Average density on Earth is

In[1]:=
```
ρ0 = 5515; (*kg/m3 = M/(4π/3 R^3) *)
```

where the radius is

In[2]:=
```
R = 6.371 × 10^3;(*km*)
```

Density in the surface is

In[3]:=
```
ρR = 1000;(*kg/m3*)
```

then $\rho_0 = 5.515\ \rho_R$.

Total mass in the Earth is

$$M_T = 4\pi \int_0^R \rho(r)\, r^2\, dr$$

acceleration in the surface is

In[4]:=
```
g = 9.81 ;(*m/s2*)
```

Assuming now that the mass density profile inside the Earth is of the form

$$\rho(r) = \rho_{av}\, b \left(1 - c\, \frac{r}{R}\right)^d$$

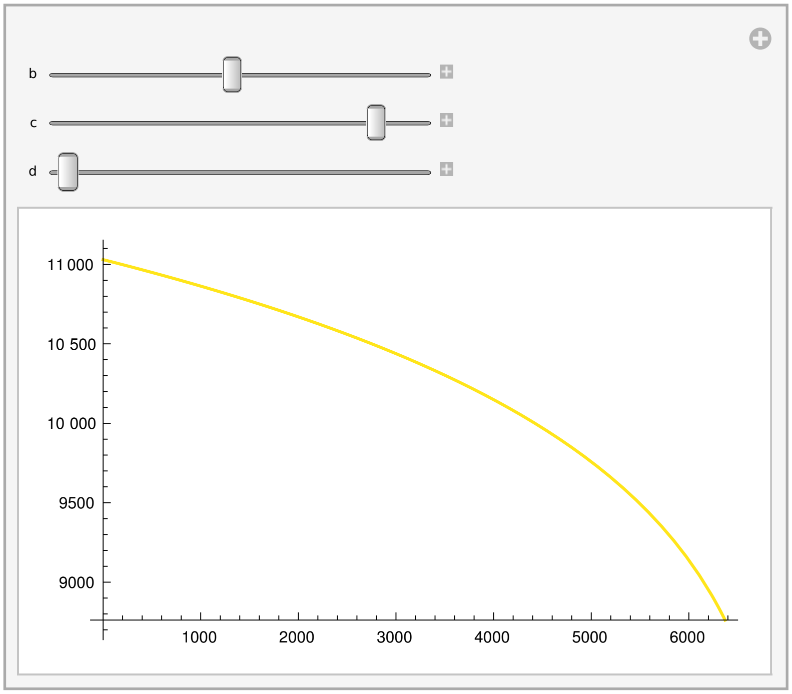
being b,c,d parameters to determine. It can be written in the code as the function

In[5]:=
```
ρ1[r_, b_, c_, d_] := ρ0 b Power[1 - c r / R, d]
```

this density have the following forms

In[6]:=
```
Manipulate[Plot[ρ1[r, b, c, d], {r, 0, R}, PlotStyle → RGBColor[b, c, d]],
 {b, 0.2, 4, 0.2}, {c, 0.1, 1, 0.1}, {d, 0.1, 1, 0.1}]
```

Out[6]=



Now, we have the next conditions to determine parameters b,c,d.

$$\rho(r = R) = b\,\rho_0\,(1 - c)^d = \rho_R$$

$$b\,(1 - c)^d = \frac{\rho_R}{\rho_0} = 0.1813 \qquad (1)$$

The second conditions is given from the total mass, we must have

$$M_T = 4\pi \int_0^R \rho(r)\, r^2\, dr = 4\pi\,\rho_0\, b \int_0^R \left(1 - c\,\frac{r}{R}\right)^d r^2\, dr$$

Doing the integral in Mathematica gives

In[ ]:=
```
Integrate[b * x ^ 2 * (1 - c * (x / R)) ^ d, {x, 0, R}]
```

··· Integrate : Invalid integration variable or limit (s) in {2.37325 , 0, 6371. }.

