Gravity Tunnel in the Earth

A novel effective function for the mass density profile inside the Earth is proposed in order to describe qualitatively and quantitatively physical aspects of the terrestrial density derived from seismic models. This was done using a decreasing potential law function dependent only on the distance to the center of the Earth and on some parameters, in such a way that central density, surface density and total mass are correctly given. This effective function was applied to gravity tunnel concepts as a test of the predictive power of the effective model, but the comparison is established with the geophysical approach predictions, in the place of real life experiments, due to the impossibility of performing this. A high correspondence between the effective model and the reference model was found.

1 Introduction

A gravity tunnel is a hypothetical means of transportation in which, to arrive to one point to another on the surface of the planet, it appeals to the acceleration of gravity in a tunnel connecting both points. The problem is clearly pedagogic and has been thoroughly discussed in journals of this character.

This has been made in a lot of different ways, considering the simplest case of constant density, proposed as exercise in classical mechanics textbooks, adding rail's friction, or considering the rotation of the Earth , and the effect of this in its geometrical form, or even in the frame of general relativity . Likewise, this has been done considering density-pressure relations based on polytrope , or in more accurate density geophysical models as the Preliminary Reference Earth Model (PREM), being this last one done numerically by Klotz , making a comparison with a constant gravity approximation; and also doing an analytical treatment using a linear, piecewise approximation which throw closer results to those of the PREM, and which get out the poorly physical situation of a constant gravity (since this implies infinitive densities at the center of the planet).

To geophysical, astrophysical, as well as purely physical purposes, knowing the density distribution inside of our planet is of great utility. In a first approximation and to uniquely illustrative aims, it can be taken to be constant, dividing total mass under its volume and getting $\rho = 5513\ \mathrm{kg} / m^3$.

With this one can obtain approximate values for the acceleration of gravity or moments of inertia. Nevertheless, it is clear that the density on Earth (and on any astronomical body) is far from being constant. Other models have been proposed in order to describe the density or the acceleration of our planet, but the most accepted model is the Preliminary Reference Earth Model, according to which the density in the interior has discontinuities between the different layers that form it: inner core, outer core, mantle and crust. Inside each region, density can be approximated with a polynomial function . This can be seen on the figure above . Notice that density increases towards the center and reaches its maximum value around $1300\,\mathrm{kg}/m^3$, while in the surface is almost $1020\,\mathrm{kg}/m^3$, which corresponds to the density of the water of the ocean.

In the present work it is intended to discuss a possible approximation to this model using an effective density function more simple to express: soft, continuous and certainly not constant. This function is contemplated as a possible replacement to the described for the Reference Model and, thus, as a simple but accurate description of Earth physical effects.

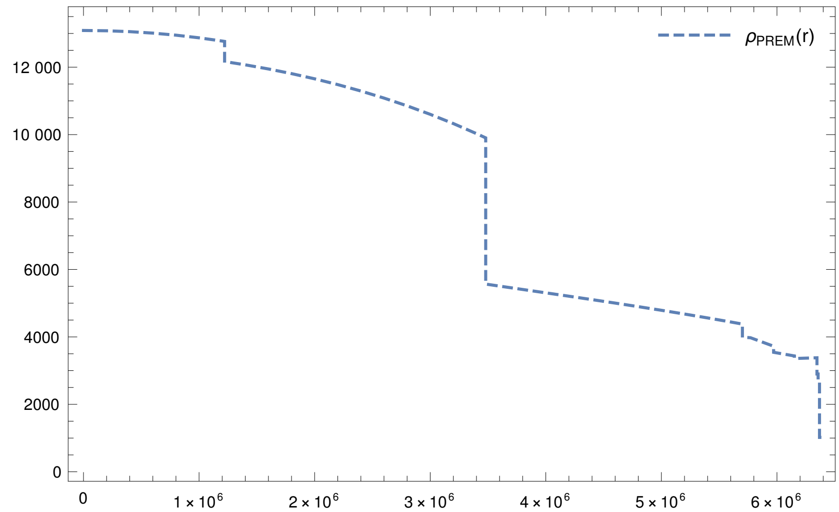
The use of this function not only represents a totally different treatment compared to those previously mentioned for the treatment of the gravity tunnel, but also serves as an example to illustrate the construction and the development of an effective model. Effective models in classical mechanics and, actually in undergraduate physics, are very rare. Nonetheless its utility in the scientific world is becoming greater and greater; hence, it is important that students at this level in a physics or engineering career get familiar with some model with a simple and fairly known physics and that doesn't contain any complicated mathematical or computational methods. The present article accomplish all those requirements due to the simple deductions, presented in the field of Newton mechanics, and to the functions used, which are integrable either in analytical form or with the conventional numerical methods.

For the effective density function construction, the density profiles are first raised in section 2, were conditions over the parameters to fix are stated. Total mass and gravity are computed as consistency checks in section 3, then, speed profiles, traversal times and brachistochrone shapes are computed and compared in 4.

Finally, in , we study the acceleration along the path, in order to give a dynamic explanation of why the times are shorter for longer paths such as brachistochrone, and not for the paths of shorter distance.

