Free fall through the rotating and inhomogeneous Earth

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Free fall through the rotating and inhomogeneous Earth

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Analytical solutions of the motion through the Earth and suitably shaped gravity tunnels are derived so that inertial forces caused by the Earth's rotation are eliminated and the journey takes place in complete weightlessness. The inhomogeneity of the Earth is represented by the one-dimensional Preliminary Reference Earth Model (PREM). A piecewise linear approximation of the model is used to obtain an analytical solution that corresponds very well to the solution obtained numerically for the PREM.
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I. INTRODUCTION

The thought experiment of a gravity tunnel dates back to the 17th century. In his *Dialogue Concerning the two Chief Systems*, published in 1632, Galileo Galilei lets his protagonist Salvati argue, "… if the terrestrial globe were perforated through the center, a cannon ball descending through the hole would have acquired at the center such an impetus from its speed that it would pass beyond the center and be driven upward through as much space as it had fallen,…"[1,2] Nowadays the analytical solution of a gravity tunnel for a homogeneous gravitating sphere is well known (brachistochrone problem)[3] and is left as an exercise for undergraduates.[4] Actually, calculations are also being carried out on how to design accelerated journeys on Earth[5] or in space[6] so that they are as comfortable as possible. And there are concepts for traffic routes in the USA, Europe and India, consisting of largely evacuated tubes through which transport capsules can move at very high speeds (Hyperloop).[7] The first test tracks are under construction.

Even if it may still be a long way to the realization of gravity tunnels or these will never be realized, it is still worth looking at realistic models of gravity tunnels, which take into account the inhomogeneous density of the Earth, inertial forces caused by the rotation of the Earth and friction. On straight paths or on a brachistochrone curve through the Earth, inertial forces cause accelerations perpendicular to the direction of motion, which can be up to 0.15 g at the Earth's center if the path is perpendicular to the Earth's axis of rotation.[8] In order to eliminate resulting frictional forces, the tunnel must describe the path of the free fall through the rotating Earth, otherwise the oscillatory motion of the gravity train is damped by friction in such a way that the other end of the tunnel is never reached. Taking into account the rotation of the Earth, however, paths can be modeled in which inertial forces do not occur any further. Corresponding analytical calculations have been carried out, e.g., for the Earth modeled as a homogeneous sphere[9,10] or as a flattened, homogeneous spheroid.[11] For more realistic models of the Earth, it is necessary to consider its inhomogeneous density distribution, which leads to different results. Calculations in this respect cannot only be performed numerically,[12,13] but can also be very well analytically approximated.[14]

Based on the realistic and widely used Preliminary Reference Earth Model (PREM),[15] the gravity profile of the earth is approximated piecewise linearly as described in Ref. 14. The parameters of this approximation are adjusted to obtain the best possible analytical approximations in the form of trigonometric and hyperbolic functions for the traversal time and the path of a free fall in the geocentric non-rotating Earth system. The analytically described path is finally transformed into the rotating terrestrial frame in a common way, which leads to a simple expression for the target points of a corresponding tunnel depending on the start coordinates at the Earth's surface. The results obtained are compared with numerically obtained results for the PREM and analytical results for a constant density model and a constant gravity model.

II. THE MODEL

We assume completely evacuated tunnels, because in hypothetical gravity tunnels significantly higher velocities (9.9 km/s near the center of the Earth)[14] are reached than in partially evacuated tunnels (350 m/s for Hyperloop).[7] Since the Earth rotates at the angular velocity $\Omega = 7.292 \times 10^{-5}\ \mathrm{s}^{-1}$, there is a specific angular momentum $\vec{h} = \vec{r} \times \vec{v}$ at each point of the Earth's surface, which is in spherical coordinates

$$\vec{h} = R\mathbf{e}_r \times v\mathbf{e}_\varphi = \Omega R^2 \cos\theta \mathbf{e}_\theta, \tag{1}$$

where $R = 6371$ km is the radius of the Earth rotating at angular velocity Ω and $v = \Omega R \cos\theta$ is the local velocity at latitude θ, always assuming spherical symmetry. In a free fall there are no external forces other than the central force of gravity, so that energy and angular momentum are conserved quantities and the motion takes place in a plane perpendicular to the specific angular momentum $\vec{h}$, as shown in Fig. 1.

In the following equations, all variables are dimensionless. In order to recover natural units, dimensionless lengths must be multiplied by R, velocities by $(gR)^{1/2}$, and times by $(R/g)^{1/2}$, where $g = 9.815\ \mathrm{ms}^{-2}$ is the gravity at the Earth's surface, determined only by the mass and the radius of the Earth.[15] In dimensionless units the absolute value of the specific angular momentum is given by $h = h_0 \cos\theta$ with $h_0 = 5.875 \times 10^{-2}$ and $\Omega = h_0$.

The equations of motion for the central force problem result directly from the conservation of energy and angular momentum

$$\left|\frac{dr}{dt}\right| = \sqrt{2(E - U(r)) - \frac{h^2}{r^2}}, \tag{2a}$$

$$\frac{d\varphi}{dt} = \frac{h}{r^2}, \tag{2b}$$

where φ and r are the polar coordinates in the plane of motion, E is the total energy, and U is the potential energy