Out[ ]=

$$\int_0^{6371.} 13.364 \times d\,2.37325$$

we have therefore the second condition:

$$3\,b\left[2 - (1 - c)^{d+1}\left(2 + 2\,c\,(1 + d) + c^2\,(2 + 3\,d + d^2)\right)\right] = c^3\,(6 + 11\,d + 6\,d^2 + d^3) \qquad (2)$$

As we want to approximate PREM density, let’s suppose d=0.6, then the other variables give

In[ ]:=

```
k = 0.6;
Remove[x, y];
NSolve[
 x * (1 - y) ^ k == 0.1813 && x * (6 - 3 * (1 - y) ^ (k + 1) * (2 + 2 * y * (1 + k) + y ^ 2 * (2 + 3 * k + k ^ 2))) ==
   (y ^ 3 * (6 + 11 * k + 6 * k ^ 2 + k ^ 3)), {x, y}]
```

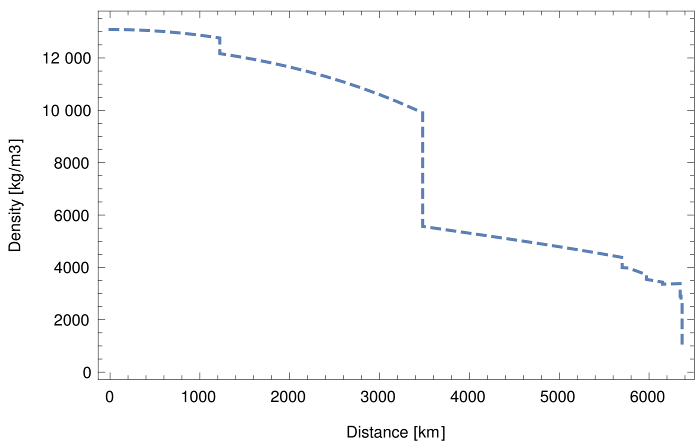
Out[ ]= {{x → 2.40802 , y → 0.986576}, {x → 0.1813 , y → 0.}, {x → 0.1813 , y → 0.}, {x → 0.1813 , y → 0.}}

This exponent was used in such a way that the density in the center coincides the better with that of prem:

In[9]:=

```
data = Import["/home/nicolas/Documents/Physics/Bachelors-Dissertation/1-Earth
     Gravity Tunnel/Numerical Data/prem-density.csv", "Table"];
density_prem = ListLinePlot[data,
  PlotStyle → Dashed,
  Frame → True,
  FrameLabel → {{"Density [kg/m3]", None}, {"Distance [km]", None}}]
```

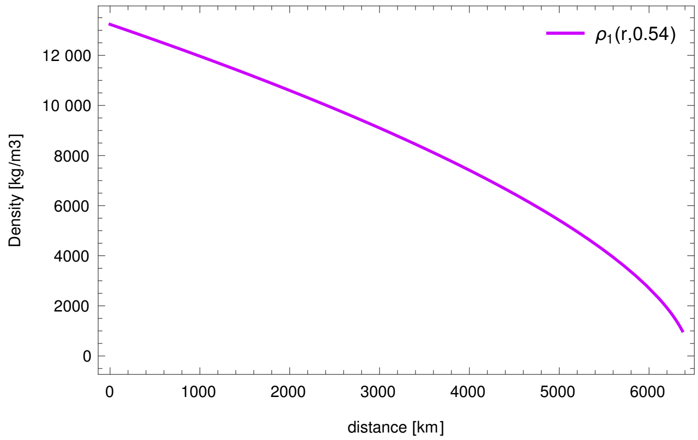
Out[10]=



the effective function with this parameter looks

In[11]:=
```
EffctiveFunction1 = Plot[{ρ1[x, 2.4, 0.9865, 0.6]}, {x, 0, R},
PlotLegends → Placed[{"ρ1(r,0.54)"}, {Right, Top}],
PlotStyle → {Thickness[0.005], Hue[0.8]},
Frame → True,
FrameLabel → {{"Density[kg/m3]", None}, {"distance[km]", None}}]
```

