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The gravity tunnel in a non-uniform Earth

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This paper examines the gravity tunnel using the internal structure of Earth as ascertained from seismic data. Numerically, it is found that the time taken to fall along the diameter is 38 min, compared to 42 min for a planet with uniform density. The time taken to fall along a straight line between any two points is no longer independent of distance but interpolates between 42 min for short trips and 38 min for long trips. The brachistochrone path (minimizing the time between any two points) is similar in shape to the uniform-density solution but tends to reach a greater maximum depth and takes less time to traverse. Although the assumption of uniform density works well in many cases, the simpler assumption of a constant gravitational field serves as a better approximation to the true results.
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I. INTRODUCTION

The idea of the gravity tunnel was proposed by Cooper in 1966 in the American Journal of Physics.[1] He showed that a projectile would take 42 min to fall through a tube drilled straight through the planet along its diameter, given some assumptions and ignoring various engineering considerations. In addition, he showed that a straight tube connecting any two points could be traversed in the same amount of time, independent of distance, and the time could be made shorter with a more efficient curved path.

In a subsequent issue of this journal, five technical comments on Cooper's original paper appeared. Kirmser[2] lamented the insufficient literature review and pointed out that the idea can be found in an 1898 engineering textbook, although the concept also appears in an 1883 French magazine.[3] Venezian,[4] Mallett,[5] and Laslett[6] each derived an expression for the brachistochrone through Earth, the path over which the total transit time is minimized. Cooper himself wrote a comment addressing these papers[7] and suggested that the length independence of chord fall times was a coincidence based on the assumption of uniform density. Some applications of the idea were discussed in subsequent issues.[8,9] While the various discussions of such tunnels maintained the assumption that Earth is of uniform density, Simoson[10] considered the time taken to fall to the center according to various (realistic and unrealistic) models of Earth's interior.

Since it is unlikely that such a tunnel will ever be excavated in the near future, the concept serves largely a pedagogical role. In introductory physics, the diameter-length gravity tunnel is used as an example of simple harmonic motion: it is much easier to derive the period of oscillations than to solve kinematical equations with a changing acceleration. In advanced mechanics, it is revisited as a problem of variational calculus: what is the path connecting two points that would take the shortest amount of time to traverse?

The key assumption made when discussing the gravity tunnel is that the density of Earth is uniform throughout. This makes the gravitational field linear with respect to radial position, which means that the falling object will undergo simple harmonic motion. This paper examines the gravity tunnel without this assumption, to quantify the assumption's accuracy as an approximation and to obtain more accurate estimates that go beyond this approximation.

The internal structure of Earth is described by the preliminary reference Earth model (PREM), based on reconstructions from seismic data.[11] The radial density profile can be used to reconstruct the radial mass and gravity profiles. A reproduction of PREM data is shown in Fig. 1. The Earth is denser towards the center (reaching 13000 kg/m^3) and exhibits a sharp discontinuity in the density at the boundary of the outer core, dropping by nearly 50%. Because of this sharp discontinuity, the gravitational field strength actually *increases* below the surface, reaching a maximum of about 1.09 g, before decreasing in a roughly linear manner through the core to the center.

Using the reconstructed gravitational field strength inside the Earth, three versions of the gravity tunnel are analyzed below: a straight line through the center of the Earth to the other side; a straight line between two non-antipodal points; and the brachistochrone path that minimizes travel time. Numerical integration is used to generate solutions to these problems.

Although the typical assumption is that the density of Earth is uniform, an even simpler assumption can be made: that the gravitational field is constant in magnitude throughout the interior of the planet, always pointing towards the center with a strength of 9.8 N/kg. A glance at Fig. 1, however, shows that this simpler assumption does not agree at all with the PREM data. Moreover, this assumption is unphysical in that it implies a singular density at the origin, and also because it leads to a sharp discontinuity in acceleration as the falling object passes the origin. Nevertheless, we will show that this assumption works quite well at predicting the travel time for a more realistic density profile.

