimated using relatively shorter and longer sample periods. Further, the choice of model for a particular time series can also depend on the sample period. For example, an AR(1) model may be appropriate for the sales of a company in one particular sample period, but an AR(2) model may be necessary for an earlier or later sample period (or for a longer or shorter sample period). Thus the choice of a sample period is an important decision in modeling a financial time series.

Unfortunately, there is usually no clear-cut basis in economic or financial theory for determining whether to use data from a longer or shorter sample period to estimate a time-series model. We can get some guidance, however, if we remember that our models are valid only for covariance-stationary time series. For example, we should not combine data from a period when exchange rates were fixed with data from a period when exchange rates were floating. The exchange rates in these two periods would not likely have the same variance because exchange rates are usually much more volatile under a floating-rate regime than when rates are fixed. Similarly, many US analysts consider it inappropriate to model US inflation or interest-rate behavior since the 1960s as a part of one sample period, because the Federal Reserve had distinct policy regimes during this period. A simple way to determine appropriate samples for time-series estimation is to look at graphs of the data to see if the time series looks stationary before estimation begins. If we know that a government policy changed on a specific date, we might also test whether the time-series relation was the same before and after that date.

In the following example, we illustrate how the choice of a longer versus a shorter period can affect the decision of whether to use, for example, a first- or second-order time-series model. We then show how the choice of the time-series model (and the associated regression coefficients) affects our forecast. Finally, we discuss which sample period, and accordingly which model and corresponding forecast, is appropriate for the time series analyzed in the example.

EXAMPLE 9

Instability in Time-Series Models of US Inflation

In Example 6, analyst Lisette Miller concluded that US CPI inflation should be modeled as an AR(2) time series. A colleague examined her results and questioned estimating one time-series model for inflation in the United States since 1984, given that the Federal Reserve responded aggressively to the financial crisis that emerged in 2007. He argues that the inflation time series from 1984 to 2013 has two **regimes** or underlying models generating the time series: one running from 1984 through 2006, and another starting in 2007. Therefore, the colleague suggests that Miller estimate a new time-series model for US inflation starting in 2007. Because of his suggestion, Miller first estimates an AR(1) model for inflation using data for a shorter sample period from 2007 to 2013. Table 8 shows her AR(1) estimates.

Table 8 Autoregression: AR(1) Model Monthly CPI Inflation at an Annual Rate, February 2007–December 2013

Regression Statistics	
R-squared	0.3070
Standard error	4.4749
Observations	83
Durbin–Watson	1.8164

	Coefficient	Standard Error	t-Statistic
Intercept	0.9585	0.5337	1.7960
Inflation_{t-1}	0.5544	0.0926	5.9870

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	t-Statistic
1	0.0878	0.1098	0.7999
2	−0.0091	0.1098	−0.0829
3	−0.0355	0.1098	−0.3234
4	0.0020	0.1098	0.0182

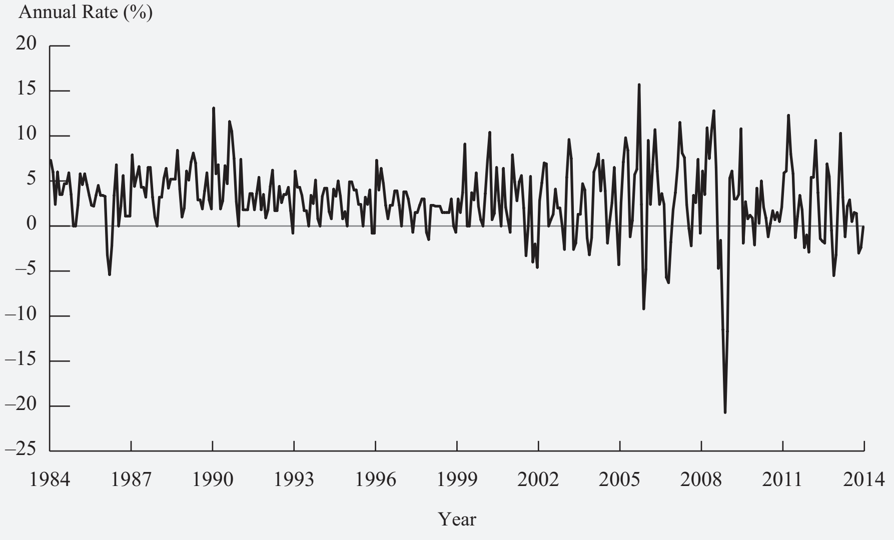
Source: US Bureau of Labor Statistics.

The bottom part of Table 8 shows that the first four autocorrelations of the residuals from the AR(1) model are quite small. None of these autocorrelations has a t-statistic larger than 1.99, the critical value for significance. Consequently, Miller cannot reject the null hypothesis that the residuals are serially uncorrelated. The AR(1) model is correctly specified for the sample period from 2007 to 2013, so there is no need to estimate the AR(2) model. This conclusion is very different from that reached in Example 6 using data from 1984 to 2013. In that example, Miller initially rejected the AR(1) model because its residuals exhibited serial correlation. When she used a larger sample, an AR(2) model initially appeared to fit the data much better than did an AR(1) model.

How deeply does our choice of sample period affect our forecast of future inflation? Suppose that in a given month, inflation was 4 percent at an annual rate, and the month before that it was 3 percent. The AR(1) model shown in Table 8 predicts that inflation in the next month will be 0.9585 + 0.5544(4) = approximately 3.17 percent. Therefore, the forecast of the next month's inflation using the 2007 to 2013 sample is 3.17 percent. Remember from the analysis following Example 6 that the AR(2) model for the 1984 to 2013 sample predicts inflation of 3.45 percent in the next month. Thus, using the correctly specified model for the shorter sample produces an inflation forecast 0.28 percentage points below the forecast made from the correctly specified model for the longer sample period. Such a difference might substantially affect a particular investment decision.

Which model is correct? Figure 9 suggests an answer. Monthly US inflation was so much more volatile during the latter part of the study period than in the earlier years that inflation is probably not a covariance-stationary time series from 1984 to 2013. Therefore, we can reasonably believe that the data have more than one regime and Miller should estimate a separate model for inflation from 2007 to 2013, as shown above. In fact, the standard deviation of annualized monthly inflation rates is just 3.24 percent for the period of 1984 to 2006 but 5.32 percent for the period of 2007 to 2013. As the example shows, experience (such as knowledge of government policy changes) and judgment play a vital role in determining how to model a time series. Simply relying on autocorrelations of the residuals from a time-series model cannot tell us the correct sample period for our analysis.

Figure 9 Monthly CPI Inflation



Source: US Bureau of Labor Statistics.

5 RANDOM WALKS AND UNIT ROOTS

So far, we have examined those time series in which the time series has a tendency to revert to its mean level as the change in a variable from one period to the next follows a mean-reverting pattern. In contrast, there are many financial time series in which the changes follow a random pattern. We discuss these "random walks" in the following section.

5.1 Random Walks

A random walk is one of the most widely studied time-series models for financial data. A **random walk** is a time series in which the value of the series in one period is the value of the series in the previous period plus an unpredictable random error. A random walk can be described by the following equation:

$$x_t = x_{t-1} + \varepsilon_t,\ E(\varepsilon_t) = 0,\ E(\varepsilon_t^2) = \sigma^2,\ \mathrm{Cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ if } t \neq s \quad \textbf{(8)}$$

Equation 8 means that the time series x_t is in every period equal to its value in the previous period plus an error term, ε_t, that has constant variance and is uncorrelated with the error term in previous periods. Note two important points. First, this equation is a special case of an AR(1) model with $b_0 = 0$ and $b_1 = 1$.[26] Second, the expected value of ε_t is zero. Therefore, the best forecast of x_t that can be made in period $t - 1$ is x_{t-1}. In fact, in this model, x_{t-1} is the best forecast of x in every period after $t - 1$.

Random walks are quite common in financial time series. For example, many studies have tested and found that currency exchange rates follow a random walk. Consistent with the second point made above, some studies have found that sophisticated exchange rate forecasting models cannot outperform forecasts made using the random walk model, and that the best forecast of the future exchange rate is the current exchange rate.

Unfortunately, we cannot use the regression methods we have discussed so far to estimate an AR(1) model on a time series that is actually a random walk. To see why this is so, we must determine why a random walk has no finite mean-reverting level or finite variance. Recall that if x_t is at its mean-reverting level, then $x_t = b_0 + b_1 x_t$, or $x_t = b_0/(1 - b_1)$. In a random walk, however, $b_0 = 0$ and $b_1 = 1$, so $b_0/(1 - b_1) = 0/0$. Therefore, a random walk has an undefined mean-reverting level.

What is the variance of a random walk? Suppose that in Period 1, the value of x_1 is 0. Then we know that $x_2 = 0 + \varepsilon_2$. Therefore, the variance of $x_2 = \mathrm{Var}(\varepsilon_2) = \sigma^2$. Now $x_3 = x_2 + \varepsilon_3 = \varepsilon_2 + \varepsilon_3$. Because the error term in each period is assumed to be uncorrelated with the error terms in all other periods, the variance of $x_3 = \mathrm{Var}(\varepsilon_2) + \mathrm{Var}(\varepsilon_3) = 2\sigma^2$. By a similar argument, we can show that for any period t, the variance of $x_t = (t - 1)\sigma^2$. But this means that as t grows large, the variance of x_t grows without an upper bound: It approaches infinity. This lack of upper bound, in turn, means that a random walk is not a covariance-stationary time series, because a covariance-stationary time series must have a finite variance.

What is the practical implication of these issues? *We cannot use standard regression analysis on a time series that is a random walk.* We can, however, attempt to convert the data to a covariance-stationary time series if we suspect that the time series is a random walk. In statistical terms, we can difference it.

26 Equation 8 with a nonzero intercept added (as in Equation 9 given later) is sometimes referred to as a random walk with drift.

We difference a time series by creating a new time series, say y_t, that in each period is equal to the difference between x_t and x_{t-1}. This transformation is called **first-differencing** because it subtracts the value of the time series in the first prior period from the current value of the time series. Sometimes the first difference of x_t is written as $\Delta x_t = x_t - x_{t-1}$. Note that the first difference of the random walk in Equation 8 yields

$$y_t = x_t - x_{t-1} = \varepsilon_t,\ E(\varepsilon_t) = 0,\ E\left(\varepsilon_t^2\right) = \sigma^2,\ \mathrm{Cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s$$

The expected value of ε_t is 0. Therefore, the best forecast of y_t that can be made in period $t - 1$ is 0. This implies that the best forecast is that there will be no change in the value of the current time series, x_{t-1}.

The first-differenced variable, y_t, is covariance stationary. How is this so? First, note that this model ($y_t = \varepsilon_t$) is an AR(1) model with $b_0 = 0$ and $b_1 = 0$. We can compute the mean-reverting level of the first-differenced model as $b_0/(1 - b_1) = 0/1 = 0$. Therefore, a first-differenced random walk has a mean-reverting level of 0. Note also that the variance of y_t in each period is $\mathrm{Var}(\varepsilon_t) = \sigma^2$. Because the variance and the mean of y_t are constant and finite in each period, y_t is a covariance-stationary time series and we can model it using linear regression.[27] Of course, modeling the first-differenced series with an AR(1) model does not help us predict the future, as $b_0 = 0$ and $b_1 = 0$. We simply conclude that the original time series is, in fact, a random walk.

Had we tried to estimate an AR(1) model for a time series that was a random walk, our statistical conclusions would have been incorrect because AR models cannot be used to estimate random walks or any time series that is not covariance stationary. The following example illustrates this issue with exchange rates.

EXAMPLE 10

The Yen/US Dollar Exchange Rate

Financial analysts often assume that exchange rates are random walks. Consider an AR(1) model for the Japanese yen/US dollar exchange rate (JPY/USD). Table 9 shows the results of estimating the model using month-end observations from January 1980 through December 2013.

Table 9 Yen/US Dollar Exchange Rate: AR(1) Model Month-End Observations, January 1980–December 2013

Regression Statistics	
R-squared	0.9902
Standard error	4.9437
Observations	408
Durbin–Watson	1.8981

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.9958	0.7125	1.3976
JPY/USD$_{t-1}$	0.9903	0.0049	202.1020

27 All the covariances are finite, for two reasons: The variance is finite, and the covariance of a time series with its own past value can be no greater than the variance of the series.

Table 9 (Continued)

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.0687	0.0495	1.3879
2	0.0384	0.0495	0.7758
3	0.0686	0.0495	1.3859
4	0.0407	0.0495	0.8222

Source: US Federal Reserve Board of Governors.

The results in Table 9 suggest that the yen/US dollar exchange rate is a random walk because the estimated intercept does not appear to be significantly different from 0 and the estimated coefficient on the first lag of the exchange rate is very close to 1. Can we use the t-statistics in Table 9 to test whether the exchange rate is a random walk? Unfortunately, no, because the standard errors in an AR model are invalid if the model is estimated using a data series that is a random walk (remember, a random walk is not covariance stationary). If the exchange rate is, in fact, a random walk, we might come to an incorrect conclusion based on faulty statistical tests and then invest incorrectly. We can use a test presented in the next section to test whether the time-series is a random walk.

Suppose the exchange rate is a random walk, as we now suspect. If so, the first-differenced series, $y_t = x_t - x_{t-1}$, will be covariance stationary. We present the results from estimating $y_t = b_0 + b_1 y_{t-1} + \varepsilon_t$ in Table 10. If the exchange rate is a random walk, then $b_0 = 0$ and $b_1 = 0$ and the error term will not be serially correlated.

Table 10 First-Differenced Yen/US Dollar Exchange Rate: AR(1) Model Month-End Observations, January 1980–December 2013

Regression Statistics	
R-squared	0.0026
Standard error	4.9611
Observations	408
Durbin–Watson	2.0010

	Coefficient	**Standard Error**	***t*-Statistic**
Intercept	−0.3128	0.2463	−1.2700
JPY/USD$_{t-1}$ − JPY/USD$_{t-2}$	0.0506	0.0494	1.0243

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.0193	0.0495	0.3899
2	0.0345	0.0495	0.6970

(continued)

Table 10 (Continued)

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
3	0.0680	0.0495	1.3737
4	0.0399	0.0495	0.8061

Source: US Federal Reserve Board of Governors.

In Table 10, neither the intercept nor the coefficient on the first lag of the first-differenced exchange rate differs significantly from 0, and no residual autocorrelations differ significantly from 0.[28] These findings are consistent with the yen/US dollar exchange rate being a random walk.

We have concluded that the differenced regression is the model to choose. Now we can see that we would have been seriously misled if we had based our model choice on an R^2 comparison. In Table 9, the R^2 is 0.9902, whereas in Table 10 the R^2 is 0.0026. How can this be, if we just concluded that the model in Table 10 is the one that we should use? In Table 9, the R^2 measures how well the exchange rate in one period predicts the exchange rate in the next period. If the exchange rate is a random walk, its current value will be an extremely good predictor of its value in the next period, and thus the R^2 will be extremely high. At the same time, if the exchange rate is a random walk, then changes in the exchange rate should be completely unpredictable. Table 10 estimates whether changes in the exchange rate from one month to the next can be predicted by changes in the exchange rate over the previous month. If they cannot be predicted, the R^2 in Table 10 should be very low. In fact, it is low (0.0026). This comparison provides a good example of the general rule that we cannot necessarily choose which model is correct solely by comparing the R^2 from the two models.

The exchange rate is a random walk, and changes in a random walk are by definition unpredictable. Therefore, we cannot profit from an investment strategy that predicts changes in the exchange rate.

To this point, we have discussed only simple random walks; that is, random walks without drift. In a random walk without drift, the best predictor of the time series in the next period is its current value. A random walk with drift, however, should increase or decrease by a constant amount in each period. The equation describing a random walk with drift is a special case of the AR(1) model:

$$\begin{aligned} x_t &= b_0 + b_1 x_{t-1} + \varepsilon_t \\ b_1 &= 1,\ b_0 \neq 0, \text{ or} \\ x_t &= b_0 + x_{t-1} + \varepsilon_t,\ E(\varepsilon_t) = 0 \end{aligned} \tag{9}$$

A random walk with drift has $b_0 \neq 0$ compared to a simple random walk, which has $b_0 = 0$.

We have already seen that $b_1 = 1$ implies an undefined mean-reversion level and thus nonstationarity. Consequently, we cannot use an AR model to analyze a time series that is a random walk with drift until we transform the time series by taking first differences. If we first-difference Equation 9, the result is $y_t = x_t - x_{t-1}$, $y_t = b_0 + \varepsilon_t$, $b_0 \neq 0$.

28 See Greene (2018) for a test of the joint hypothesis that both regression coefficients are equal to 0.

5.2 The Unit Root Test of Nonstationarity

In this section, we discuss how to use random walk concepts to determine whether a time series is covariance stationary. This approach focuses on the slope coefficient in the random-walk-with-drift case of an AR(1) model in contrast with the traditional autocorrelation approach which we discuss first.