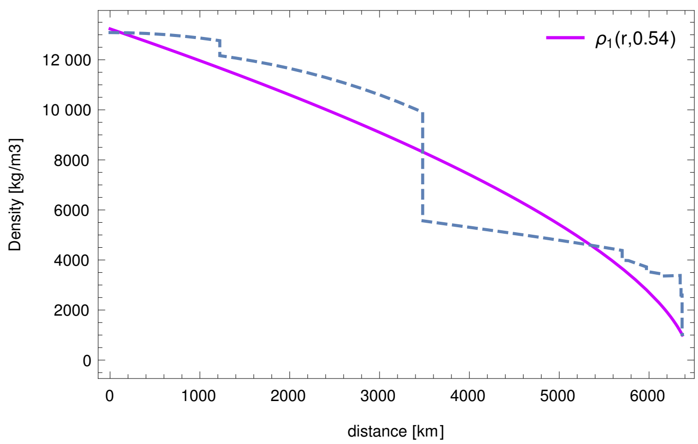
Out[11]=



We can now plot them together.

In[13]:=
```
Show[EffctiveFunction1 , density_prem]
```

Out[13]=



Let's include, however, the value on the center as a third conditions to solve the system

$$\rho(0) = \rho_0 \, b = 13\,088.5 \tag{3}$$

then

In[ ]:=
```
b = 13 088.5 / ρ0
```

Out[ ]= 2.37325

the system to solve is now

$$(1-c)^d = \frac{0.1813}{2.37325} = 0.076393$$

$$7.1197\left[2 - (1-c)^{d+1}\left(2 + 2c(1+d) + c^2\left(2 + 3d + d^2\right)\right)\right] = c^3\left(6 + 11d + 6d^2 + d^3\right)$$

this would be the code on Mathematica to solve the system, but it takes too long

In[ ]:=
```
Remove[x, y];
NSolve[{(1 - x) ^ y == 0.0763 , 7.1197 (2 - (1 - x) ^ (y + 1) * (2 + 2 * x * (1 + y) + x ^ 2 * (2 + 3 * y + x ^ 2))) /
      (x ^ 3 * (6 + 11 * y + 6 * y ^ 2 + y ^ 3)) == 1}, {x, y}]
```

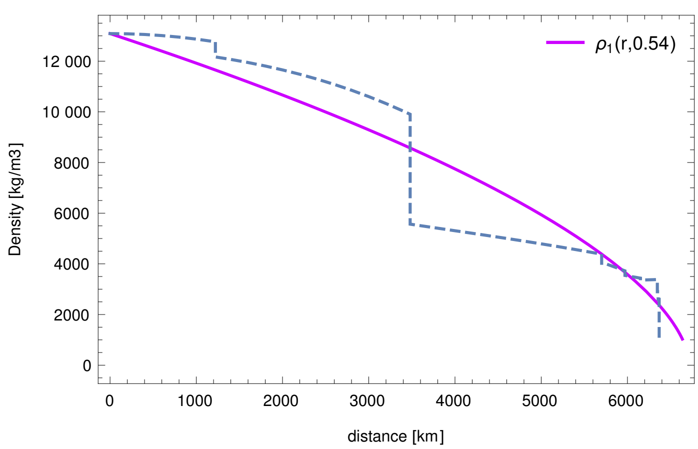
On the other hand, the next code on Python finds the answer in a few seconds:

In[ ]:=
```python
from scipy.optimize  import  root

def equations(p):
    x, y  = p
    eq1 = (1-x)**y  - 0.0763
    eq2 = 7.12*(2  - (1  - x)**(y  + 1)*(2  + 2*x*(1  + y)  + x**2*(2  + 3*y  + y**2)))  /
(x**3*(6  + 11*y  + 6*y**2  + y**3))  - 1
    return  ( eq1  , eq2  )

sol =  root(equations,  (0.1,  0.1),  method='lm',  jac=None,  tol=None,
callback=None,  options={'col_deriv':  0, 'xtol':  1.49012e-08,  'ftol':  1.49012e-08,
'gtol':  0.0, 'maxiter':  0, 'eps':  0.0, 'factor':  100, 'diag':  None})


x = sol.x[0]
y = sol.x[1]

print(equations((x,  y)))
print(x)
print(y)
```