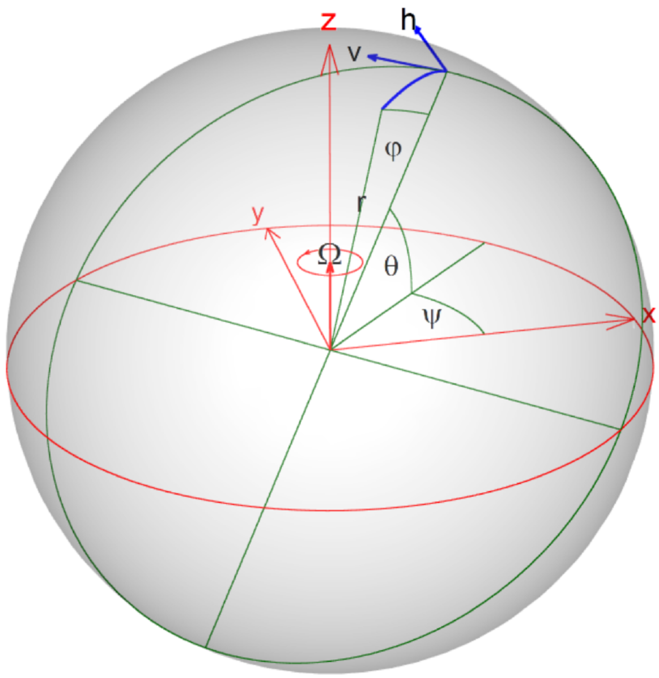

Fig. 1. The motion of a free fall takes place in the non-rotating plane perpendicular to the local specific angular momentum $\vec{h}$ at latitude θ and can be described by the polar coordinates r and φ.

per mass unit. We assume zero radial velocity at the surface and define a radial velocity profile $v(r)$, given the velocity at distance r in the case of $h = 0$

$$\frac{v^2(r)}{2} = \int_r^1 dr' g(r') = -U(r), \tag{3}$$

where $g(r)$ is the gravity profile of the Earth. Due to the angular velocity h and zero potential at the surface, the total energy per mass unit is given by $E = h^2/2$. With this Eq. (2a) can be rewritten as

$$\left|\frac{dr}{dt}\right| = \sqrt{v^2(r) - h^2\left(\frac{1}{r^2} - 1\right)}. \tag{4}$$

Dziewonski and Anderson[15] describe a model (PREM) in which the Earth is divided into regions within which the density $\varrho(r)$ varies smoothly. In the mathematical abstraction of this model, the density is approximated by low-order polynomials (up to order 3) in the normalized radius. The normalized gravity profile of the Earth can be calculated analytically via

$$g(r) = \frac{4\pi R^3}{Mr^2}\int_0^r dr' r'^2 \varrho(r'), \tag{5a}$$

$$\frac{M}{4\pi R^3} = \frac{\bar{\varrho}}{3} = \int_0^1 dr' r'^2 \varrho(r'). \tag{5b}$$

Here, $\bar{\varrho}$ is the mean density of the Earth with mass M so that g is normalized by $g(1) = 1$.

The velocity profile of Eq. (3) can be calculated by further analytical integration. In Ref. 14, it is shown that traversal times through the Earth with a density profile given by the PREM can be determined very accurately by a two-parameter piecewise linear approximation of the gravity profile

$$g_{\text{appr.}}(r) = \begin{cases} 1 + \mu(1 - r) & 1 \geq r \geq \hat{r}, \\ \dfrac{\hat{g}}{\hat{r}} r & \hat{r} \geq r \geq 0, \end{cases}$$
$$\text{where } \mu = \frac{\hat{g} - 1}{1 - \hat{r}}. \tag{6}$$

In this linear approximation, gravity at the surface increases linearly with the gradient $\mu > 0$ from its value 1 to a maximum value $\hat{g}$ at a distance $\hat{r}$ in the region of the outer core of the Earth and then drops linearly to zero as indicated in Fig. 2.

By means of the linear approximation of the gravity profile, we obtain the approximation of the velocity profile by Eq. (3)

$$v^2_{\text{appr.}}(r) = \begin{cases} (1 - r)\big(2 + \mu(1 - r)\big) & 1 \geq r \geq \hat{r}, \\ v_0^2 - \omega^2 r^2 & \hat{r} \geq r \geq 0, \end{cases}$$
$$\text{where } v_0^2 = 1 + \hat{g} - \hat{r},\ \omega^2 = \frac{\hat{g}}{\hat{r}}. \tag{7}$$

In the case of $h = 0$, v_0 is the maximum velocity at the center of the Earth. The potential U is represented by the velocity profile in Eq. (7) and it is known that the central force problem represented in Eqs. (2), (4), and (7) is integrable.[16] In the region $1 \geq r \geq \hat{r}$ the potential $A - Br - Cr^2$ is solvable by elliptic integrals, while the solution of the potential $D - Er^2$ in the region $\hat{r} \geq r \geq 0$ (with arbitrary positive constants A, B, C, D and E) is given by circular functions with frequency ω, which can easily be expressed in Cartesian coordinates.[4,17] By setting $\hat{g} = \hat{r} = 1$ one can identify the case of the spatial oscillator with frequency $\omega = 1$ and its solution with $x = \cos t$ and $y = h \sin t$ for corresponding initial conditions $(x(0) = 1, \dot{x}(0) = 0, y(0) = 0, \dot{y}(0) = h)$. It should be noted that $v(r)$ and its approximation is continuously differentiable.

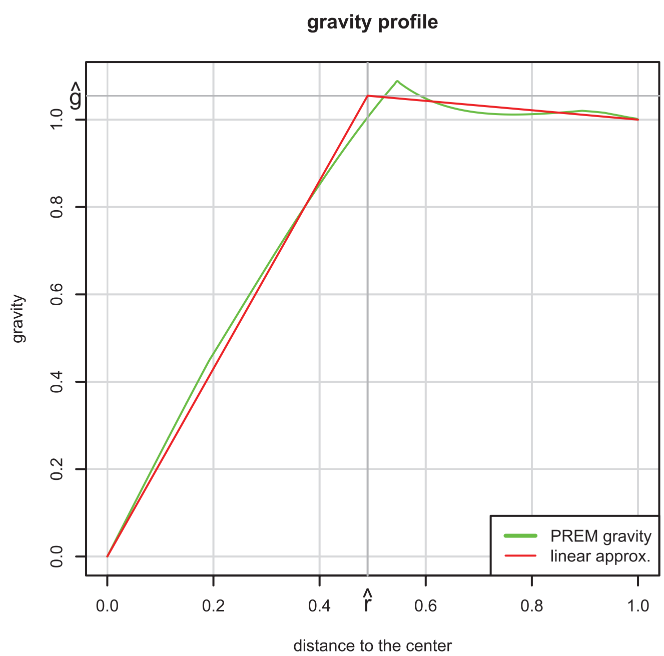

Fig. 2. The gravity profile of the Earth deduced from the PREM and a linear approximation with parameters $\hat{r}$ and $\hat{g}$. Distance to the center and gravity are measured in normalized units.