The examination of the autocorrelations of a time series at various lags is a well-known prescription for inferring whether or not a time series is stationary. Typically, for a stationary time series, either autocorrelations at all lags are statistically indistinguishable from zero, or the autocorrelations drop off rapidly to zero as the number of lags becomes large. Conversely, the autocorrelations of a nonstationary time series do not exhibit those characteristics. However, this approach is less definite than a currently more popular test for nonstationarity known as the Dickey–Fuller test for a unit root.

We can explain what is known as the unit root problem in the context of an AR(1) model. If a time series comes from an AR(1) model, then to be covariance stationary the absolute value of the lag coefficient, b_1, must be less than 1.0. We could not rely on the statistical results of an AR(1) model if the absolute value of the lag coefficient were greater than or equal to 1.0 because the time series would not be covariance stationary. If the lag coefficient is equal to 1.0, the time series has a **unit root**: it is a random walk and is not covariance stationary.[29] By definition, all random walks, with or without a drift term, have unit roots.

How do we test for unit roots in a time series? If we believed that a time series, x_t, was a random walk with drift, it would be tempting to estimate the parameters of the AR(1) model $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$ using linear regression and conduct a t-test of the hypothesis that $b_1 = 1$. Unfortunately, if $b_1 = 1$, then x_t is not covariance stationary and the t-value of the estimated coefficient, $\hat{b}_1$, does not actually follow the t-distribution; consequently, a t-test would be invalid.

Dickey and Fuller (1979) developed a regression-based unit root test based on a transformed version of the AR(1) model $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$. Subtracting x_{t-1} from both sides of the AR(1) model produces

$$x_t - x_{t-1} = b_0 + (b_1 - 1)x_{t-1} + \varepsilon_t$$

or

$$x_t - x_{t-1} = b_0 + g_1 x_{t-1} + \varepsilon_t,\ E(\varepsilon_t) = 0 \qquad \textbf{(10)}$$

where $g_1 = (b_1 - 1)$. If $b_1 = 1$, then $g_1 = 0$ and thus a test of $g_1 = 0$ is a test of $b_1 = 1$. If there is a unit root in the AR(1) model, then g_1 will be 0 in a regression where the dependent variable is the first difference of the time series and the independent variable is the first lag of the time series. The null hypothesis of the Dickey–Fuller test is H_0: $g_1 = 0$—that is, that the time series has a unit root and is nonstationary—and the alternative hypothesis is H_a: $g_1 < 0$, that the time series does not have a unit root and is stationary.

To conduct the test, one calculates a t-statistic in the conventional manner for $\hat{g}_1$ but instead of using conventional critical values for a t-test, one uses a revised set of values computed by Dickey and Fuller; the revised set of critical values are larger in absolute value than the conventional critical values. A number of software packages incorporate Dickey–Fuller tests.[30]

29 When b_1 is greater than 1 in absolute value, we say that there is an explosive root. For details, see Diebold (2008).

30 Dickey and Fuller developed three separate tests of the hypothesis that $g_1 = 0$ assuming the following models: random walk, random walk with drift, or random walk with drift and trend. The critical values for the Dickey–Fuller tests for the three models are different. For more on this topic, see Greene (2018) or Tsay (2010).

EXAMPLE 11

AstraZeneca's Quarterly Sales (1)

In January 2012, equity analyst Aron Berglin is building a time-series model for the quarterly sales of AstraZeneca, a British-Swedish biopharmaceutical company headquartered in London, UK. He is using AstraZeneca's quarterly sales in US dollars during January 2000 to December 2011 and any lagged sales data that he may need prior to 2000 to build this model. He finds that a log-linear trend model seems better suited for modeling AstraZeneca's sales than does a linear trend model. However, the Durbin–Watson statistic from the log-linear regression is just 0.7064, which causes him to reject the hypothesis that the errors in the regression are serially uncorrelated. He concludes that he cannot model the log of AstraZeneca's quarterly sales using only a time-trend line. He decides to model the log of AstraZeneca's quarterly sales using an AR(1) model. He uses $\ln \text{Sales}_t = b_0 + b_1 \ln \text{Sales}_{t-1} + \varepsilon_t$.

Before he estimates this regression, the analyst should use the Dickey–Fuller test to determine whether there is a unit root in the log of AstraZeneca's quarterly sales. If he uses the sample of quarterly data on AstraZeneca's sales from the first quarter of 2000 through the fourth quarter of 2011, takes the natural log of each observation, and computes the Dickey–Fuller *t*-test statistic, the value of that statistic might cause him to fail to reject the null hypothesis that there is a unit root in the log of AstraZeneca's quarterly sales.

If a time series appears to have a unit root, how should we model it? One method that is often successful is to model the first-differenced series as an autoregressive time series. The following example demonstrates this method.

EXAMPLE 12

AstraZeneca's Quarterly Sales (2)

The plot of the log of AstraZeneca's quarterly sales is shown as Figure 10. By looking at the plot, Berglin is convinced that the log of quarterly sales is not covariance stationary (that it has a unit root).

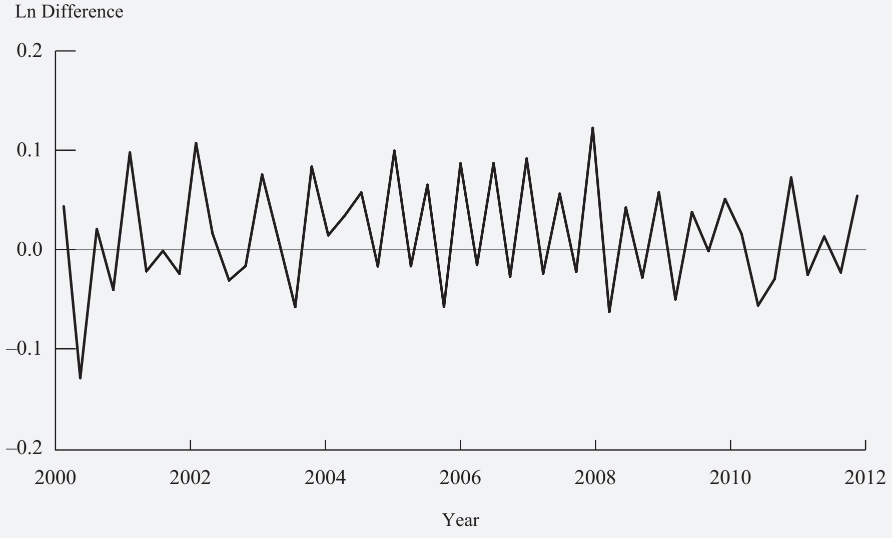
Figure 10 Log of AstraZeneca's Quarterly Sales

Source: Compustat.

So he creates a new series, y_t, that is the first difference of the log of AstraZeneca's quarterly sales. Figure 11 shows that series.

Berglin compares Figure 11 to Figure 10 and notices that first-differencing the log of AstraZeneca's quarterly sales eliminates the strong upward trend that was present in the log of AstraZeneca's sales. Because the first-differenced series has no strong trend, Berglin is better off assuming that the differenced series is covariance stationary rather than assuming that AstraZeneca's sales or the log of AstraZeneca's sales is a covariance-stationary time series.

Figure 11 Log Difference, AstraZeneca's Quarterly Sales



Source: Compustat.

Now suppose Berglin decides to model the new series using an AR(1) model. Berglin uses ln (Sales$_t$) – ln (Sales$_{t-1}$) = b_0 + b_1[ln (Sales$_{t-1}$) – ln (Sales$_{t-2}$)] + ε_t. Table 11 shows the results of that regression.

Table 11 Log Differenced Sales: AR(1) Model of AstraZeneca Quarterly Observations, January 2000–December 2011

Regression Statistics	
R-squared	0.3005
Standard error	0.0475
Observations	48
Durbin–Watson	1.6874

	Coefficient	Standard Error	t-Statistic
Intercept	0.0222	0.0071	3.1268
ln Sales$_{t-1}$ – ln Sales$_{t-2}$	–0.5493	0.1236	–4.4442

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	t-Statistic
1	0.2809	0.1443	1.9466
2	–0.0466	0.1443	–0.3229
3	0.0081	0.1443	0.0561
4	0.2647	0.1443	1.8344

Source: Compustat.

The lower part of Table 11 suggests that the first four autocorrelations of residuals in this model are not statistically significant. With 48 observations and two parameters, this model has 46 degrees of freedom. The critical value for a t-statistic in this model is above 2.0 at the 0.05 significance level. None of the t-statistics for these autocorrelations has an absolute value larger than 2.0. Therefore, we fail to reject the null hypotheses that each of these autocorrelations is equal to 0 and conclude instead that no significant autocorrelation is present in the residuals.

This result suggests that the model is well specified and that we could use the estimates. Both the intercept ($\hat{b}_0$ = 0.0222) and the coefficient ($\hat{b}_1$ = –0.5493) on the first lag of the new first-differenced series are statistically significant.

1 Explain how to interpret the estimated coefficients in the model.

Solution to 1:

The value of the intercept (0.0222) implies that if sales have not changed in the current quarter (y_t = ln Sales$_t$ – ln Sales$_{t-1}$ = 0), sales will grow by 2.22 percent next quarter.[31] If sales have changed during this quarter, however, the model predicts that sales will grow by 2.22 percent minus 0.5493 times the sales growth in this quarter.

2 AstraZenca's sales in the third and fourth quarters of 2011 were $8,405 million and $8,872 million, respectively. If we use the above model soon after the end of the fourth quarter of 2011, what will be the predicted value of AstraZenca's sales for the first quarter of 2012?

Solution to 2:

Let us say that t is the fourth quarter of 2011, so t – 1 is the third quarter of 2011 and t + 1 is the first quarter of 2012. Then we would have to compute $\hat{y}_{t+1}$ = 0.0222 – 0.5493y_t. To compute $\hat{y}_{t+1}$, we need to know y_t = ln Sales$_t$ – ln Sales$_{t-1}$. In the third quarter of 2011, AstraZenca's sales were $8,405 million, so ln (Sales$_{t-1}$) = ln 8,405 = 9.0366. In the fourth quarter of 2011, AstraZenca's sales were $8,872 million, so ln (Sales$_t$) = ln 8,872 = 9.0907. Thus y_t = 9.0907 – 9.0366 = 0.0541. Therefore, $\hat{y}_{t+1}$ = 0.0222 – 0.5493(0.0541) = –0.0075. If $\hat{y}_{t+1}$ = –0.0075, then –0.0075 = ln (Sales$_{t+1}$) – ln (Sales$_t$) = ln (Sales$_{t+1}$/Sales$_t$). If we exponentiate both sides of this equation, the result is

$$e^{-0.0075} = \left(\frac{\text{Sales}_{t+1}}{\text{Sales}_t}\right)$$

$$\begin{aligned}\text{Sales}_{t+1} &= \text{Sales}_t e^{-0.0075}\\ &= \$8{,}872 \text{ million} \times 0.9925\\ &= \$8{,}805 \text{ million}\end{aligned}$$

Thus, based on fourth quarter sales for 2011, this model would have predicted that AstraZenca's sales in the first quarter of 2012 would be $8,805 million. This sales forecast might have affected our decision to buy AstraZenca's stock at the time.

6 MOVING-AVERAGE TIME-SERIES MODELS

So far, many of the forecasting models we have used have been autoregressive models. Because most financial time series have the qualities of an autoregressive process, autoregressive time-series models are probably the most frequently used time-series models in financial forecasting. Some financial time series, however, seem to follow more closely another kind of time-series model called a moving-average model. For example, as we will see later, returns on the S&P BSE 100 Index can be better modeled as a moving-average process than as an autoregressive process.

31 Note that 2.22 percent is the exponential growth rate, not [(Current quarter sales/Previous quarter sales) – 1]. The difference between these two methods of computing growth is usually small.

In this section, we present the fundamentals of moving-average models so that you can ask the right questions when considering their use. We first discuss how to smooth past values with a moving average and then how to forecast a time series using a moving-average model. Even though both methods include the words "moving average" in the name, they are very different.

6.1 Smoothing Past Values with an *n*-Period Moving Average

Suppose you are analyzing the long-term trend in the past sales of a company. In order to focus on the trend, you may find it useful to remove short-term fluctuations or noise by smoothing out the time series of sales. One technique to smooth out period-to-period fluctuations in the value of a time series is an ***n*-period moving average**. An n-period moving average of the current and past $n - 1$ values of a time series, x_t, is calculated as

$$\frac{x_t + x_{t-1} + \cdots + x_{t-(n-1)}}{n} \tag{11}$$

The following example demonstrates how to compute a moving average of AstraZeneca's quarterly sales.

EXAMPLE 13

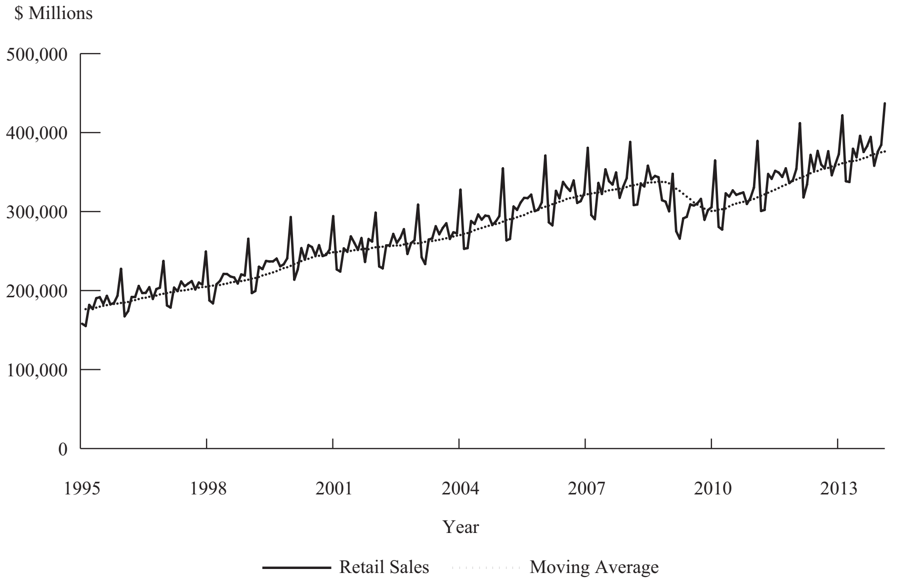
AstraZeneca's Quarterly Sales (3)

Suppose we want to compute the four-quarter moving average of AstraZeneca's sales as of the beginning of the first quarter of 2012. AstraZeneca's sales in the previous four quarters were 1Q:2011, $8,490 million; 2Q:2011, $8,601 million; 3Q:2011, $8,405 million; and 4Q:2011, $8,872 million. The four-quarter moving average of sales as of the beginning of the first quarter of 2012 is thus (8,490 + 8,601 + 8,405 + 8,872)/4 = $8,592 million.

We often plot the moving average of a series with large fluctuations to help discern any patterns in the data. Figure 12 shows monthly retail sales for the United States from January 1995 to December 2013, along with a 12-month moving average of the data.[32]

32 A 12-month moving average is the average value of a time series over each of the last 12 months. Although the sample period starts in 1995, data from 1994 are used to compute the 12-month moving average for the months of 1994.

Figure 12 Monthly US Real Retail Sales and 12-Month Moving Average of Retail Sales

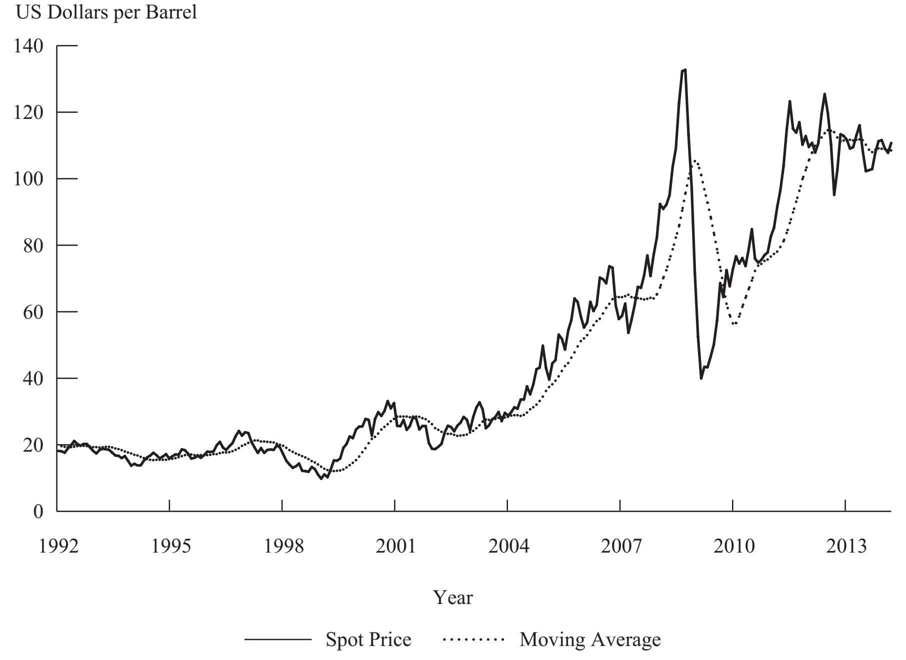


Source: US Department of Commerce, Census Bureau.