II. DERIVATIONS AND CALCULATIONS

A. Falling through the center of Earth

The time taken to fall through Earth along its diameter is usually calculated under the assumption that the Earth is of uniform density ρ. Under this approximation, the force of gravity F_G acting on a test mass m at radial position r comes from the mass of the sphere below the object, due to the shell theorem

$$F_G(r) = -\frac{Gm\left(4\pi\rho r^3/3\right)}{r^2} = -mg\frac{r}{R}. \tag{1}$$

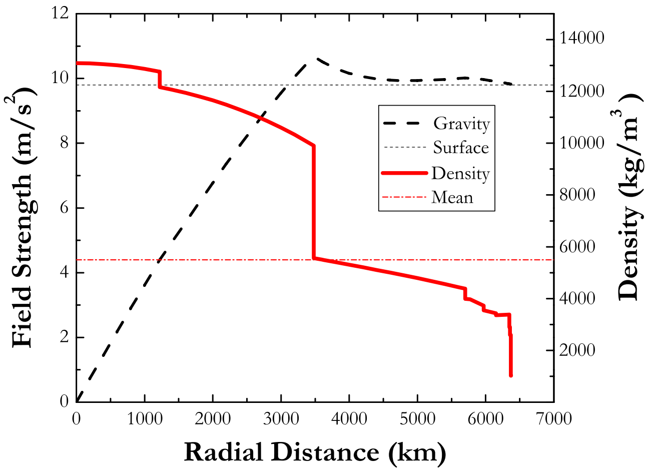

Fig. 1. The gravitational field strength (dashed) and density (solid) as functions of radius inside Earth according to the PREM (Ref. 11). The horizontal lines indicate the surface gravity and mean density.

Here, $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$ is the gravitational constant, R is the radius of Earth, and $g = 9.8\,\mathrm{N/kg}$ (or $9.8\,\mathrm{m/s^2}$) is the gravitational field strength at the surface of Earth ($r = R$). Because the force in Eq. (1) is linear in r, the test mass undergoes simple harmonic motion with angular frequency

$$\omega = \sqrt{\frac{4\pi}{3} G\rho} = \sqrt{\frac{g}{R}}, \tag{2}$$

so the period of oscillation would be

$$\tau_\rho = \frac{\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}} \approx 84\,\mathrm{min}. \tag{3}$$

Therefore, it would take 42 min to fall through a uniform Earth, and the peak velocity at the center would be near 8 km/s, over 30 times the speed of a typical transatlantic aircraft. Interestingly, the period of oscillation and the peak velocity are the same for a circular orbit at Earth's surface (a grazing orbit).[12]

If, on the other hand, the gravitational field is constant (in magnitude) inside the Earth, then the time taken to fall through the Earth can be found by simple kinematics

$$\tau_g = 2\sqrt{\frac{2R}{g}} \approx 38\,\mathrm{min}. \tag{4}$$

This is shorter than the uniform-density time by a factor of $\pi/\sqrt{8}$, or about 11%.

To calculate the time taken to fall to the center of the Earth given the PREM radial gravity profile (Fig. 1), the kinematical equation is integrated numerically, with the gravitational strength at any radial position calculated by linear interpolation between the two closest PREM reference points (we use MATLAB for the calculations). The result is that the time taken to fall through a tunnel through the center of the real Earth is almost exactly what it would be if the gravitational field strength were uniform throughout the Earth (see Fig. 2), about 38 min. At any given time during the fall, the position of an object falling through the Earth would be very close to its position as predicted by $R - gt^2/2$, with deviations of up to 50 m. To the nearest second, the fall times are:

- PREM: 38 min 11 s;
- constant gravity: 38 min 0 s;
- uniform density: 42 min 12 s.

The reason for the remarkable accuracy of the constant-gravity result is discussed in Sec. III.

B. The chord path

The uniform-density gravity tunnel has the interesting property that any chord path, along a straight line between any two points on the surface, will be traversed in the same amount of time. Does the non-uniform density of the planet still preserve this feature? To answer this question, we consider the kinematics of an object falling on a chord path under a central gravitational pull, using the coordinate system in Fig. 3.

An object falling along a non-central linear path under the influence of central gravity experiences an acceleration

$$a = \frac{d^2y}{dt^2} = -g(\theta)\sin\theta. \tag{5}$$

The position at any time can be written as (see Fig. 3)

$$y = R\cos\theta_0 \tan\theta, \tag{6}$$

and when substituted into Eq. (5) leads to

$$R\cos\theta_0 \sec^2\theta\,[\ddot{\theta} + 2\dot{\theta}^2 \tan\theta] = -g(\theta)\sin\theta, \tag{7}$$

where a dot represents a time derivative. At this point, the gravitational field strength $g(\theta)$ in Eq. (7) is arbitrary; it can either be the constant surface gravity, the realistic internal gravity from the PREM data, or the radially linear gravity gr/R from the uniform-density assumption. The time taken to fall to the middle of a linear path can be found numerically by integrating Eq. (7) from θ_0 to zero using the Runge–Kutta