III. TRAVERSAL TIME FOR A FREE FALL

It is shown in Ref. 14 that time-integrals containing the velocity profile of the PREM are accurately approximated by the velocity profile given in Eq. (7) and choice of parameters $\hat{r}_t = 0.49$ and $\hat{g}_t = 1.055$. The subscript t indicates that the parameters belong to the approximation of traversal times and originate from the least squares matching of the inverse approximated velocity profile of Eq. (7) to the inverse PREM velocity profile. The matching of the inverse profiles takes into account that lower velocities at the surface make a greater contribution to the accumulated time, which is reflected by the fact that integrals containing the inverse velocity profile have an integrable square root singularity at $r = 1$, since $v(r) = \sqrt{2(1-r)}(1 + O(1-r))$ regardless of the value of μ.

We solve the time integral by inserting $v^2_{\text{appr.}}$ in Eq. (4) and taking into account that r varies between 1 and $r_{\min}$ at which the radial velocity is zero and changes its sign

$$t(r) = \int_{r \geq r_{\min}}^{1} \frac{dr}{\sqrt{v^2_{\text{appr.}}(r) - h^2\left(\frac{1}{r^2} - 1\right)}}. \tag{8}$$

The time $t_2(r) \geq t(r_{\min})$ for the second half of the trajectory after passing the center of the earth is obtained by $t_2(r) = 2t(r_{\min}) - t(r)$. The minimal distance $r_{\min}$ to the center of the Earth is calculated by setting in Eq. (4) $dr/dt = 0$ and $v^2(r) = v_0^2 - \omega^2 r^2$

$$r_{\min} = \sqrt{\frac{v_0^2 + h^2 - \sqrt{\left(v_0^2 + h^2\right)^2 - (2h\omega)^2}}{2\omega^2}} = \frac{h}{v_0}\left(1 + \frac{(\omega^2 - v_0^2)h^2}{2v_0^4} + O(h^4)\right). \tag{9}$$

The minimal distance to the center of the Earth is h/v_0 with a relative error less than 10^{-3}. According to the approximation of the velocity profile, the computation of the integral of Eq. (8) has to be split into the parts for the upper half $t_u(r)$ with $1 \geq r \geq \hat{r}$ and the lower half $t_l(r)$ with $\hat{r} \geq r \geq r_{\min}$. As mentioned in Sec. II, the solution for the upper range can be expressed in terms of elliptic integrals, but they have a complex analytical structure due to the non-trivial zeros of the square root of the integrand. However, since we are only interested in values for small h, we can expand the integrand in h so that we only have to deal with integrands containing inverse square roots of second degree polynomials whose solution is given by trigonometric or hyperbolic functions. Thus we get for the upper half $1 \geq r \geq \hat{r}$

$$\begin{aligned} t_u(r) &= \frac{2}{\sqrt{\mu}}\sinh^{-1}\sqrt{\frac{\mu}{2}(1-r)} + h^2 f(r) + O(h^4), \\ f(r) &= \frac{\left(2 + \mu(1+r) + 2\mu^2 r\right)\sqrt{1-r}}{2(2+\mu)^2 r\sqrt{2 + \mu(1-r)}} \\ &\quad + \frac{3(1+\mu)}{(2+\mu)^{5/2}}\sinh^{-1}\sqrt{\frac{(2+\mu)(1-r)}{2r}}. \end{aligned} \tag{10}$$

The function $f(r)$ looks complicated but is bounded between 0 and 1 with its maximum value $f(\hat{r}) = 0.72$.

For the lower half $\hat{r} \geq r \geq r_{\min}$, the motion is driven by a harmonic oscillator potential. Hence, we know that the solution is part of a closed ellipse with semi-minor axes $b = r_{\min}$ and semi-major axis $a = h/(\omega b) = v_0/\omega + O(h^2)$. By adjusting the ellipse to $\hat{r}$, one can determine the time dependence for the lower half by means of $r^2(t) = a^2\cos^2\omega t + b^2\sin^2\omega t$. Alternatively one can solve the integral by simple Euler substitution and of course get the same result[18]

$$\begin{aligned} t_l(r) &= s(r) - s(\hat{r}), \\ s(r) &= \frac{1}{2\omega}\sin^{-1}\left(\frac{v_0^2 + h^2 - 2\omega^2 r^2}{\sqrt{\left(v_0^2 + h^2\right)^2 - 4\omega^2 h^2}}\right). \end{aligned} \tag{11}$$

Here, we count the time $t_l(r)$ starting from zero at $\hat{r}$, so that $t(r < \hat{r}) = t_u(\hat{r}) + t_l(r)$. Possibly one recognizes the integrals concerning the two-dimensional oscillator from a course on classical mechanics.[19] Considering $s(r_{\min}) = \pi/4\omega$ we can expand $t_l(r_{\min})$ in h

$$\begin{aligned} t_l(r_{\min}) &= \frac{1}{\omega}\sin^{-1}\left(\frac{\omega\hat{r}}{v_0}\right) \\ &\quad - h^2\frac{v_0^2 - (2\omega^2 - v_0^2)\hat{r}^2}{2v_0^4\hat{r}\sqrt{v_0^2 - \omega^2\hat{r}^2}} + O(h^4) \\ &= \frac{1}{\omega}\sin^{-1}\left(\frac{\omega\hat{r}}{v_0}\right) - 0.37h^2 + O(h^4). \end{aligned} \tag{12}$$

The leading term gives just the time of a linear oscillation of the frequency ω and amplitude $a = v_0/\omega$ to get from $\hat{r}$ to 0. The traversal time $T = 2(t_u(\hat{r}) + t_l(r_{\min}))$ for a free fall as a function of latitude θ is thus given by

$$\begin{aligned} T &= \frac{4}{\sqrt{\mu}}\sinh^{-1}\sqrt{\frac{\mu}{2}(1-\hat{r})} \\ &\quad + \frac{2}{\omega}\sin^{-1}\left(\frac{\omega\hat{r}}{v_0}\right) + 0.7(h_0\cos\theta)^2. \end{aligned} \tag{13}$$

For a free fall between the poles ($\theta = \pi/2$), we obtain the known analytical result[14] of $T = 2.8453$ (38′12″ in natural units), which is very well in agreement with the value of 2.8442 obtained by numerical integration of the PREM velocity profile. In a free fall through the equatorial plane ($\theta = 0$), the traversal time is only slightly increased by $0.7h_0^2$ to $T = 2.8477$ (for PREM $T = 2.8467$, obtained by numerical integration), so that the journey takes just 2 s longer.