As Figure 12 shows, each year has a very strong peak in retail sales (December) followed by a sharp drop in sales (January). Because of the extreme seasonality in the data, a 12-month moving average can help us focus on the long-term movements in retail sales instead of seasonal fluctuations. Note that the moving average does not have the sharp seasonal fluctuations of the original retail sales data. Rather, the moving average of retail sales grows steadily, for example, from 1995 through the second half of 2008, then declines for about a year, and grows steadily thereafter. We can see that trend more easily by looking at a 12-month moving average than by looking at the time series itself.

Figure 13 shows monthly Europe Brent Crude Oil spot prices along with a 12-month moving average of oil prices. Although these data do not have the same sharp regular seasonality displayed in the retail sales data in Figure 12, the moving average smooths out the monthly fluctuations in oil prices to show the longer-term movements.

Figure 13 Monthly Europe Brent Crude Oil Price and 12-Month Moving Average of Prices



Source: US Energy Information Administration.

Figure 13 also shows one weakness with a moving average: It always lags large movements in the actual data. For example, when oil prices rose quickly in late 2007 and the first half of 2008, the moving average rose only gradually. When oil prices fell sharply toward the end of 2008, the moving average also lagged. Consequently, a simple moving average of the recent past, though often useful in smoothing out a time series, may not be the best predictor of the future. A main reason for this is that a simple moving average gives equal weight to all the periods in the moving average. In order to forecast the future values of a time series, it is often better to use a more sophisticated moving-average time-series model. We discuss such models below.

6.2 Moving-Average Time-Series Models for Forecasting

Suppose that a time series, x_t, is consistent with the following model:

$$x_t = \varepsilon_t + \theta\varepsilon_{t-1},\ E(\varepsilon_t) = 0,\ E\left(\varepsilon_t^2\right) = \sigma^2,$$
$$\text{Cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s \quad (12)$$

This equation is called a moving-average model of order 1, or simply an MA(1) model. Theta (θ) is the parameter of the MA(1) model.[33]

33 Note that a moving-average time-series model is very different from a simple moving average, as discussed in Section 6.1. The simple moving average is based on observed values of a time series. In a moving-average time-series model, we never directly observe, ε_t or any other ε_{t-j}, but we can infer how a particular moving-average model will imply a particular pattern of serial correlation for a time series, as we discuss below.

Equation 12 is a moving-average model because in each period, x_t is a moving average of ε_t and ε_{t-1}, two uncorrelated random variables that each have an expected value of zero. Unlike the simple moving-average model of Equation 11, this moving-average model places different weights on the two terms in the moving average (1 on ε_t, and θ on ε_{t-1}).

We can see if a time series fits an MA(1) model by looking at its autocorrelations to determine whether x_t is correlated only with its preceding and following values. First, we examine the variance of x_t in Equation 12 and its first two autocorrelations. Because the expected value of x_t is 0 in all periods and ε_t is uncorrelated with its own past values, the first autocorrelation is not equal to 0, but the second and higher autocorrelations are equal to 0. Further analysis shows that all autocorrelations except for the first will be equal to 0 in an MA(1) model. Thus for an MA(1) process, any value x_t is correlated with x_{t-1} and x_{t+1} but with no other time-series values; we could say that an MA(1) model has a memory of one period.

Of course, an MA(1) model is not the most complex moving-average model. A qth order moving-average model, denoted MA(q) and with varying weights on lagged terms, can be written as

$$x_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q},\ E(\varepsilon_t) = 0,\ E\left(\varepsilon_t^2\right) = \sigma^2,$$
$$\mathrm{Cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t\varepsilon_s) = 0 \text{ for } t \neq s \tag{13}$$

How can we tell whether an MA(q) model fits a time series? We examine the autocorrelations. For an MA(q) model, the first q autocorrelations will be significantly different from 0, and all autocorrelations beyond that will be equal to 0; an MA(q) model has a memory of q periods. This result is critical for choosing the right value of q for an MA model. We discussed this result above for the specific case of q = 1 that all autocorrelations except for the first will be equal to 0 in an MA(1) model.

How can we distinguish an autoregressive time series from a moving-average time series? Once again, we do so by examining the autocorrelations of the time series itself. The autocorrelations of most autoregressive time series start large and decline gradually, whereas the autocorrelations of an MA(q) time series suddenly drop to 0 after the first q autocorrelations. We are unlikely to know in advance whether a time series is autoregressive or moving average. Therefore, the autocorrelations give us our best clue about how to model the time series. Most time series, however, are best modeled with an autoregressive model.

EXAMPLE 14

A Time-Series Model for Monthly Returns on the S&P BSE 100 Index

The S&P BSE 100 index is designed to reflect the performance of India's top 100 large-cap companies listed on the BSE Ltd. (formerly Bombay Stock Exchange). Are monthly returns on the S&P BSE 100 index autocorrelated? If so, we may be able to devise an investment strategy to exploit the autocorrelation. What is an appropriate time-series model for S&P BSE 100 monthly returns?

Table 12 shows the first six autocorrelations of returns to the S&P BSE 100 using monthly data from January 2000 through December 2013. Note that all of the autocorrelations are quite small. Do they reach significance? With 168 observations, the critical value for a t-statistic in this model is about 1.98 at the 0.05 significance level. None of the autocorrelations has a t-statistic larger in absolute value than the critical value of 1.98. Consequently, we fail to reject the null hypothesis that those autocorrelations, individually, do not differ significantly from 0.

Table 12 Annualized Monthly Returns to the S&P BSE 100 January 2000–December 2013

	Autocorrelations		
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.1103	0.0772	1.4288
2	–0.0045	0.0772	–0.0583
3	0.0327	0.0772	0.4236
4	0.0370	0.0772	0.4793
5	–0.0218	0.0772	–0.2824
6	0.0191	0.0772	0.2474
Observations	168		

Source: BSE Ltd.

If returns on the S&P BSE 100 were an MA(q) time series, then the first q autocorrelations would differ significantly from 0. None of the autocorrelations is statistically significant, however, so returns to the S&P BSE 100 appear to come from an MA(0) time series. An MA(0) time series in which we allow the mean to be nonzero takes the following form:[34]

$$x_t = \mu + \varepsilon_t,\ E(\varepsilon_t) = 0,\ E(\varepsilon_t^2) = \sigma^2,$$
$$\text{Cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s \qquad (14)$$

which means that the time series is not predictable. This result should not be too surprising, as most research suggests that short-term returns to stock indexes are difficult to predict.

We can see from this example how examining the autocorrelations allowed us to choose between the AR and MA models. If returns to the S&P BSE 100 had come from an AR(1) time series, the first autocorrelation would have differed significantly from 0 and the autocorrelations would have declined gradually. Not even the first autocorrelation is significantly different from 0, however. Therefore, we can be sure that returns to the S&P BSE 100 do not come from an AR(1) model—or from any higher-order AR model, for that matter. This finding is consistent with our conclusion that the S&P BSE 100 series is MA(0).

7 SEASONALITY IN TIME-SERIES MODELS

As we analyze the results of the time-series models in this reading, we encounter complications. One common complication is significant **seasonality**, a case in which the series shows regular patterns of movement within the year. At first glance,

34 On the basis of investment theory and evidence, we expect that the mean monthly return on the S&P BSE 100 is positive ($\mu > 0$). We can also generalize Equation 13 for an MA(q) time series by adding a constant term, μ. Including a constant term in a moving-average model does not change the expressions for the variance and autocovariances of the time series. A number of early studies of weak-form market efficiency used Equation 14 as the model for stock returns. See Garbade (1982).

seasonality might appear to rule out using autoregressive time-series models. After all, autocorrelations will differ by season. This problem can often be solved, however, by using seasonal lags in an autoregressive model.

A seasonal lag is usually the value of the time series one year before the current period, included as an extra term in an autoregressive model. Suppose, for example, that we model a particular quarterly time series using an AR(1) model, $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$. If the time series had significant seasonality, this model would not be correctly specified. The seasonality would be easy to detect because the seasonal autocorrelation (in the case of quarterly data, the fourth autocorrelation) of the error term would differ significantly from 0. Suppose this quarterly model has significant seasonality. In this case, we might include a seasonal lag in the autoregressive model and estimate

$$x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-4} + \varepsilon_t \quad (15)$$

to test whether including the seasonal lag would eliminate statistically significant autocorrelation in the error term.

In Examples 15 and 16, we illustrate how to test and adjust for seasonality in a time-series model. We also illustrate how to compute a forecast using an autoregressive model with a seasonal lag.

EXAMPLE 15

Seasonality in Sales at Starbucks

Earlier, we concluded that we could not model the log of Starbucks' quarterly sales using only a time-trend line (as shown in Example 3) because the Durbin–Watson statistic from the regression provided evidence of positive serial correlation in the error term. Based on methods presented in this reading, we might next investigate using the first difference of log sales to remove an exponential trend from the data to obtain a covariance stationary time series.

Using quarterly data from the first quarter of 1995 to the last quarter of 2012, we estimate the following AR(1) model using ordinary least squares: $(\ln \text{Sales}_t - \ln \text{Sales}_{t-1}) = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + \varepsilon_t$. Table 13 shows the results of the regression.

Table 13 Log Differenced Sales: AR(1) Model Starbucks, Quarterly Observations, 1995–2013

Regression Statistics	
R-squared	0.1548
Standard error	0.0762
Observations	74
Durbin–Watson	1.9165

	Coefficient	Standard Error	t-Statistic
Intercept	0.0669	0.0101	6.6238
$\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}$	−0.3813	0.1050	−3.6314

(continued)

Table 13 (Continued)

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	−0.0141	0.1162	−0.1213
2	−0.0390	0.1162	−0.3356
3	0.0294	0.1162	0.2530
4	0.7667	0.1162	6.5981

Source: Compustat.

The first thing to note in Table 13 is the strong seasonal autocorrelation of the residuals. The bottom portion of the table shows that the fourth autocorrelation has a value of 0.7667 and a *t*-statistic of 6.60. With 74 observations and two parameters, this model has 72 degrees of freedom.[35] The critical value for a *t*-statistic is about 1.99 at the 0.05 significance level. Given this value of the *t*-statistic, we must reject the null hypothesis that the fourth autocorrelation is equal to 0 because the *t*-statistic is larger than the critical value of 1.99.

In this model, the fourth autocorrelation is the seasonal autocorrelation because this AR(1) model is estimated with quarterly data. Table 13 shows the strong and statistically significant seasonal autocorrelation that occurs when a time series with strong seasonality is modeled without taking the seasonality into account. Therefore, the AR(1) model is misspecified, and we should not use it for forecasting.

Suppose we decide to use an autoregressive model with a seasonal lag because of the seasonal autocorrelation. We are modeling quarterly data, so we estimate Equation 15: $(\ln \text{Sales}_t - \ln \text{Sales}_{t-1}) = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + b_2(\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}) + \varepsilon_t$. Adding the seasonal difference $\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}$ is an attempt to remove a consistent quarterly pattern in the data and could also eliminate a seasonal non-stationarity if one existed. The estimates of this equation appear in Table 14.

Table 14 Log Differenced Sales: AR(1) Model with Seasonal Lag Starbucks, Quarterly Observations, 1995–2013

Regression Statistics	
R-squared	0.8163
Standard error	0.03405
Observations	71
Durbin–Watson	2.0791

35 In this example, we restrict the start of the sample period to the beginning of 1995, and we do not use prior observations for the lags. Accordingly, the number of observations decreases with an increase in the number of lags. In Table 13, the first observation is for the third quarter of 1995 because we use up to two lags. In Table 14, the first observation is for the second quarter of 1996 because we use up to five lags.

Table 14 (Continued)

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0084	0.0059	1.4237
ln Sales_{t-1} − ln Sales_{t-2}	−0.0602	0.0540	−1.1148
ln Sales_{t-4} − ln Sales_{t-5}	0.8048	0.0524	15.3588

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	−0.0441	0.1187	−0.3715
2	0.0675	0.1187	0.5687
3	0.0749	0.1187	0.6310
4	−0.2091	0.1187	−1.7616

Source: Compustat.

Note the autocorrelations of the residual shown at the bottom of Table 14. When we include a seasonal lag in the regression, the coefficient on the first difference in log sales, with a t-statistic of −1.1148, is no longer statistically significant. However, none of the t-statistics on the first four autocorrelations is now significant. Because the overall regression is highly significant (an F-test is significant at the 0.01 level), we can take an AR(1) model with a seasonal lag as a reasonable working model for Starbucks sales. (A model having only a seasonal lag term was investigated and not found to improve on this model.)

How can we interpret the coefficients in this model? To predict the current quarter's sales growth at Starbucks, we need to know two things: sales growth in the previous quarter and sales growth four quarters ago. If sales remained constant in each of those two quarters, the model in Table 14 predicts that sales will grow by 0.0084 (0.84 percent) in the current quarter. If sales grew by 1 percent last quarter and by 2 percent four quarters ago, then the model predicts that sales growth this quarter will be 0.0084 − 0.0602(0.01) + 0.8048(0.02) = 0.0239 or 2.39 percent.[36] Notice also that the R^2 in the model with the seasonal lag (0.8163 in Table 14) was more than five times higher than the R^2 in the model without the seasonal lag (0.1548 in Table 13). Again, the seasonal lag model does a much better job of explaining the data.

EXAMPLE 16

Retail Sales Growth

We want to predict the growth in monthly retail sales of Canadian furniture and home furnishing stores so that we can decide whether to recommend the shares of these stores. We decide to use non-seasonally adjusted data on retail sales. To begin with, we estimate an AR(1) model with observations on the

36 Note that all of these growth rates are exponential growth rates.

annualized monthly growth in retail sales from January 1995 to December 2012. We estimate the following equation: Sales growth$_t$ = b_0 + b_1 Sales growth$_{t-1}$ + ε_t. Table 15 shows the results from this model.

The autocorrelations of the residuals from this model, shown at the bottom of Table 15, indicate that seasonality is extremely significant in this model. With 216 observations and two parameters, this model has 214 degrees of freedom. At the 0.05 significance level, the critical value for a t-statistic is about 1.97. The 12th-lag autocorrelation (the seasonal autocorrelation, because we are using monthly data) has a value of 0.7620 and a t-statistic of 11.21. The t-statistic on this autocorrelation is larger than the critical value (1.97) implying that we can reject the null hypothesis that the 12th autocorrelation is 0. Note also that many of the other t-statistics for autocorrelations shown in the table differ significantly from 0. Consequently, the model shown in Table 15 is misspecified, so we cannot rely on it to forecast sales growth.

Suppose we add the seasonal lag of sales growth (the 12th lag) to the AR(1) model to estimate the equation Sales growth$_t$ = b_0 + b_1(Sales growth$_{t-1}$) + b_2(Sales growth$_{t-12}$) + ε_t.[37] Table 16 presents the results of estimating this equation. The estimated value of the seasonal autocorrelation (the 12th autocorrelation) has fallen to −0.1168. None of the first 12 autocorrelations has a t-statistic with an absolute value greater than the critical value of 1.97 at the 0.05 significance level. We can conclude that there is no significant serial correlation in the residuals from this model. Because we can reasonably believe that the model is correctly specified, we can use it to predict retail sales growth. Note that the R^2 in Table 16 is 0.6724, much larger than the R^2 in Table 15 (computed by the model without the seasonal lag).

Table 15 Monthly Retail Sales Growth of Canadian Furniture and Home Furnishing Stores: AR(1) Model January 1995–December 2012

Regression Statistics	
R-squared	0.0509
Standard error	1.8198
Observations	216
Durbin–Watson	2.0956

	Coefficient	Standard Error	t-Statistic
Intercept	1.0518	0.1365	7.7055
Sales growth$_{t-1}$	−0.2252	0.0665	−3.3865

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	t-Statistic
1	−0.0109	0.0680	−0.1603
2	−0.1949	0.0680	−2.8662
3	0.1173	0.0680	1.7250

37 In this example, although we state that the sample period begins in 1995, we use prior observations for the lags. This results in the same number of observations irrespective of the number of lags.