Plotting with this result gives

```
EffctiveFunction2  = Plot[{ρ1[x, 2.3732 , 0.9875 , 0.58]}, {x, 0, R},
PlotLegends → Placed[{"ρ1(r,0.54)"}, {Right , Top}],
PlotStyle → {Thickness [0.005], Hue[0.8]},
Frame → True ,
FrameLabel → {{"Density [kg/m3]", None}, {"distance [km]", None}}];
Show[EffctiveFunction2 , density_prem]
```

Out[ ]=



The density function to work from here on is then

In[14]:=
```
b = 2.37325;
c = 0.9875;
d = 0.5867;
ρ[r_] := ρ1[r, b, c, d]
```

Numerical checks

1. Mass

In[ ]:=
```
M[r_] := 4 π 10^9 NIntegrate[ρ[x] x^2, {x, 0, r}]
```

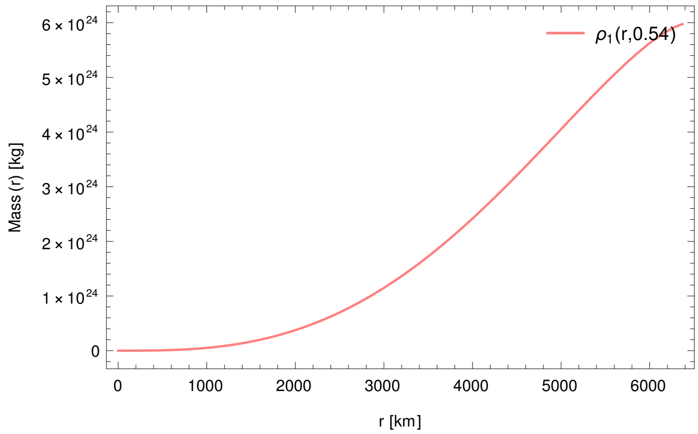
Total mass

In[ ]:=
```
M[R]
```

Out[ ]= 5.97603×10^{24}

Distribution inside the planet

```
Masses = Plot[{M[r]}, {r, 0, R},
    PlotLegends → Placed[{"ρ1(r,0.54)", "ρ2(r,0.55,4.26)"}, {Right, Top}],
PlotStyle → {{Thickness[0.004], Pink},
Frame → True,
FrameLabel → {{"Mass(r) [kg]", None}, {"r [km]", None}}]
```

Out[ ]=


2. Acceleration of gravity

Computation of the acceleration inside the planet

```
Integrate[b * x ^ 2 * (1 - c * (x / R)) ^ d, {x, 0, r}]
```

$$\text{ConditionalExpression}\left[\frac{b\left(2 R^3 + \left(1 - \frac{c r}{R}\right)^d (c r - R)\left(c^2 (1 + d)(2 + d) r^2 + 2 c (1 + d) r R + 2 R^2\right)\right)}{c^3 (1 + d)(2 + d)(3 + d)}, \operatorname{Re}\left[\frac{R}{c r}\right] > 1 \,||\, \operatorname{Re}\left[\frac{R}{c r}\right] < 0 \,||\, \frac{R}{c r} \notin \text{Reals}\right]$$

In[23]:=
```
G = 6.674 × 10 ^ -11;
a[r_] := 4 Pi 10 ^ 3 G NIntegrate[ρ[x] x ^ 2, {x, 0, r}] / r ^ 2
```

In[25]:=
```
a[R]
```

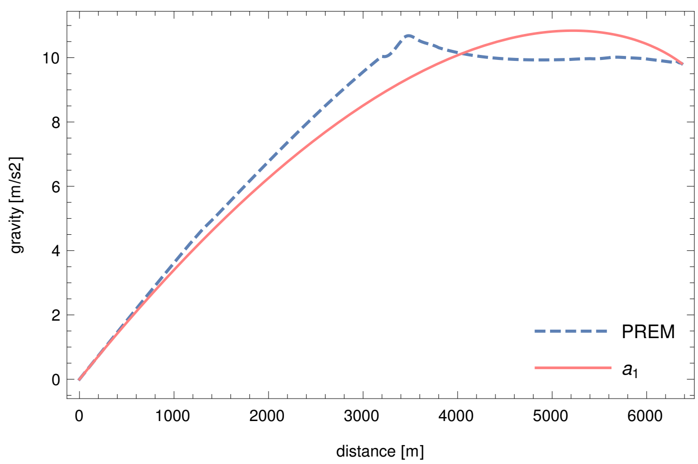
Out[25]= 9.8237

In[28]:=
```
gravity_prem =
  Import["/home/nicolas/Documents/Physics/Bachelors-Dissertation/1-Earth Gravity
     Tunnel/Numerical Data/gravity_prem.csv", "Table"];
g_prem = Interpolation[gravity_prem, InterpolationOrder → 5]
```