To see PREM density we need to import the data

Out[ ]=




```
Export["/home/nicolas/Documents/Physics/Bachelors-Dissertation/1-Earth
    Gravity Tunnel/Plots/density_prem.pdf", density_prem];
```

Before continue, let's define the value of the constants that we are going to use: Acceleration of gravity, radius of the Earth, mean density of the Earth and gravity constant:

In[1]:=

```
g = 9.8156; (*m/s2*)
R = 6.371 × 10^6; (*m*)
ρ0 = 5513 ; (*kg/m3*)
G = 6.67 × 10^(-11) ;(* N m2/kg^2*)
```

2 Density Profile

In order to account for a density that replaces the one in fig. above, with a continuous and soft form, but such that it reproduces the following geophysical situations

1. Density in the center of the Earth:

$$\rho\,(0) = \rho_0\, b\; = 13\,088.5$$

2. Density at the surface must be that of the water (1020 kg / m^3):

$$\rho\,(r\; = \;R) \; = \; b\,\rho_0\,(1 - c\,)^d \; = \; \rho_s$$

3. The third conditions is given from the total mass, we must have

$$M_T \; = \; 4\,\pi \int_0^R \rho(r)\, r^2\, d\,r \; = \; 4\,\pi\,\rho_0\, b \int_0^R \left(1 \; - \; c\,\frac{r}{R}\right)^d r^2\, d\,r$$

and such that it allows to make predictions, we have considered the following decreasing power law binomial function of three parameters:

$$\rho 1\,(r;\; b,\; c,\; d) = \rho_0\, b \left(1 - c\,\frac{r}{R}\right)^d$$

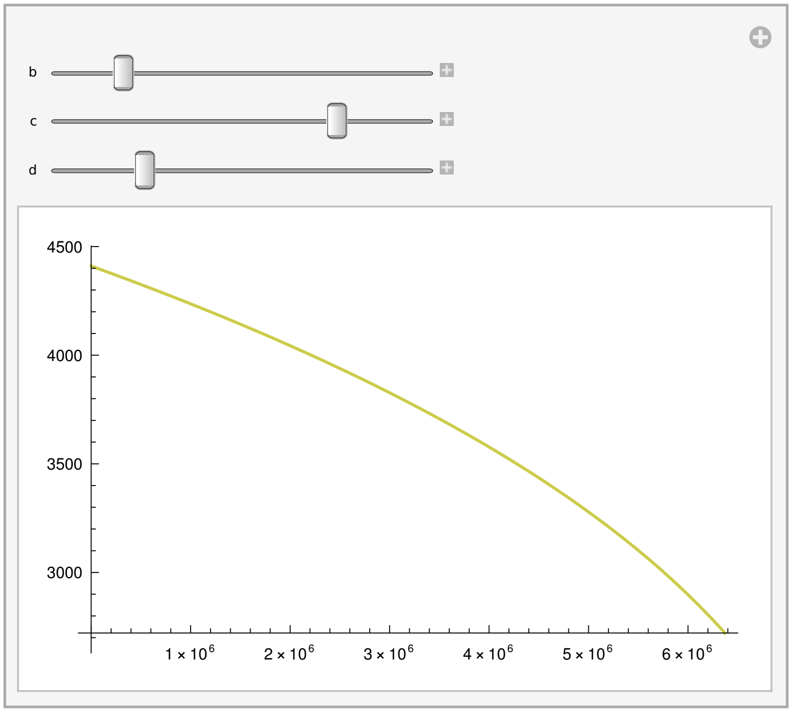
being b,c,d parameters to determine. It can be written in the code as the function

In[24]:=
```
ρ_1[r_, b_, c_, d_] := ρ0 b Power[1 - c r / R, d]
```

this density have the following forms

In[29]:=
```
Manipulate[Plot[ρ_1[r, b, c, d], {r, 0, R}, PlotStyle → RGBColor[b, c, d]],
 {b, 0.2, 4, 0.2}, {c, .1, 1, 0.1}, {d, .1, 1, 0.1}]
```

Out[29]=



2.1 Parameter for the effective density function

Lets now use the conditions stated above to find the parameters *b, c, d:*

1. Density in the center

In[ ]:=
```
b = 13 088.5 / ρ0;
Print[Style["b = ", 20, FontFamily → "Utopia"],
 Style[SetPrecision[b, 15], 20, FontFamily → "Utopia"]]
```

b = 2.37411572646472

2. Density at the surface, with the value of *b* above

$$(1-c)^d = \frac{0.1815}{2.37325} = 0.0779309$$

3. The integral for the third conditions, done with *Mathematica* gives the analytical expression

```
Clear[b, R]
 Integrate[b * x ^ 2 * (1 - c * (x / R)) ^ d, {x, 0, R}]
```

Out[ ]=
$$\frac{b\left(2+(1-c)^{d}(-1+c)(2+c(1+d)(2+c(2+d)))\right)R^{3}}{c^{3}(1+d)(2+d)(3+d)} \quad \text{if } \operatorname{Re}[c]\leq 1 \,||\, c\notin\mathbb{R}$$

so we can express the third condition as:

$$3\,b\left[2-(1-c)^{d+1}\left(2+2\,c\,(1+d)+c^{2}\left(2+3\,d+d^{2}\right)\right)\right]=c^{3}\left(6+11\,d+6\,d^{2}+d^{3}\right)$$

To solve the system of equations, this would be the code on Mathematica to solve the system, but it takes too long

In[ ]:=
```
Remove[x, y];
NSolve[{(1 - x)^y == 0.0763, 7.1197 (2 - (1 - x)^(y + 1) * (2 + 2 * x * (1 + y) + x^2 * (2 + 3 * y + x^2))) /
    (x^3 * (6 + 11 * y + 6 * y^2 + y^3)) == 1}, {x, y}]
```

On the other hand, the next code on Python finds the answer in a few seconds:

```
from scipy.optimize  import  root

def equations(p):
    x, y  = p
    eq1 = (1-x)**y  - 0.0779310081369141
    eq2 = 7.12235*(2  - (1 - x)**(y  + 1)*(2  + 2*x*(1  + y)  + x**2*(2  + 3*y  + y**2)))  /
(x**3*(6  + 11*y  + 6*y**2  + y**3))  - 1
    return  ( eq1  , eq2 )

sol =  root(equations,  (0.1,  0.1), method='lm',  jac=None,  tol=None,  callback=None,
options={'col_deriv':   0, 'xtol':  1.49012e-08,  'ftol':  1.49012e-08,  'gtol':  0.0,
'maxiter':  0, 'eps':  0.0, 'factor':  100, 'diag':  None})


x = sol.x[0]
y = sol.x[1]

#print(equations((x,   y)))
print('x  = ', x)
print('y  = ', y)
```

