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The relativistic gravity train

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The Relativistic Gravity Train

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ABSTRACT

The gravity train that takes 42.2 minutes from any point A to any other point B that is connected by a straight-line tunnel through Earth has captured the imagination more than most other applications in calculus or introductory physics courses. Brachystochron and, most recently, non-linear density solutions have been discussed. Here relativistic corrections are presented. It is discussed how the corrections affect the time to fall through earth, the sun, a white dwarf, a neutron star, and - the ultimate limit - the difference in time measured by a moving, a stationary and the fiducial observer at infinity if the density of the sphere approaches the density of a black hole. The relativistic gravity train can serve as a problem with approximate and exact analytic solutions and as numerical exercise in any introductory course on relativity.

I. INTRODUCTION

Ever since Cooper published *Through the Earth in Forty Minutes* [1] the idea of the gravity train caught the public imagination [2] beyond its role as a pedagogical example in an introductory physics course. Most recently, Klotz published *The gravity tunnel in a non-uniform Earth* [3] with an extensive literature review (going back to 1883! [4]) and a treatment of the brachystochron (minimal time) path and the use of the internal structure of earth as obtained from seismic data. With all this attention it's somewhat surprising that the relativistic case has not been discussed - that is until most recently [5] when an ingenious experiment has been proposed to measure the gravitational constant G employing the gravity train mechanism in deep space. For this experiment, relativistic effects have been estimated to order $1/c^2$ and were found to be negligible. The story of *two* trains falling through earth ("boomeranging through the Earth") has been used as pedagogical example by Wheeler to illustrate the action of curvature on nearby test masses. [6]

In this note we compare different methods of calculating the relativistic effects for the gravity train. In each case we compare the time the train needs for a full round trip to the period of a near-surface satellite which, in the classical case, is found to be the same. For the metrics considered it is shown that this synchronicity is preserved when relativistic effects are considered.

First, the gravitational effects are calculated in the weak field limit. [7,8,9] For earth the relativistic effects are found to be of the same order of magnitude as for a GPS satellite, but with *opposite* sign: the clock on the train ticks slower, the stationary observer ages faster than the train or satellite passenger.

Second, instead of using the Schwarzschild interior metric [10] a space-time metric for the gravitational field inside a nonrotating homogeneous spherical ball is used which has been proposed by Ellis [11] and which makes the analysis of geodesics and computations of travel times inside the ball relatively straightforward. [12,13]

Third, for a satellite skimming the surface, the Ellis solutions for the proper and coordinate time are compared to the Schwarzschild exterior solutions calculated for a satellite skimming the surface of the sphere.

Earth, sun, white dwarf, and neutron star serve as numerical applications. The differences in round trip time measurements by a train passenger and a stationary observer when accumulated per day range from -30 μs for earth (correction factor per period -3.48×10^{-10} compared to $+4.45 \times 10^{-10}$ for a GPS satellite) to -8543 sec for a neutron star. In the Ellis metric, when the radius of the spherical ball approaches the Schwarzschild radius R_S, the clock of the train passenger ticks slower and slower and approaches stand-still, whereas the stationary observer measures a finite roundtrip time of $2\pi R_S/c$ or, more precisely, $\frac{\pi R_S}{2c}$ for the time to reach the center of the sphere when the train's maximum velocity of c would be approached. In the Schwarzschild metric, the limiting values are reached for a spherical ball with $R \rightarrow \frac{3}{2} R_S$. For the fiducial observer at infinity in the Ellis metric, the measured trip time goes to infinity, the observer would never see the train return.

II. CLASSIC SOLUTION

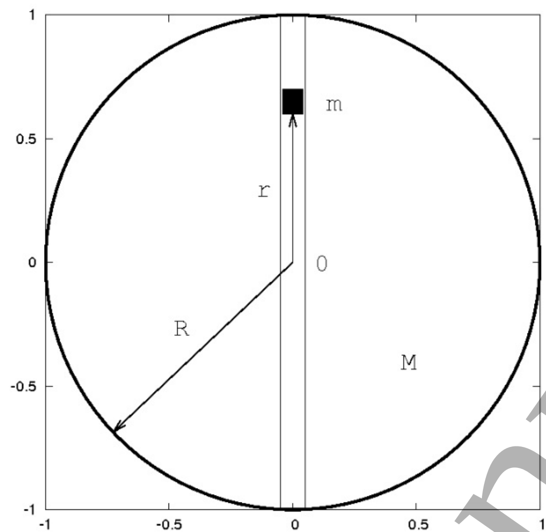


Fig. 1. Notation used to describe a mass m oscillating in a straight-line tunnel through the center of a uniform sphere with mass M and radius R. r(t=0) = R, v(t=0) = 0.

