Sliding along a Chord through a Rotating Earth

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Connect any two cities A and B on the Earth's surface by a straight-line subway rail. If a train, powered only by gravity, runs along this track without resistance, how long will it take the train to go from A to B? Remember that the Earth rotates. A little bit of linear algebra elegantly handles this new twist to this classic problem and gives some surprising answers.

To solve this problem, position the Earth at the origin $\mathbf{O}$ so that the Equator lies in the x-y plane and the North Pole is on the positive z-axis. Let M be the rotation matrix $M = M(t) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $\phi = \phi(t) = \frac{2\pi t}{Q}$ with Q being the period of the Earth about its axis and t being time. Interpret $\mathbf{A}$ and $\mathbf{B}$ as vectors giving the coordinates of our two cities at time 0. Let $\mathbf{u}$ be a unit vector in the $\mathbf{A} - \mathbf{B}$ direction. Let $\mathbf{w}$ be a unit vector pointing from the origin to $\mathbf{N}$, the point on the chord nearest to $\mathbf{O}$. When the Earth is viewed as a perfect sphere, $\mathbf{N}$ is the midpoint of the chord between $\mathbf{A}$ and $\mathbf{B}$. Note that $\mathbf{u}$ and $\mathbf{w}$ are perpendicular to each other. Condense the train to a single point. Let $\mathbf{P}$ be the position of the train along the chord at time t. Let c, r and s be the respective distances from $\mathbf{O}$ to $\mathbf{N}$, from $\mathbf{O}$ to $\mathbf{P}$ and from $\mathbf{N}$ to $\mathbf{P}$. We allow s to be positive when $\mathbf{P}$ is nearer to city A and negative when $\mathbf{P}$ is nearer to city B. As the Earth rotates under the isometry M, the two cities are at the positions $M\mathbf{A}$ and $M\mathbf{B}$; the vectors $M\mathbf{u}$ and $M\mathbf{w}$ remain unit vectors and point in the respective directions from city B to city A and from the origin to the midpoint of the two cities. See Figure 1. Unless otherwise stated, we assume that the Earth is a sphere of radius $R = 6400$ km and that the acceleration due to gravity at r units from the Earth is $f(r) = -kr$ where $k = \frac{g}{R}$, with $g = 9.8$ m/s^2.

The train's position is given by

$$\mathbf{P} = cM\mathbf{w} + sM\mathbf{u}. \tag{1}$$

Since c, $\mathbf{u}$ and $\mathbf{w}$ are constant, then the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ of the train are given by

$$\mathbf{v} = cM'\mathbf{w} + s'M\mathbf{u} + sM'\mathbf{u}$$

and

$$\mathbf{a} = cM''\mathbf{w} + s''M\mathbf{u} + 2s'M'\mathbf{u} + sM''\mathbf{u}, \tag{2}$$

where $s' = \frac{ds}{dt}$, $s'' = \frac{d^2s}{dt^2}$, $M' = \frac{2\pi}{Q}\begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $M'' = (\frac{2\pi}{Q})^2\begin{bmatrix} -\cos\phi & \sin\phi & 0 \\ -\sin\phi & -\cos\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