Table 15 (Continued)

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
4	−0.0756	0.0680	−1.1118
5	−0.1270	0.0680	−1.8676
6	−0.1384	0.0680	−2.0353
7	−0.1374	0.0680	−2.0206
8	−0.0325	0.0680	−0.4779
9	0.1207	0.0680	1.7750
10	−0.2197	0.0680	−3.2309
11	−0.0342	0.0680	−0.5029
12	0.7620	0.0680	11.2059

Source: Statistics Canada (Government of Canada).

How can we interpret the coefficients in the model? To predict growth in retail sales in this month, we need to know last month's retail sales growth and retail sales growth 12 months ago. If retail sales remained constant both last month and 12 months ago, the model in Table 16 predicts that retail sales will grow at an annual rate of about 23.7 percent this month. If retail sales grew at an annual rate of 10 percent last month and at an annual rate of 5 percent 12 months ago, the model in Table 16 predicts that retail sales will grow in the current month at an annual rate of 0.2371 − 0.0792(0.10) + 0.7798(0.05) = 0.2682 or 26.8 percent.

Table 16 Monthly Retail Sales Growth of Canadian Furniture and Home Furnishing Stores: AR(1) Model with Seasonal Lag January 1995–December 2012

Regression Statistics	
R-squared	0.6724
Standard error	1.0717
Observations	216
Durbin–Watson	2.1784

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.2371	0.0900	2.6344
Sales growth$_{t-1}$	−0.0792	0.0398	−1.9899
Sales growth$_{t-12}$	0.7798	0.0388	20.0979

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	−0.0770	0.0680	−1.1324
2	−0.0374	0.0680	−0.5500
3	0.0292	0.0680	0.4294

(continued)

Table 16 (Continued)

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
4	–0.0358	0.0680	–0.5265
5	–0.0399	0.0680	–0.5868
6	0.0227	0.0680	0.3338
7	–0.0967	0.0680	–1.4221
8	0.1241	0.0680	1.8250
9	0.0499	0.0680	0.7338
10	–0.0631	0.0680	–0.9279
11	0.0231	0.0680	0.3397
12	–0.1168	0.0680	–1.7176

Source: Statistics Canada (Government of Canada).

8 AUTOREGRESSIVE MOVING-AVERAGE MODELS

So far, we have presented autoregressive and moving-average models as alternatives for modeling a time series. The time series we have considered in examples have usually been explained quite well with a simple autoregressive model (with or without seasonal lags).[38] Some statisticians, however, have advocated using a more general model, the autoregressive moving-average (ARMA) model. The advocates of ARMA models argue that these models may fit the data better and provide better forecasts than do plain autoregressive (AR) models. However, as we discuss later in this section, there are severe limitations to estimating and using these models. Because you may encounter ARMA models, we provide a brief overview below.

An ARMA model combines both autoregressive lags of the dependent variable and moving-average errors. The equation for such a model with p autoregressive terms and q moving-average terms, denoted ARMA(p, q), is

$$\begin{aligned} x_t &= b_0 + b_1 x_{t-1} + \cdots + b_p x_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \\ E(\varepsilon_t) &= 0,\ E\left(\varepsilon_t^2\right) = \sigma^2,\ \mathrm{Cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s \end{aligned} \tag{16}$$

where $b_1, b_2, \ldots, b_p$ are the autoregressive parameters and $\theta_1, \theta_2, \ldots, \theta_q$ are the moving-average parameters.

Estimating and using ARMA models has several limitations. First, the parameters in ARMA models can be very unstable. In particular, slight changes in the data sample or the initial guesses for the values of the ARMA parameters can result in very different final estimates of the ARMA parameters. Second, choosing the right ARMA model is more of an art than a science. The criteria for deciding on p and q for a particular time series are far from perfect. Moreover, even after a model is selected, that model may not forecast well.

38 For the returns on the S&P BSE 100 (see Example 14), we chose a moving-average model over an autoregressive model.

To reiterate, ARMA models can be very unstable, depending on the data sample used and the particular ARMA model estimated. Therefore, you should be skeptical of claims that a particular ARMA model provides much better forecasts of a time series than any other ARMA model. In fact, in most cases, you can use an AR model to produce forecasts that are just as accurate as those from ARMA models without nearly as much complexity. Even some of the strongest advocates of ARMA models admit that these models should not be used with fewer than 80 observations, and they do not recommend using ARMA models for predicting quarterly sales or gross margins for a company using even 15 years of quarterly data.

AUTOREGRESSIVE CONDITIONAL HETEROSKEDASTICITY MODELS 9

Up to now, we have ignored any issues of heteroskedasticity in time-series models and have assumed homoskedasticity. **Heteroskedasticity** is the dependence of the error term variance on the independent variable; **homoskedasticity** is the independence of the error term variance from the independent variable. We have assumed that the error term's variance is constant and does not depend on the value of the time series itself or on the size of previous errors. At times, however, this assumption is violated and the variance of the error term is not constant. In such a situation, the standard errors of the regression coefficients in AR, MA, or ARMA models will be incorrect, and our hypothesis tests would be invalid. Consequently, we can make poor investment decisions based on those tests.

For example, suppose you are building an autoregressive model of a company's sales. If heteroskedasticity is present, then the standard errors of the regression coefficients of your model are incorrect. It is likely that due to heteroskedasticity, one or more of the lagged sales terms may appear statistically significant when in fact they are not. Therefore, if you use this model for your decision making, you may make some suboptimal decisions.

In work responsible in part for his shared Nobel Prize in Economics for 2003, Robert F. Engle in 1982 first suggested a way of testing whether the variance of the error in a particular time-series model in one period depends on the variance of the error in previous periods. He called this type of heteroskedasticity autoregressive conditional heteroskedasticity (ARCH).

As an example, consider the ARCH(1) model

$$\varepsilon_t \sim N\left(0, a_0 + a_1\varepsilon_{t-1}^2\right) \quad (17)$$

where the distribution of ε_t, conditional on its value in the previous period, ε_{t-1}, is normal with mean 0 and variance $a_0 + a_1\varepsilon_{t-1}^2$. If $a_1 = 0$, the variance of the error in every period is just a_0. The variance is constant over time and does not depend on past errors. Now suppose that $a_1 > 0$. Then the variance of the error in one period depends on how large the squared error was in the previous period. If a large error occurs in one period, the variance of the error in the next period will be even larger.

Engle shows that we can test whether a time series is ARCH(1) by regressing the squared residuals from a previously estimated time-series model (AR, MA, or ARMA) on a constant and one lag of the squared residuals. We can estimate the linear regression equation

$$\hat{\varepsilon}_t^2 = a_0 + a_1\hat{\varepsilon}_{t-1}^2 + u_t \quad (18)$$

where u_t is an error term. If the estimate of a_1 is statistically significantly different from zero, we conclude that the time series is ARCH(1). If a time-series model has ARCH(1) errors, then the variance of the errors in period $t + 1$ can be predicted in period t using the formula $\hat{\sigma}_{t+1}^2 = \hat{a}_0 + \hat{a}_1 \hat{\varepsilon}_t^2$.

EXAMPLE 17

Testing for ARCH(1) in Monthly Inflation

Analyst Lisette Miller wants to test whether monthly data on CPI inflation contain autoregressive conditional heteroskedasticity. She could estimate Equation 18 using the residuals from the time-series model. Based on the analyses in Examples 6 through 9, she has concluded that if she modeled monthly CPI inflation from 1984 to 2013, there is not much difference in the performance of AR(1) and AR(2) models in forecasting inflation. The AR(1) model is clearly better for the period of 2007–2013. She decides to further explore the AR(1) model for the entire period of 1984 to 2013. Table 17 shows the results of testing whether the errors in that model are ARCH(1). Because the test involves the first lag of residuals of the estimated time series model, the number of observations in the test is one less than that in the model.

The *t*-statistic for the coefficient on the previous period's squared residuals is greater than 4.1. Therefore, Miller easily rejects the null hypothesis that the variance of the error does not depend on the variance of previous errors. Consequently, the test statistics she computed in Table 5 are not valid, and she should not use them in deciding her investment strategy.

Table 17 Test for ARCH(1) in an AR(1) Model Residuals from Monthly CPI Inflation at an Annual Rate February 1984–December 2013

Regression Statistics	
R-squared	0.0467
Standard error	24.3630
Observations	358
Durbin–Watson	2.1046

	Coefficient	Standard Error	*t*-Statistic
Intercept	9.1666	1.4228	6.4426
$\hat{\varepsilon}_{t-1}^2$	0.2161	0.0518	4.1718

Source: US Bureau of Labor Statistics.

It is possible Miller's conclusion—that the AR(1) model for monthly inflation has ARCH in the errors—may have been due to the sample period employed (1984 to 2013). In Example 9, she used a shorter sample period of 2007 to 2013 and concluded that monthly CPI inflation follows an AR(1) process. (These results were shown in Table 8.) Table 17 shows that errors for a time-series model of inflation for the entire sample (1984 to 2013) have ARCH errors. Do the errors estimated with a shorter sample period (2007 to 2013) also display

ARCH? For the shorter sample period, Miller estimated an AR(1) model using monthly inflation data. Now she tests to see whether the errors display ARCH. Table 18 shows the results.

In this sample, the coefficient on the previous period's squared residual is quite small and has a *t*-statistic of only 1.3713. Consequently, Miller fails to reject the null hypothesis that the errors in this regression have no autoregressive conditional heteroskedasticity. This is additional evidence that the AR(1) model for 2007 to 2013 is a good fit. The error variance appears to be homoskedastic, and Miller can rely on the *t*-statistics. This result again confirms that a single AR process for the entire 1984–2013 period is misspecified (it does not describe the data well).

Table 18 Test for ARCH(1) in an AR(1) Model Monthly CPI Inflation at an Annual Rate February 2007–December 2013

Regression Statistics	
R-squared	0.0230
Standard error	35.218
Observations	82
Durbin–Watson	2.0521

	Coefficient	Standard Error	*t*-Statistic
Intercept	16.6213	4.4632	3.7241
$\hat{\varepsilon}_{t-1}^2$	0.1518	0.1107	1.3713

Source: US Bureau of Labor Statistics.

Suppose a model contains ARCH(1) errors. What are the consequences of that fact? First, if ARCH exists, the standard errors for the regression parameters will not be correct. We will need to use generalized least squares[39] or other methods that correct for heteroskedasticity to correctly estimate the standard error of the parameters in the time-series model. Second, if ARCH exists and we have it modeled, for example as ARCH(1), we can predict the variance of the errors. Suppose, for instance, that we want to predict the variance of the error in inflation using the estimated parameters from Table 17: $\hat{\sigma}_t^2 = 9.1666 + 0.2161\hat{\varepsilon}_{t-1}^2$. If the error in one period were 0 percent, the predicted variance of the error in the next period would be 9.1666 + 0.2161(0) = 9.1666. If the error in one period were 1 percent, the predicted variance of the error in the next period would be 9.1666 + 0.2161(1^2) = 9.3827.

Engle and other researchers have suggested many generalizations of the ARCH(1) model, including ARCH(p) and generalized autoregressive conditional heteroskedasticity (GARCH) models. In an ARCH(p) model, the variance of the error term in the current period depends linearly on the squared errors from the previous p periods: $\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \cdots + a_p\varepsilon_{t-p}^2$. GARCH models are similar to ARMA models

39 See Greene (2018).

of the error variance in a time series. Just like ARMA models, GARCH models can be finicky and unstable: Their results can depend greatly on the sample period and the initial guesses of the parameters in the GARCH model. Financial analysts who use GARCH models should be well aware of how delicate these models can be, and they should examine whether GARCH estimates are robust to changes in the sample and the initial guesses about the parameters.[40]

10 REGRESSIONS WITH MORE THAN ONE TIME SERIES

Up to now, we have discussed time-series models only for one time series. Although in the readings on correlation and regression and on multiple regression we used linear regression to analyze the relationship among different time series, in those readings we completely ignored unit roots. A time series that contains a unit root is not covariance stationary. If any time series in a linear regression contains a unit root, ordinary least squares estimates of regression test statistics may be invalid.

To determine whether we can use linear regression to model more than one time series, let us start with a single independent variable; that is, there are two time series, one corresponding to the dependent variable and one corresponding to the independent variable. We will then extend our discussion to multiple independent variables.

We first use a unit root test, such as the Dickey–Fuller test, for each of the two time series to determine whether either of them has a unit root.[41] There are several possible scenarios related to the outcome of these tests. One possible scenario is that we find that neither of the time series has a unit root. Then we can safely use linear regression to test the relations between the two time series. Otherwise, we may have to use additional tests, as we discuss later in this section.

EXAMPLE 18

Unit Roots and the Fisher Effect

In Example 8 in the reading on multiple regression, we examined the Fisher effect by estimating the regression relation between expected inflation and US Treasury bill (T-bill) returns. We used 181 quarterly observations on expected inflation rates and T-Bill returns from the sample period extending from the fourth quarter of 1968 through the fourth quarter of 2013. We used linear regression to analyze the relationship between the two time series. The results of this regression would be valid if both the time series are covariance stationary; that is, neither of the two time series has a unit root. So, if we compute the Dickey–Fuller *t*-test statistic of the hypothesis of a unit root separately for each time series and find that we can reject the null hypothesis that the T-bill return series has a unit root and the null hypothesis that the expected inflation time series has a unit root, then we can use linear regression to analyze the relation between the two series. In that case, the results of our analysis of the Fisher effect would be valid.

40 For more on ARCH, GARCH, and other models of time-series variance, see Hamilton (1994).

41 For theoretical details of unit root tests, see Greene (2018) or Tsay (2010). Unit root tests are available in some econometric software packages, such as EViews.

A second possible scenario is that we reject the hypothesis of a unit root for the independent variable but fail to reject the hypothesis of a unit root for the dependent variable. In this case, the error term in the regression would not be covariance stationary. Therefore, one or more of the following linear regression assumptions would be violated: 1) that the expected value of the error term is 0, 2) that the variance of the error term is constant for all observations, and 3) that the error term is uncorrelated across observations. Consequently, the estimated regression coefficients and standard errors would be inconsistent. The regression coefficients might appear significant, but those results would be spurious.[42] Thus we should not use linear regression to analyze the relation between the two time series in this scenario.

A third possible scenario is the reverse of the second scenario: We reject the hypothesis of a unit root for the dependent variable but fail to reject the hypothesis of a unit root for the independent variable. In this case also, like the second scenario, the error term in the regression would not be covariance stationary, and we cannot use linear regression to analyze the relation between the two time series.

EXAMPLE 19

Unit Roots and Predictability of Stock Market Returns by Price-to-Earnings Ratio

Johann de Vries is analyzing the performance of the South African stock market. He examines whether the percentage change in the Johannesburg Stock Exchange (JSE) All Share Index can be predicted by the price-to-earnings ratio (P/E) for the index. Using monthly data from January 1994 to December 2013, he runs a regression using $(P_t - P_{t-1})/P_{t-1}$ as the dependent variable and P_{t-1}/E_{t-2} as the independent variable, where P_t is the value of the JSE index at time t and E_t is the earnings on the index. De Vries finds that the regression coefficient is negative and statistically significant and the value of the R-squared for the regression is quite high. What additional analysis should he perform before accepting the regression as valid?

De Vries needs to perform unit root tests for each of the two time series. If one of the two time series has a unit root, implying that it is not stationary, the results of the linear regression are not meaningful and cannot be used to conclude that stock market returns are predictable by P/E.[43]

The next possibility is that both time series have a unit root. In this case, we need to establish whether the two time series are **cointegrated** before we can rely on regression analysis.[44] Two time series are cointegrated if a long-term financial or economic relationship exists between them such that they do not diverge from each other without bound in the long run. For example, two time series are cointegrated if they share a common trend.

In the fourth scenario, both time series have a unit root but are not cointegrated. In this scenario, as in the second and third scenarios above, the error term in the linear regression will not be covariance stationary, some regression assumptions will be violated, the regression coefficients and standard errors will not be consistent, and we cannot use them for hypothesis tests. Consequently, linear regression of one variable on the other would be meaningless.

42 The problem of spurious regression for nonstationary time series was first discussed by Granger and Newbold (1974).

43 Barr and Kantor (1999) contains evidence that the P/E time series is nonstationary.

44 Engle and Granger (1987) first discussed cointegration.

Finally, the fifth possible scenario is that both time series have a unit root, but they are cointegrated. In this case, the error term in the linear regression of one time series on the other will be covariance stationary. Accordingly, the regression coefficients and standard errors will be consistent, and we can use them for hypothesis tests. However, we should be very cautious in interpreting the results of a regression with cointegrated variables. The cointegrated regression estimates the long-term relation between the two series but may not be the best model of the short-term relation between the two series. Short-term models of cointegrated series (error correction models) are discussed in Engle and Granger (1987) and Tsay (2010), but these are specialist topics.

