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A note on a straight gravity tunnel through a rotating body

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It is well-known that a straight gravity tunnel between any two different positions on a non-rotating uniform density Earth is traversable, i.e., an object initially at rest will reach its destination through the gravity tunnel in both directions. Moreover, the time taken to fall is independent of the tunnel endpoints. These facts are no longer true if rotation is allowed. The aim of this note is to derive the necessary and sufficient condition for traversability of straight gravity tunnels through a rotating physical body with a spherically symmetric gravitational field. Fall-through times are expressed in a closed form for linear and constant gravitational fields. These models are then compared to numerically obtained data using the internal structure of the Earth.
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I. INTRODUCTION

The concept of the gravity tunnel refers to the travel along frictionless subterranean passages from place A to place B, which are positioned on the surface of a massive physical body, e.g., the Earth, with no expenditure of energy for locomotion. Although the corresponding vehicle called the gravity train is probably in the domain of science fiction[1] only due to obvious engineering problems, this thought experiment is still an active field of research, which also paves the way for undergraduate physics. A classical result by Cooper[2] asserts that a gravity train has a simple harmonic motion and always needs a constant amount of time (roughly 42 min) to fall through a chord path (a straight gravity tunnel) between any two different positions on a non-rotating Earth, which has a uniform density, i.e., a linear gravitational field. Recently, several authors have discussed other planets' interiors and gravitational fields,[3–5] e.g., constant gravity, the gravitational strength predicted by the Preliminary Reference Earth Model[6] (PREM) and polytropes, as well as gravity tunnels with relativistic[7] and friction[8] effects. The brachistochrone path, i.e., a path which minimizes the time between any two points, is also studied in some of these papers. Some authors consider gravity tunnels to refer only to an object's free fall trajectories since then the gravity train is no longer subject to frictional and inertial forces. The results in this direction[9] are known for rotating homogeneous spheres,[11] rotating homogeneous flattened spheroids,[12] and rotating PREM Earth.[13] However, in this note, only straight gravity tunnels are considered. Some references to the history of gravity tunnels can be found in Selmke's note.[14]

The most common example of the gravity tunnel is through the center of a non-rotating Earth. Taking angular frequency $\omega_\oplus = \sqrt{g_\oplus/R_\oplus}$, where $g_\oplus = 9.807\,\mathrm{ms}^{-2}$ is the gravitational field strength at the surface of the Earth with the radius $R_\oplus = 6.371 \times 10^6\,\mathrm{m}$, it follows by simple kinematics that the half-period of oscillation in the linear gravitational field is $\pi\omega_\oplus^{-1} \approx 42.19\,\mathrm{min}$, while in the constant gravitational field, it is $2\sqrt{2}\omega_\oplus^{-1} \approx 38\,\mathrm{min}$. Note that Cooper's result about the constant fall-through time T is not true in the constant gravitational field. Moreover, T only depends on the distance of the gravity tunnel from the center of the Earth and $2\sqrt{2}\omega_\oplus^{-1} \le T < \pi\omega_\oplus^{-1}$.

Surprisingly, none of the above authors considered the rotating case for straight gravity tunnels. In this note, we would like to emphasize that rotation and the spherically symmetric gravitational field imply traversability of straight gravity tunnels. A traversable gravity tunnel means that the gravity train, which initially rests at A, will reach its destination B and vice versa. It turns out that there are two necessary and sufficient conditions: the first one asserts that the gravitational energy is large enough while the second one asserts that the absolute value of A's latitude is the same as the absolute value of B's latitude. The latter is more important because the first condition is, under reasonable circumstances, always satisfied by constant and linear gravitational fields as well as by the PREM. The gravity tunnels, which go through the center of a massive body or are perpendicular to the rotational axis, are rather specific.

Although Simoson[15] addresses this problem by providing several examples in linear gravitational fields, he does not state this general condition. In Sec. II, we derive Simoson's equation of motion, and in Sec. III, the general condition. In Sec. IV, exact formulas for fall-through times in linear and constant gravitational fields are provided and these results are then compared to the PREM.