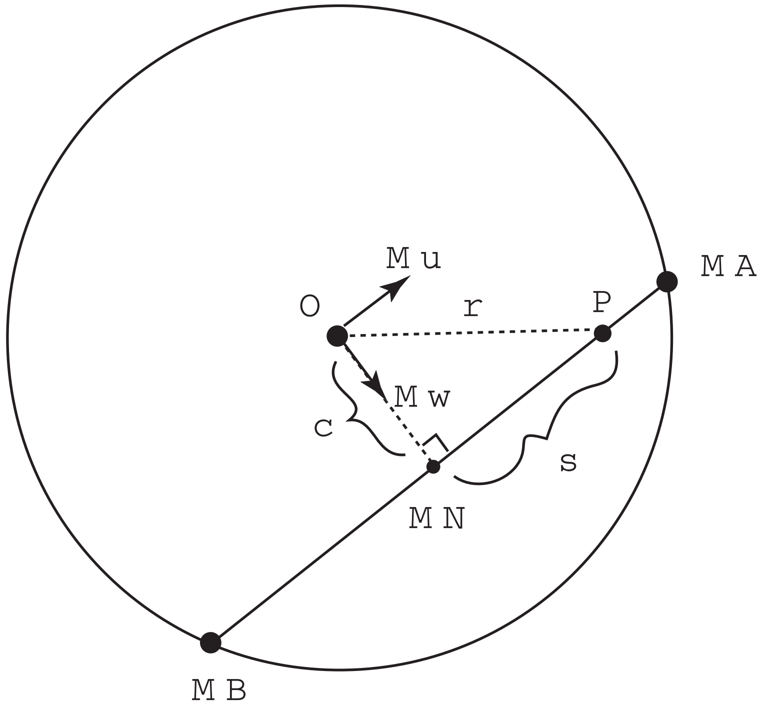

FIGURE 1. A singular train sliding along a chord in a rotating Earth

The effective acceleration acting on the train along the chord as given by (2) is the magnitude of the projection of $\mathbf{a}$ onto the unit vector $M\mathbf{u}$, which is found simply by dotting (2) with $M\mathbf{u}$. For the moment, let us naïvely think that the only acceleration in this system is that due to gravity. As such the effective acceleration along the chord should also be given by the projection of $-kr$ onto the chord. That is, the acceleration of the train along the chord should satisfy the equation

$$(cM''\mathbf{w} + s''M\mathbf{u} + 2s'M'\mathbf{u} + sM''\mathbf{u}) \cdot M\mathbf{u} = -ks. \tag{3}$$

Let T denote the *transpose* operator. Since the dot product $\mathbf{x} \cdot \mathbf{y}$ is $\mathbf{x}^T\mathbf{y}$ and since $(XY)^T = Y^TX^T$, (3) simplifies as

$$s'' - (\frac{2\pi}{Q})^2(c\mathbf{w}^T + s\mathbf{u}^T)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{u} = -ks, \tag{4}$$

because $M^TM = I$, the identity matrix, $(M')^TM = \mathbf{0}$, the zero matrix, and $(M'')^TM = -(\frac{2\pi}{Q})^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Before further simplifying (4), let us pause to see that it truly is the equation we desire. Knowing some physics, we expect that the acceleration on a train in our subway should be the sum of its gravitational acceleration and its centripetal acceleration. From any physics text, the centripetal acceleration on a particle at position $\mathbf{P}$ is $\omega^2\mathbf{r}$, where ω is the rotation rate and $\mathbf{r}$ is the vector of displacement of the particle from the axis of rotation. See FIGURE 2.

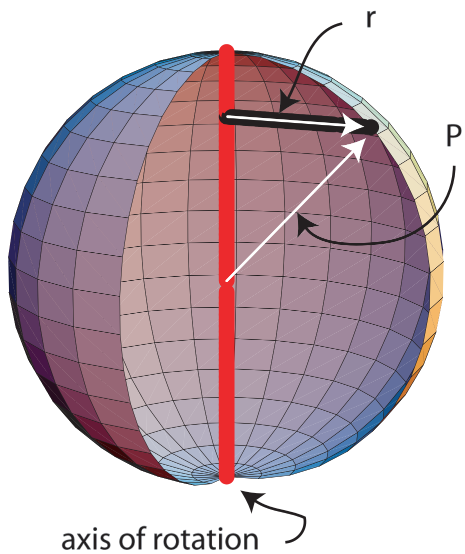

FIGURE 2. Cutaway of a spherical shell.

In our case, $\omega = \dfrac{2\pi}{Q}$ and $\mathbf{r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{P}$. Thus the projection of $\omega^2\mathbf{r}$ onto the direction of the tunnel is

$$\omega^2\mathbf{r} \cdot M\mathbf{u} = (\frac{2\pi}{Q})^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (cM\mathbf{w} + sM\mathbf{u}) \cdot M\mathbf{u},$$

which simplifies to

$$(\frac{2\pi}{Q})^2 (c\mathbf{w}^T + s\mathbf{u}^T) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{u}.$$

That is, s'' as given by (4) is indeed the sum of gravitational and centripetal acceleration.

When the dust settles in expanding (4) for particular cities A and B, the equation becomes

$$s'' + \lambda s = \mu, \tag{5}$$

where λ and μ are constants. In particular, if $\mathbf{u} = (u_1,\, u_2,\, u_3)$ and $\mathbf{w} = (w_1,\, w_2,\, w_3)$, then $\lambda = k - \frac{4\pi^2}{Q^2}(u_1^2 + u_2^2)$ and $\mu = \frac{4c\pi^2}{Q^2}(u_1 w_1 + u_2 w_2)$.