IV. PATH IN THE INERTIAL FRAME

In the inertial frame of the stationary plane perpendicular to the initial specific angular momentum at latitude θ the path is given by $d\varphi/dr = \dot{\varphi}/\dot{r}$, $\dot{\varphi}$ in Eq. (2b) and $\dot{r}$ in Eq. (4)

$$\varphi(r) = \int_{r \geq r_{\min}}^{1} dr \frac{h}{r^2\sqrt{v^2_{\text{appr.}}(r) - h^2\left(\frac{1}{r^2} - 1\right)}}, \tag{14}$$

where the second half of the trajectory is given by $\varphi_2(r) = 2\varphi(r_{\min}) - \varphi(r)$.

The largest contribution of the integrand is obtained from the range $r \simeq r_{\min}$ contrary to the time integral, which get the largest contribution at $r = 1$. Therefore, we select a

Table I. Parameters of velocity profiles.

Parameter	t - approximation	φ - approximation
$\hat{r}$	0.490	0.451
$\hat{g}$	1.055	1.024
$\mu = (\hat{g}-1)/(1-\hat{r})$	0.107	0.043
$v_0 = \sqrt{1+\hat{g}-\hat{r}}$	1.251	1.254
$\omega = \sqrt{\hat{g}/\hat{r}}$	1.467	1.506

different approximation of the velocity profile to the PREM, as discussed in Ref. 14, so that the integral of Eq. (14) is approximated as well as possible. We perform a nonlinear least square approximation by minimizing the function

$$\sigma^2(\hat{r},\hat{g}) = \int_0^1 dr\left(\frac{1}{r^2 v_{\text{appr.}}(r,\hat{r},\hat{g})} - \frac{1}{r^2 v_{\text{PREM}}(r)}\right)^2, \quad (15)$$

which measures the mean square error of the integrand in Eq. (14) if one replaces v_{PREM} by $v_{\text{appr.}}$ neglecting the value of h. This approximation forces a coincidence of $v_{\text{appr.}}$ and v_{PREM} at $r=0$, otherwise the integral diverges, and a very good approximation for $r \ll 1$. We can therefore eliminate one of the two parameters $\hat{r}$ and $\hat{g}$ via the equation $v_0^2 = 1+\hat{g}-\hat{r} = v_{\text{PREM}}^2(0) = 1.5722$. Using a standard routine for nonlinear optimization, we obtain $\hat{r}_\varphi = 0.451$ and $\hat{g}_\varphi = 1.024$. All relevant parameters of the approximations are summarized in Table I.

Figure 3 shows the comparison of the two different approximations and the exact match for the velocity profile with parameters $\hat{r}_\varphi$ and $\hat{g}_\varphi$ at the center of the Earth, whereas the velocity profiles of models of constant gravity and constant density significantly differ at the center. All models coincide near the surface where $1-r \ll 1$ applies and the $\hat{r}_t$, $\hat{g}_t$—approximation fits better in the upper half where the contribution to the time—integral is dominant.

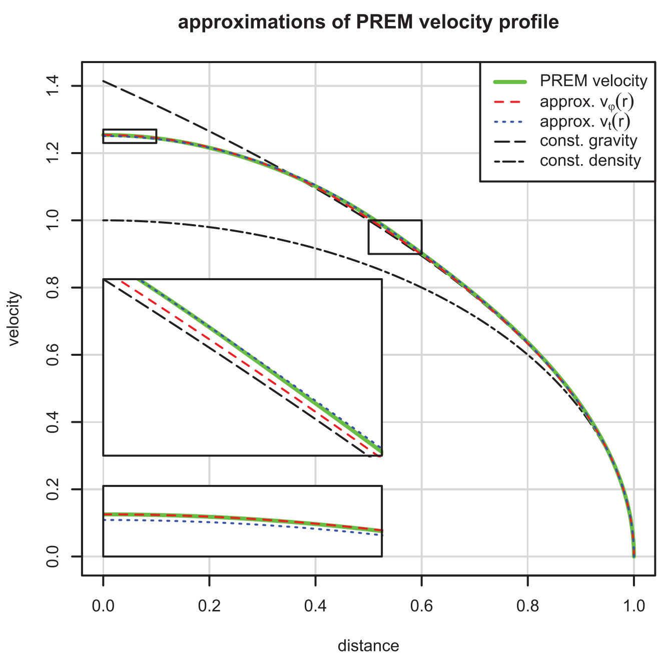

Fig. 3. The comparison of the approximations shows the better match of the analytical approximation for the φ - integral at the center, while the velocity approximation for the t- integral matches better in the upper half.