Assuming a uniform density ρ for a sphere with mass M and radius R, the non-relativistic equation of motion for the gravity train depicted as mass m in Fig.1 is simply

$$d^2r/dt^2 = - k\, r \quad \text{with} \quad k = GM/R^3 = 4/3\, \pi\, G\, \rho = v_m^2 / R^2 \quad . \tag{1}$$

G is the gravitational constant (6.67×10^{-11} $m^3\, kg^{-1}\, s^{-2}$) and $v_m = \sqrt{GM/R}$ the maximum velocity reached by the train at the center of the sphere at r = 0 and $v = \pm\sqrt{k(R^2 - r^2)}$. The classical period T_0 is given by $2\pi/\sqrt{k} = 2\pi\, (R^3/GM)^{1/2}$ and, as can be seen from Eq. (1), it depends only on the density of the sphere. Using M = 5.972×10^{24} kg and R = 6371 km for earth, one obtains for v_m 7.9 km/s or 28,440 km/h. The half-period of oscillation $1/2T_0$ is 42.2 min and, as can be shown easily, is the same for any straight-line tunnel through earth. T_0 is also the period of a near-earth satellite with orbital velocity $v = v_m$. This synchronicity is not a coincidence but a consequence of the basic relationship between uniform circular motion and simple harmonic motion: the projection of the uniform circular motion on the diameter of the circle (orbit of the satellite) undergoes the simple harmonic motion found for the gravity train.

III. RELATIVISTIC CORRECTIONS

Einstein's field equations of general relativity are ten nonlinear partial differential equations.To find solutions for certain physical situations *approximations* are often introduced. For example, for weak gravitational fields the nonlinear contributions from the curved space-time metric are ignored and the metric is written as sum of the flat Minkowski metric and a small perturbation due to a weak gravitational potential. The solution of the linearized Einstein field equations describing the geometry generated by such a weak field with a Newtonian potential Φ leads to the following line element or, with $ds = c\, d\tau$, proper time form [7]

$$ds^2 = c^2\, d\tau^2 = \left(1 + \frac{2\,\Phi}{c^2}\right)(cdt)^2 - \left(1 - \frac{2\,\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2) \qquad (2)$$

One can also derive special solutions under *simplifying assumptions*: for a spherical mass in vacuum, Schwarzschild derived already in 1916 an external solution [14] to Einstein's field equations wich exhibits a coordinate singularity for $r = R_S = 2GM/c^2 = 2m$ (the "Schwarzschild radius"). The metric takes the proper time form

$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{c^2}\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - \frac{1}{c^2}\, r^2 d\Omega^2 \qquad (3)$$

Later in the same year, he published an "interior" solution [10] for a sphere of incompressible fluid which matches the external solution at the surface r = R:

$$d\tau^2 = [\ \tfrac{3}{2}\,(1- \tfrac{2m}{R}\,)^{1/2} - \tfrac{1}{2}\,(1 - \tfrac{2m}{R}\tfrac{r^2}{R^2}\,)^{1/2}\ ]^{\ 2}\ dt^2\ - \tfrac{1}{c^2}\,(1 - \tfrac{2m}{R}\tfrac{r^2}{R^2}\,)^{-1}\ dr^2 - \tfrac{1}{c^2}\ r^2 d\Omega^2 \qquad (4)$$

A variety of choices for both the time and the spatial coordinates has been made. For example the Painlevé - Gullstrand coordinates [15, 16] with a new time coordinate $t_r = t - f(r)$ avoid the coordinate singularity at the Schwarzschild radius. They have been employed to describe a gravitationally collapsing body.[17, 18] Ellis adopts a similar approach to describe gravity inside a nonrotating homogeneous spherical body with the following metric [11]

$$d\tau^2 = [1\text{-}f^{\,2}(\rho)]dt^2 - \tfrac{1}{c^2}\,[1\text{-}f^{\,2}(\rho)]^{-1}d\rho^2 - \tfrac{1}{c^2}\,r^2(\rho)d\Omega^2 \qquad (5)$$

He finds a solution in closed form without singularities that is matched as smoothly as possible at the surface to the Schwarzschild exterior solution.[12, 13] This allows him to derive trajectories of particles travelling inside the sphere. In the following it is attempted to explain some basic properties of the variables used in the calculation but the main emphasis of this paper is simply to use the weak field limit and the Ellis solution and calculate the relativistic effects for the gravity train in these metrics.

III. A. Weak field limit

In the weak field limit the space-time metric has the approximate form given by equ. (2). τ is the proper time of the moving observer, the coordinate time t is the time of a clock at rest far away from the massive body's gravity (stationary observer at infinity). For the problem at hand it is advantageous to define a coordinate time t' kept by a clock *at the surface* of the sphere that is related to t by t = (1- Φ(R)/c^2) t' where Φ(R) is the gravitational potential at the surface of the body with radius R.[8, 9] One then obtains (always neglecting terms smaller than 1/c^2)

$$d\tau^2 = \left(1 + \frac{2\,\Phi}{c^2}\right)\left(1 - \frac{2\,\Phi(R)}{c^2} + O\left[\frac{1}{c^4}\right]\right)(dt')^2 - \frac{1}{c^2}\left(1 - \frac{2\,\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2) \qquad (6)$$

