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Two-density model of the Earth

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A model of the Earth is discussed in which a spherical and homogeneous core is surrounded by the mantle, taken to be a spherical shell of constant density. Four pedagogical exercises are worked out using this model: the variation of gravitational acceleration with radius, the moment of inertia of the Earth, the pressure at the center of the Earth, and the tunneling time through the Earth. The results are compared to actual geophysical values and also to the predictions of the single-density model.

I. INTRODUCTION

The density of the Earth as a function of radius is quite well known today and is plotted in Fig. 1. The most pronounced feature of the density variation is the very sharp rise in density at the boundary between the mantle and the core. This paper considers a simple approximation to this density variation in which the core and the mantle each have a separate and constant density; it is shown that this two-density model leads to four instructive exercises at the level of introductory calculus-based physics. Two-density models played a very important historical role in the discovery of the Earth's core in the late 19th century; that history has recently been described at length by Brush.[1] This paper will consider only the pedagogical role such models can play in opening up some interesting topics in geophysics to first-year physics students. The four geophysical parameters considered here are the variation of the gravitational acceleration with radius, the tunneling time through the Earth, the moment of inertia of the Earth, and the pressure at the center of the Earth. The results are compared to those obtained in standard textbook presentations which ignore the Earth's geophysical structure completely by assuming a single constant density throughout the interior.

Specifically then, we take the Earth's core to be a sphere whose radius r_c is 3490 km (0.548 times the radius of the Earth) and whose constant density D_c is 11.0 g/cm^3. Surrounding the core and extending out to $R = 6371$ km is the mantle, taken to be a spherically symmetric shell of constant density $D_m = 4.437$ g/cm^3. The densities used here are the r^2-weighted averages of the density distributions in Bullen's A'' model, as presented by Jeffreys.[2] D_c and D_m are only slightly model dependent; the other distributions listed by Jeffreys yield values of D_c and D_m that differ by not more than 1%. D_c and D_m are, of course, connected by

$$DR^3 = D_m(R^3 - r_c^3) + D_c r_c^3 \tag{1}$$

to Earth's overall average density $D = 5.517$ g/cm^3, which (unlike D_c and D_m separately) has long been known to high accuracy.

II. VARIATION OF g WITH RADIUS

In the context of the model described above one can easily find $g(r)$, the gravitational acceleration as a function of distance from the center of the Earth, using only the fact that a spherically symmetric mass shell produces no gravitational field inside the shell, and outside the shell produces a field equal to that of a point mass at the center. The result is

if $r < r_c$,

$$g(r) = \tfrac{4}{3}\pi G D_c r = 1958\ r/R\ \text{cm/s}^2; \tag{2a}$$

if $r_c < r < R$,

$$g(r) = \frac{4}{3}\pi G\left[D_c\frac{r_c^3}{r^2} + D_m\left(r - \frac{r_c^3}{r^2}\right)\right]$$
$$= 192\left(\frac{R}{r}\right)^2 + 790\ \frac{r}{R}\ \text{cm/s}^2. \tag{2b}$$

Figure 2 shows this gravity dependence which follows from the simple two-density model along with Earth's actual $g(r)$, with which it is in reasonable agreement. The model results exactly match the actual values (neglecting effects due to spherical asymmetry) at $r = 0$, $r = R$, and at the core–mantle interface $r = r_c$, where the model predicts a maximum field that is stronger than the field at the surface by a factor $r_c D_c / RD$. It is customary in elementary textbooks to discuss the *single*-density model of the Earth and

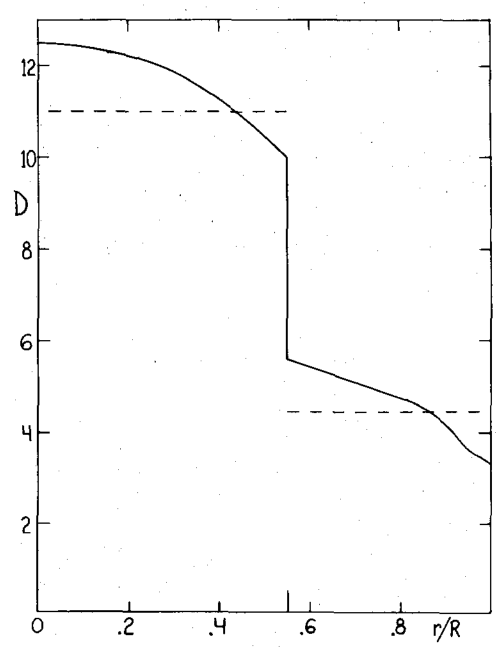

Fig. 1. Density, D, in g/cm^3 versus fractional radius. The core–mantle interface at $r_c/R = 0.548$ is marked. Solid line, actual geophysical values from Jeffreys[2]; dashed line, the two-density model discussed in this paper.