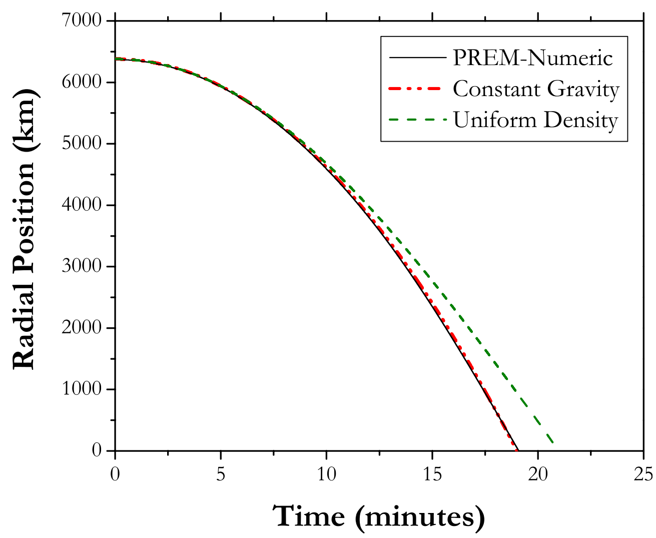

Fig. 2. Radial position versus time when falling to the center of Earth according to the PREM data and according to the uniform-density and constant-gravity approximations. The PREM curve is not identical to the constant-gravity curve, but the differences are difficult to distinguish by eye.

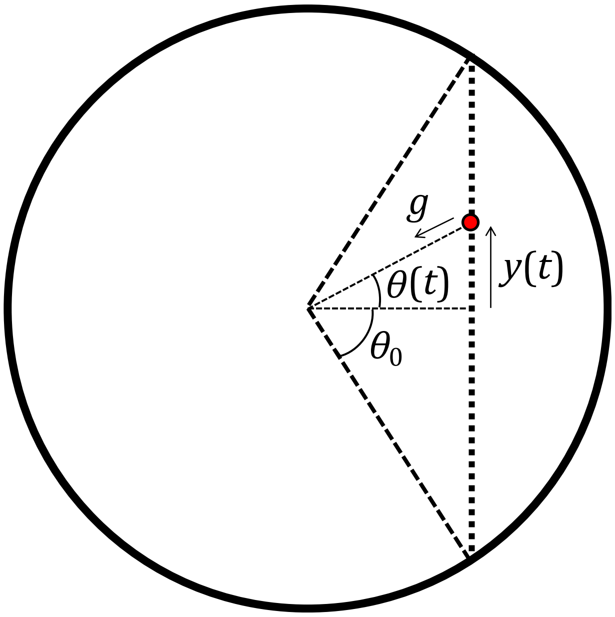

Fig. 3. Diagram of the coordinate system used to find the time taken to fall through a chord path.

method and recording the number of time steps in the integration.

Figure 4 shows the chord path fall times as a function of surface distance. The numerical results replicate the distance independence for the uniform-density case, whereas the PREM time decreases with increasing path length, from near 42 min for short paths where the gravity does not deviate much from its surface value, to 38 min as the path approaches the full diameter. The constant-gravity solution is again similar to the PREM solution, following the same trend with respect to distance. An exact solution exists for the constant-gravity chord fall time that can be expressed in terms of elliptic integrals (see Appendix A), or as a Taylor series about τ_ρ. The series is similar to the correction to the simple pendulum period for the initial angle dependence,[13] the main difference being that the leading-order correction is negative. The time can be approximated well at next-to-leading order

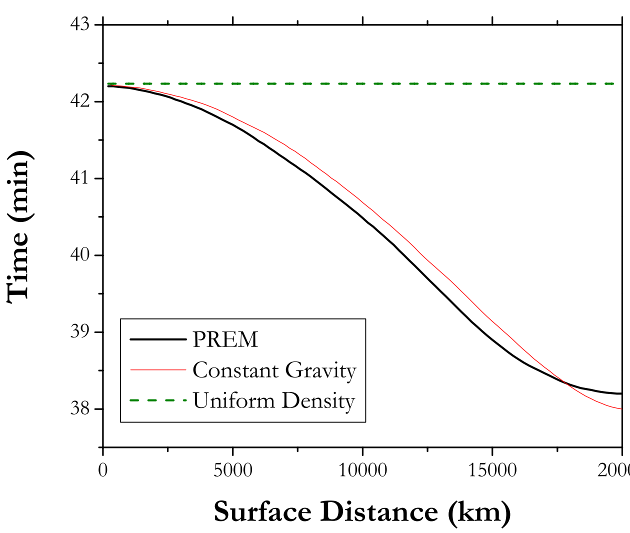

Fig. 4. Time taken to fall through a chord path between two points at a given distance along the surface.