As for the t - integral we split the integral in Eq. (14) into the parts for the upper half $\varphi_u(r)$ with $1 \geq r \geq \hat{r}$ and the lower half $\varphi_l(r)$ with $\hat{r} \geq r \geq r_{\min}$. To avoid elliptic integrals of complex analytical form, the integrand for the upper half is expanded in h, neglecting orders greater than $O(h^2)$ which are beyond the accuracy of the approximation

$$\varphi_u(r) = h\left(\frac{\sqrt{(1-r)(2+\mu(1-r))}}{(2+\mu)r} + \frac{1+\mu}{(2+\mu)^{3/2}}\cosh^{-1}\left(\frac{1+(1+\mu)(1-r)}{r}\right)\right) + O(h^3). \quad (16)$$

For the analytical solution of the lower half, we know that the path is part of an ellipse and one can make use of its equation in polar coordinates $r^2(\varphi) = a^2b^2/(b^2\cos^2\varphi + a^2\sin^2\varphi)$ with axis a and b, where $ab = h/\omega$ and $a^2+b^2 = (v_0^2+h^2)/\omega^2$. The same result[18] is obtained by solving the standard integral in Eq. (14)[19]

$$\varphi_l(r) = k(\hat{r}) - k(r),$$
$$k(r) = \frac{1}{2}\sin^{-1}\left(\frac{(v_0^2+h^2)r^2 - 2h^2}{r^2\sqrt{(v_0^2+h^2)^2 - 4\omega^2h^2}}\right), \quad (17)$$

and

$$\varphi_l(r_{\min}) = \frac{\pi}{2} - h\frac{\sqrt{v_0^2 - \omega^2\hat{r}^2}}{\hat{r}v_0^2} + O(h^3), \quad (18)$$

where $\varphi_l(\hat{r}) = 0$, $\varphi(r \geq \hat{r}) = \varphi_u(r)$ and $\varphi(r < \hat{r}) = \varphi_u(\hat{r}) + \varphi_l(r)$. The path $\varphi_2(r)$ for the second half after passing $r_{\min}$ can be obtained via $\varphi_2(r) = 2\varphi(r_{\min}) - \varphi(r)$.

Now we are prepared to calculate the deviation $\delta\varphi$ from a straight line. Inserting the values of the parameters of Table I we obtain up to the order $O(h^3)$, $\varphi_u(\hat{r}) = 1.829h$, $\varphi_l(r_{\min}) = \pi/2 - 1.485h$ and

$$\delta\varphi = 2(\varphi_u(\hat{r}) + \varphi_l(r_{\min})) - \pi$$
$$= 0.687h + O(h^3) = 0.040\cos\theta + O(h^3). \quad (19)$$

The angle $\delta\varphi$ is just twice the angle by which the elliptical part of the trajectory with semi-minor axis $b = r_{\min} \simeq h/v_0$ and semi-major axis $a = h/(\omega r_{\min}) \simeq v_0/\omega$ has to be rotated to match the trajectory at $\hat{r}$. Such a patching of an elliptical trajectory is also described in Ref. 20. Figure 4 shows exemplarily for $\theta = 0$ the very good agreement of the paths obtained from direct integration of the PREM gravitation profile with the analytical approximations given here.

The polar coordinates $x = r\cos\varphi(r)$ and $y = r\sin\varphi(r)$ are plotted, with r varying between 1 and $r_{\min} \simeq h_0/v_0 = 4.69\times10^{-2}$ and back to 1 starting at $\varphi = \pi$. For a free fall through the equatorial plane, the minimal distance to the center in natural units is $r_{\min} = 299$ km. For the constant density model (isotropic harmonic oscillator), it is known that the elliptical orbit is closed with $\delta\varphi = 0$,[4] which can also be seen by Eq. (18) for $v_0 = \omega = \hat{r} = 1$.

To compare these results with the constant gravity model, we evaluate the integral in Eq. (14) with $\mu = 0$ and

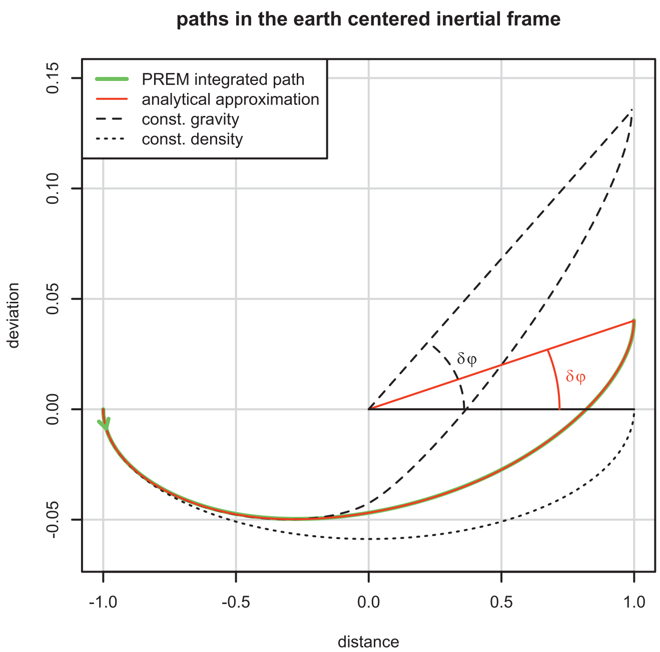

Fig. 4. Path of a free fall in the inertial frame of a stationary plane for latitude $\theta = 0$ starting at $\varphi = \pi$. The deviation for the PREM and its analytical approximation for $\theta = 0$ is $\delta\varphi = 0.04$ whereas for the constant gravity model it is $\delta\varphi = 0.136$.

$v^2_{\mathrm{appr.}}(r) = 2(1 - r)$ and a minimal distance r_0 to the center of the Earth

$$r_0 = \frac{h}{4}\sqrt{8 + h^2} + \frac{h^2}{4}, \tag{20}$$

and get

$$\begin{aligned}\delta\varphi_0 &= 2\sqrt{2}h\sqrt{\frac{1 - r_0}{1 - h^2}}\Pi\left(1 - r_0, \frac{1 - r_0}{\sqrt{1 - h^2}}\right) - \pi \\ &= \frac{h}{\sqrt{2}}\left(\frac{7\log 2 - 4}{2} - \log(h)\right) + O\big(h^3\log(h)\big).\end{aligned} \tag{21}$$

$\Pi(n, k)$ is the complete elliptical integral of the third kind following the notation of Gradshteyn and Ryzhik,[21] where k is called the modulus of the integral and n is the parameter of the integral of the third kind. Evaluating Eq. (21) at $h = h_0$ we get a deviation $\delta\varphi_0 = 0.136$ for the constant gravity model at latitude $\theta = 0$, which is much larger than for the PREM and its analytical approximation, and is also shown in Fig. 4. The large deviation is represented by the log term of the approximation in Eq. (21), which is caused by the linear velocity profile near the center and again reflects the unphysical state of constant gravity at the center of the Earth.