When λ is positive, the reader will recognize that the solution of differential equation (5) is

$$s = \frac{\mu}{\lambda} + \alpha \cos(\sqrt{\lambda} t + \beta), \tag{6}$$

where α and β are constants. As initial conditions, the train leaves the Earth's surface from rest; that is, $s(0) = \gamma$ and $s'(0) = 0$ where γ corresponds with the signed distance from N

to city A. As the reader may verify, $\beta = 0$, $\alpha = \gamma - \frac{\mu}{\lambda}$, which means that (6) becomes

$$s = \frac{\mu}{\lambda} + (\gamma - \frac{\mu}{\lambda})\cos(\sqrt{\lambda}\, t). \tag{7}$$

If the train ever ventures beyond the surface of the Earth under the motion as given by (7), then thereafter equation (7) no longer models the train's behavior. Note that if $\mu \neq 0$ then the simple harmonic motion of (7) either results in the train failing to reach city B before returning to city A or the train proceeds beyond city B, at which point it launches itself as a projectile. Note that at time $t = 0$, the train is at city A which is γ distance from $\mathbf{N}$. At time $t_1 = \frac{\pi}{\sqrt{\lambda}}$, the train will be at its opposite extreme as given by (7); that is, at t_1 the train should be at position $-\gamma + \frac{2\mu}{\lambda}$ by (7). Thus the train's shortfall or surplus distance in reaching city B is given by

$$\frac{2\mu}{\lambda}. \tag{8}$$

We explore this possibility in an example below. Finally, the period of (7) varies from a minimum of $\frac{2\pi}{\sqrt{k}}$ to a maximum of $\frac{2\pi}{\sqrt{k - \frac{4\pi^2}{Q^2}}}$. Note that the period of the train as given by (7) tends to increase as the rotation rate of the Earth quickens. This result should make intuitive sense because as the rotation rate of a body is continually increased, the centripetal acceleration eventually overcomes the gravitional acceleration at any point not on the axis of rotation. For instance, imagine that gravitational and centripetal acceleration in the direction of the tunnel sums to zero for a train at the start of the tunnel; such a train fails to slide down the tunnel. Thus when gravitational acceleration slightly edges centripetal acceleration, the train proceeds down the tunnel, but its period will be fairly long.

In particular, when $\lambda = 0$ the train remains static, an event which will occur only if the Earth's rotation rate is speeded up dramatically. Similarly when λ is negative, the solution is hyperbolic, which means that the train leaves the surface of the Earth.

Now let's consider some specific chords on the Earth.

Example I: *Chords parallel to the z-axis.* In this case note that $\mathbf{u}$ is $(0, 0, 1)$, which means that (4) becomes

$$s'' = -ks.$$

Trains sliding along these chords follow simple harmonic motion with periods of $\frac{2\pi}{\sqrt{k}}$ which is about 84.6 minutes. This result is the same as that for any chord in the classic problem for a nonrotating Earth, a standard problem which appears in many introductory physics texts such as [**2**, pp. 258-9] and recreational mathematics books such as [**1**].

Example II: *Chords lying in the Equatorial plane.* For any chord in the Equatorial plane, the corresponding vectors $\mathbf{u}$ and $\mathbf{w}$ both have third components of 0, which means that (4) becomes

$$s'' = (-k + \frac{4\pi^2}{Q^2})s.$$

That is, trains running along these chords follow simple harmonic motion with periods of $\frac{2\pi}{\sqrt{k - \frac{4\pi^2}{Q^2}}}$, provided the argument of the radical is positive. For our Earth with $Q = 24 \cdot 60 \cdot 60$ seconds $= 86400$ seconds, the period is about 9 seconds longer than along the chords in example I. Contrary to what might be expected, in general these chords are not minimum time curves in the Equatorial plane for travel from A to B; see [**3**, p. 180, FIGURE 5] for example.

If the period of the Earth is shortened, the period for the train increases along chords in the Equatorial plane. For example if Q is shortened to 2 hours, then the period of the train is about 119 minutes. Further shortening Earth's period soon results in the train, as well as everything else, flying from the Earth's surface.