$$\tau_g(\theta_0) = \pi\sqrt{\frac{R}{g}}\left[1 - \frac{1}{16}\theta_0^2 + \frac{19}{3072}\theta_0^4 + O\left(\theta_0^6\right)\right]. \tag{8}$$

C. The brachistochrone

The brachistochrone (from the Greek for "shortest time") is the path that takes the least amount of time to fall between two points. The simple brachistochrone in a uniform vertical gravitational field was issued as a challenge by Johann Bernoulli in 1696, leading to the development of variational calculus. The brachistochrone path for a gravity tunnel inside a uniform Earth was considered numerically by Cooper[1] and solved analytically by Venezian and others.[4] Here, the brachistochrone path for an arbitrary spherical mass distribution will be derived, in order to numerically find the path through the Earth according to the PREM.

Conservation of energy dictates that the sum of kinetic and gravitational potential energies is constant. For a given radial mass distribution $M(r)$, the gravitational potential energy at a given point r is the radially integrated gravitational field from the potential zero-point R_0 to the point of interest r. Following the convention set (implicitly) by Venezian,[4] the potential zero-point can be placed at the center of the Earth ($R_0 = 0$) so that the potential energy is positive everywhere. This convention may be unacceptable for extreme mass distributions such as that of a point-mass, in which case the zero can be taken at infinity. Ultimately, it is only the difference in potential between the surface and a given radial height that is relevant.

The total energy E (kinetic plus potential) at any radial position r must equal the potential energy at the surface R, where the velocity is zero

$$E = \frac{1}{2}mv^2 + \int_{R_0}^{r}\frac{GM(r)m}{r^2}dr = \int_{R_0}^{R}\frac{GM(r)m}{r^2}dr. \tag{9}$$

Solving for the velocity then gives

$$v(r) = \sqrt{2GI(r)}, \tag{10}$$

where we have defined

$$I(r) \equiv \int_r^R \frac{M(r)}{r^2}dr. \tag{11}$$

The brachistochrone path between points A and B is found by minimizing the time integral τ in polar coordinates

$$\tau = \int_A^B \frac{ds}{v} = \int_0^{\theta_{AB}} \frac{\sqrt{dr^2 + r^2 d\theta^2}}{v(r)} = \int_0^{\theta_{AB}} \sqrt{\frac{r'^2 + r^2}{2GI(r)}}\,d\theta, \tag{12}$$

where the prime denotes a derivative with respect to θ. The integrand $f(r, r')$ does not explicitly depend on the θ, so the integral can be minimized using the Beltrami identity (also known as the second form of Euler's condition)[15]

$$f - r'\frac{\partial f}{\partial r'} = C, \tag{13}$$

where C is a constant. Evaluating this minimization condition gives

$$\frac{r^2}{\sqrt{2GI(r)(r^2+r'^2)}} = C. \tag{14}$$

Now, when the path reaches its maximum depth (at $r = R_d$) we have $r'(\theta) = 0$. Thus, Eq. (14) simplifies to

$$\frac{R_d}{\sqrt{2GI(R_d)}} = C. \tag{15}$$

Finally, combining Eqs. (14) and (15) leads to an equation that can be solved to give the brachistochrone

$$\frac{dr}{d\theta} = \frac{r}{R_d}\sqrt{\frac{r^2 I(R_d) - R_d^2 I(r)}{I(r)}}. \tag{16}$$

Without knowing the radial mass profile, we can go no further. For the case of uniform density, Eqs. (11) and (16) lead to

$$\frac{dr}{d\theta} = \frac{rR}{R_d}\sqrt{\frac{r^2 - R_d{}^2}{R^2 - r^2}}. \tag{17}$$

Similarly, for the case of constant gravity, we find

$$\frac{dr}{d\theta} = \frac{r}{R_d}\sqrt{\frac{(r-R_d)[r(R-R_d)+RR_d]}{R-r}}. \tag{18}$$

The known analytic solution to Eq. (17) is that of a hypocycloid curve—the shape traced by a circle of diameter $(R - R_d)$ rolling inside a circle of radius R. Equation (18) is similar to the classic brachistochrone proposed by the Bernoullis, but the polar geometry makes it more difficult to solve. The solutions to these models are discussed in Appendix A.