Although the path of free fall in the constant gravity model corresponds to the path of the PREM to a distance of approximately 0.3 to the center of the Earth, the different gravity near the center of the Earth causes considerable differences in the further course. The change of the angle φ at the center of the Earth is approximately $\dot{\varphi} = v_0^2/h$, so that here with the clearly different maximum velocities v_0 and the characteristics near the center of the earth (linear in the constant gravity model, otherwise quadratic) also clearly different orbitals arise in the different models. For models with linear decreasing gravity from a distance $\hat{r}$ to the center, the resulting deviation $\delta\varphi$ is completely determined by matching the elliptical trajectory, i.e., by the angle $\varphi(\hat{r})$ and by the semi-axes $a \simeq v_0/\omega$ and $b = r_{\min}$. This differs from the time dependence, which is very insensitive to the shape of gravity in the core and does not depend on $b = r_{\min}$ up to order h^2 (see Eq. (12)).

If the path is continued in the inertial system, it does not return to the starting point as in the constant density model, but precesses by an angle $\varphi = 2\delta\varphi = 4.6^\circ \cos\theta$, as shown in Fig. 5. In the constant gravity model, the angle of precession does not depend linearly on $\cos\theta$ and for $\theta = 0$ it is $\varphi_0 = 2\delta\varphi_0 = 15.5^\circ$. The minimum distance to the center of the Earth is $r_{\min} \simeq h/v_0$, where v_0 is the maximal velocity of the different models at the center of the Earth. Figure 5 also shows the nearly identical traversal time (path length) for the constant gravity model and the PREM with its analytical approximation, whereas in the constant density model one returns exactly to the starting point after a time of $t = 2\pi$.

V. PATH IN THE ROTATING EARTH FRAME

To ensure that the free fall can take place undisturbed in the rotating Earth, the path in the inertial frame must be transformed into the rotating Earth frame so that each point $\vec{r}$, after rotation by the angle $\Omega t(\vec{r})$ about the z-axis, matches the path in the inertial system. Figure 6 shows schematically that the transformed path of the tunnel in the rotating Earth frame in general (except for $\theta = 0$) does not take place in a plane.

The path in the inertial frame is given by its polar coordinates $x_{\mathrm{I}} = r\cos\varphi(r)$ and $y_{\mathrm{I}} = r\sin\varphi(r)$ with $\varphi(r)$ given in Eqs. (16)–(19). To get the appropriate coordinates $(x_{\mathrm{R}}, y_{\mathrm{R}}, z_{\mathrm{R}})$ in the rotating Earth frame the coordinates of the path in the inertial frame are first rotated around the y-axis by the angle θ

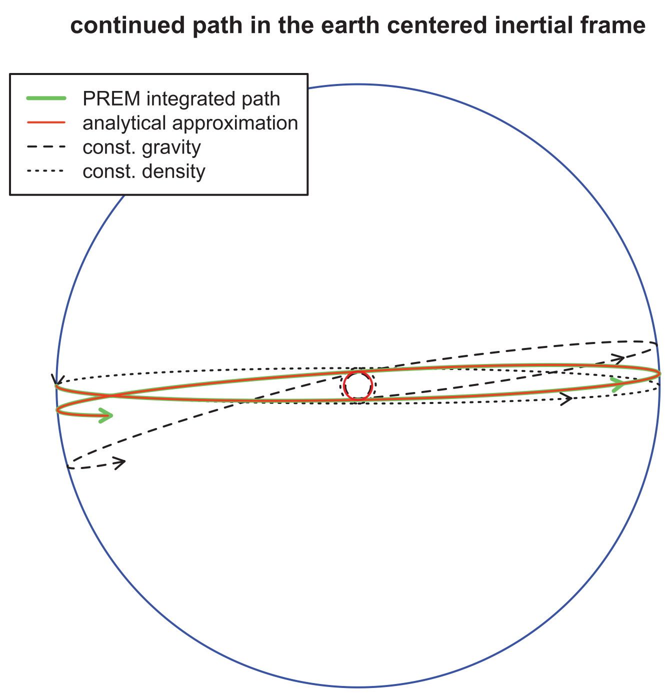

Fig. 5. Comparison of different models of a continued free fall with travel time $t = 2\pi$ in the inertial frame for latitude $\theta = 0$. The minimum distance $r_{\min}$ to the center of the Earth is indicated for the different models by the inner circles.

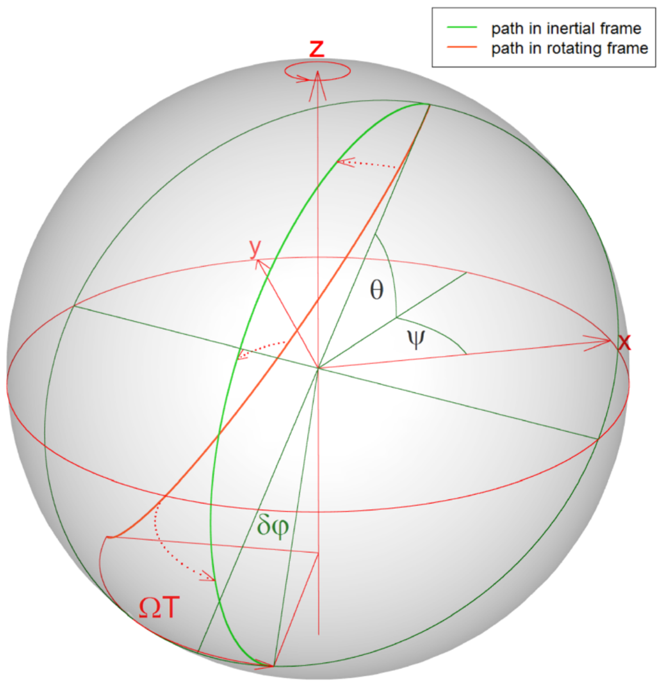

Fig. 6. The path of the gravity tunnel in the rotating Earth must be constructed so that each point on the path starting at latitude θ and longitude ψ lies on the corresponding point of the path of a free fall in the inertial system at time t and rotation by an angle Ωt around the z-axis.