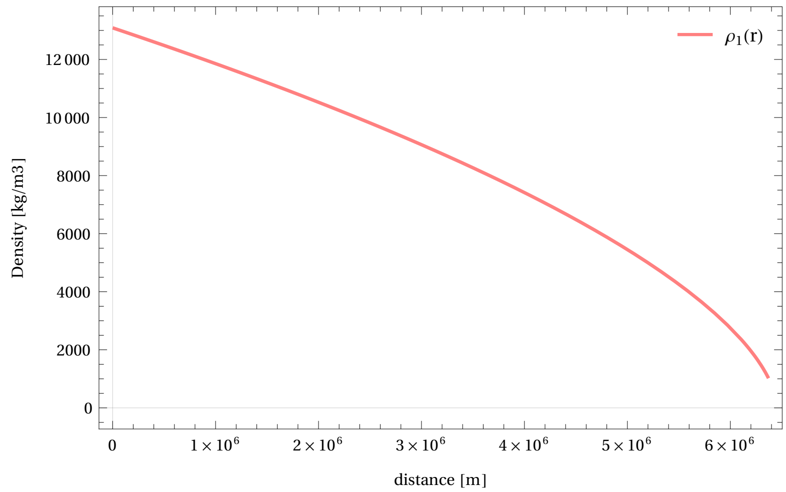
```
x = 0.98695
y = 0.588137
```

For simplicity, the density function to work from here on is then

In[7]:=
```
b = 2.37411572646472 ;
c = 0.986950462357128 ;
d = 0.5881369116467612 ;
ρ[r_] := ρ_1[r, b, c, d]
```

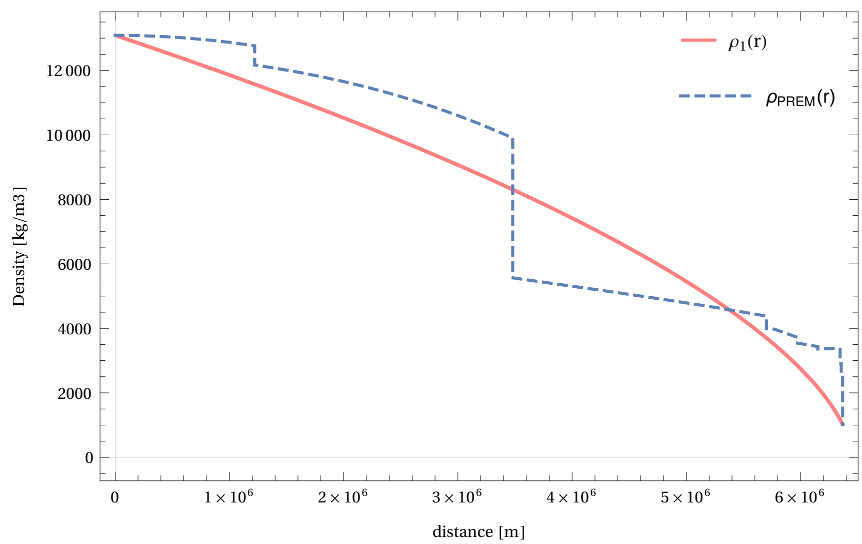
Plotting with this result gives

Out[∘]=



To compare, we can plot them together

Out[∘]=



we see that actually the proposed functions fit between the data.

3 Numerical consistency checks

In this section, we want to use the effective density function found to check if the numerical computation of some quantities is correct, giving us a first clue of the correctness of the method elaborated.

Densities

The test of the first two conditions is trivial and it’s just the minimum requirement. With all the digits we have checked that the condition values are returned.

- In the surface we find:

In[ ]:=
```
Print[Style["ρ(R) = ", 20, FontFamily → "Utopia"],
 Style[ρ[R], 20, FontFamily → "Utopia"],
 Style[" kg/m³ ", 20, Italic, FontFamily → "Utopia"]]
```

$$\rho(\mathrm{R}) = 1020.\ kg/m^3$$

- In the center

In[ ]:=
```
Print[Style["ρ(0) = ", 20, FontFamily → "Utopia"],
 Style[ρ[0], 20, FontFamily → "Utopia"],
 Style[" kg/m³ ", 20, Italic, FontFamily → "Utopia"]]
```

$$\rho(0) = 13\,088.5\ kg/m^3$$

Masses

Let's consider now the integration of the density to find the mass, which is less immediate. With any integration algorithm, it can be seen that the following integral
Now, we want to determine the mass as a function of the radius. At $r = R$ we would like to have $M(R) = M \sim 6\,\mathrm{x}10^{25}$ kg ,