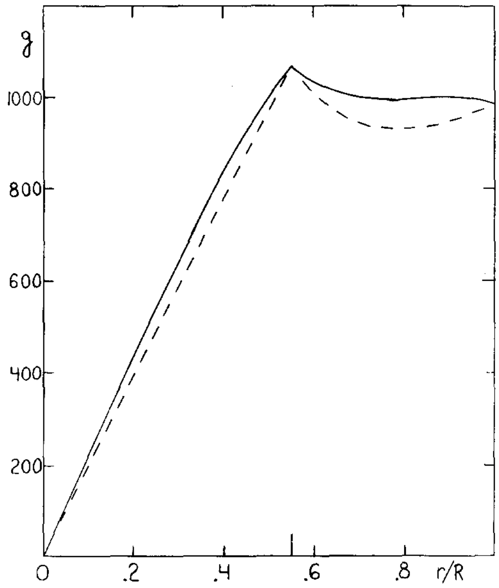

Fig. 2. Gravitational acceleration g in cm/s^2 versus fractional radius. The core–mantle interface at $r_c/R = 0.548$ is marked. Solid line, actual geophysical values from Jeffreys[2]; dashed line, the two-density model discussed in this paper.

to derive its consequence, that $g(r)$ increases linearly all the way from $r = 0$ to $r = R$. As Fig. 2 shows, this is even qualitatively incorrect; introducing the two-density model thus leads away from this misconception to a familiarity with a basic fact of geophysics.

III. TUNNELING TIME

In textbook treatments of the single-density model it is also customary to include a calculation of the time required for a particle to fall through a hypothetical tunnel dug through the Earth along a diameter. Utilizing the fact that the motion is simple harmonic, the analysis is appealingly direct and leads to the well-known result $\pi(R/g)^{1/2} = 42.2$ min. This calculation can also be carried out for the two-density model, but not with the same analytical ease, since the motion in the mantle is not simple harmonic [cf. Eq. 2(b)]. Some students may find it an interesting exercise to numerically integrate the acceleration given by Eqs. (2a) and (2b); the result for the two-density tunneling time is found to be 39.0 min, close to the 38.1 min which results from integration of the complete model A'' Earth.

Alternatively, a closed-form solution for the tunneling time can still be obtained with elementary methods if we introduce a further approximation. Figure 2 shows that in the mantle, $g(r)$ is surprisingly flat; Jeffreys[2] remarks, "the approximate constancy of gravity in the (mantle) is a curious feature, first noticed by A. E. Benfield." A tractable approximation to the tunneling time results if we treat $g(r)$ as completely constant in the mantle so that the familiar constant acceleration formulas can be used. Then t_m, the time to traverse the mantle once from R to r_c, is obtained from

$$R - r_c = 0.5\bar{g}_m t_m^2, \tag{3}$$

where $\bar{g}_m$ is an average acceleration in the mantle. In the interest of pedagogical simplicity the arithmetic average $\bar{g}_m = 0.5[g(R) + g(r_c)] = 1027$ cm/s^2 will be used; a time-averaged value would be somewhat more physical, but more cumbersome. Equation (3) now yields 12.5 min for t_m. The particle crosses the mantle–core interface with a speed $V = (2(R - r_c)\bar{g}_m)^{1/2}$. Once inside the core, the motion is simple harmonic [cf. Eq. (2a)], and the analysis proceeds along the familiar lines; the travel time through the core is 12.8 min. Thus the total tunneling time through the Earth is 37.8 min in good agreement with the time obtained by numerical methods.

In passing, it is interesting to note one other tunneling time: that which would result if all of the Earth's mass were concentrated at its center. This time, which forms a lower limit to the tunneling times possible with any spherically symmetric arrangement of Earth's interior density, is $\pi(R/2g)^{1/2} = 29.8$ min.

IV. MOMENT OF INERTIA

The moment of inertia of the Earth is an important geophysical parameter that is simple and instructional to calculate in the two-density model. Since a uniform sphere has $I = 8/15\,\pi DR^5$, we have for the two-density Earth,

$$\begin{aligned} I = I_c + I_m &= 8/15\pi[D_c r_c^5 + D_m(R^5 - r_c^5)] \\ &= \tfrac{2}{5}MR^2[D_c(r_c/R)^5 + D_m(1 - r_c^5/R^5)]/D \\ &= 0.345MR^2, \end{aligned} \tag{4}$$

where M is the total mass of the Earth. For comparison, the empirical value of I is $0.3308\,MR^2$. A single-density model, of course, predicts $0.400\,MR^2$; the two-density model thus removes 80% of the error in that result. I is still overestimated by a significant amount because the contribution of shells at radius r is proportional to r^4, so that the outermost parts of the mantle, with actual densities lower than the assumed average D_m, play an important role.