Equation (16) was solved numerically for the PREM model using the following scheme. Starting with $r = R_d$ (with a miniscule offset to allow a nonzero derivative) and $\theta = 0$, the path was calculated by Euler integration using the known derivative. The gravitational field integral $I(r)$ was calculated numerically from the radial mass distribution, and at each radius the integral was calculated based on linear interpolation between the two closest reference points. The radius was then incremented according to its derivative until it reached or exceeded the total radius of the Earth. The time taken to fall through this path was calculated by solving the time integral (12), again with Euler integration, using the calculated values for r as a function of θ. The final integration step was numerically unstable because the derivative diverges as the path becomes vertical; this instability was handled by halting the integration at the penultimate step and integrating to the surface assuming a vertical path. The same procedure was repeated for the uniform-density case (17) and the constant-gravity case (18). As a validation of this scheme, the times calculated for the uniform-density case as a function of distance can be compared to the known analytic solution (see Appendix A).

The times taken to traverse the brachistochrone paths as functions of surface distance are shown in Fig. 5. In each case, the time taken to traverse the brachistochrone path

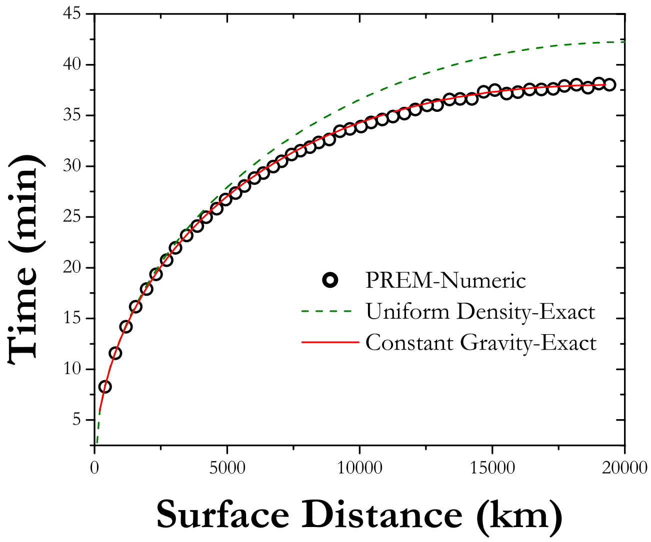

Fig. 5. Minimum time taken to fall through a tunnel connecting two points, as a function of distance along the surface. The constant gravity and PREM curves appear similar by eye but are distinct.

increases from zero to the diameter-length distance, be it 38 or 42 min, and the relationship between surface distance and time was similar. The brachistochrone time for the PREM solution is very close to that predicted by the constant-gravity assumption, as it is for the diameter and chord paths.

The paths connecting sets of two points of varying separation are shown in Fig. 6. For the shortest paths shown, about 2000 km in separation (approximately the distance between Paris and Athens), the PREM path and those of the constant gravity and uniform-density assumptions are all similar; this is not surprising since the gravitational field near the surface is similar for all three models. For the medium-length paths shown, about 10000 km in separation (approximately the distance between Tokyo and Chicago), the PREM path runs deeper than the uniform-density path, while the constant-gravity path is again similar to the PREM path. For the longest paths shown, about 17000 km (approximately the distance between Montreal and Melbourne), the PREM path runs deeper than the uniform-density path, and the constant-gravity path is deeper still.

III. DISCUSSION

It is clear from these results that the approximation of a constant magnitude gravitational field is more accurate than the approximation of a constant mass density. Why does the

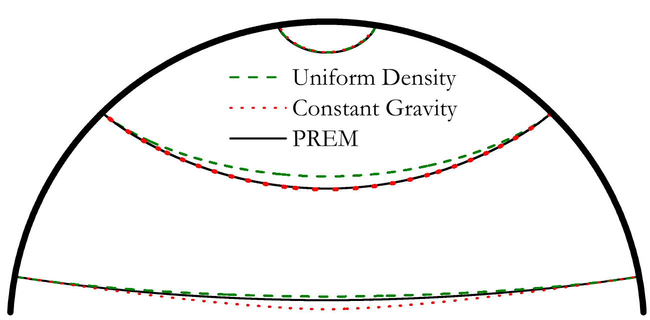

Fig. 6. Diagrams of brachistochrone paths through Earth. Paths of three separations (surface distance approximately 2000 km, 10000 km, and 17000 km) are shown, with the brachistochrone path calculated for the PREM model and the uniform density and constant-gravity assumptions.

constant-gravity approximation work so well? Heuristically, the gravitational field strength inside the Earth does not deviate far from its surface value until more than halfway towards the center (see Fig. 1). By the time the falling object reaches these weaker gravitational fields, it is traveling sufficiently fast that the time spent in these regions is minimal; most of the time is spent in regions where the acceleration is close to g. Empirically, a constant gravitational field would imply that the mass enclosed within a given volume is quadratic with radius, in order to exactly cancel Newtonian gravitation. A power-law fit to the PREM mass profile for $M(r) = M(r/R)^{\alpha}$ yields a scaling exponent $\alpha = 1.97 \pm 0.02$, which is very close to two.