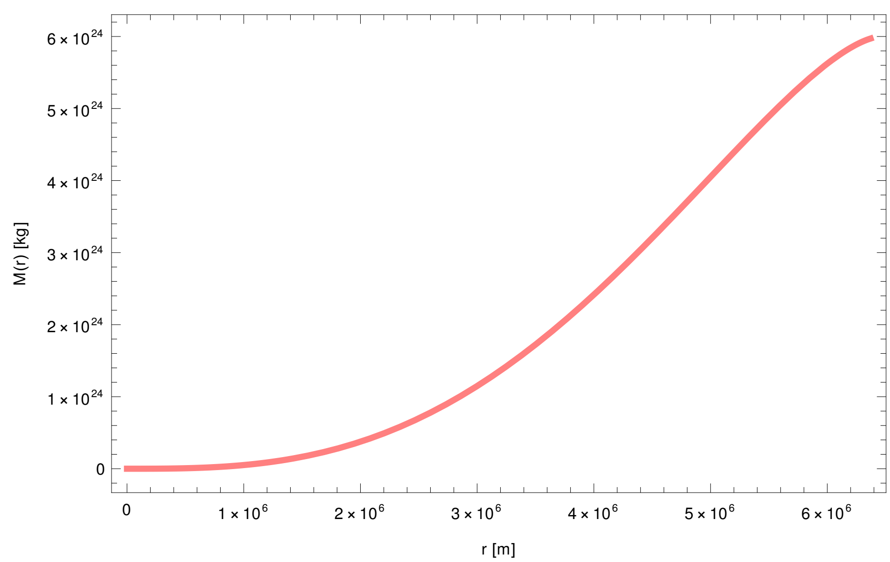
In[11]:=
```
M[r_] := 4 π NIntegrate[ρ[x] x², {x, 0, r}]
```

the output at the radius is

In[ ]:=
```
Print[Style["M_T = ", 20, FontFamily → "Utopia"],
 Style[M[R], 20, FontFamily → "Utopia"], Style[" kg", 20, Italic, FontFamily → "Utopia"]]
```

$$\mathrm{M_T} = 5.97172 \times 10^{24}\ kg$$

the expected one. To see the behaviour at any other point let's plot this functions

Out[ ]=


Gravity

The acceleration of gravity in the surface is another physical constant that must be appropriately given by our effective model. This is, however, not a prediction. Thanks to Newton's shell theorem, total mass is what gives gravity its constant value on the surface, regardless of the distribution of mass. This fact can also be seen in the mathematical expression derivable from Gauss law (the integrals are the same):

$$a(r) = \frac{4\pi G}{r^2} \int_0^r \rho(r')\, r'^2\, dr'$$

The interest we have in the form of the acceleration, not only in the surface but in the inside as well, is that, to perform the predictions of the model within the context of the gravity tunnel, we will need this as a function of the radius. Furthermore, there are two ways to address this and some of the next computations. One is integrating by brute force, as we did with the expression of M_T.

```
In[ ]:= Clear[b, c, d, R]
Integrate[x^2*(1-c*(x/R))^d, {x, 0, r}]
```

Out[ ]=

$$\frac{2 R^3 + \left(1-\frac{c\, r}{R}\right)^d (c\, r - R)\left(c^2 (1+d)(2+d)\, r^2 + 2 c (1+d)\, r R + 2 R^2\right)}{c^3 (1+d)(2+d)(3+d)} \quad \text{if} \quad \text{condition}$$

Now, let's define the function as

```
In[ ]:= a_analytic [r_] := 3 g b (2 R^2 - (1 - c r / R)^(d + 1) (c^2 (1 + d) (2 + d) r^2 + 2 c (1 + d) r R + 2 R^2)) /
    (c^3 r^2 (6 + 11 d + 6 d^2 + d^3))
```

At the radius the gravity is

$$g_{\text{analytic}} = 9.8156\ m/s^2$$

As it should be. The other is re-writing the density using Newton's binomial theorem

$$a(r) = \frac{4\pi G}{r^2} \rho_0\, b \int_0^r \left(1 - c\,\frac{r'}{R}\right)^d r'^2\, dr' = \frac{4\pi G}{r^2}\,\frac{3M}{4\pi R^3} \sum_{n=0}^{\infty} \binom{d}{n} \frac{(-c)^n}{R^n} \int_0^r r^{n+2}\, dr$$

$$= 3\, g\, b \sum_{n=0}^{\infty} \binom{d}{n} \frac{(-c)^n}{n+3} \left(\frac{r}{R}\right)^{n+1}$$

in the code

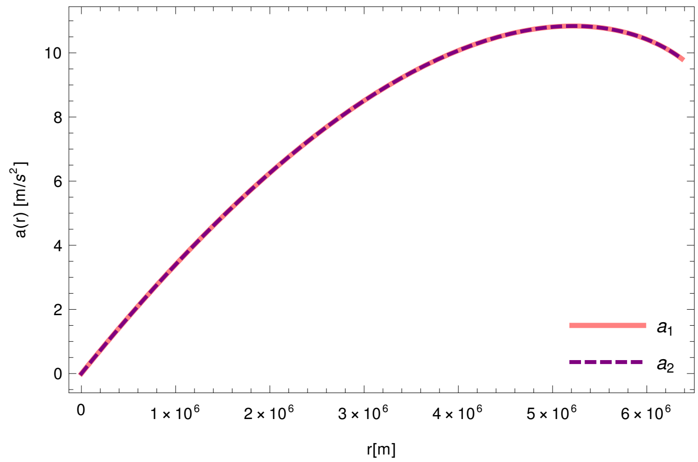
```
In[ ]:= a[r_] := 3 b g Sum[QBinomial[d, n, 1] (-c)^n (r / R)^(n + 1) / (n + 3), {n, 0, Infinity}]
```

at the surface we have

$$g_{\text{Binom}} = 9.8156\ m/s^2$$

which is a little above the expected value.
We can both functions in the hole range by making plots