Now let us look at how we can test for cointegration between two time series that each have a unit root as in the last two scenarios above.[45] Engle and Granger suggest this test: If y_t and x_t are both time series with a unit root, we should do the following:

1 Estimate the regression $y_t = b_0 + b_1 x_t + \varepsilon_t$.

2 Test whether the error term from the regression in Step 1 has a unit root using a Dickey–Fuller test. Because the residuals are based on the estimated coefficients of the regression, we cannot use the standard critical values for the Dickey–Fuller test. Instead, we must use the critical values computed by Engle and Granger, which take into account the effect of uncertainty about the regression parameters on the distribution of the Dickey–Fuller test.

3 If the (Engle–Granger) Dickey–Fuller test fails to reject the null hypothesis that the error term has a unit root, then we conclude that the error term in the regression is not covariance stationary. Therefore, the two time series are not cointegrated. In this case any regression relation between the two series is spurious.

4 If the (Engle–Granger) Dickey–Fuller test rejects the null hypothesis that the error term has a unit root, then we may assume that the error term in the regression is covariance stationary and that the two time series are cointegrated. The parameters and standard errors from linear regression will be consistent and will let us test hypotheses about the long-term relation between the two series.

45 Consider a time series, x_t, that has a unit root. For many such financial and economic time series, the first difference of the series, $x_t - x_{t-1}$, is stationary. We say that such a series, whose first difference is stationary, has a *single* unit root. However, for some time series, even the first difference may not be stationary and further differencing may be needed to achieve stationarity. Such a time series is said to have *multiple* unit roots. In this section, we consider only the case in which each nonstationary series has a single unit root (which is quite common).

EXAMPLE 20

Testing for Cointegration between Intel Sales and Nominal GDP

Suppose we want to test whether the natural log of Intel's sales and the natural log of GDP are cointegrated (that is, whether there is a long-term relation between GDP and Intel sales). We want to test this hypothesis using quarterly data from the first quarter of 1995 through the fourth quarter of 2013. Here are the steps:

1 Test whether the two series each have a unit root. If we cannot reject the null hypothesis of a unit root for both series, implying that both series are nonstationary, we must then test whether the two series are cointegrated.

2 Having established that each series has a unit root, we estimate the regression ln (Intel Sales$_t$) = $b_0 + b_1$ ln GDP$_t$ + ε_t, then conduct the (Engle–Granger) Dickey–Fuller test of the hypothesis that there is a unit root in the error term of this regression using the residuals from the estimated regression. If we reject the null hypothesis of a unit root in the error term of the regression, we reject the null hypothesis of no cointegration. That is, the two series would be cointegrated. If the two series are cointegrated, we can use linear regression to estimate the long-term relation between the natural log of Intel Sales and the natural log of GDP.

We have so far discussed models with a single independent variable. We now extend the discussion to a model with two or more independent variables, so that there are three or more time series. The simplest possibility is that none of the time series in the model has a unit root. Then, we can safely use multiple regression to test the relation among the time series.

EXAMPLE 21

Unit Roots and Returns to the Fidelity Select Technology Fund

In Example 3 in the reading on multiple regression, we used multiple linear regression to examine whether returns to either the S&P 500 Growth Index or the S&P 500 Value Index explain returns to the Fidelity Select Technology Portfolio using 60 monthly observations between January 2009 and December 2013. Of course, if any of the three time series has a unit root, then the results of our regression analysis may be invalid. Therefore, we could use a Dickey–Fuller test to determine whether any of these series has a unit root.

If we reject the hypothesis of unit roots for all three series, we can use linear regression to analyze the relation among the series. In that case the results of our analysis of the factors affecting returns to the Fidelity Select Technology Portfolio would be valid.

If at least one time series (the dependent variable or one of the independent variables) has a unit root while at least one time series (the dependent variable or one of the independent variables) does not, the error term in the regression cannot be covariance stationary. Consequently, we should not use multiple linear regression to analyze the relation among the time series in this scenario.

Another possibility is that each time series, including the dependent variable and each of the independent variables, has a unit root. If this is the case, we need to establish whether the time series are cointegrated. To test for cointegration, the procedure is similar to that for a model with a single independent variable. First, estimate the regression $y_t = b_0 + b_1x_{1t} + b_2x_{2t} + \ldots + b_kx_{kt} + \varepsilon_t$. Then conduct the (Engle–Granger) Dickey–Fuller test of the hypothesis that there is a unit root in the errors of this regression using the residuals from the estimated regression.

If we cannot reject the null hypothesis of a unit root in the error term of the regression, we cannot reject the null hypothesis of no cointegration. In this scenario, the error term in the multiple regression will not be covariance stationary, so we cannot use multiple regression to analyze the relationship among the time series.

If we can reject the null hypothesis of a unit root in the error term of the regression, we can reject the null hypothesis of no cointegration. However, modeling three or more time series that are cointegrated may be difficult. For example, an analyst may want to predict a retirement services company's sales based on the country's GDP and the total population over age 65. Although the company's sales, GDP, and the population over 65 may each have a unit root and be cointegrated, modeling the cointegration of the three series may be difficult, and doing so is beyond the scope of this volume. Analysts who have not mastered all these complex issues should avoid forecasting models with multiple time series that have unit roots: The regression coefficients may be inconsistent and may produce incorrect forecasts.

11 OTHER ISSUES IN TIME SERIES

Time-series analysis is an extensive topic and includes many highly complex issues. Our objective in this reading has been to present those issues in time series that are the most important for financial analysts and can also be handled with relative ease. In this section, we briefly discuss some of the issues that we have not covered but could be useful for analysts.

In this reading, we have shown how to use time-series models to make forecasts. We have also introduced the RMSE as a criterion for comparing forecasting models. However, we have not discussed measuring the uncertainty associated with forecasts made using time-series models. The uncertainty of these forecasts can be very large, and should be taken into account when making investment decisions. Fortunately, the same techniques apply to evaluating the uncertainty of time-series forecasts as apply to evaluating the uncertainty about forecasts from linear regression models. To accurately evaluate forecast uncertainty, we need to consider both the uncertainty about the error term and the uncertainty about the estimated parameters in the time-series model. Evaluating this uncertainty is fairly complicated when using regressions with more than one independent variable.

In this reading, we used the US CPI inflation series to illustrate some of the practical challenges analysts face in using time-series models. We used information on US Federal Reserve policy to explore the consequences of splitting the inflation series in two. In financial time-series work, we may suspect that a time series has more than one regime but lack the information to attempt to sort the data into different regimes. If you face such a problem, you may want to investigate other methods, especially switching regression models, to identify multiple regimes using only the time series itself.

If you are interested in these and other advanced time-series topics, you can learn more in Diebold (2008) and Tsay (2010).

SUGGESTED STEPS IN TIME-SERIES FORECASTING

12

The following is a step-by-step guide to building a model to predict a time series.

1 Understand the investment problem you have, and make an initial choice of model. One alternative is a regression model that predicts the future behavior of a variable based on hypothesized causal relationships with other variables. Another is a time-series model that attempts to predict the future behavior of a variable based on the past behavior of the same variable.

2 If you have decided to use a time-series model, compile the time series and plot it to see whether it looks covariance stationary. The plot might show important deviations from covariance stationarity, including the following:
- a linear trend;
- an exponential trend;
- seasonality; or
- a significant shift in the time series during the sample period (for example, a change in mean or variance).

3 If you find no significant seasonality or shift in the time series, then perhaps either a linear trend or an exponential trend will be sufficient to model the time series. In that case, take the following steps:
- Determine whether a linear or exponential trend seems most reasonable (usually by plotting the series).
- Estimate the trend.
- Compute the residuals.
- Use the Durbin–Watson statistic to determine whether the residuals have significant serial correlation. If you find no significant serial correlation in the residuals, then the trend model is sufficient to capture the dynamics of the time series and you can use that model for forecasting.

4 If you find significant serial correlation in the residuals from the trend model, use a more complex model, such as an autoregressive model. First, however, reexamine whether the time series is covariance stationary. Following is a list of violations of stationarity, along with potential methods to adjust the time series to make it covariance stationary:
- If the time series has a linear trend, first-difference the time series.
- If the time series has an exponential trend, take the natural log of the time series and then first-difference it.
- If the time series shifts significantly during the sample period, estimate different time-series models before and after the shift.
- If the time series has significant seasonality, include seasonal lags (discussed in Step 7).

5 After you have successfully transformed a raw time series into a covariance-stationary time series, you can usually model the transformed series with a short autoregression.[46] To decide which autoregressive model to use, take the following steps:

46 Most financial time series can be modeled using an autoregressive process. For a few time series, a moving-average model may fit better. To see if this is the case, examine the first five or six autocorrelations of the time series. If the autocorrelations suddenly drop to 0 after the first q autocorrelations, a moving-average model (of order q) is appropriate. If the autocorrelations start large and decline gradually, an autoregressive model is appropriate.

- Estimate an AR(1) model.
- Test to see whether the residuals from this model have significant serial correlation.
- If you find no significant serial correlation in the residuals, you can use the AR(1) model to forecast.

6 If you find significant serial correlation in the residuals, use an AR(2) model and test for significant serial correlation of the residuals of the AR(2) model.

- If you find no significant serial correlation, use the AR(2) model.
- If you find significant serial correlation of the residuals, keep increasing the order of the AR model until the residual serial correlation is no longer significant.

7 Your next move is to check for seasonality. You can use one of two approaches:

- Graph the data and check for regular seasonal patterns.
- Examine the data to see whether the seasonal autocorrelations of the residuals from an AR model are significant (for example, the fourth autocorrelation for quarterly data) and whether the autocorrelations before and after the seasonal autocorrelations are significant. To correct for seasonality, add seasonal lags to your AR model. For example, if you are using quarterly data, you might add the fourth lag of a time series as an additional variable in an AR(1) or an AR(2) model.

8 Next, test whether the residuals have autoregressive conditional heteroskedasticity. To test for ARCH(1), for example, do the following:

- Regress the squared residual from your time-series model on a lagged value of the squared residual.
- Test whether the coefficient on the squared lagged residual differs significantly from 0.
- If the coefficient on the squared lagged residual does not differ significantly from 0, the residuals do not display ARCH and you can rely on the standard errors from your time-series estimates.
- If the coefficient on the squared lagged residual does differ significantly from 0, use generalized least squares or other methods to correct for ARCH.

9 Finally, you may also want to perform tests of the model's out-of-sample forecasting performance to see how the model's out-of-sample performance compares to its in-sample performance.

Using these steps in sequence, you can be reasonably sure that your model is correctly specified.

SUMMARY

- The predicted trend value of a time series in period t is $\hat{b}_0 + \hat{b}_1 t$ in a linear trend model; the predicted trend value of a time series in a log-linear trend model is $e^{\hat{b}_0 + \hat{b}_1 t}$.
- Time series that tend to grow by a constant amount from period to period should be modeled by linear trend models, whereas time series that tend to grow at a constant rate should be modeled by log-linear trend models.

- Trend models often do not completely capture the behavior of a time series, as indicated by serial correlation of the error term. If the Durbin–Watson statistic from a trend model differs significantly from 2, indicating serial correlation, we need to build a different kind of model.
- An autoregressive model of order *p*, denoted AR(*p*), uses *p* lags of a time series to predict its current value: $x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-2} + \ldots + b_p x_{t-p} + \varepsilon_t$.
- A time series is covariance stationary if the following three conditions are satisfied: First, the expected value of the time series must be constant and finite in all periods. Second, the variance of the time series must be constant and finite in all periods. Third, the covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods. Inspection of a nonstationary time-series plot may reveal an upward or downward trend (nonconstant mean) and/or nonconstant variance. The use of linear regression to estimate an autoregressive time-series model is not valid unless the time series is covariance stationary.
- For a specific autoregressive model to be a good fit to the data, the autocorrelations of the error term should be 0 at all lags.
- A time series is mean reverting if it tends to fall when its level is above its long-run mean and rise when its level is below its long-run mean. If a time series is covariance stationary, then it will be mean reverting.
- The one-period-ahead forecast of a variable x_t from an AR(1) model made in period *t* for period *t* + 1 is $\hat{x}_{t+1} = \hat{b}_0 + \hat{b}_1 x_t$. This forecast can be used to create the two-period ahead forecast from the model made in period *t*, $\hat{x}_{t+2} = \hat{b}_0 + \hat{b}_1 x_{t+1}$. Similar results hold for AR(*p*) models.
- In-sample forecasts are the in-sample predicted values from the estimated time-series model. Out-of-sample forecasts are the forecasts made from the estimated time-series model for a time period different from the one for which the model was estimated. Out-of-sample forecasts are usually more valuable in evaluating the forecasting performance of a time-series model than are in-sample forecasts. The root mean squared error (RMSE), defined as the square root of the average squared forecast error, is a criterion for comparing the forecast accuracy of different time-series models; a smaller RMSE implies greater forecast accuracy.
- Just as in regression models, the coefficients in time-series models are often unstable across different sample periods. In selecting a sample period for estimating a time-series model, we should seek to assure ourselves that the time series was stationary in the sample period.
- A random walk is a time series in which the value of the series in one period is the value of the series in the previous period plus an unpredictable random error. If the time series is a random walk, it is not covariance stationary. A random walk with drift is a random walk with a nonzero intercept term. All random walks have unit roots. If a time series has a unit root, then it will not be covariance stationary.
- If a time series has a unit root, we can sometimes transform the time series into one that is covariance stationary by first-differencing the time series; we may then be able to estimate an autoregressive model for the first-differenced series.
- An *n*-period moving average of the current and past $(n - 1)$ values of a time series, x_t, is calculated as $[x_t + x_{t-1} + \ldots + x_{t-(n-1)}]/n$.
- A moving-average model of order *q*, denoted MA(*q*), uses *q* lags of a random error term to predict its current value.

- The order q of a moving average model can be determined using the fact that if a time series is a moving-average time series of order q, its first q autocorrelations are nonzero while autocorrelations beyond the first q are zero.
- The autocorrelations of most autoregressive time series start large and decline gradually, whereas the autocorrelations of an MA(q) time series suddenly drop to 0 after the first q autocorrelations. This helps in distinguishing between autoregressive and moving-average time series.
- If the error term of a time-series model shows significant serial correlation at seasonal lags, the time series has significant seasonality. This seasonality can often be modeled by including a seasonal lag in the model, such as adding a term lagged four quarters to an AR(1) model on quarterly observations.
- The forecast made in time t for time $t + 1$ using a quarterly AR(1) model with a seasonal lag would be $x_{t+1} = \hat{b}_0 + \hat{b}_1 x_t + \hat{b}_2 x_{t-3}$.
- ARMA models have several limitations: the parameters in ARMA models can be very unstable; determining the AR and MA order of the model can be difficult; and even with their additional complexity, ARMA models may not forecast well.
- The variance of the error in a time-series model sometimes depends on the variance of previous errors, representing autoregressive conditional heteroskedasticity (ARCH). Analysts can test for first-order ARCH in a time-series model by regressing the squared residual on the squared residual from the previous period. If the coefficient on the squared residual is statistically significant, the time-series model has ARCH(1) errors.
- If a time-series model has ARCH(1) errors, then the variance of the errors in period $t + 1$ can be predicted in period t using the formula $\hat{\sigma}_{t+1}^2 = \hat{a}_0 + \hat{a}_1 \hat{\varepsilon}_t^2$.
- If linear regression is used to model the relationship between two time series, a test should be performed to determine whether either time series has a unit root:
  - If neither of the time series has a unit root, then we can safely use linear regression.
  - If one of the two time series has a unit root, then we should not use linear regression.
  - If both time series have a unit root and the time series are cointegrated, we may safely use linear regression; however, if they are not cointegrated, we should not use linear regression. The (Engle–Granger) Dickey–Fuller test can be used to determine if time series are cointegrated.

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PRACTICE PROBLEMS

Note: In the Problems and Solutions for this reading, we use the hat (ˆ) to indicate an estimate if we are trying to differentiate between an estimated and an actual value. However, we suppress the hat when we are clearly showing regression output.

1 The civilian unemployment rate (UER) is an important component of many economic models. Table 1 gives regression statistics from estimating a linear trend model of the unemployment rate: $\text{UER}_t = b_0 + b_1 t + \varepsilon_t$.