There are other complications that this article has ignored. The asphericity of the Earth, leading to differences in the radius and gravitational field with respect to latitude,[14] gives corrections of only about ten seconds. Planetary rotation, again imparting latitude dependence, is insignificant as long as the fall time is much less than the rotation period; for Earth, the correction amounts to about four seconds over the forty-minute trip.[1]

It was shown that the PREM and constant-gravity paths run deeper than the uniform-density path, and for the longest paths the constant-gravity path runs the deepest. Qualitatively, the path that minimizes the time is one that reaches great depths where velocities are highest, while still keeping the total arc-length of the path short. For example, a path that follows a straight line to the center, changes directions, and returns to the surface in another straight line, reaches the greatest velocity but is not ideal as it takes as long as the diameter path. Similarly, the cord path is geometrically the shortest but takes even longer to traverse because high velocities are not reached. In the PREM and constant-gravity models, because the gravitational field does not decrease as much or at all with depth compared to that of the uniform-density model, the PREM and constant-gravity paths can run deeper, trading a longer path for greater velocity. Within the core region, the gravitational field in the PREM model also decreases strongly with depth, while the constant gravitational field continues to run deeper without compromising fall time.

In his second paper,[7] Cooper discusses whether it is coincidental that the all linear paths (as well as the orbit half-period) take the same time. He argues that it is coincidental based on his assumption that Earth's density is uniform. Relaxing that assumption, it is seen that this equivalence is now merely a similarity: chord fall times vary by up to 11% from the surface orbit time. Cooper surmised that the true fall time would be "very much different" given a non-uniform density, but in light of the analysis presented in this paper, the actual answer appears closer to his approximation than he thought.

A question central to this paper is whether the assumption of uniform density is justified. The results show that such an assumption works reasonably well at approximating the PREM solution; deviations rarely exceed ten percent. However, if simplifying assumptions are to be made, then the assumption of constant gravity is more accurate. Pedagogically, this assumption would invalidate the gravity tunnel as an introductory problem for simple harmonic motion, as the fall is described by constant-acceleration kinematics. The brachistochrone curve and time under this assumption may be solvable, but it is more challenging than the uniform-density assumption. These hallmark textbook problems should not necessarily change in light of these computations, although a discussion of the validity of the assumptions may be appropriate.

IV. CONCLUSION

Numerical analysis has been used to study the dynamics of an object falling through a gravity tunnel without the assumption that Earth has a uniform mass density. It was found that the assumption of a constant gravitational field serves as a better approximation to the empirically based PREM result than the assumption of uniform density. For an object falling along the diameter of the Earth, the uniform-density assumption overpredicts the time taken by 4 min, or 11%. The most significant deviation from the uniform-density predictions is that the fall time for a chord path is no longer independent of surface distance. The brachistochrone curves do not deviate significantly from the analytic uniform-density result, although they tend to reach a greater depth. These analyses show that the assumption of constant gravity is more accurate than that of uniform density, though the latter remains useful pedagogically.

APPENDIX A: SOLUTIONS TO THE SIMPLIFIED BRACHISTOCHRONES

This appendix discusses closed-form solutions to the brachistochrone curves for constant gravity and uniform density. The computer algebra software package MAPLE 9 was used to find many of these expressions. Throughout this section, Φ_F, Φ_E, and Φ_Π are the *incomplete* elliptic integrals of the first, second, and third kind, respectively, while Φ_K and Φ_G, are the *complete* elliptic integrals of the first and second.

The constant-gravity brachistochrone is traversed in time [see Eqs. (11) and (12)]

$$\tau = \int \frac{\sqrt{dr^2 + r^2 d\theta^2}}{\sqrt{2g(R-r)}}, \tag{A1}$$

and the minimization condition is dependent on the radius at maximum depth [see Eq. (18)]

$$\frac{dr}{d\theta} = \frac{r}{R_d}\sqrt{\frac{(r-R_d)[r(R-R_d)+RR_d]}{R-r}}. \tag{A2}$$

The path can be found by isolating $d\theta$ in the above expression and integrating to get

$$\theta(r) = -i\frac{2}{R(2R-R_d)} \times \left[\left(R_d^2 + 2R^2 - 3R_dR\right)\Pi + \left(2R_dR - R_d^2\right)F\right], \tag{A3}$$

where the symbols Π and F are defined as

$$\Pi = \Phi_\Pi\left(\sqrt{\frac{rR - rR_d + RR_d}{R_d\,(2R-R_d)}}, \frac{2R-R_d}{R}, \sqrt{\frac{R_d\,(2R-R_d)}{R^2}}\right) \tag{A4}$$

and

$$F = \Phi_F\left(\sqrt{\frac{rR - rR_d + RR_d}{R_d}}, \sqrt{\frac{R_d\,(2R - R_d)}{R^2}}\right). \qquad \text{(A5)}$$