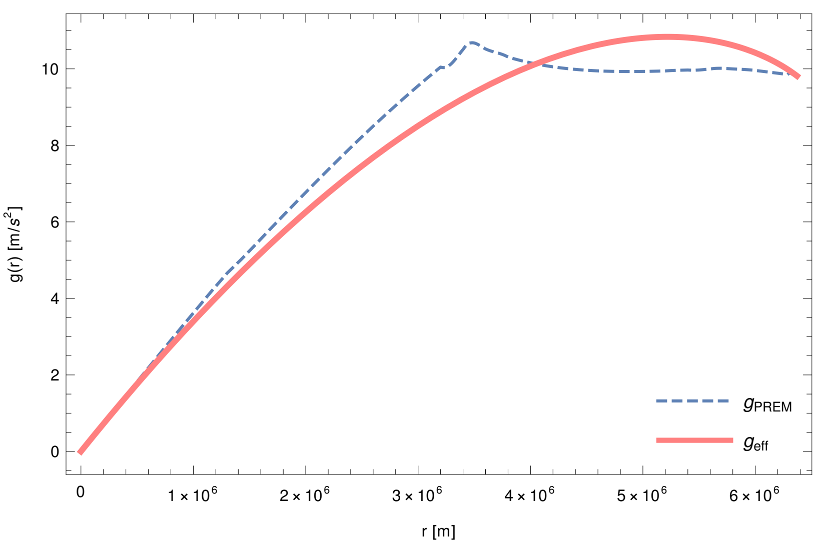
Out[•]=



We see now that they are indeed the same. Now, lets import the data from PREM and plot together the founded profiles along with this one:

In[•]:=
```
gravity_prem =
  Import["/home/nicolas/Documents/Physics/Bachelors-Dissertation/1-Earth Gravity
     Tunnel/Numerical Data/gravity_prem.csv", "Table"];
g_prem = Interpolation[gravity_prem, InterpolationOrder → 5];
gravity_prem_ad =
  Import["/home/nicolas/Documents/Physics/Bachelors-Dissertation/1-Earth
     Gravity Tunnel/Numerical Data/prem-grav-ad.csv", "Table"];
g_prem_ad = Interpolation[gravity_prem_ad, InterpolationOrder → 5];
```

Out[•]=



4. Numerical Predictions

With the confidence on our effective density function gained above, and with the help of some physical quantities already covered, we can test the prediction power of the model . Keep in mind that, in contrast with effective model in real world, our predictions can not be tested experimentally, so we rely on the same quantities computed directly from PREM . In the next subsections we use both and compare the results .

4.1 Velocity Profiles

Once the mass distribution within the physical sphere to be considered is known, the speed followed by a test particle while inside it can be found from energy considerations as

$$v(r) = \sqrt{8\pi G \int_r^R \int_0^y \rho(x)\, x^2\, dx \frac{1}{y^2}\, dy}$$

or in terms of the integral of the acceleration. For the PREM profile, we have

```
In[ ]:= vprem[r_?NumberQ] := Sqrt[2 * NIntegrate[g_prem[x], {x, r, R}]]
vprem_ad[r_?NumberQ] := Sqrt[2 / 3 * NIntegrate[g_prem_ad[x], {x, r, 1}]]
```

For the case of the effective density function, this will be the direct integration of the analytical acceleration

```
In[ ]:= v_analytic [r_] :=
  Sqrt[8 Pi G ρ0 NIntegrate[b Power[1 - c x / R, d] x^2 / y^2, {y, r, R}, {x, 0, y}]]
```

while this is an still analytical expression using the binomial theorem expression:

```
In[ ]:= v_ad[r_] :=
  Sqrt[2 b Sum[QBinomial[d, k, 1] * (-c)^k (1 - r^(k + 2)) / ((k + 2) (k + 3)), {k, 0, Infinity}]]
v[r_] := Sqrt[8 Pi G ρ0 R^2]
   Sqrt[b Sum[QBinomial[d, k, 1] * (-c)^k (1 - r^(k + 2)) / ((k + 2) (k + 3)), {k, 0, Infinity}]]
```

We can see that they exactly the same and very close to the value predicted by PREM distribution

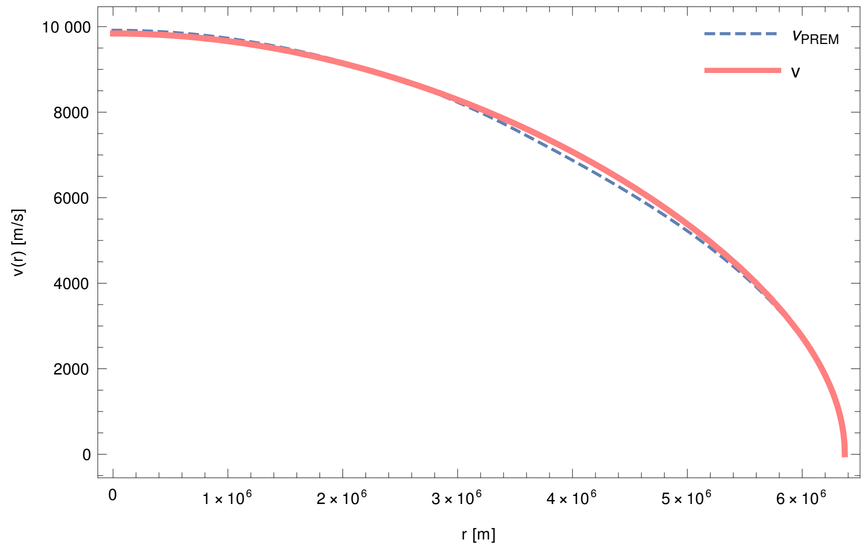
$$v_{\text{PREM}}\,(0) = 9914.72\ m/s\ ,\ v_{\text{analytic}}\,(0) = 9839.49\ m/s\ ,\ v_{\text{Binom}}\,(0) = 9839.49\ m/s$$