Out[29]= InterpolatingFunction[Domain: $\{\{0., 6.37 \times 10^3\}\}$ Output: scalar]

In[ ]:=
```
accelerations = Plot[{g_prem[r], a[r]}, {r, 0, R},
PlotLegends → Placed[{"PREM", "a1"}, {Right, Bottom}],
PlotStyle → {Dashed, {Thickness[0.004], Pink}},
Frame → True,
FrameLabel → {{"gravity[m/s2]", None}, {"distance[m]", None}}]
```

Out[ ]=



Predictions

1. Velocity

```
Integrate[
 (2 * R ^ 3 + (1 - c * x / R) ^ d * (c * x - R) * (2 * R ^ 2 + 2 * c * (1 + d) * R * x + c ^ 2 * (2 + 3 * d + d ^ 2) * x ^ 2)) /
  x ^ 2, {x, R, r}]
```

$$\text{ConditionalExpression}\left[-\left(-2+(1-c)^{2+d}\,(2+c+c\,d)\right)R^2+\frac{-2\,R^3+\left(1-\frac{c\,r}{R}\right)^d(-c\,r+R)^2\,(c\,(1+d)\,r+2\,R)}{r},\right.$$
$$\left(\frac{\mathrm{Im}[c]\left(\mathrm{Im}[R]^2+\mathrm{Re}[R]^2\right)}{\mathrm{Re}[c]\,(\mathrm{Im}[R]\,\mathrm{Re}[r]-\mathrm{Im}[r]\,\mathrm{Re}[R])+\mathrm{Im}[c]\left(-\mathrm{Im}[r]\,\mathrm{Im}[R]+\mathrm{Im}[R]^2-\mathrm{Re}[r]\,\mathrm{Re}[R]+\mathrm{Re}[R]^2\right)}\geq 1\;||\right.$$
$$\left(\mathrm{Im}[r]\,\mathrm{Re}[c]\,\mathrm{Re}[R]+\mathrm{Im}[c]\,(\mathrm{Im}[r]\,\mathrm{Im}[R]+\mathrm{Re}[r]\,\mathrm{Re}[R])\geq \mathrm{Im}[R]\,\mathrm{Re}[c]\,\mathrm{Re}[r]+\mathrm{Im}[c]\left(\mathrm{Im}[R]^2+\mathrm{Re}[R]^2\right)\right.\;\&\&$$
$$\left(\mathrm{Im}[c]\left(\mathrm{Im}[R]^2+\mathrm{Re}[R]^2\right)\geq 0\;||\;\mathrm{Im}[c]\left(\mathrm{Im}[R]^2-\mathrm{Im}[c]\,\mathrm{Im}[R]\,\mathrm{Re}[r]+\mathrm{Re}[R]\,(-\mathrm{Re}[r]+\mathrm{Re}[R])\,+\right.\right.$$
$$\left.\left.\left.\mathrm{Im}[r]\,(-\mathrm{Im}[R]+\mathrm{Im}[c]\,\mathrm{Re}[R])\right)\leq(-1+\mathrm{Re}[c])\,\mathrm{Re}[c]\,(\mathrm{Im}[R]\,\mathrm{Re}[r]-\mathrm{Im}[r]\,\mathrm{Re}[R])\right)\right)\;||$$
$$\left(\mathrm{Im}[r]\,\mathrm{Re}[c]\,\mathrm{Re}[R]+\mathrm{Im}[c]\,(\mathrm{Im}[r]\,\mathrm{Im}[R]+\mathrm{Re}[r]\,\mathrm{Re}[R])\leq \mathrm{Im}[R]\,\mathrm{Re}[c]\,\mathrm{Re}[r]+\mathrm{Im}[c]\left(\mathrm{Im}[R]^2+\mathrm{Re}[R]^2\right)\right.\;\&\&$$
$$\left(\mathrm{Im}[c]\left(\mathrm{Im}[R]^2+\mathrm{Re}[R]^2\right)\leq 0\;||\;\mathrm{Im}[c]\left(\mathrm{Im}[R]^2-\mathrm{Im}[c]\,\mathrm{Im}[R]\,\mathrm{Re}[r]+\mathrm{Re}[R]\,(-\mathrm{Re}[r]+\mathrm{Re}[R])\,+\right.\right.$$
$$\left.\left.\left.\left.\mathrm{Im}[r]\,(-\mathrm{Im}[R]+\mathrm{Im}[c]\,\mathrm{Re}[R])\right)\geq(-1+\mathrm{Re}[c])\,\mathrm{Re}[c]\,(\mathrm{Im}[R]\,\mathrm{Re}[r]-\mathrm{Im}[r]\,\mathrm{Re}[R])\right)\right)\right)\;\&\&$$
$$\left(\frac{R}{r-R}\notin \text{Reals}\;||\;\mathrm{Re}\left[\frac{R}{r-R}\right]<-1\;||\left(\frac{R}{r-R}\neq 0\;\&\&\;\mathrm{Re}\left[\frac{R}{r-R}\right]\geq 0\right)\right)\;\&\&$$
$$\left(\left(\left(\mathrm{Re}\left[\frac{R-c\,R}{c\,r-c\,R}\right]\geq 1\;||\;\mathrm{Re}\left[\frac{R-c\,R}{c\,r-c\,R}\right]\leq 0\right)\;\&\&\;\frac{(-1+c)\,R}{c\,(r-R)}\notin \text{Reals}\right)\;||\right.$$
$$\left.\left.\frac{R-c\,R}{c\,r-c\,R}\notin \text{Reals}\;||\;\mathrm{Re}\left[\frac{R-c\,R}{c\,r-c\,R}\right]>1\;||\;\mathrm{Re}\left[\frac{R-c\,R}{c\,r-c\,R}\right]<0\right)\right]$$