Table 1 Estimating a Linear Trend in the Civilian Unemployment Rate Monthly Observations, January 1996–December 2000

Regression Statistics	
R-squared	0.9314
Standard error	0.1405
Observations	60
Durbin–Watson	0.9099

	Coefficient	Standard Error	t-Statistic
Intercept	5.5098	0.0367	150.0363
Trend	–0.0294	0.0010	–28.0715

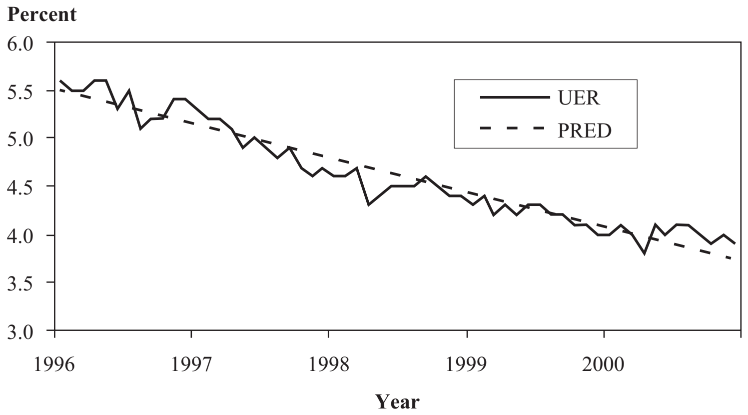
A Using the regression output in the above table, what is the model's prediction of the unemployment rate for July 1996?

B How should we interpret the Durbin–Watson (DW) statistic for this regression? What does the value of the DW statistic say about the validity of a t-test on the coefficient estimates?

2 Figure 1 compares the predicted civilian unemployment rate (PRED) with the actual civilian unemployment rate (UER) from January 1996 to December 2000. The predicted results come from estimating the linear time trend model $\text{UER}_t = b_0 + b_1 t + \varepsilon_t$.

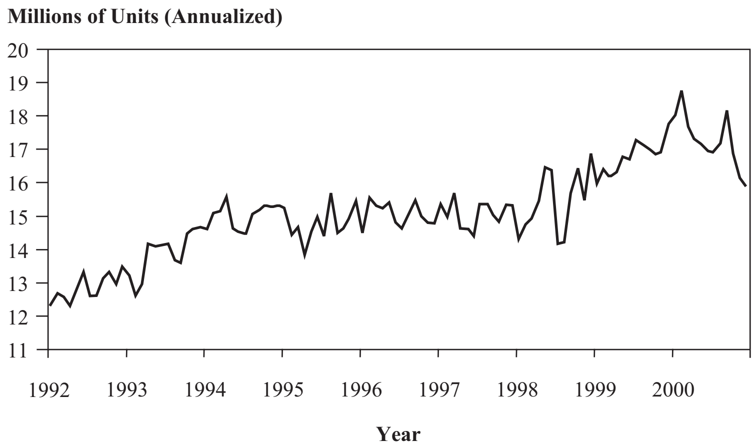
What can we conclude about the appropriateness of this model?

Figure 1 Predicted and Actual Civilian Unemployment Rates



3 You have been assigned to analyze automobile manufacturers and as a first step in your analysis, you decide to model monthly sales of lightweight vehicles to determine sales growth in that part of the industry. Figure 2 gives lightweight vehicle monthly sales (annualized) from January 1992 to December 2000.

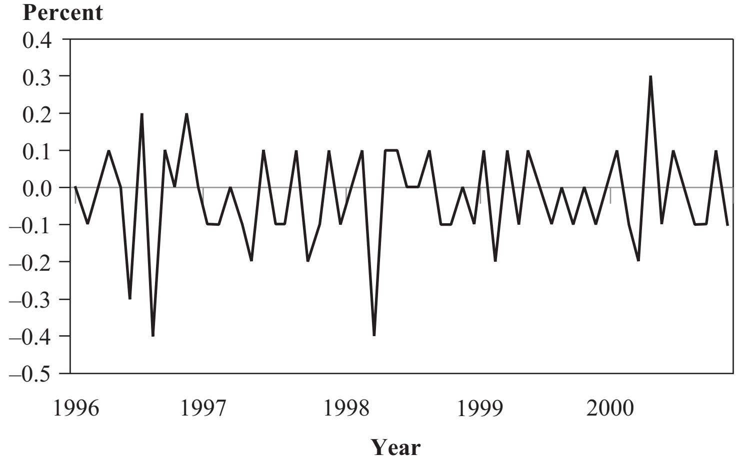
Figure 2 Lightweight Vehicle Sales



Monthly sales in the lightweight vehicle sector, $Sales_t$, have been increasing over time, but you suspect that the growth rate of monthly sales is relatively constant. Write the simplest time-series model for $Sales_t$ that is consistent with your perception.

4 Figure 3 shows a plot of the first differences in the civilian unemployment rate (UER) between January 1996 and December 2000, $\Delta UER_t = UER_t - UER_{t-1}$.

Figure 3 Change in Civilian Unemployment Rate



A Has differencing the data made the new series, ΔUER_t, covariance stationary? Explain your answer.

B Given the graph of the change in the unemployment rate shown in the figure, describe the steps we should take to determine the appropriate autoregressive time-series model specification for the series ΔUER_t.

5 Table 2 gives the regression output of an AR(1) model on first differences in the unemployment rate. Describe how to interpret the DW statistic for this regression.

Table 2 Estimating an AR(1) Model of Changes in the Civilian Unemployment Rate Monthly Observations, March 1996–December 2000

Regression Statistics	
R-squared	0.2184
Standard error	0.1202
Observations	58
Durbin–Watson	2.1852

	Coefficient	**Standard Error**	***t*-Statistic**
Intercept	–0.0405	0.0161	–2.5110
ΔUER_{t-1}	–0.4674	0.1181	–3.9562

6 Assume that changes in the civilian unemployment rate are covariance stationary and that an AR(1) model is a good description for the time series of changes in the unemployment rate. Specifically, we have $\Delta UER_t = -0.0405 - 0.4674\Delta UER_{t-1}$ (using the coefficient estimates given in the previous problem). Given this equation, what is the mean-reverting level to which changes in the unemployment rate converge?

7 Suppose the following model describes changes in the civilian unemployment rate: $\Delta UER_t = -0.0405 - 0.4674\Delta UER_{t-1}$. The current change (first difference) in the unemployment rate is 0.0300. Assume that the mean-reverting level for changes in the unemployment rate is –0.0276.

A What is the best prediction of the next change?

B What is the prediction of the change following the next change?

C Explain your answer to Part B in terms of equilibrium.

8 Table 3 gives the actual sales, log of sales, and changes in the log of sales of Cisco Systems for the period 1Q:2001 to 4Q:2001.

Table 3

Date Quarter: Year	Actual Sales ($ Millions)	Log of Sales	Changes in Log of Sales $\Delta\ln(\text{Sales}_t)$
1Q:2001	6,519	8.7825	0.1308
2Q:2001	6,748	8.8170	0.0345
3Q:2001	4,728	8.4613	–0.3557
4Q:2001	4,298	8.3659	–0.0954
1Q:2002			
2Q:2002			

Forecast the first- and second-quarter sales of Cisco Systems for 2002 using the regression $\Delta\ln(\text{Sales}_t) = 0.0661 + 0.4698\Delta\ln(\text{Sales}_{t-1})$.

9 Table 4 gives the actual change in the log of sales of Cisco Systems from 1Q:2001 to 4Q:2001, along with the forecasts from the regression model $\Delta\ln(\text{Sales}_t) = 0.0661 + 0.4698\Delta\ln(\text{Sales}_{t-1})$ estimated using data from 3Q:1991 to 4Q:2000. (Note that the observations after the fourth quarter of 2000 are out of sample.)

Table 4

Date	Actual Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$	Forecast Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$
1Q:2001	0.1308	0.1357
2Q:2001	0.0345	0.1299
3Q:2001	–0.3557	0.1271
4Q:2001	–0.0954	0.1259

A Calculate the RMSE for the out-of-sample forecast errors.

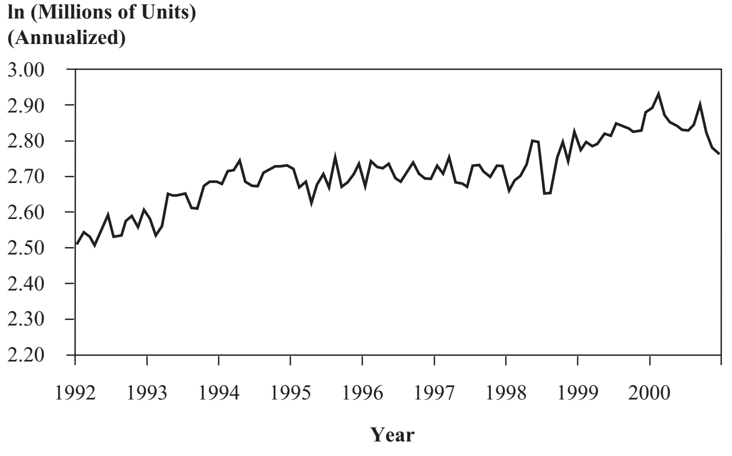
B Compare the forecasting performance of the model given with that of another model having an out-of-sample RMSE of 20 percent.

10 **A** The AR(1) model for the civilian unemployment rate, $\Delta UER_t = -0.0405 - 0.4674\Delta UER_{t-1}$, was developed with five years of data. What would be the drawback to using the AR(1) model to predict changes in the civilian unemployment rate 12 months or more ahead, as compared with one month ahead?

B For purposes of estimating a predictive equation, what would be the drawback to using 30 years of civilian unemployment data rather than only five years?

11 Figure 4 shows monthly observations on the natural log of lightweight vehicle sales, ln ($Sales_t$), for the period January 1992 to December 2000.

Figure 4 Lightweight Vehicle Sales

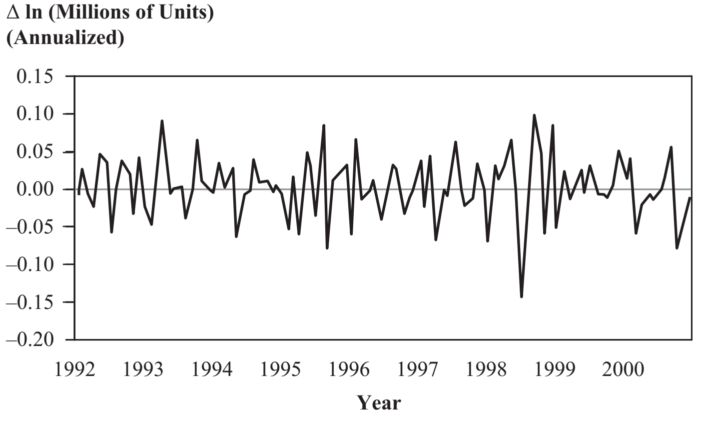


A Using the figure, comment on whether the specification ln ($Sales_t$) = b_0 + b_1[ln ($Sales_{t-1}$)] + ε_t is appropriate.

B State an appropriate transformation of the time series.

12 Figure 5 shows a plot of first differences in the log of monthly lightweight vehicle sales over the same period as in Problem 11. Has differencing the data made the resulting series, Δln ($Sales_t$) = ln ($Sales_t$) – ln ($Sales_{t-1}$), covariance stationary?

Figure 5 Change in Natural Log of Lightweight Vehicle Sales



13 Using monthly data from January 1992 to December 2000, we estimate the following equation for lightweight vehicle sales: Δln ($Sales_t$) = 2.7108 + 0.3987Δln ($Sales_{t-1}$) + ε_t. Table 5 gives sample autocorrelations of the errors from this model.

Table 5 Different Order Autocorrelations of Differences in the Logs of Vehicle Sales

Lag	Autocorrelation	Standard Error	t-Statistic
1	0.9358	0.0962	9.7247
2	0.8565	0.0962	8.9005
3	0.8083	0.0962	8.4001
4	0.7723	0.0962	8.0257
5	0.7476	0.0962	7.7696
6	0.7326	0.0962	7.6137
7	0.6941	0.0962	7.2138
8	0.6353	0.0962	6.6025
9	0.5867	0.0962	6.0968
10	0.5378	0.0962	5.5892
11	0.4745	0.0962	4.9315
12	0.4217	0.0962	4.3827

A Use the information in the table to assess the appropriateness of the specification given by the equation.

B If the residuals from the AR(1) model above violate a regression assumption, how would you modify the AR(1) specification?

14 Figure 6 shows the quarterly sales of Cisco Systems from 1Q:1991 to 4Q:2000.

Figure 6 Quarterly Sales at Cisco

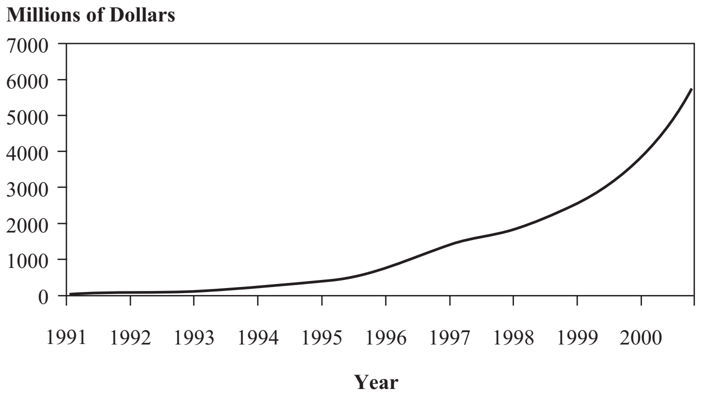


Table 6 gives the regression statistics from estimating the model $\Delta\ln(\text{Sales}_t) = b_0 + b_1\Delta\ln(\text{Sales}_{t-1}) + \varepsilon_t$.

Table 6 Change in the Natural Log of Sales for Cisco Systems Quarterly Observations, 3Q:1991–4Q:2000

Regression Statistics	
R-squared	0.2899
Standard error	0.0408
Observations	38
Durbin–Watson	1.5707

	Coefficient	Standard Error	t-Statistic
Intercept	0.0661	0.0175	3.7840
$\Delta\ln$ (Sales$_{t-1}$)	0.4698	0.1225	3.8339

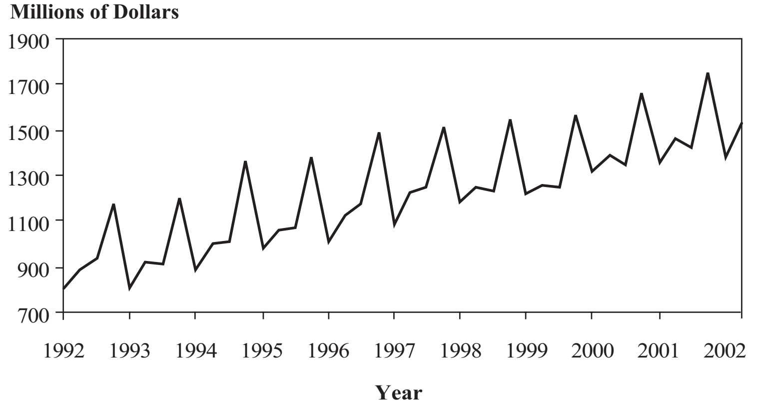
A Describe the salient features of the quarterly sales series.

B Describe the procedures we should use to determine whether the AR(1) specification is correct.

C Assuming the model is correctly specified, what is the long-run change in the log of sales toward which the series will tend to converge?

15 Figure 7 shows the quarterly sales of Avon Products from 1Q:1992 to 2Q:2002. Describe the salient features of the data shown.

Figure 7 Quarterly Sales at Avon



16 Table 7 below shows the autocorrelations of the residuals from an AR(1) model fit to the changes in the gross profit margin (GPM) of The Home Depot, Inc.

Table 7 Autocorrelations of the Residuals from Estimating the Regression $\Delta GPM_t = 0.0006 - 0.3330_1 \Delta GPM_{t-1} + \varepsilon_t$ 1Q:1992–4Q:2001 (40 Observations)

Lag	Autocorrelation
1	−0.1106
2	−0.5981

Table 7 (Continued)	
Lag	**Autocorrelation**
3	–0.1525
4	0.8496
5	–0.1099

Table 8 shows the output from a regression on changes in the GPM for Home Depot, where we have changed the specification of the AR regression.

Table 8 Change in Gross Profit Margin for Home Depot 1Q:1992–4Q:2001

Regression Statistics	
R-squared	0.9155
Standard error	0.0057
Observations	40
Durbin–Watson	2.6464

	Coefficient	Standard Error	t-Statistic
Intercept	–0.0001	0.0009	–0.0610
ΔGPM_{t-1}	–0.0608	0.0687	–0.8850
ΔGPM_{t-4}	0.8720	0.0678	12.8683

A Identify the change that was made to the regression model.

B Discuss the rationale for changing the regression specification.

17 Suppose we decide to use an autoregressive model with a seasonal lag because of the seasonal autocorrelation in the previous problem. We are modeling quarterly data, so we estimate Equation 15: (ln Sales$_t$ – ln Sales$_{t-1}$) = b_0 + b_1(ln Sales$_{t-1}$ – ln Sales$_{t-2}$) + b_2(ln Sales$_{t-4}$ – ln Sales$_{t-5}$) + ε_t. Table 9 shows the regression statistics from this equation.