With ΔΦ = Φ(r) - Φ(R) and $v^2 = (dx^2 + dy^2 + dz^2) / (dt')^2$ one obtains

$$d\tau^2 = \left[1 + \frac{2\,\Delta\Phi}{c^2} - \left(1 - \frac{2\,\Phi}{c^2}\right)\frac{v^2}{c^2}\right] dt'^2 \quad , \qquad (7)$$

or, again neglecting corrections of order smaller than 1/c^2 ,

$$d\tau \approx \left(1 + \frac{\Delta\Phi}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right) dt' \quad . \qquad (8)$$

v is the velocity measured by a stationary observer at the surface of the sphere, Φ is the gravitational potential of a uniform sphere with radius R and mass M

$$\Phi(r) = \begin{cases} -\frac{GM}{r}, & r \geq R \\ -GM\frac{3R^2 - r^2}{2R^3}, & r \leq R \end{cases} \quad , \qquad (9)$$

Equ. (8) is the basic equation to calculate the relativistic effects that need to be taken into account for a GPS satellite [8, 9] : the difference in gravitational potential between the surface of the earth and the satellite orbit and the time dilation known from special relativity. For a GPS satellite at 20,184 km above earth $\frac{\Delta\Phi}{c^2}$ = [Φ(r = 26,562 km) - Φ(R = 6,371 km)]/c^2 = 5.291×10^{-10} (if other effects like non-spherical mass distribution and earth rotation are neglected) and $-\frac{1}{2}\frac{v^2}{c^2} = -0.835\times10^{-10}$. The gravitational field effect is larger than the special relativity effect, the GPS satellite clock is ticking faster and would be off by 38.5 μs after one day if not corrected.

Using equ.s (8) and (9), one obtains with $\Delta \Phi/c^2 = - GM(R^2 - r^2)/(2c^2R^3)$ and $v=\sqrt{k(R^2-r^2)} = \sqrt{GM/R^3}\ \sqrt{(R^2-r^2)}$ for the period τ_{train}, i.e., the proper time measured by a passenger moving on the gravity train (T is the time for a round trip from R to -R and back, $v_m = \sqrt{GM/R}$ and $GM=c^2R_S/2$)

$$\tau_{train} = \int^T \left(1 + \frac{\Delta\Phi}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right) dt' = \int^{Path} \left(1 + \frac{\Delta\Phi}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right) \frac{dt'}{dr} dr =$$

$$= 2\int_{-R}^{R} \left(1 + \frac{\Delta\Phi}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right) \frac{1}{v} dr =$$

$$= 2\int_{-R}^{R} \left(1 - \frac{1}{2c^2}\frac{GM}{R^3}(R^2-r^2) - \frac{1}{2c^2}\frac{GM}{R^3}(R^2-r^2)\right)\left(\sqrt{GM/R^3}\ \sqrt{(R^2-r^2)}\right)^{-1} dr =$$

$$= 2\sqrt{R^3/GM}\left(\int_{-R}^{R}\frac{1}{\sqrt{R^2-r^2}} dr - \frac{1}{c^2}\frac{GM}{R^3}\int_{-R}^{R}\sqrt{(R^2-r^2)}\, dr\right) =$$

$$= 2\sqrt{R^3/GM}\left(\pi - \frac{1}{c^2}\frac{GM}{R^3}\frac{\pi R^2}{2}\right) = 2\pi\sqrt{R^3/GM}\left(1 - \frac{1}{2}\frac{v_m^2}{c^2}\right) =$$

$$= T_0\left(1 - \frac{1}{2}\frac{v_m^2}{c^2}\right) = T_0\left(1 - \frac{1}{4}\frac{R_S}{R}\right). \qquad (10)$$

The gravitational field effect is exactly the same as the special relativity effect: the combined correction leads to a slowing down of the clock carried on the train. The roundtrip time τ_{train} measured by the moving observer is shorter than the round trip time T_0 measured by the stationary observer on the surface. For earth, the correction factor per period is $- 3.48\times10^{-10}$. In one day the train passengers would stay younger by 30.0 μs.

The physical interpretation of equ. (10) is that, in the weak field limit, one recovers simple harmonic motion for the gravity train, but with a reduced period. Since the harmonic motion can be seen as projection of a uniform circular motion of a surface-skimming satellite, one can expect that the synchronicity from the Newtonian case should be preserved. This is demonstrated in the next paragraph.

For a satellite skimming the surface of the earth $\Delta\Phi = 0$ and v = constant = $v_m = \sqrt{GM/R}$. One obtains

$$\tau_{sat} = \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) T_{sat} = \left(1 - \frac{1}{2}\frac{v_m^2}{c^2}\right)\frac{2\pi R}{v_m} = 2\pi\sqrt{R^3/GM}\left(1 - \frac{1}{2}\frac{v_m^2}{c^2}\right) =$$

$$= T_0\left(1 - \frac{1}{2}\frac{v_m^2}{c^2}\right) = T_0\left(1 - \frac{1}{4}\frac{R_S}{R}\right), \qquad (11)$$

i.e., the same result as obtained for the gravity train in equ. (10).