clockwise and then around the z-axis by the angle $\psi - \Omega t$ counterclockwise. θ is the latitude, ψ is the longitude of the starting point, and Ω is the angular velocity of the Earth. The time $t(r)$ is given in Eqs. (10)–(13). The coordinates of the path in the rotating Earth frame are thus given by

$$\begin{pmatrix} x_R \\ y_R \\ z_R \end{pmatrix} = \begin{pmatrix} \cos(\psi-\Omega t) & -\sin(\psi-\Omega t) & 0 \\ \sin(\psi-\Omega t) & \cos(\psi-\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x_I \\ y_I \\ 0 \end{pmatrix}. \tag{22}$$

Performing the matrix multiplications, we obtain

$$x_R = \cos(\psi-\Omega t)\cos(\theta)x_I - \sin(\psi-\Omega t)y_I, \tag{23a}$$

$$y_R = \sin(\psi-\Omega t)\cos(\theta)x_I + \cos(\psi-\Omega t)y_I, \tag{23b}$$

$$z_R = \sin(\theta)x_I. \tag{23c}$$

One identifies the same equations as in Ref. 11 by substituting $\psi=0$, $x_I = \cos t$ and $y_I = h\sin t$ in Eq. (23) and $\omega_1 = \omega_2 = 1$, $\lambda = \theta$ and $\alpha\cos\lambda = h$ in Eq. (11) in Ref. 11, which gives the equation of gravity tunnels in a rotating homogeneous sphere starting at longitude $\psi=0$ and latitude θ.

To better identify the shape of the tunnel, we transform the coordinates (x_R, y_R, z_R) into the base (x_{IR}, y_{IR}, z_{IR}) of the non-rotating plane by means of an inverse rotation by the angle θ, so that the coordinates (x_{IR}, y_{IR}) correspond to the projection of the coordinates (x_R, y_R) onto the plane inclined by the angle θ and z_{IR} corresponds to the distance to the plane

$$x_{IR} = r\cos(\psi + \varphi(r) - \Omega t\cos\theta) + O(\Omega^3), \tag{24a}$$

$$y_{IR} = r\sin(\psi + \varphi(r) - \Omega t\cos\theta) + O(\Omega^3), \tag{24b}$$

$$z_{IR} = (\Omega t/2)^2\sin(2\theta)x_I - \Omega t\sin(\theta)y_I + O(\Omega^3). \tag{24c}$$

Considering in Eq. (24c) that y_I is of order $h_0 = \Omega$, we see that the distance to the plane of the inertial system is only of order h_0^2 and is maximal at $t \simeq T$ with $z_{IR\max} \simeq -h_0^2\sin(2\theta)$ (and $z_{IR\max} \simeq -(\pi h_0/2)^2\sin(2\theta)$ for the model of constant density). Although the distance is quite small, it is never zero except for $\theta=0$ (and for $\theta \pm \pi/2$ and $y_I=0$ it is a straight line connecting the poles) with the trajectory in the equatorial plane given by

$$x_R = r\cos(\psi + \varphi(r) - \Omega t(r)), \tag{25a}$$

$$y_R = r\sin(\psi + \varphi(r) - \Omega t(r)). \tag{25b}$$

Equations (25) correspond to Eq. (23) in Ref. 9 (and Eq. (23), Chap. VIII in Ref. 10), which is given for a free fall through the equatorial plane of a homogeneous Earth. One recognizes that the trajectory starting at latitude θ in the rotating earth frame is very similar to the trajectory in the equatorial plane if it is rotated by an angle θ and h_0 and Ω are replaced by $h_0\cos\theta$.

From Eqs. (23), we can calculate the r-dependent spherical coordinates $\psi(r)$ and $\theta(r)$ of the path in the rotating Earth frame by $\psi(r) = \tan^{-1}(y_R/x_R)$ and $\theta(r) = \sin^{-1}(\sin\theta\cos\varphi(r))$. To calculate the angle ψ correctly, the sign of x_R and y_R must be taken into account. In particular, the end coordinates ψ_e, θ_e of a free fall tunnel can be specified depending on the start coordinates ψ_s, θ_s. Here, we take into account that $\varphi_e = \pi + \delta\varphi$ and $\Omega t_e = h_0 T$ with $\delta\varphi$ from Eq. (19) and T from Eq. (13) and get

$$\psi_e = (\psi_s - \Delta\varphi)\bmod 2\pi - \pi, \tag{26a}$$

$$\theta_e = -\theta_s + \Delta\theta, \tag{26b}$$

where

$$\Delta\varphi = h_0 T - 0.687h_0 + O(h_0^3) \simeq 0.127, \tag{27a}$$

$$\Delta\theta = \frac{(0.687h_0)^2}{4}\sin 2\theta_s + O(h_0^4) \simeq 4\times 10^{-4}\sin 2\theta_s. \tag{27b}$$

The modulo function in Eq. (26a) is introduced because the longitude ψ varies between $-180°$ and $180°$. The path is always deviated by an angle of $7.26°$ to the west compared to a straight line, whereas the deviation of the latitude θ is always in the direction of the equator and with a maximum of $\pm 1.4'$ ($0.023° \equiv 2.5$ km) at latitude $\theta_s = \pm 45°$ almost negligible, but in contrast to the model of constant density, where the distance to the equator does not change. The path through the rotating equatorial plane given in Eq. (25) is shown in Fig. 7 and is compared with models of constant gravity ($\Delta\varphi = \Omega T - 0.136 = 0.031$) and constant density ($\Delta\varphi = \Omega\pi = 0.185$). The strange path in the constant gravity model is due to the rapid change in polar angle near the center.