This represents a percentual error of

$$\text{Error} = 0.758851\,\%$$

In the hole range, the distribution of velocities looks as

Out[ ]=



4.2 Times for the Chord Path

We have arrived to the most important part of this work, the prediction of the traversal time of the train through the chord path. In the case between antipodes, the times are given by

$$T = 2\int_0^R \frac{1}{v(r)}\, dr.$$

in a more general chord path, characterized by a parameter d (distance from the polar axis), they are

$$T(d) = 2\int_d^R \frac{r}{v(r)}\frac{1}{\sqrt{r^2 - d^2}}\, dr.$$

With the velocity profiles founded above, we can define three functions for computing times; one for PREM case and two for the effective density function. The reason for having taken all this time the two methods is that *analytic* functions were faster to plot above, but the integration of *binomial* case in this case results better, so we will define them but keep from here on, only the second case

In[ ]:=
```
Tprem[d_] := NIntegrate[x / (vprem[x] * Sqrt[x ^ 2 - d ^ 2]), {x, d, R}]
Tprem_ad[d_] := NIntegrate[x / (vprem_ad[x] * Sqrt[x ^ 2 - d ^ 2]), {x, d, 1}]
T1[d_] := NIntegrate[x / (v_analytic[x] * Sqrt[x ^ 2 - d ^ 2]), {x, d, R}]
T_ad[d_] := NIntegrate[x / (v_ad[x] * Sqrt[x ^ 2 - d ^ 2]), {x, d, 1}]
T[d_] := R NIntegrate[x / (v[x] * Sqrt[x ^ 2 - d ^ 2]), {x, d, 1}]
```

Now, we can compute the output of this functions in minutes:

$$T_{\text{PREM}} = 38.1878\ min \quad , \quad T_{\text{Eff}} = 37.7661\ min$$

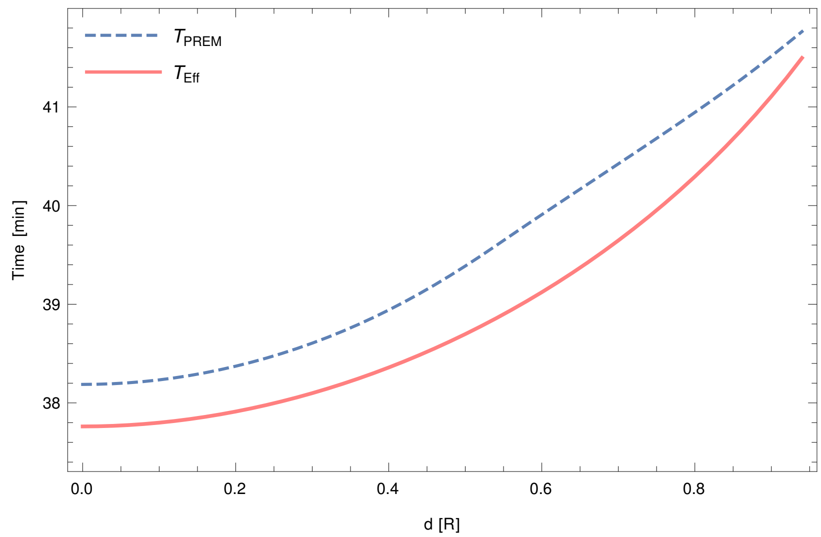
this means an error of

$$\text{Error} = 1.1043\ \%$$

But, again, the best way to compare the results is with a plot

```
ListTimesPrem = Table[{d, Tprem_ad[d] * Sqrt[1 / 3 * R / g] / 30}, {d, 0, 0.95, 0.02}];
ListTimesEff = Table[{x, T_ad[x] * Sqrt[1 / 3 * R / g] / 30}, {x, 0., 0.95, 0.02}];
```

Out[ ]=



4.3 Brachistochrone Path

Shapes of the brachistochrone

The other important part of this work is the time for the path of fastest descent and its shape. To find it we will need to perform the following integration of the trajectory:

$$\theta(r) = \int_d^r I(x, d)\, dx$$

with the integrand

$$I(r, d) = \left[\frac{r^4}{d^2}\left(\frac{v(d)}{v(r)}\right)^2 - r^2\right]^{-1/2}$$

The next is the code to define auxiliary functions for the integration,

In[ ]:=

```
I1[x_, d_] := 1 / (Sqrt[x ^ 4 / d ^ 2 * (v_ad[d] / v_ad[x]) ^ 2 - x ^ 2])
I_prem[x_, d_] := 1 / (Sqrt[x ^ 4 / d ^ 2 * (vprem_ad[d] / vprem_ad[x]) ^ 2 - x ^ 2])
```

With them we can proceed to do the integration in a valid range

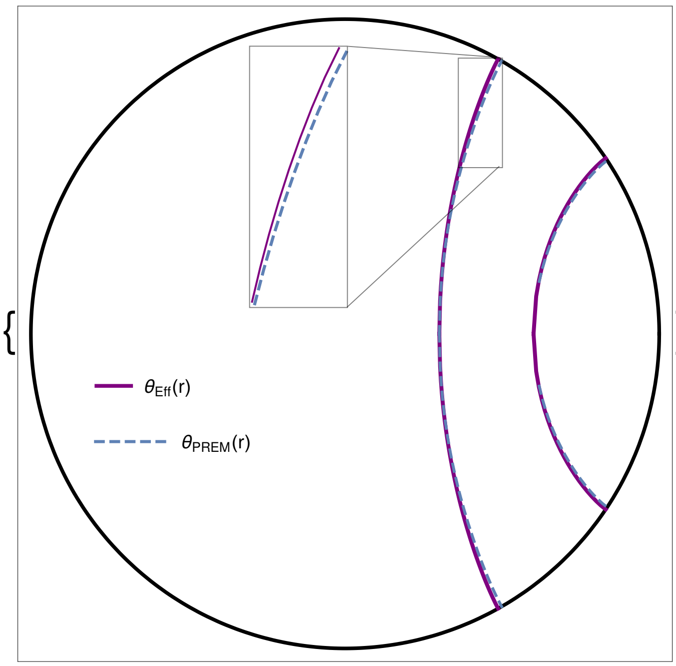
In[ ]:=

```
θ[r_?NumericQ, d_?NumericQ] := If[r > d, NIntegrate[I1[x, d], {x, d, r}], 0.0]
θ_prem[r_?NumericQ, d_?NumericQ] :=
 Piecewise[{{0.0, r ≤ d}, {NIntegrate[I_prem[x, d], {x, d, r}], r > d}}]
```