The angle subtended over the path from a given R_d to the surface can be found by substituting $r = R$ into the above Eq. (A3), and the total angle subtended by the path can be found by doubling that result. The point-to-point surface distance traversed S can be found by multiplying that angle by the radius of the Earth. The total time can be found by substituting the minimal path derivative (A2) into the time integral, giving

$$\begin{aligned} \tau &= 2\int_{R_d}^{R} \frac{\sqrt{1 + (r/r')^2}}{\sqrt{2g(R - r)}}\, dr = i\sqrt{\frac{8}{g(R - R_d)}} \\ &\quad \times \Big\{ R\big[\Phi_G(\delta) - \Phi_K(\delta) + \Phi_F(\delta^{-1}, \delta) - \Phi_E(\delta^{-1}, \delta)\big] \\ &\quad + R_d\big[\Phi_K(\delta) - \Phi_F(\delta^{-1}, \delta)\big]\Big\}, \qquad \text{(A6)} \end{aligned}$$

where

$$\delta = \sqrt{\frac{R_d(2R - R_d)}{R^2}}. \qquad \text{(A7)}$$

The factor of i in the equations above cancels an imaginary term from the elliptic functions. Because elliptic functions are periodic over the complex plane, some implementations may apply unwanted complex rotations. Using MAPLE, for example, requires angles to be calculated by adding π to the real part of the output of Eq. (A3).

By comparison, the brachistochrone for the uniform-density Earth, as derived by Venezian,[4] is much simpler

$$\begin{aligned} \theta(r) &= \arctan\left(\frac{R}{R_d}\sqrt{\frac{r^2 - R_d{}^2}{R^2 - r^2}}\right) \\ &\quad - \frac{R_d}{R}\arctan\left(\sqrt{\frac{r^2 - R_d{}^2}{R^2 - r^2}}\right). \qquad \text{(A8)} \end{aligned}$$

Because the hypocycloid curve is defined by a small circle rolling inside a larger one, the ratio between surface distance S and maximum depth $R - R_d$ is simply π. The time taken to traverse the path is can thus be written

$$\tau = \pi\sqrt{\frac{R^2 - R_d^2}{Rg}} = \sqrt{\frac{S}{R}\left(\frac{2\pi R - S}{g}\right)}. \qquad \text{(A9)}$$

While the surface distance S is a more natural variable with which to consider these paths, the minimum radius R_d is typically used in calculations because of the simpler boundary conditions at that point.

The time taken to fall along a linear path under constant gravity can be found using the same method as above

$$\begin{aligned} \tau_{\text{line}} &= \int_{-\theta_0}^{\theta_0} \frac{R\cos\theta_0 \sec^2\theta}{\sqrt{2gR(1 - \cos\theta_0\sec\theta)}}\, d\theta, \\ &= \sqrt{\frac{8R}{g(1 + \cos\theta_0)}}\left[\Phi_G\left(\sqrt{\frac{1 - \cos\theta_0}{1 + \cos\theta_0}}\right)\right. \\ &\quad \left.\times (1 + \cos\theta_0) + \Phi_K\left(\sqrt{\frac{1 - \cos\theta_0}{1 + \cos\theta_0}}\right)\cos\theta_0\right]. \qquad \text{(A10)} \end{aligned}$$

To validate the numerical scheme used to generate the PREM brachistochrone, these analytic expressions are compared in Fig. 7 to numerical solutions with the same potential.

APPENDIX B: SAMPLE QUESTIONS

Because the traditional uniform-density gravity tunnel is often discussed as an exercise in undergraduate physics classes, some additional questions about the more realistic case may be a useful addition to a discussion of the subject. Some sample questions are suggested here but are by no means exhaustive, and are appropriate for an advanced mechanics class.