The length $l(\theta)$ of the path is only slightly less than $2R$ and can be geometrically approximated by

$$l(\theta) \simeq 2\sqrt{\left(1 + r_{\min}^2 - 2r_{\min}\cos\left(\frac{\pi}{2} - \frac{\Delta\varphi}{2}\cos\theta\right)\right)} = 2 - 3.74\times 10^{-3}\cos^2\theta \tag{28}$$

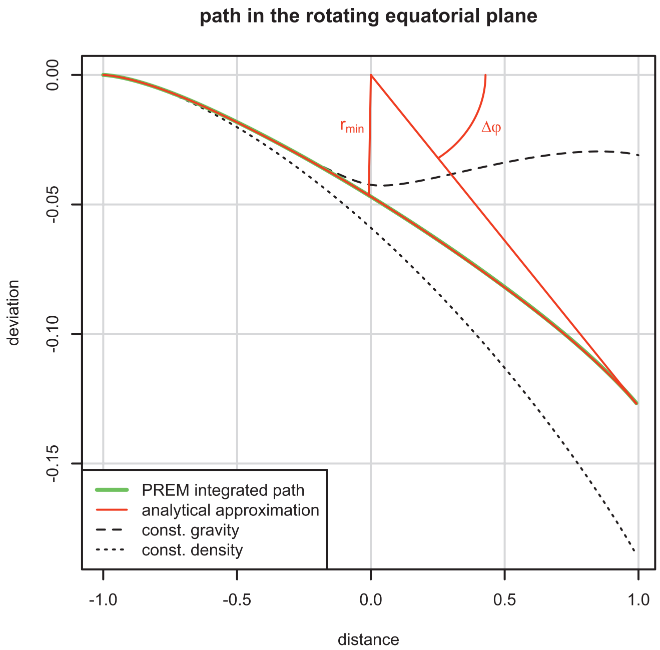

Fig. 7. The path through the rotating equatorial plane according to Eq. (25) shows the deviation $\Delta\varphi = 0.127$.

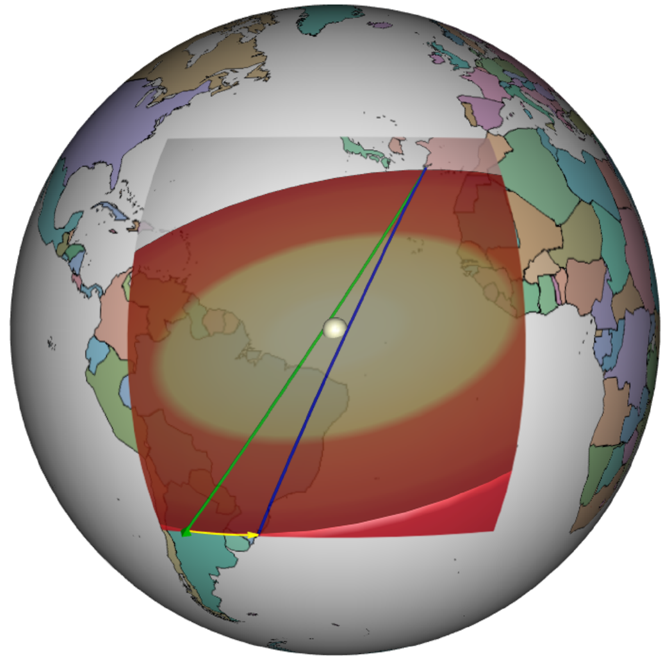

Fig. 8. Tunnels for a free fall from Shanghai to Argentina and back from Brazil to Shanghai, passing the center of the Earth at a distance of 255 km. For an animation see supplementary material at http://dx.doi.org/10.1042/1.5674822.

and is 1.9963 for $\theta = 0$ compared with the numerically calculated value of 1.9964, i.e., only a maximum of 23 km shorter than a straight path.

VI. CONCLUSION

The complexity of the 30 parameter density model of the Earth given by the PREM has been reduced to a model with two parameters, which have been determined in such a way that the traversal time and the trajectory of a free fall through the rotating Earth can be completely analytically determined and correspond accurately to the numerical solutions of the PREM. But it should also be mentioned that the problems of drilling such a tunnel and its evacuation are completely unsolved.

While the travel time for a free fall hardly differs in comparison to a straight line neglecting friction (increases from $38'12''$ to a maximum of $38'14''$), the path is completely different and neither the model of constant gravity nor the model of constant density are good approximations. And corrections to the model of a homogeneous Earth, which are caused by the inhomogeneous density distribution, are four orders of magnitude larger than corrections caused by the flattening of an Earth assumed to be homogeneous.[11,22]

After a time of about 19 min, the center of the Earth is passed at a distance of $r_{\min} \simeq h/v_0 \equiv 298\,\text{km} \times \cos(\theta_s)$ with velocity $r_{\min}\dot{\varphi} \simeq v_0 \equiv 9.9\,\text{km/s}$. At the other end of the tunnel, the Earth's surface is reached again with a deviation of $7.26^\circ \equiv 809\,\text{km} \times \cos(\theta_s)$ towards the west. Unfortunately, the same tunnel cannot be used for the way back and most of the time one arrives in the sea.

But there are also hypothetical tunnels that connect continents. Starting at Shanghai in China (31.23° N, 121.47° E), one will pass the center of the Earth at a distance of 255 km with a velocity of 9.9 km/s. At the other end one reaches Argentina about 150 km west of Cordoba (31.21° S, 65.79° W). To return to Shanghai via a free fall, one has to travel 14.5° (1380 km) east to Puerto Alegre in Brazil (31.25° S, 51.27° W). These tunnels are modeled in Fig. 8, which like all other figures has been created with the support of R.[23]

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Induction Coil

My notes from the time that I photographed this induction coil at Washington and Lee University tell me that it was made by Ritchie of Boston. Examination of my Ritchie catalogues suggests that this item was made about 1870. The coil is a low-end product, capable of producing a spark only one or two inches in length, but enough to light up a Geissler tube for demonstrating the properties of cathode rays. (Picture and text by Thomas B. Greenslade, Jr., Kenyon College)