In order to plot the trajectories and visualise the motion in a better way, we should make a polar plot. As this is not so immediate, even in Mathematica, we have to make some lists with the data points, which is what the next cell does

```
In[ ]:= Step = 0.02;
Theta3 = Table[{θ[i, 0.3], i}, {i, 0.3, 1, Step}];
Theta3_minus = Table[{- θ[i, 0.3], i}, {i, 0.3, 1, Step}];
Theta6 = Table[{θ[i, 0.6], i}, {i, 0.6, 1, Step}];
Theta6_minus = Table[{- θ[i, 0.6], i}, {i, 0.6, 1, Step}];
ThetaPrem3 = Table[{θ_prem[i, 0.3], i}, {i, 0.3, 1, Step}];
ThetaPrem3_minus = Table[{- θ_prem[i, 0.3], i}, {i, 0.3, 1, Step}];
ThetaPrem6 = Table[{θ_prem[i, 0.6], i}, {i, 0.6, 1, Step}];
ThetaPrem6_minus = Table[{- θ_prem[i, 0.6], i}, {i, 0.6, 1, Step}];
```

With that data, we plot the Earth as the Black circle, and the trajectories inside it.

{  }

Times for the brachistochrone paths

In this subsection we are going to find the times for the brachistochrone paths plotted above. As this are by definition the paths of fastest descent, the times should be the minimal times. The formula to compute this times is better given by

$$T_{\mathrm{BRAQ}} = \int_d^r \frac{\sqrt{1 + x^2\, I^2(x)}}{v\,(x)}\, dx$$

Where $I\,(r) = \partial_r\, \theta(r,\, d)$.

and then the times for a given maximum approaching point d

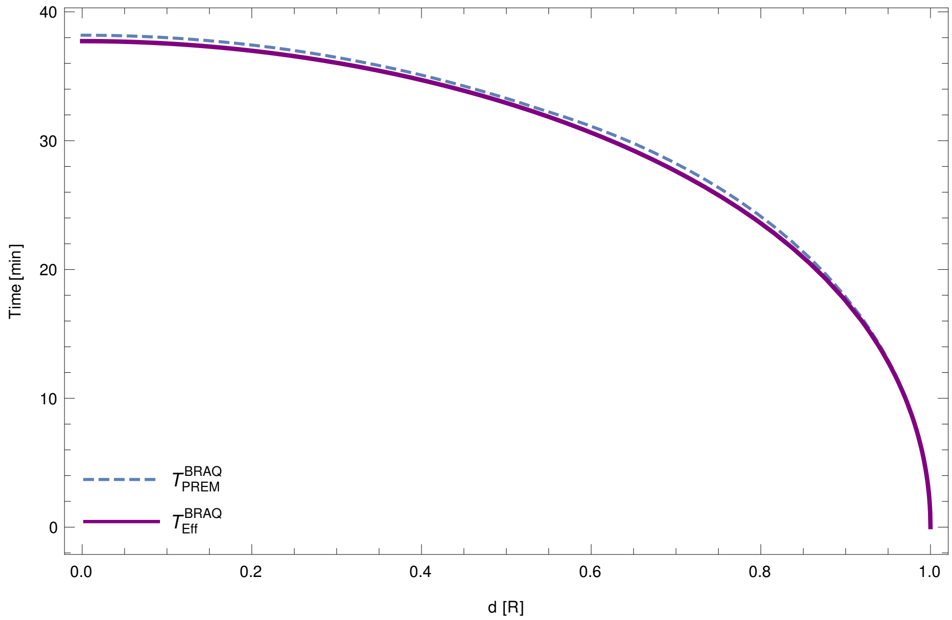
```
Tbraq[d_?NumericQ] := NIntegrate[Sqrt[1 + r^2 I1[r, d]^2] / v_ad[r], {r, d, 1}]
Tbraq_prem[d_?NumericQ] := NIntegrate[Sqrt[1 + r^2 I_prem[r, d]^2] / vprem_ad[r], {r, d, 1}]
```

let's check that the times taken when d=0 are the same as for the chord path

$$T^{BRAQ}_{PREM} = 38.1875 \text{ min} \quad , \quad T^{BRAQ}_{Eff} = 37.7288 \text{ min}$$

They are almost the same, maybe it’s a question of precision, but we can trust in this results. We can see the complete dependence of the times on the position of the tunnel in the following plot

Out[ ]=



5. Accelerations

In this last section, we want give a dynamical reason for the difference between chord path times and brachistochrone path times. First, define some auxiliary functions

In[ ]:=
```
dist[d_] := Sqrt[1 - (Sin[θ[1, d]])^2]
dist_prem[d_] := Sqrt[1 - (Sin[θ_prem[1, d]])^2]
```

with the radial accelerations and the these auxiliary functions we can define the acceleration in the direction of motion for the chord path