III. B. Geodesics inside a nonrotating homogeneous sphere

It is difficult to summarize in succinct form the Ellis derivation. A minimum number of variables used for the calculation of the relativistic corrections needs to be introduced. Ellis[11] replaced Schwarzschild's interior metric [10] with the space-time metric given in equ. (4) that allows computations of travel times inside a nonrotating homogeneous sphere with radius R (provided with a tunnel for the test object). From gravitational field equations Ellis derives relations between the radial coordinate ρ and the Schwarzschild coordinate r and f(ρ) in the metric in equ. (4): [12, 13]

$$1 - f^2(\rho) = \frac{1}{\lambda^2}\left(1 + \frac{\lambda m}{R^3} r^2(\rho)\right) \qquad \text{with } r(\rho) = \lambda(\rho - \rho_0). \qquad (12)$$

From the requirement that at the surface of the sphere r(ρ) and ρ should coincide (r(ρ) = R when ρ = R) follows $\rho_0 = \frac{1}{\lambda}(\lambda - 1)R$. The parameter λ is determined by matching $f^2(R)$ of the interior solution to 2m/R of the exterior solution at the surface ρ = r = R

$$\lambda = \frac{m+\sqrt{m^2+4R(R-2m)}}{2(R-2m)} . \tag{13}$$

$\lambda \to 1$ for $R \gg R_S$ and $\lambda \to \infty$ for $R \to R_S$ ($m = ½ R_S$). In addition there are three constants of motion that exist for every affinely parametrized geodesic path in a sphere with fixed longitude φ (equ.s (7), (8) and (10) in ref. 13):

$$h := -\frac{1}{c} r^2 \dot{\theta} , \tag{14a}$$

$$\kappa := (1 - f^2)\, \dot{t} , \tag{14b}$$

$$\varepsilon := (1 - f^2)\, \dot{t}^2 - \frac{1}{c^2}\frac{1}{1-f^2}\, \dot{\rho}^2 - \frac{1}{c^2} r^2\dot{\theta}^2 . \tag{14c}$$

· denotes derivative with respect to the proper time τ and ε = 1,0, -1 for time-like, light-like and space-like geodesics.

From these equations and equ. (12) the following equation is derived for time-like geodesics (equ. (32) in ref. 13)

$$\dot{\rho}^2 = \frac{c^2 m}{\lambda R^3 r^2}(r^2 - a^2)(b^2 - r^2) \tag{15}$$

where $a = \sqrt{\alpha - \beta}$, $b = \sqrt{\alpha + \beta}$, and $\alpha = \frac{(\lambda^2 \kappa^2 - 1)R^3 - \lambda m h^2}{2\lambda m}$ and $\beta = (\alpha^2 - R^3h^2/\lambda m)^{1/2}$.

The equation for a gravity train is obtained for h = 0, because then a = 0 and equ. (14) describes a particle whose position oscillates along a diameter of the sphere between extremes at r = b if b ≤ R. With $\dot{\rho} = \dot{r} / \lambda$ and b = R at the surface of the sphere one obtains

$$\dot{r}^2 = \lambda\, GM/R^3\, (R^2 - r^2)\,. \tag{16}$$

That is, in the Ellis metric the equation for simple harmonic motion is recovered and one can therefore expect that synchronicity between the periods of the gravity train and near-surface satellite is preserved.

The proper time τ elapsed for a round trip on the clock of an observer on the gravity train is

$$\tau = 2\int_{\tau_{-R}}^{\tau_R} d\tau = 2\int_{-R}^{R} \frac{d\tau}{dr}\,\mathrm{dr} = 2\int_{-R}^{R} \frac{1}{\dot{r}}\,\mathrm{dr} =$$

$$= \frac{2}{\sqrt{\lambda}}\sqrt{R^3/GM}\int_{-R}^{R}\frac{1}{\sqrt{R^2 - r^2}}\,\mathrm{dr} = \frac{2\pi}{\sqrt{\lambda}}\sqrt{R^3/GM} = \frac{1}{\sqrt{\lambda}}\,T_0\,. \tag{17}$$

The proper time T measured for the round trip by a stationary observer on the surface of the sphere is

$$T = 2\,(1 - f^2(R))^{1/2}\int_{t_{-R}}^{t_R} dt = 2\,\frac{1}{\lambda}\sqrt{1 + \lambda m/R}\int_{\tau_{-R}}^{\tau_R}\frac{dt}{d\tau}\,d\tau \tag{18}$$