Table 9 Log Differenced Sales: AR(1) Model with Seasonal Lag Johnson & Johnson Quarterly Observations, January 1985–December 2001

Regression Statistics	
R-squared	0.4220
Standard error	0.0318

(continued)

Table 9 (Continued)

Regression Statistics

Observations	68
Durbin–Watson	1.8784

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0121	0.0053	2.3055
Lag 1	–0.0839	0.0958	–0.8757
Lag 4	0.6292	0.0958	6.5693

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0572	0.1213	0.4720
2	–0.0700	0.1213	–0.5771
3	0.0065	0.1213	–0.0532
4	–0.0368	0.1213	–0.3033

A Using the information in Table 9, determine if the model is correctly specified.

B If sales grew by 1 percent last quarter and by 2 percent four quarters ago, use the model to predict the sales growth for this quarter.

18 Describe how to test for autoregressive conditional heteroskedasticity (ARCH) in the residuals from the AR(1) regression on first differences in the civilian unemployment rate, $\Delta UER_t = b_0 + b_1\Delta UER_{t-1} + \varepsilon_t$.

19 Suppose we want to predict the annualized return of the five-year T-bill using the annualized return of the three-month T-bill with monthly observations from January 1993 to December 2002. Our analysis produces the data shown in Table 10.

Table 10 Regression with 3-Month T-Bill as the Independent Variable and 5-Year Treasury Bill as the Dependent Variable Monthly Observations, January 1993 to December 2002

Regression Statistics

R-squared	0.5829
Standard error	0.6598
Observations	120
Durbin–Watson	0.1130

	Coefficient	Standard Error	*t*-Statistic
Intercept	3.0530	0.2060	14.8181
Three-month	0.5722	0.0446	12.8408

Can we rely on the regression model in Table 10 to produce meaningful predictions? Specify what problem might be a concern with this regression.

The following information relates to Questions 20–26

Angela Martinez, an energy sector analyst at an investment bank, is concerned about the future level of oil prices and how it might affect portfolio values. She is considering whether to recommend a hedge for the bank portfolio's exposure to changes in oil prices. Martinez examines West Texas Intermediate (WTI) monthly crude oil price data, expressed in US dollars per barrel, for the 181-month period from August 2000 through August 2015. The end-of-month WTI oil price was $51.16 in July 2015 and $42.86 in August 2015 (Month 181).

After reviewing the time-series data, Martinez determines that the mean and variance of the time series of oil prices are not constant over time. She then runs the following four regressions using the WTI time-series data.

- Linear trend model: Oil $\text{price}_t = b_0 + \text{b}_1 t + e_t$
- Log-linear trend model: ln Oil $\text{price}_t = b_0 + b_1 t + e_t$
- AR(1) model: Oil $\text{price}_t = b_0 + b_1\text{Oil price}_{t-1} + e_t$
- AR(2) model: Oil $\text{price}_t = b_0 + b_1\text{Oil price}_{t-1} + b_2\text{Oil price}_{t-2} + e_t$

Exhibit 1 presents selected data from all four regressions, and Exhibit 2 presents selected autocorrelation data from the AR(1) models.

Exhibit 1 Crude Oil Price per Barrel, August 2000–August 2015

	Regression Statistics (t-statistics for coefficients are reported in parentheses)			
	Linear	**Log-Linear**	**AR(1)**	**AR(2)**
R^2	0.5703	0.6255	0.9583	0.9656
Standard error	18.6327	0.3034	5.7977	5.2799
Observations	181	181	180	179
Durbin–Watson	0.10	0.08	1.16	2.08
RMSE			2.0787	2.0530
Coefficients:				
Intercept	28.3278	3.3929	1.5948	2.0017
	(10.1846)	(74.9091)	(1.4610)	(1.9957)
t (Trend)	0.4086	0.0075		
	(15.4148)	(17.2898)		
Oil Price_{t-1}			0.9767	1.3946
			(63.9535)	(20.2999)
Oil Price_{t-2}				–0.4249
				(–6.2064)

In Exhibit 1, at the 5% significance level, the lower critical value for the Durbin–Watson test statistic is 1.75 for both the linear and log-linear regressions.

Exhibit 2 Autocorrelations of the Residual from AR(1) Model

Lag	Autocorrelation	*t*-Statistic
1	0.4157	5.5768
2	0.2388	3.2045
3	0.0336	0.4512
4	–0.0426	–0.5712

Note: At the 5% significance level, the critical value for a *t*-statistic is 1.97.

After reviewing the data and regression results, Martinez draws the following conclusions.

Conclusion 1 The time series for WTI oil prices is covariance stationary.

Conclusion 2 Out-of-sample forecasting using the AR(1) model appears to be more accurate than that of the AR(2) model.

20 Based on Exhibit 1, the predicted WTI oil price for October 2015 using the linear trend model is *closest* to:

A $29.15.

B $74.77.

C $103.10.

21 Based on Exhibit 1, the predicted WTI oil price for September 2015 using the log-linear trend model is *closest* to:

A $29.75.

B $29.98.

C $116.50.

22 Based on the regression output in Exhibit 1, there is evidence of positive serial correlation in the errors in:

A the linear trend model but not the log-linear trend model.

B both the linear trend model and the log-linear trend model.

C neither the linear trend model nor the log-linear trend model.

23 Martinez's Conclusion 1 is:

A correct.

B incorrect because the mean and variance of WTI oil prices are not constant over time.

C incorrect because the Durbin–Watson statistic of the AR(2) model is greater than 1.75.

24 Based on Exhibit 1, the forecasted oil price in September 2015 based on the AR(2) model is *closest* to:

A $38.03.

B $40.04.

C $61.77.

25 Based on the data for the AR(1) model in Exhibits 1 and 2, Martinez can conclude that the:

A residuals are not serially correlated.

B autocorrelations do not differ significantly from zero.

C standard error for each of the autocorrelations is 0.0745.

26 Based on the mean-reverting level implied by the AR(1) model regression output in Exhibit 1, the forecasted oil price for September 2015 is *most likely* to be:

A less than \$42.86.

B equal to \$42.86.

C greater than \$42.86.

The following information relates to Question 27–35

Max Busse is an analyst in the research department of a large hedge fund. He was recently asked to develop a model to predict the future exchange rate between two currencies. Busse gathers monthly exchange rate data from the most recent 10-year period and runs a regression based on the following AR(1) model specification:

Regression 1: $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$, where x_t is the exchange rate at time t.

Based on his analysis of the time series and the regression results, Busse reaches the following conclusions:

Conclusion 1 The variance of x_t increases over time.

Conclusion 2 The mean-reverting level is undefined.

Conclusion 3 b_0 does not appear to be significantly different from 0.

Busse decides to do additional analysis by first-differencing the data and running a new regression.

Regression 2: $y_t = b_0 + b_1 y_{t-1} + \varepsilon_t$, where $y_t = x_t - x_{t-1}$.

Exhibit 1 shows the regression results.

Exhibit 1 First-Differenced Exchange Rate AR(1) Model: Month-End Observations, Last 10 Years

Regression Statistics	
R^2	0.0017
Standard error	7.3336
Observations	118
Durbin–Watson	1.9937

	Coefficient	Standard Error	*t*-Statistic
Intercept	–0.8803	0.6792	–1.2960
$x_{t-1} - x_{t-2}$	0.0412	0.0915	0.4504

(continued)

Exhibit 1 (Continued)

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0028	0.0921	0.0300
2	0.0205	0.0921	0.2223
3	0.0707	0.0921	0.7684
4	0.0485	0.0921	0.5271

Note: The critical *t*-statistic at the 5% significance level is 1.98.

Busse decides that he will need to test the data for nonstationarity using a Dickey–Fuller test. To do so, he knows he must model a transformed version of Regression 1.

Busse's next assignment is to develop a model to predict future quarterly sales for PoweredUP, Inc., a major electronics retailer. He begins by running the following regression:

Regression 3: $\ln \text{Sales}_t - \ln \text{Sales}_{t-1} = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + \varepsilon_t$.

Exhibit 2 presents the results of this regression.

Exhibit 2 Log Differenced Sales: AR(1) Model PoweredUP, Inc., Last 10 Years of Quarterly Sales

Regression Statistics

R^2	0.2011
Standard error	0.0651
Observations	38
Durbin–Watson	1.9677

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0408	0.0112	3.6406
$\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}$	−0.4311	0.1432	−3.0099

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0146	0.1622	0.0903
2	−0.1317	0.1622	−0.8119
3	−0.1123	0.1622	−0.6922
4	0.6994	0.1622	4.3111

Note: The critical *t*-statistic at the 5% significance level is 2.02.

Because the regression output from Exhibit 2 raises some concerns, Busse runs a different regression. These regression results, along with quarterly sales data for the past five quarters, are presented in Exhibits 3 and 4, respectively.

Exhibit 3 Log Differenced Sales: AR(1) Model with Seasonal Lag PoweredUP, Inc., Last 10 Years of Quarterly Sales

Regression Statistics	
R^2	0.6788
Standard error	0.0424
Observations	35
Durbin–Watson	1.8799

	Coefficient	Standard Error	t-Statistic
Intercept	0.0092	0.0087	1.0582
$\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}$	−0.1279	0.1137	−1.1252
$\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}$	0.7239	0.1093	6.6209

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	t-Statistic
1	0.0574	0.1690	0.3396
2	0.0440	0.1690	0.2604
3	0.1923	0.1690	1.1379
4	−0.1054	0.1690	−0.6237

Note: The critical t-statistic at the 5% significance level is 2.03.

Exhibit 4 Most Recent Quarterly Sales Data (in billions)

Dec 2015 (Sales_{t-1})	\$3.868
Sept 2015 (Sales_{t-2})	\$3.780
June 2015 (Sales_{t-3})	\$3.692
Mar 2014 (Sales_{t-4})	\$3.836
Dec 2014 (Sales_{t-5})	\$3.418

After completing his work on PoweredUP, Busse is asked to analyze the relationship of oil prices and the stock prices of three transportation companies. His firm wants to know whether the stock prices can be predicted by the price of oil. Exhibit 5 shows selected information from the results of his analysis.

Exhibit 5 Analysis Summary of Stock Prices for Three Transportation Stocks and the Price of Oil

	Unit Root?	Linear or Exponential Trend?	Serial Correlation of Residuals in Trend Model?	ARCH(1)?	Comments
Company #1	Yes	Exponential	Yes	Yes	Not co-integrated with oil price
Company #2	Yes	Linear	Yes	No	Co-integrated with oil price

(continued)

Exhibit 5 (Continued)

	Unit Root?	Linear or Exponential Trend?	Serial Correlation of Residuals in Trend Model?	ARCH(1)?	Comments
Company #3	No	Exponential	Yes	No	Not co-integrated with oil price
Oil Price	Yes				

To assess the relationship between oil prices and stock prices, Busse runs three regressions using the time series of each company's stock prices as the dependent variable and the time series of oil prices as the independent variable.

27 Which of Busse's conclusions regarding the exchange rate time series is consistent with both the properties of a covariance-stationary time series and the properties of a random walk?

- **A** Conclusion 1
- **B** Conclusion 2
- **C** Conclusion 3

28 Based on the regression output in Exhibit 1, the first-differenced series used to run Regression 2 is consistent with:

- **A** a random walk.
- **B** covariance stationarity.
- **C** a random walk with drift.

29 Based on the regression results in Exhibit 1, the *original* time series of exchange rates:

- **A** has a unit root.
- **B** exhibits stationarity.
- **C** can be modeled using linear regression.

30 In order to perform the nonstationarity test, Busse should transform the Regression 1 equation by:

- **A** adding the second lag to the equation.
- **B** changing the regression's independent variable.
- **C** subtracting the independent variable from both sides of the equation.

31 Based on the regression output in Exhibit 2, what should lead Busse to conclude that the Regression 3 equation is not correctly specified?

- **A** The Durbin–Watson statistic
- **B** The t-statistic for the slope coefficient
- **C** The t-statistics for the autocorrelations of the residual

32 Based on the regression output in Exhibit 3 and sales data in Exhibit 4, the forecasted value of quarterly sales for March 2016 for PoweredUP is *closest* to:

- **A** \$4.193 billion.
- **B** \$4.205 billion.
- **C** \$4.231 billion.

33 Based on Exhibit 5, Busse should conclude that the variance of the error terms for Company #1:

- **A** is constant.

- **B** can be predicted.
- **C** is homoskedastic.

34 Based on Exhibit 5, for which company would the regression of stock prices on oil prices be expected to yield valid coefficients that could be used to estimate the long-term relationship between stock price and oil price?

- **A** Company #1
- **B** Company #2
- **C** Company #3

35 Based on Exhibit 5, which single time-series model would *most likely* be appropriate for Busse to use in predicting the future stock price of Company #3?

- **A** Log-linear trend model
- **B** First-differenced AR(2) model
- **C** First-differenced log AR(1) model

SOLUTIONS

1 **A** The estimated forecasting equation is $UER_t = 5.5098 - 0.0294(t)$. The data begin in January 1996, and July 1996 is period 7. Thus the linear trend model predicts the unemployment rate to be $UER_7 = 5.5098 - 0.0294\ (7) = 5.3040$ or approximately 5.3 percent.

B The DW statistic is designed to detect positive serial correlation of the errors of a regression equation. Under the null hypothesis of no positive serial correlation, the DW statistics is 2.0. Positive serial correlation will lead to a DW statistic that is less than 2.0. From the table in Problem 1, we see that the DW statistic is 0.9099. To see whether this result is significantly less than 2.0, refer to the Durbin–Watson table in Appendix E at the end of this volume, in the column marked $k = 1$ (one independent variable) and the row corresponding to 60 observations. We see that $d_l = 1.55$. Because our DW statistic is clearly less than d_l, we reject the null hypothesis of no serial correlation at the 0.05 significance level.

The presence of serial correlation in the error term violates one of the regression assumptions. The standard errors of the estimated coefficients will be biased downward, so we cannot conduct hypothesis testing on the coefficients.

2 The difference between UER and its forecast value, PRED, is the forecast error. In an appropriately specified regression model, the forecast errors are randomly distributed around the regression line and have a constant variance. We can see that the errors from this model specification are persistent. The errors tend first to be above the regression line and then, starting in 1997, they tend to be below the regression line until 2000 when they again are persistently above the regression line. This persistence suggests that the errors are positively serially correlated. Therefore, we conclude that the model is not appropriate for making estimates.

3 A log-linear model captures growth at a constant rate. The log-linear model $\ln(\text{Sales}_t) = b_0 + b_1 t + \varepsilon_t$ would be the simplest model consistent with a constant growth rate for monthly sales. Note that we would need to confirm that the regression assumptions are satisfied before accepting the model as valid.

4 **A** The plot of the series ΔUER_t seems to fluctuate around a constant mean; its volatility appears to be constant throughout the period. Our initial judgment is that the differenced series is covariance stationary.

B The change in the unemployment rate seems covariance stationary, so we should first estimate an AR(1) model and test to see whether the residuals from this model have significant serial correlation. If the residuals do not display significant serial correlation, we should use the AR(1) model. If the residuals do display significant serial correlation, we should try an AR(2) model and test for serial correlation of the residuals of the AR(2) model. We should continue this procedure until the errors from the final AR(p) model are serially uncorrelated.

5 The DW statistic cannot be appropriately used for a regression that has a lagged value of the dependent variable as one of the explanatory variables. To test for serial correlation, we need to examine the autocorrelations.

6 When a covariance-stationary series is at its mean-reverting level, the series will tend not to change until it receives a shock (ε_t). So, if the series ΔUER_t is at the mean-reverting level, $\Delta UER_t = \Delta UER_{t-1}$. This implies that $\Delta UER_t = -0.0405$

$-0.4674\Delta UER_t$, so that $(1 + 0.4674)\ \Delta UER_t = -0.0405$ and $\Delta UER_t = -0.0405/(1 + 0.4674) = -0.0276$. The mean-reverting level is –0.0276. In an AR(1) model, the general expression for the mean-reverting level is $b_0/(1 - b_1)$.

7 **A** The predicted change in the unemployment rate for next period is –5.45 percent, found by substituting 0.0300 into the forecasting model: $-0.0405 - 0.4674(0.03) = -0.0545$.

B If we substitute our one-period-ahead forecast of –0.0545 into the model (using the chain rule of forecasting), we get a two-period ahead forecast of –0.0150 or –1.5 percent.

C The answer to Part B is quite close to the mean-reverting level of – 0.0276. A stationary time series may need many periods to return to its equilibrium, mean-reverting level.

8 The forecast of sales is $4,391 million for the first quarter of 2002 and $4,738 million for the second quarter of 2002, as the following table shows.