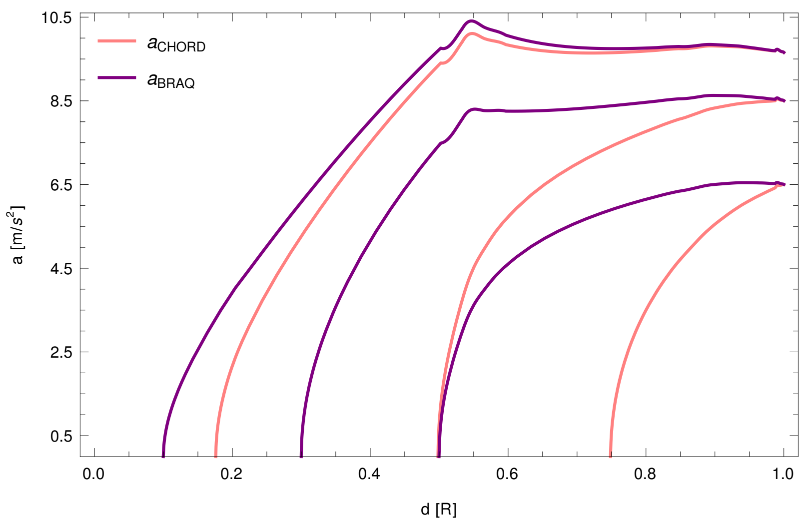
In[ ]:=
```
a_chord[r_, d_] := Re[a_analytic[r R] / g Sqrt[r^2 - (dist[d])^2] / r]
a_prem_chord[x_, d_] := Re[g_prem_ad[x] * Sqrt[x^2 - (dist_prem[d])^2] / x]
```

as well as the acceleration in the direction of motion for the brachistochrone paths

In[ ]:=
```
a_braq[x_, d_] := Re[a_analytic[x R] / g * Sin[θ[x, d]]]
a_braq_prem[x_, d_] := Re[g_prem_ad[x] * Sin[θ_prem[x, d]]]
```

Now, let’s plot

Out[ ]=



6. Exercise Proposed: Moment of Inertia

In this section we develop the proposed exercise: **Calculate, with the effective density function found, the moment of inertia of the Earth around its axis and compare it with the one of the case of constant density.**

The moments of inertia can be defined for any object, with respect to rotation in each axis, as

$$I_{xy} = \int_V \rho(r)\left(x^2 + y^2\right) d\,V$$
$$I_{xz} = \int_V \rho(r)\left(x^2 + z^2\right) d\,V$$
$$I_{yz} = \int_V \rho(r)\left(y^2 + z^2\right) d\,V$$

in the case of a sphere with density changing only around the radial direction, the three are equal, therefore, adding up we have

$$I_{xy} + I_{yz} + I_{xz} = 3\,I = \int_V \rho\,(r)\left(2\,x^2 + 2\,y^2 + 2\,z^2\right) \mathrm{dV}$$

hence, writing the explicit volume differential

$$I = \frac{8\,\pi}{3} \int_0^R \rho(r)\, r^4\, \mathrm{dr}$$
$$I = \frac{8\,\pi}{3}\, \rho_0\, b \int_0^R \left(1 - c\,\frac{r}{R}\right)^d r^4\, \mathrm{dr}$$

With the effective density function found, this can be done analytically in the both ways mentioned: directly, by the method of substitution or by expansion in form of the binomial theorem (of course, numerically is also, as always, an option):

Analytically

The important part of the integral, namely:

$$\int_0^R \left(1 - c\,\frac{r}{R}\right)^d r^4\, \mathbf{dr} = R^5 \int_0^1 (1 - c\,x)^d\, dx$$

Can be expressed as

In[◦]:=
```
Integrate[(1 - c x)^d x^4, {x, 0, 1}]
```

Out[◦]=

$$\frac{24}{c^5 (1+d)(2+d)(3+d)(4+d)(5+d)} + \frac{(1-c)^{1+d}\left(-\frac{1}{1+d} + \frac{4-4c}{2+d} - \frac{6(-1+c)^2}{3+d} - \frac{4(-1+c)^3}{4+d} - \frac{(-1+c)^4}{5+d}\right)}{c^5} \quad \text{if } \mathrm{Re}[c] \le 1 \,\|\, c \notin \mathbb{R}$$

then,

$$I = \frac{8\pi}{3}\left(\frac{3M}{4\pi R^3}\right) b\, R^5 [\,]$$

Using Binomial Theorem

In this case, we have

$$\int_0^R \left(1 - c\,\frac{r}{R}\right)^d r^4\, \mathbf{dr} = R^5 \int_0^1 (1 - c\,x)^d\, dx = R^5 \sum_{n=0}^{\infty} \binom{d}{n} (-c)^n \int_0^1 x^{n+4}\, dx = R^5 \sum_{n=0}^{\infty} \binom{d}{n} (-c)^n \frac{1}{n+5}$$

so we get

$$I = \frac{8\pi}{3}\left(\frac{3M}{4\pi R^3}\right) b\, R^5 \sum_{n=0}^{\infty} \binom{d}{n} \frac{(-c)^n}{n+5}$$

$$= 2\,M\,R^2\, b \sum_{n=0}^{\infty} \binom{d}{n} \frac{(-c)^n}{n+5}$$

in this case is trivial to see that when $d = 0$, this reduces to the moment of inertia of a uniform sphere. Now, let's put the parameters to get the answer

In[21]:=
```
Ibinom = 2 M[R] R^2 b Sum[QBinomial[d, n, 1] (-c)^n/(n+5), {n, 0, Infinity}];
Print[Style["I_BINOM = ", 20, FontFamily → "Utopia"],
 Style[Ibinom, 20, FontFamily → "Utopia"],
 Style[" kg m^2", 20, Italic, FontFamily → "Utopia"]]
```

$$I_{\mathrm{BINOM}} = 7.72448 \times 10^{37}\ kg\ m^2$$

Numerically

Just to check, let's make a last computation of the moment of inertia, directly numerically:

$$I_{\mathrm{Numeric}} = 7.72448 \times 10^{37}\ kg\ m^2$$