From the equations of motions one obtains (with $\dot{\rho} = 0$ and $\dot{\theta} = 0$ for a stationary observer)

$$\kappa = (1 - f^2(R))^{1/2} = \frac{1}{\lambda}\sqrt{1 + \lambda m/R} \quad \text{and}$$

$$\dot{t} = \kappa / (1 - f^2(r)) = \lambda\sqrt{1 + \lambda m/R}\,\left(1 + \frac{\lambda m}{R^3}\,r^2\right)^{-1} \tag{19}$$

and

$$\int_{t_{-R}}^{t_R} dt = \int_{\tau_{-R}}^{\tau_R}\frac{dt}{d\tau}\,d\tau = \int_{-R}^{R}\frac{dt}{d\tau}\frac{d\tau}{dr}\,\mathrm{dr} = \int_{-R}^{R}\dot{t}\,\frac{1}{\dot{r}}\,\mathrm{dr} =$$

$$= \lambda \sqrt{1 + \lambda m/R}\, \frac{1}{\sqrt{\lambda}}\, \sqrt{R^3/GM} \quad \int_{-R}^{R} \frac{1}{\sqrt{R^2 - r^2}} \left(1 + \frac{\lambda m}{R^3} r^2\right)^{-1} dr =$$

$$= \sqrt{\lambda}\, \sqrt{1 + \lambda m/R}\, \sqrt{R^3/GM}\, \frac{\pi}{\sqrt{1+\lambda m/R}} = \sqrt{\lambda}\, \sqrt{R^3/GM}\, \pi \qquad (20)$$

Therefore

$$T = 2 \frac{1}{\lambda} \sqrt{1 + \lambda m/R}\, \sqrt{\lambda}\, \sqrt{R^3/GM}\, \pi = \frac{1}{\sqrt{\lambda}} \sqrt{1 + \lambda m/R}\, T_0 . \qquad (21)$$

Comparing the periods measured by the stationary and moving observer

$$T/\tau = \sqrt{1 + \lambda m/R} = \sqrt{1 + \frac{1}{2} \lambda R_S/R} > 1 \qquad (22)$$

we find again that the clock carried on the train slows down.

If h ≠ 0 Ellis derives closed-form solutions for the integrals over proper and coordinate times of timelike geodesics. As illustrative example he discusses a satellite skimming the surface of the earth ("Newton's cannon ball"). In this case one has $a = b = R$, $r(\rho) = \rho = R$ and $h = \sqrt{\lambda m R} = \frac{1}{c} \sqrt{\lambda G M R}$. One obtains (equ. (40) and (46) in ref. 13 with $\delta = \pi$)

$$\tau = \frac{2R^2}{ch}\pi = \frac{1}{\sqrt{\lambda}} T_0 \qquad , \qquad (23)$$

$$T = \frac{1}{\lambda} \sqrt{1 + \lambda m/R}\, \frac{2\lambda R^2}{ch}\pi = \sqrt{1 + \lambda m/R}\, \tau = \sqrt{1 + \lambda m/R}\, \frac{1}{\sqrt{\lambda}} T_0 \qquad (24)$$

$$T/\tau = \sqrt{1 + \lambda m/R} = \sqrt{1 + \frac{1}{2} \lambda R_S/R} > 1, \qquad (25)$$

i.e., the same results as obtained before for a full round trip of the gravity train (see equ. 22). As in the weak field limit, the synchronicity between the train and the satellite stays preserved because, in the Ellis metric, the train's motion is a simple harmonic motion.

For completeness, the roundtrip time measured by an inertial observer free falling from rest at $\rho=\infty$ ("fiducial observer at infinity") becomes (equ. 51 of ref. 13 with a=R)

$$\overline{T} = t_B - t_A = \sqrt{\lambda}\, T_0 \quad . \tag{26}$$

III. C. Comparison of "Ellis metric" results with Schwarzschild solutions

Having established the equivalence of the solution for the gravity train and the satellite orbiting along the surface we can use the existing results for circular orbits in the external Schwarzschild metric with $r \rightarrow R$, the surface of the sphere. One obtains [19] for τ, the period as measured by an astronaut inside a satellite in a circular orbit with radius r

$$\tau = 2\pi r^2/c\, (\tfrac{1}{2} R_S\, r)^{-1/2}\, (1- \tfrac{3}{2} R_S/r)^{1/2} = 2\pi\, (r^3/GM)^{1/2}\, (1- \tfrac{3}{2} R_S/r)^{1/2} = (1- \tfrac{3}{2} R_S/r)^{1/2}\, T_0\,, \tag{27}$$

and for T, the time measured by a stationary astronaut outside the satellite at fixed r

$$T = 2\pi r^2/c\, (\tfrac{1}{2} R_S\, r)^{-1/2}\, (1- R_S/r)^{1/2} = (1- R_S/r)^{1/2}\, T_0\,, \tag{28}$$

and therefore, for $r \rightarrow R$,

$$T/\tau = \sqrt{\frac{1-\frac{R_S}{R}}{1-\frac{3}{2}\frac{R_S}{R}}} > 1\,. \tag{29}$$

Of course, conclusions of a comparison between Ellis metric and Schwarzschild metric results apply strictly only to the satellite motion, transference to the gravity train remains speculative until an exact solution is found for the train in the interior Schwarzschild metric.