Example III: *Chords between the South Pole and the Equator.* Let's take A as Entebbe, Uganda. Without loss of generality, we let $\mathbf{A} = (R,\, 0,\, 0)$. Note that $\mathbf{B} = (0,\, 0,\, -R)$. Thus $\mathbf{u} = (1,\, 0,\, 1)/\sqrt{2}$, $\mathbf{w} = (1,\, 0,\, -1)/\sqrt{2}$ and $c = R/\sqrt{2}$. If we start the train at A, our initial conditions are $s(0) = R/\sqrt{2}$ and $s'(0) = 0$. With these values, by (8) our polar express fails to reach the South Pole, turning back at $\frac{2\sqrt{2}\pi^2 R}{kQ^2-2\pi^2}$ km shy of the mark, about 15.7 km too short.

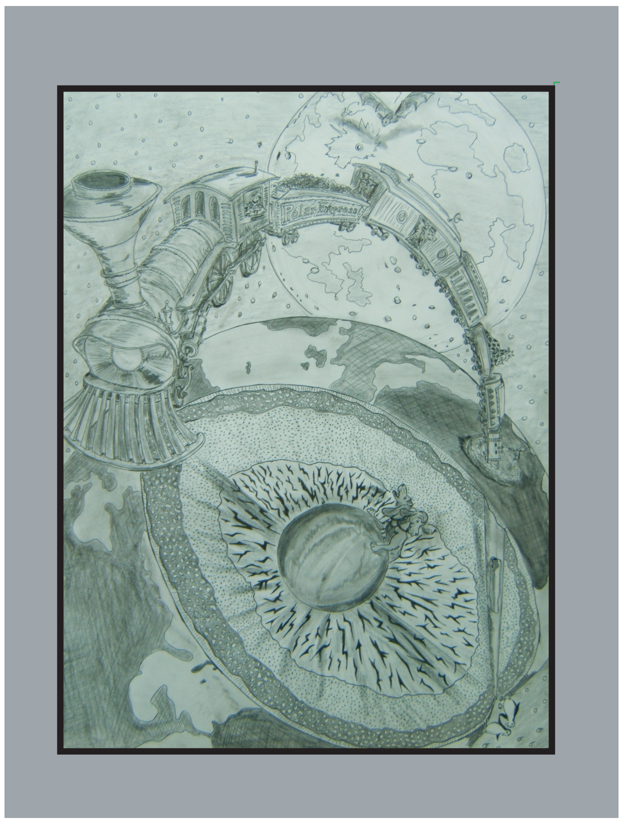

FIGURE 3. The Polar Express.

On the other hand, if we start the train at the South Pole, it glides on past Entebbe and becomes a projectile, crashing ultimately about 22 km to the north. FIGURE 3 is a whimsical sketch of such a train ride, not drawn to scale.

Example IV: *The Polar Express for our Earth.* If we remember that the Earth is an oblate spheroid, bulging at the equator by $\delta \approx 7.1$ km and flattened at the poles by $\epsilon \approx 14.2$ km, the results of Example III change. Take $R \approx 6371$ km, so that Entebbe is now at $\mathbf{A} = (R + \delta,\, 0,\, 0)$ and $\mathbf{B} = (0,\, 0,\, R - \epsilon)$. Note that $\mathbf{N}$ is no longer the midpoint of $\mathbf{A}$ and $\mathbf{B}$. This time $\gamma \approx 4517.6$ km where the distance from $\mathbf{N}$ to the South Pole is $\Gamma \approx 4487.4$

km, and $\dfrac{2\mu}{\lambda} \approx 15.5$ km. That is, when the train starts at Entebbe, it wants to turn back about 15.5 km shy of $-\gamma$. But $\gamma - \Gamma \approx 30.2$ km, which means that the train wants to turn back about 14.7 km beyond the South Pole!

Readers may enjoy determining the itineraries for similar train rides between any two of their favorite cities.

References

1. R. B. Banks, *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics*, Princeton University Press, Princeton, 1999.

2. G. Shortley and D. Williams, *Elements of Physics*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

3. A. J. Simoson, Falling down a hole through the Earth, *Mathematics Magazine,* 77:3 (2004) 171-189.