Date	Sales ($ Millions)	Log of Sales	Actual Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$	Forecast Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$
1Q:2001	6,519	8.7825	0.1308	
2Q:2001	6,748	8.8170	0.0345	
3Q:2001	4,728	8.4613	–0.3557	
4Q:2001	4,298	8.3659	–0.0954	
1Q:2002	4,391	8.3872		0.0213
2Q:2002	4,738	8.4633		0.0761

We find the forecasted change in the log of sales for the first quarter of 2002 by inputting the value for the change in the log of sales from the previous quarter into the equation $\Delta\ln(\text{Sales}_t) = 0.0661 + 0.4698\Delta\ln(\text{Sales}_{t-1})$. Specifically, $\Delta\ln(\text{Sales}_t) = 0.0661 + 0.4698(-0.0954) = 0.0213$, which means that we forecast the log of sales in the first quarter of 2002 to be 8.3659 + 0.0213 = 8.3872.

Next, we forecast the change in the log of sales for the second quarter of 2002 as $\Delta\ln(\text{Sales}_t) = 0.0661 + 0.4698(0.0213) = 0.0761$. Note that we have to use our first-quarter 2002 estimated value of the change in the log of sales as our input for $\Delta\ln(\text{Sales}_{t-1})$ because we are forecasting past the period for which we have actual data.

With a forecasted change of 0.0761, we forecast the log of sales in the second quarter of 2002 to be 8.3872 + 0.0761 = 8.4633.

We have forecasted the log of sales in the first and second quarters of 2002 to be 8.3872 and 8.4633, respectively. Finally, we take the antilog of our estimates of the log of sales in the first and second quarters of 2002 to get our estimates of the level of sales: $e^{8.3872} = 4{,}391$ and $e^{8.4633} = 4{,}738$, respectively, for sales of $4,391 million and $4,738 million.

9 **A** The RMSE of the out-of-sample forecast errors is approximately 27 percent. Out-of-sample error refers to the difference between the realized value and the forecasted value of $\Delta\ln(\text{Sales}_t)$ for dates beyond the estimation period. In this case, the out-of-sample period is 1Q:2001 to 4Q:2001. These are the four quarters for which we have data that we did not use to obtain the estimated model $\Delta\ln(\text{Sales}_t) = 0.0661 + 0.4698\Delta\ln(\text{Sales}_{t-1})$.

The steps to calculate RMSE are as follows:

i. Take the difference between the actual and the forecast value. This is the error.

ii. Square the error.

iii. Sum the squared errors.

iv. Divide by the number of forecasts.

v. Take the square root of the average.

We show the calculations for RMSE in the table below.

Actual Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$	Forecast Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$	Error (Column 1 – Column 2)	Squared Error (Column 3 Squared)
0.1308	0.1357	–0.0049	0.0000
0.0345	0.1299	–0.0954	0.0091
–0.3557	0.1271	–0.4828	0.2331
–0.0954	0.1259	–0.2213	0.0490
		Sum	0.2912
		Mean	0.0728
		RMSE	0.2698

B The lower the RMSE, the more accurate the forecasts of a model in forecasting. Therefore, the model with the RMSE of 20 percent has greater accuracy in forecasting than the model in Part A, which has an RMSE of 27 percent.

10 A Predictions too far ahead can be nonsensical. For example, the AR(1) model we have been examining, $\Delta\text{UER}_t = -0.0405 - 0.4674\Delta\text{UER}_{t-1}$, taken at face value, predicts declining civilian unemployment into the indefinite future. Because the civilian unemployment rate will probably not go below 3 percent frictional unemployment and cannot go below 0 percent unemployment, this model's long-range forecasts are implausible. The model is designed for short-term forecasting, as are many time-series models.

B Using more years of data for estimation may lead to nonstationarity even in the series of first differences in the civilian unemployment rate. As we go further back in time, we increase the risk that the underlying civilian unemployment rate series has more than one regime (or true model). If the series has more than one regime, fitting one model to the entire period would not be correct. Note that when we have good reason to believe that a time series is stationary, a longer series of data is generally desirable.

11 A The graph of $\ln(\text{Sales}_t)$ appears to trend upward over time. A series that trends upward or downward over time often has a unit root and is thus not covariance stationary. Therefore, using an AR(1) regression on the undifferenced series is probably not correct. In practice, we need to examine regression statistics to confirm visual impressions such as this.

B The most common way to transform a time series with a unit root into a covariance-stationary time series is to difference the data—that is, to create a new series $\Delta\ln(\text{Sales}_t) = \ln(\text{Sales}_t) - \ln(\text{Sales}_{t-1})$.

12 The plot of the series $\Delta\ln(\text{Sales}_t)$ appears to fluctuate around a constant mean; its volatility seems constant throughout the period. Differencing the data appears to have made the time series covariance stationary.

13 A In a correctly specified regression, the residuals must be serially uncorrelated. We have 108 observations, so the standard error of the autocorrelation is $1/\sqrt{T}$, or in this case $1/\sqrt{108} = 0.0962$. The t-statistic for each lag is significant at the 0.01 level. We would have to modify the model specification before continuing with the analysis.

B Because the residuals from the AR(1) specification display significant serial correlation, we should estimate an AR(2) model and test for serial correlation of the residuals of the AR(2) model. If the residuals from the AR(2) model are serially uncorrelated, we should then test for seasonality and ARCH behavior. If any serial correlation remains in the residuals, we should estimate an AR(3) process and test the residuals from that specification for serial correlation. We should continue this procedure until the errors from the final AR(p) model are serially uncorrelated. When serial correlation is eliminated, we should test for seasonality and ARCH behavior.

14 A The series has a steady upward trend of growth, suggesting an exponential growth rate. This finding suggests transforming the series by taking the natural log and differencing the data.

B First, we should determine whether the residuals from the AR(1) specification are serially uncorrelated. If the residuals are serially correlated, then we should try an AR(2) specification and then test the residuals from the AR(2) model for serial correlation. We should continue in this fashion until the residuals are serially uncorrelated, then look for seasonality in the residuals. If seasonality is present, we should add a seasonal lag. If no seasonality is present, we should test for ARCH. If ARCH is not present, we can conclude that the model is correctly specified.

C If the model $\Delta\ln(\text{Sales}_t) = b_0 + b_1[\Delta\ln(\text{Sales}_{t-1})] + \varepsilon_t$ is correctly specified, then the series $\Delta\ln(\text{Sales}_t)$ is covariance stationary. So, this series tends to its mean-reverting level, which is $b_0/(1 - b_1)$ or $0.0661/(1 - 0.4698) = 0.1247$.

15 The quarterly sales of Avon show an upward trend and a clear seasonal pattern, as indicated by the repeated regular cycle.

16 A A second explanatory variable, the change in the gross profit margin lagged four quarters, ΔGPM_{t-4}, was added.

B The model was augmented to account for seasonality in the time series (with quarterly data, significant autocorrelation at the fourth lag indicates seasonality). The standard error of the autocorrelation coefficient equals 1 divided by the square root of the number of observations: $1/\sqrt{40}$ or 0.1581. The autocorrelation at the fourth lag (0.8496) is significant: $t = 0.8496/0.1581 = 5.37$. This indicates seasonality, and accordingly we added ΔGPM_{t-4}. Note that in the augmented regression, the coefficient on ΔGPM_{t-4} is highly significant. (Although the autocorrelation at second lag is also significant, the fourth lag is more important because of the rationale of seasonality. Once the fourth lag is introduced as an independent variable, we might expect that the second lag in the residuals would not be significant.)

17 A In order to determine whether this model is correctly specified, we need to test for serial correlation among the residuals. We want to test whether we can reject the null hypothesis that the value of each autocorrelation is 0 against the alternative hypothesis that each is not equal to 0. At the 0.05 significance level, with 68 observations and three parameters, this model has 65 degrees of freedom. The critical value of the t-statistic needed to reject the null hypothesis is thus about 2.0. The absolute value of the t-statistic for each autocorrelation is below 0.60 (less than 2.0), so we cannot reject the null hypothesis that each autocorrelation is not significantly different from 0. We have determined that the model is correctly specified.

B If sales grew by 1 percent last quarter and by 2 percent four quarters ago, then the model predicts that sales growth this quarter will be 0.0121 – 0.0839 ln(1.01) + 0.6292 ln(1.02) = $e^{0.02372} - 1$ = 2.40%.

18 We should estimate the regression $\Delta UER_t = b_0 + b_1 \Delta UER_{t-1} + \varepsilon_t$ and save the residuals from the regression. Then we should create a new variable, $\hat{\varepsilon}_t^2$, by squaring the residuals. Finally, we should estimate $\hat{\varepsilon}_t^2 = a_0 + a_1 \hat{\varepsilon}_{t-1}^2 + u_t$ and test to see whether a_1 is statistically different from 0.

19 To determine whether we can use linear regression to model more than one time series, we should first determine whether any of the time series has a unit root. If none of the time series has a unit root, then we can safely use linear regression to test the relations between the two time series. Note that if one of the two variables has a unit root, then our analysis would not provide valid results; if both of the variables have unit roots, then we would need to evaluate whether the variables are cointegrated.

20 C is correct. The predicted value for period t from a linear trend is calculated as $\hat{y}_t = \hat{b}_0 + \hat{b}_1(t)$.

October 2015 is the second month out of sample, or t = 183. So, the predicted value for October 2015 is calculated as

$$\hat{y}_t = 28.3278 + 0.4086(183) = \$103.10.$$

Therefore, the predicted WTI oil price for October 2015 based on the linear trend model is $103.10.

21 C is correct. The predicted value for period t from a log-linear trend is calculated as $\ln \hat{y}_t = \hat{b}_0 + \hat{b}_1(t)$.

September 2015 is the first month out of sample, or t = 182. So, the predicted value for September 2015 is calculated as follows:

$$\ln \hat{y}_t = 3.3929 + 0.0075(182)$$

$$\ln \hat{y}_t = 4.7579$$

$$\hat{y}_t = e^{4.7579} = \$116.50$$

Therefore, the predicted WTI oil price for September 2015, based on the log-linear trend model, is $116.50.

22 B is correct. The Durbin–Watson statistic for the linear trend model is 0.10 and, for the log-linear trend model, 0.08. Both of these values are below the critical value of 1.75. Therefore, we can reject the hypothesis of no positive serial correlation in the regression errors in both the linear trend model and the log-linear trend model.

23 B is correct. There are three requirements for a time series to be covariance stationary. First, the expected value of the time series must be constant and finite in all periods. Second, the variance of the time series must be constant and finite in all periods. Third, the covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods. Martinez concludes that the mean and variance of the time series of WTI oil prices are not constant over time. Therefore, the time series is not covariance stationary.

24 B is correct. The last two observations in the WTI time series are July and August 2015, when the WTI oil price was \$51.16 and \$42.86, respectively. Therefore, September 2015 represents a one-period-ahead forecast. The one-period-ahead forecast from an AR(2) model is calculated as

$$\hat{x}_{t+1} = \hat{b}_0 + \hat{b}_1 x_t + \hat{b}_2 x_{t+1}$$

So, the one-period-ahead (September 2015) forecast is calculated as

$$\hat{x}_{t+1} = 2.0017 + 1.3946(\$42.86) - 0.4249(\$51.16) = \$40.04.$$

Therefore, the September 2015 forecast based on the AR(2) model is \$40.04.

25 C is correct. The standard error of the autocorrelations is calculated as $\frac{1}{\sqrt{T}}$, where T represents the number of observations used in the regression.

Therefore, the standard error for each of the autocorrelations is $\frac{1}{\sqrt{180}} = 0.0745$.

Martinez can conclude that the residuals are serially correlated and are significantly different from zero because two of the four autocorrelations in Exhibit 2 have a t-statistic in absolute value that is greater than the critical value of 1.97.

Choices A and B are incorrect because two of the four autocorrelations have a t-statistic in absolute value that is greater than the critical value of the t-statistic of 1.97.

26 C is correct. The mean-reverting level from the AR(1) model is calculated as

$$\hat{x}_t = \frac{b_0}{1 - b_1} = \frac{1.5948}{1 - 0.9767} = \$68.45$$

Therefore, the mean-reverting WTI oil price from the AR(1) model is \$68.45. The forecasted oil price in September 2015 will likely be greater than \$42.86 because the model predicts that the price will rise in the next period from the August 2015 price of \$42.86.

27 C is correct. A random walk can be described by the equation $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$, where $b_0 = 0$ and $b_1 = 1$. So $b_0 = 0$ is a characteristic of a random walk time series. A covariance-stationary series must satisfy the following three requirements:

1. The expected value of the time series must be constant and finite in all periods.
2. The variance of the time series must be constant and finite in all periods.
3. The covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods.

$b_0 = 0$ does not violate any of these three requirements and is thus consistent with the properties of a covariance-stationary time series.

28 B is correct. The critical t-statistic at a 5% confidence level is 1.98. As a result, neither the intercept nor the coefficient on the first lag of the first-differenced exchange rate in Regression 2 differs significantly from zero. Also, the residual autocorrelations do not differ significantly from zero. As a result, Regression 2 can be reduced to $y_t = \varepsilon_t$ with a mean-reverting level of $b_0/(1 - b_1) = 0/1 = 0$. Therefore, the variance of y_t in each period is $\text{Var}(\varepsilon_t) = \sigma^2$. The fact that the residuals are not autocorrelated is consistent with the covariance of the times series, with itself being constant and finite at different lags. Because the variance and the mean of y_t are constant and finite in each period, we can also conclude that y_t is covariance stationary.

29 A is correct. If the exchange rate series is a random walk, then the first-differenced series will yield $b_0 = 0$ and $b_1 = 0$, and the error terms will not be serially correlated. The data in Exhibit 1 show that this is the case: Neither the intercept nor the coefficient on the first lag of the first-differenced exchange rate in Regression 2 differs significantly from zero because the t-statistics of both coefficients are less than the critical t-statistic of 1.98. Also, the residual autocorrelations do not differ significantly from zero because the t-statistics of all autocorrelations are less than the critical t-statistic of 1.98. Therefore, because all random walks have unit roots, the exchange rate time series used to run Regression 1 has a unit root.

30 C is correct. To conduct the Dickey–Fuller test, one must subtract the independent variable, x_{t-1}, from both sides of the original AR(1) model. This results in a change of the dependent variable (from x_t to $x_t - x_{t-1}$) and a change in the regression's slope coefficient (from b_1 to $b_1 - 1$) but not a change in the independent variable.

31 C is correct. The regression output in Exhibit 2 suggests there is serial correlation in the residual errors. The fourth autocorrelation of the residual has a value of 0.6994 and a t-statistic of 4.3111, which is greater than the t-statistic critical value of 2.02. Therefore, the null hypothesis that the fourth autocorrelation is equal to zero can be rejected. This indicates strong and significant seasonal autocorrelation, which means the Regression 3 equation is misspecified.

32 C is correct. The quarterly sales for March 2016 is calculated as follows:

$$\begin{aligned}
\ln \text{Sales}_t - \ln \text{Sales}_{t-1} &= b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + b_2(\ln \text{Sales}_{t-4} \\
&\quad - \ln \text{Sales}_{t-5}).\\
\ln \text{Sales}_t - \ln 3.868 &= 0.0092 - 0.1279(\ln 3.868 - \ln 3.780) + 0.7239(\ln \\
&\quad 3.836 - \ln 3.418).\\
\ln \text{Sales}_t - 1.35274 &= 0.0092 - 0.1279(1.35274 - 1.32972) + \\
&\quad 0.7239(1.34443 - 1.22906).\\
\ln \text{Sales}_t &= 1.35274 + 0.0092 - 0.1279(0.02301) + \\
&\quad 0.7239(0.11538).\\
\ln \text{Sales}_t &= 1.44251.\\
\text{Sales}_t &= e^{1.44251} = 4.231.
\end{aligned}$$

33 B is correct. Exhibit 5 shows that the time series of the stock prices of Company #1 exhibits heteroskedasticity, as evidenced by the fact that the time series is ARCH(1). If a time series is ARCH(1), then the variance of the error in one period depends on the variance of the error in previous periods. Therefore, the variance of the errors in period $t + 1$ can be predicted in period t using the formula

$$\hat{\sigma}_{t+1}^2 = \hat{a}_0 + \hat{a}_1 \hat{\varepsilon}_t^2$$

34 B is correct. When two time series have a unit root but are co-integrated, the error term in the linear regression of one time series on the other will be covariance stationary. Exhibit 5 shows that the series of stock prices of Company #2 and the oil prices both contain a unit root, and the two time series are co-integrated. As a result, the regression coefficients and standard errors are consistent and can be used for hypothesis tests. Although the co-integrated regression estimates the long-term relation between the two series, it may not be the best model of the short-term relationship.

35 C is correct. As a result of the exponential trend in the time series of stock prices for Company #3, Busse would want to take the natural log of the series and then first-difference it. Because the time series also has serial correlation in the residuals from the trend model, Busse should use a more complex model, such as an autoregressive (AR) model.