IV. APPLICATIONS AND CONCLUDING DISCUSSION

First, for $R >> R_S$, we compare the results for $D = T/\tau$ obtained in the Ellis metric (equ. 25) and Schwarzschild metric (equ. 29) to the weak field solution D_0 (equ. (10)):

$$D_0 = T_0/\tau = \left(1 - \frac{1}{4}\frac{R_S}{R}\right)^{-1} \approx 1 + \frac{1}{4}\frac{R_S}{R} \quad , \tag{30}$$

$$D_{Ellis} = \sqrt{1 + \frac{1}{2}\lambda R_S/R} \approx 1 + \frac{1}{4}\frac{R_S}{R} \qquad (\lambda \to 1 \text{ for } R >> R_S) , \tag{31}$$

$$D_{Schwarz} = = \sqrt{\frac{1 - \frac{R_S}{R}}{1 - \frac{3}{2}\frac{R_S}{R}}} = \left[1 - \frac{1}{2}\frac{R_S}{R}\left(1 - \frac{R_S}{R}\right)^{-1}\right]^{-1/2} \approx 1 + \frac{1}{4}\frac{R_S}{R} \quad , \tag{32}$$

i.e., both converge to the same weak field limit.

The convergence of the numerical values for the relativistic corrections for earth, sun, and white dwarf for $R >> R_S$ is also demonstrated in Table I. The respective R_S/R values are 1.4×10^{-9}, 4.2×10^{-6}, and 2.1×10^{-4} . For a neutron star, $R_S/R = 3.5\times10^{-1}$ and the weak field approximation is no longer valid.

The graphs in Fig.2 (a) for $10 \leq x = R/R_S \leq 100$ and in Fig. 2(b) for $2.8 \leq x = R/R_S \leq 10$ illustrate how the values begin to differ when x approaches 2.8, the value for a neutron star. Fig. 1(c) shows the asymptotic limit for $D_{Schwarz}$ for $x \to 3/2$ and D_{Ellis} for $x \to 1$, i.e., when $R \to 3/2\, R_S$ and R_S, respectively. The corresponding asymptotic limits for the proper time τ and T, the time measured by the stationary observer, are analyzed in more detail for both metrics in Fig. 3.

Table I. Numerical values for the relativistic corrections for earth, sun, white dwarf, and neutron star. The respective R_S/R values are 1.4×10^{-9}, 4.2×10^{-6}, 2.1×10^{-4}, and 3.5×10^{-1}. For the neutron star, the weak field approximation is no longer valid.

	Earth	Sun	White Dwarf	Neutron Star
R [m]	6.371×10^6	6.957×10^8	6.957×10^6	11650.
v_m [m/s]	7.90968×10^3	4.36819×10^5	3.088781×10^6	1.26303×10^8
T_0 [s]	5060.907508432309	10006.90502145614	14.15190079872270	0.0005795509778777
τ_{weak} [s]	5060.907506670839	10006.89439881368	14.15114966447092	
T_{Ellis} [s]	5060.907507551575	10006.89971012504	14.15152519670854	0.0005488200795561
τ_{Ellis} [s]	5060.907505790105	10006.88908747131	14.15077402258411	0.0004915144667232
$T_{Schwarz}$ [s]	5060.907504909371	10006.88377614867	14.1503984504754	0.0004654517359592
$\tau_{Schwarz}$ [s]	5060.907503147901	10006.87315347801	14.14964721653448	0.0003962679772347
$\overline{T}_{Ellis}$ [s]	5060.907511074514	10006.92095546635	14.15302766458250	0.0006833559512463
Corrections/day [s]				
weak field	−0.0000300718848603	−0.09171630056404008	−4.585815027748674	
Ellis	−0.0000300718693334	−0.09171639792072339	−4.586058457725915	−8543.174177518998
Schwarzschild	−0.0000300718848603	−0.09171654396360095	−4.586423648750452	−10313.97923904066
D_0	1.000000000348054	1.0000010615312565	1.000053076562821	
D_{Ellis}	1.000000000348054	1.0000010615340735	1.000053083606821	1.11658988028350
$D_{Schwarz}$	1.000000000348054	1.0000010615374544	1.000053092061551	1.17458831573338