In[30]:=
```
vprem[r_?NumberQ] := 10^(3/2) Sqrt[2 * NIntegrate[g_prem[x], {x, r, R}]]
v[r_?NumberQ] := Sqrt[2 * 4 Pi 10 ^ 6 G NIntegrate[ ρ[x] x ^ 2 / y ^ 2, {y, r, R}, {x, 0, y}]]
```

In[ ]:=
```
{v[R], v[0], vprem[0]}
```

Out[ ]=
```
{0., 9842.53, 9914.73}
```

Check of times for constant density case

```
Integrate[1 / Sqrt[R ^ (2) - x ^ (2)], {x, 0, R}]
```

$$\text{ConditionalExpression}\left[\frac{\pi}{2},\ \mathrm{Re}[R]>0\ \&\&\ \mathrm{Im}[R]==0\right]$$

2. Traversal times for chord path.

```
Remove[c, d, R];
Integrate[
 1 / Sqrt[(2 + (1 - c) ^ {2 + d} * (2 + c + c * d)) * R ^ 2 - (2 * R / x + (1 - c * x / R) ^ {2 + d} * (2 * R / x + c + c * d))],
 {x, 0, R}]
```

```
Remove[c, k, R, x];
Integrate[1 / Sqrt[1 - (1 / k) * (2 * R / x - (1 - c * x / R) ^ 2)], {x, 0, R}]
```

$$\int_0^R \frac{1}{\sqrt{1 - \frac{\frac{2R}{x} - \left(1 - \frac{c\,x}{R}\right)^2}{k}}}\, dx$$

In[34]:=
```
Tprem = 10 ^ 3 NIntegrate[1 / vprem[x], {x, 0, R}] / 30
Teff = 10 ^ 3 NIntegrate[1 / v[x], {x, 0, R}] / 30
```

Out[34]= 38.1875

Out[35]= 37.7428

In[37]:=
```
Error = (Tprem - Teff) / Tprem * 100
```

Out[37]= 1.16436