Historically, the empirical value of I was known from astronomical measurements very early, and so calculations of I played an important role in early geophysical attempts to determine the density distribution of the Earth by modeling. An exercise which illustrates these connections can be constructed in the present framework: If I, M, and r_c/R are known, Eqs. (1) and (4) can be solved to determine D_m and D_c. Using the modern value of $r_c/R = 0.548$ (which was not accurately known at the time of Wiechert's early work[1]), one thus obtains $D_m = 4.15$ g/cm^3 and $D_c = 12.4$ g/cm^3. Although these results are not especially accurate by modern standards, due to the heavy weighting of I by mass at large r mentioned above, this calculation can still be a useful pedagogical exercise.

V. CENTRAL PRESSURE

The pressure at the center of the Earth is 3.62×10^{11} Pa $\pm$ 0.5% according to the three Earth models presented by Jeffreys.[2] One can obtain an estimate of this value in a simple single-density model by using the equation of static equilibrium and recognizing that g depends linearly on r in such a model:

$$\frac{dP}{dr} = Dg = \frac{4}{3}\pi GD^2 r, \tag{5}$$

so that at the center, $P_0 = \frac{2}{3}\pi GD^2R^2 = 1.73\times10^{11}$ Pa using the values of D and R cited above.

We now calculate an estimate of P_0 using a core-plusmantle model. In order to keep the calculation at an elementary level and to obtain a simple result in closed form, we will use the subsidiary approximation of a constant-gravity mantle (cf. Sec. III). Then the pressure increases from (say) zero at the surface to $D_m \bar{g}_m (R - r_c)$ at the mantle–core interface. It then increases by a further amount $\frac{2}{3}\pi G D_c^2 r_c^2$ in the core itself, so that

$$P_0 = D_m \bar{g}_m (R - r_c) + \tfrac{2}{3}\pi G D_c^2 r_c^2 = 3.4 \times 10^{11} \text{ Pa}, \quad (6)$$

in good agreement with the geophysical value cited above.

[1]S. G. Brush, Am. J. Phys. 48, 705 (1980).
[2]H. Jeffreys, *The Earth* (Cambridge U.P., New York, 1976), 6th ed.

How Olympic records depend on location

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The summer Olympics at Mexico City in 1968 produced a number of anomalous records in track and field, some of which can be explained by the effects of low air density and gravity at Mexico City, while others were the result of outstanding athletic performances.

I. INTRODUCTION

In the study of Newtonian mechanics, students find the standard textbook problems of blocks on inclined planes, etc., to be dull and of little relevance to the "real world" outside the physics classroom. Therefore, it is not surprising that many teachers use athletic events as examples of applications of elementary mechanics in order to generate student interest. An indication of the popularity of applying mechanics to sports is given by the large number of papers in this area which have been published in this Journal and elsewhere in recent years.[1–33]

This paper presents an analysis of several Olympic track and field events with emphasis on the effect of gravity and air resistance on performance. This topic has been extremely well received by students in introductory physics courses at the University of Guelph, and by students in nearby high schools.

II. DATA

Table I presents data on gold medal performances for several Olympic events at sites from Helsinki in 1952 to Los Angeles in 1984. Even a cursory glance will show that there are a number of anomalies in the records for the 1968 Mexico City Olympics. World and Olympic records which still stand unbroken were set in the 400-m dash and the long jump; the performances in the 100-m dash and the triple jump are Olympic records yet. The 200-m dash time remained a world record until 1979, and an Olympic record until 1984. The high jump performance, while not as striking as that in the other events, is still one of only two in the high jump list which was not surpassed at the next Olympics. However, the times for the 5000-m run and the marathon were the worst Olympic times in 16 and 12 yr, respectively.

Students will readily provide explanations for these anomalous records. Because of the high altitude of Mexico City, both the air density and gravity are reduced. The lower gravity produces higher and longer jumps, but has negligible effect on the running events. The reduced air resistance results in enhanced performances in athletic events which are of short duration, such as sprints and jumps. These events are anaerobic in nature: The energy is available from the muscles without the presence of oxygen. However, as the duration of the event increases, the aerobic (oxygen requiring) metabolism provides an increasing fraction of the energy, and the reduced air density becomes an increasing liability because of the lower availability of oxygen. At a time of 20 s into the event, approximately 3% of the energy is being provided aerobically; at 120 s, this has increased to about 75%.[34]

These qualitative explanations, while generally correct, do not, of course, give any indication of the magnitude of the effects of lower air density and gravity. The remainder of the paper considers this topic in some detail; it will become evident that performances in different events are affected to very different degrees by changes in gravity and air density.

III. GRAVITY, AIR DENSITY

The local gravitational acceleration g varies with location because of three major factors: nonsphericity of the Earth, altitude above sea level, and the centrifugal effect. There are additional minor variations due to differences in crustal composition, etc.

The nonsphericity of the Earth and the centrifugal effect produce a sea-level g which varies with latitude, maximum at the poles, and minimum at the equator. The difference between the polar and equatorial g values is 0.18% because of the nonsphericity[35–37] and 0.35% because of the centrifugal effect.[38,39] The effect of altitude on g is well known[40,41]; the decrease in g is roughly 1.97×10^{-3} m/s^2 for every km of land elevation.