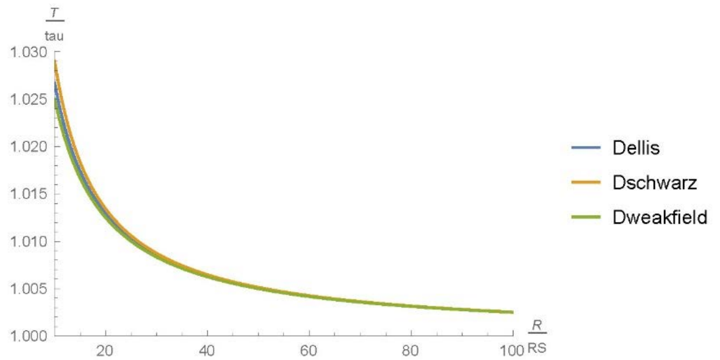
Both show the same qualitative behavior but in the Ellis metric, the singularity is reached for $R \rightarrow R_S$ ($\lambda \rightarrow \infty$, $\tau \rightarrow 0$, $T \rightarrow \sqrt{1/2}\, T_0 = 0.707\, T_0$; see. equ.s (24) and (25)) whereas in the Schwarzschild metric, the singularity is reached for $R \rightarrow 3/2\, R_S$ ($\tau \rightarrow 0$, $T \rightarrow \sqrt{1/3}\, T_0 = 0.577\, T_0$; see equ.s (28) and (29)). The explicit limiting values for T are

$$T_{Ellis}\,(R \rightarrow R_S) = 1/\sqrt{2}\; 2\pi R_S\, /\, (GM/R_S)^{1/2} = 2\pi R_S\, /c\, , \tag{34}$$

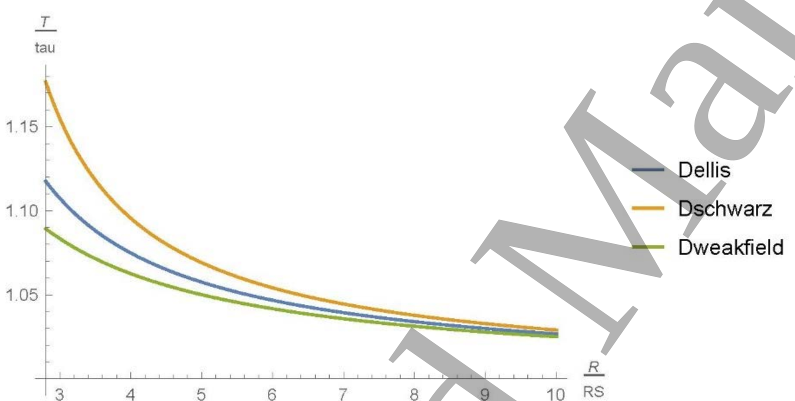
$$T_{Schwarz}(R \rightarrow \tfrac{3}{2} R_S) = 1/\sqrt{3}\; 2\pi\, (\tfrac{3}{2} R_S)\, /\, (GM/(\tfrac{3}{2}\, R_S)\,)^{1/2} = 2\pi(\tfrac{3}{2}\, R_S)\, /c\, . \tag{35}$$

In both cases, the limiting velocity is the speed of light which would correspond to the orbital velocity of the satellite or the maximum speed of the gravity train reached at the center of the sphere. The clock of the passengers on the satellite (train) ticks slower and slower and approaches stand-still. On the other hand, the stationary observer measures a finite period of $2\pi R_S$ /c on the surface of a sphere of radius R_S in the Ellis metric and of $3\pi R_S$ /c on a sphere of radius 3/2 R_S in the Schwarzschild metric.

(a)



(b)



(c)

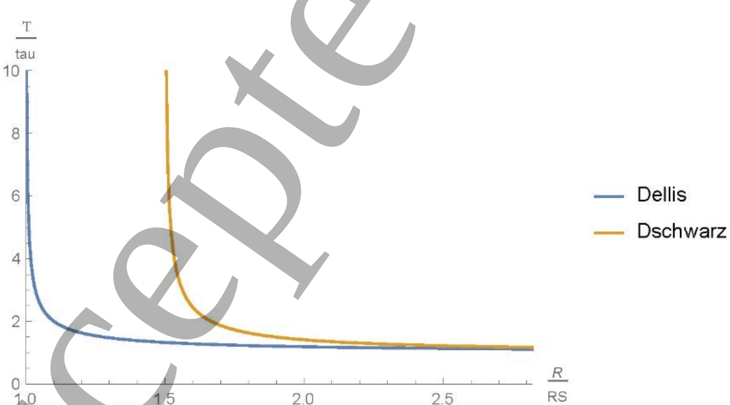
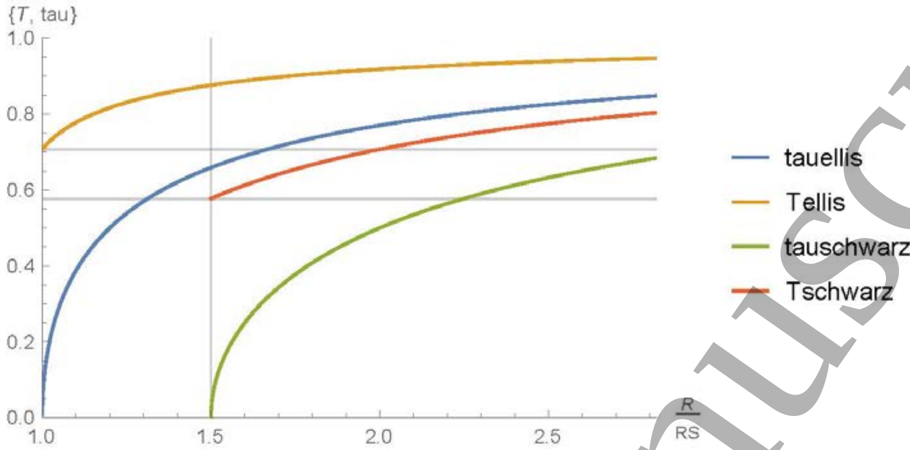