1. What is the time needed to fall through the moon and other planets in the solar system, under the uniform-density and constant-gravity assumptions? Given what is known about their internal structure, can you find a more realistic time?
2. Show whether or not the chord paths and brachistochrone paths are also tautochrones (they take the same amount of time to traverse regardless of initial position). Consider both the uniform-density and constant-gravity cases.
3. Examining the gravitational field profile in Fig. 1, create a piece-wise (or continuous) model describing the gravity (or density) inside the Earth. How long does it take to fall according to your model?
4. Cooper[1] considered another type of path where straight lines from two points meet below their surface midpoint. According to each model, what is the ideal angle at which the two paths intersect, and how long does the transit take.
5. Does the equivalence principle apply to an observer falling through a gravity tunnel? If you are standing on a scale in an elevator falling through such a tunnel, what would it read?
6. How long would it take to fall along a chord path outside a point mass?

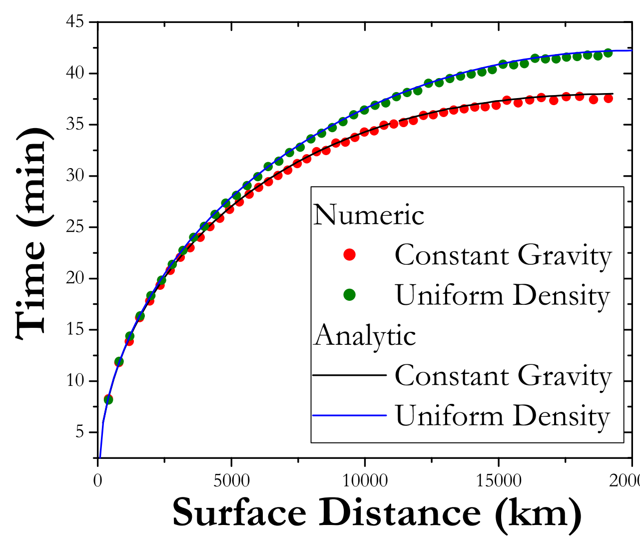

Fig. 7. Comparison of analytic and numeric solutions to the brachistochrone travel time for the constant-gravity and uniform-density approximations.

7. Consider the first-order relativistic correction to the harmonic oscillator: $T = (2\pi/\omega)[1 + \frac{3}{16}(\omega R/c)^2]$. How would this correction affect the time taken to fall through a white dwarf and a neutron star?

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[1]P. W. Cooper, "Through the Earth in forty minutes," Am. J. Phys. **34**, 68–70 (1966).

[2]P. G. Kirmser, "An example of the need for adequate references," Am. J. Phys. **34**, 701 (1966).

[3]A. Redier, "De Paris à Rio de Janeiro en 42 minutes et 11 secondes," La Nature **520** (May 19, 1883).

[4]G. Venezian, "Terrestrial brachistochrone," Am. J. Phys. **34**, 701 (1966); available at http://sciences.gloubik.info/spip.php?article1223.

[5]R. L. Mallett, "Comments on 'through the Earth in forty minutes'," Am. J. Phys. **34**, 702 (1966).

[6]L. J. Laslett, "Trajectory for minimum transit time through the earth," Am. J. Phys. **34**, 702–703 (1966).

[7]P. W. Cooper, "Further commentary on 'through the Earth in forty minutes'," Am. J. Phys. **34**, 703–704 (1966).

[8]S. M. Lee, "The isochronous problem inside the spherically uniform earth," Am. J. Phys. **40**, 315–318 (1972).

[9]J. E. Prussing, "Brachistochrone-tautochrone problem in a homogeneous sphere," Am. J. Phys. **44**, 304–305 (1976).

[10]A. J. Simoson, "The gravity of Hades," Math. Mag. **75**, 335–350 (2002).

[11]A. M. Dziewonski and D. L. Anderson, "Preliminary reference earth model," Phys. Earth Planet. Inter. **25**, 297–356 (1981).

[12]R. H. Romer, "The answer is forty-two—Many mechanics problems, only one answer," Phys. Teach. **41**, 286–290 (2003).

[13]R. A. Nelson and M. G. Olsson, "The pendulum: Rich physics from a simple system," Am. J. Phys. **54**(2), 112–121 (1986).

[14]H. Moritz, "Geodetic reference system 1980," J. Geod. **54**, 395–405 (1980).

[15]G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic Press, San Diego, 1985), p. 928.

Papin's Digester

The ancestor of this device was developed in 1690 by Denis Papin (1647-1712) and is still in use as the modern pressure-cooker. The apparatus in the picture is listed at $17.50 in the 1916 catalogue of the E.L. Knott Apparatus Company of Boston as a device to illustrate the rise in the boiling point of water as the pressure is increased in the sealed vessel. A thermometer sits in the well projecting upward at the back of the top plate and a bath of oil keeps it in contact with the water inside. Note the essential safety valve arm projecting to the right. A Bunsen burner beneath the apparatus supplies the necessary energy to heat the water. This apparatus is a Vassar College. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)