Figure 2. Results for D=T/τ obtained in the weak field [equ. (30)], Ellis [equ.(31)], and Schwarzschild metric [equ. (32)] as function of $x = R/R_S$.

(a)



(b)

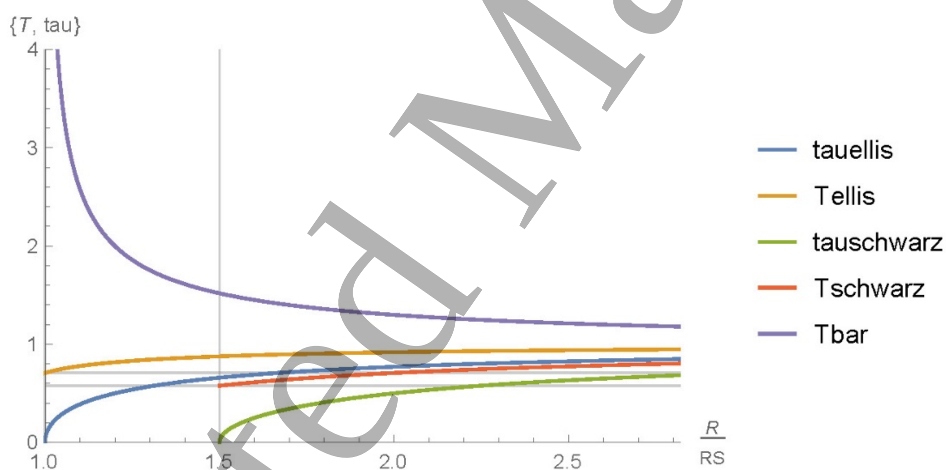


Figure 3. (a) Proper time τ/T_0 and coordinate time T/T_0 of the stationary observer for $x = R/R_S$ approaching the limiting value $R = \frac{3}{2} R_S$ (Schwarzschild) and $R = R_S$ (Ellis). (b) Same as (a) but also showing $\overline{T}/T_0$ as measured by the fiducial observer at ∞ in the Ellis metric.

As for the roundtrip time measured by the fiducial observer at infinity in the Ellis metric, $\overline{T}=\sqrt{\lambda}\,T_0$ approaches ∞ for $R\rightarrow R_S$ ($\lambda\rightarrow\infty$), i.e., the fiducial observer would never see the train or satellite return (see Fig. 3(b)).

The difference in the limiting values for D obtained from the solution of the interior Ellis metric and the solution of the exterior Schwarzschild metric can be explained by comparing the radial equation of motion in both metrics at the boundary (surface of the sphere): one finds [12] continuity of the gravitational component of the radial acceleration but discontinuity of the centrifugal component inasmuch as $R/\lambda \neq R\text{-}3/2\, R_S$ (comparing equ.s (40) and (41) in ref. 12).

In summary, relativistic corrections have been presented for the gravity train. For earth, they are of the same order of magnitude as those for a GPS satellite but with opposite sign. The stationary observer on earth is aging by 30 μs per day more than passengers continuously riding the train. In the Ellis metric, which allows for a straightforward calculation of travel times along geodesics through the interior of a homogeneous sphere, the proper time, the coordinate time, and the time measured by a fiducial observer at infinity are calculated for both the train and a satellite skimming the surface of the sphere. It is found that the synchronicity of the periods for the train and the satellite derived in Newtonian mechanics is preserved when relativistic effects are taken into account in the weak field limit and in the Ellis metric since one recovers simple harmonic motion for the gravity train albeit with reduced period. For the satellite skimming the surface one can compare the Ellis interior solution to the exterior Schwarzschild solution for $r \rightarrow R$ at the surface of the sphere. Transference of the Schwarzschild solution to the train problem due to synchronicity is speculative since one does not have the solution for the gravity train in the interior Schwarzschild metric. Qualitatively, both solutions show the same behavior but with quantitatively different limiting values. The singularity is reached for $R\rightarrow R_S$ (Ellis) and for R

$\rightarrow 3/2\ R_S$ (Schwarzschild) when the velocity of the train or satellite approaches the speed of light and the time for passengers on the train or satellite would stop flowing, the stationary observer on the surface measured a finite round trip time approaching $2\pi R_S$ /c (Ellis) and $3\pi R_S$ /c (Schwarzschild), and the fiducial observer never saw the train return.

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