II. THE EQUATION OF MOTION

We follow Simoson's paper,[15] although our derivation of the equation of motion differs from Simoson's in the sense that we use a non-inertial rotating reference frame instead of applying rotational transformation to an inertial one. The reader is advised to see Fig. 1.

Assume that our physical body is a rotating ball with radius R and rotational period Q. Take an orthogonal coordinate system $(x, y, z) \in \mathbb{R}^3$ with the origin O at the center of the ball such that the z-axis is the rotational axis. This gives rise to the body's inertial frame $\mathcal{I} = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Let $x = R\cos\varphi\cos\lambda$, $y = R\cos\varphi\sin\lambda$ and $z = R\sin\varphi$ be a

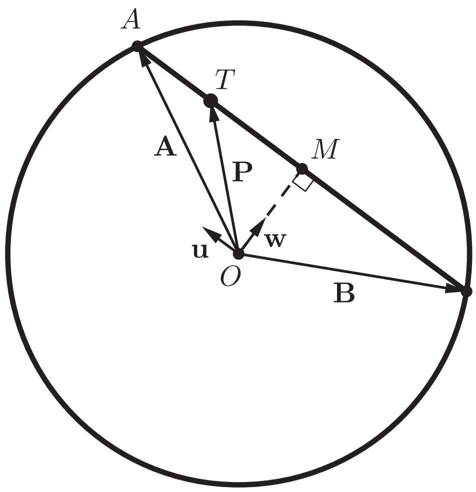

Fig. 1. The gravity train T that moves along a straight tunnel between A and B.

parametrization of the surface of this ball with latitude $\varphi \in [-\pi/2, \pi/2]$ and longitude $\lambda \in [-\pi, \pi)$.

Let A and B be two distinct points on the surface of this ball and let $\mathbf{A}$ and $\mathbf{B}$ be the corresponding position vectors. Identify point A with spherical coordinates (φ_0, λ_0) and B with (φ, λ). Take the vector $\mathbf{u} = (\mathbf{A} - \mathbf{B})/|\mathbf{A} - \mathbf{B}|$ and let $c = \frac{1}{2}|\boldsymbol{A} + \boldsymbol{B}|$ be the distance between O and M, the midpoint of the segment AB. We have

$$c = \frac{R}{\sqrt{2}}\sqrt{1 + \sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos(\lambda - \lambda_0)},$$

$$\mathbf{u} = \frac{-1}{2\sqrt{1-(c/R)^2}}(\cos\varphi\cos\lambda - \cos\varphi_0\cos\lambda_0, \cos\varphi\sin\lambda - \cos\varphi_0\sin\lambda_0, \sin\varphi - \sin\varphi_0).$$

Points A and B are antipodal points if and only if $c = 0$. If $c \neq 0$, then we take the vector $\mathbf{w} = (\mathbf{A} + \mathbf{B})/|\mathbf{A} + \mathbf{B}|$, which is

$$\mathbf{w} = \frac{R}{2c}(\cos\varphi\cos\lambda + \cos\varphi_0\cos\lambda_0, \cos\varphi\sin\lambda + \cos\varphi_0\sin\lambda_0, \sin\varphi + \sin\varphi_0).$$

If $c = 0$, then we take $\mathbf{w} = -(\sin\varphi_0\cos\lambda_0, \sin\varphi_0\sin\lambda_0, -\cos\varphi_0)$ if $\varphi_0 \notin \{-\pi/2, \pi/2\}$, and $\mathbf{w} = (1, 0, 0)$ otherwise. Observe that vectors $\mathbf{u}$ and $\mathbf{w}$ are always unit vectors and $\mathbf{u}$ is parallel to the line AB while $\mathbf{w}$ is perpendicular to it. We can take $\mathcal{F} = \{O; \mathbf{u}, \mathbf{w}, \mathbf{u} \times \mathbf{w}\}$ as a non-inertial frame which rotates with the angular velocity $\boldsymbol{\Omega} = (2\pi/Q)\mathbf{k}$ relative to $\mathcal{I}$. One can then describe the position of the gravity train T in the frame $\mathcal{F}$, which moves along the straight tunnel from A to B as a function $\mathbf{P}(t) = c\mathbf{w} + s(t)\mathbf{u}$ dependent on time t such that $\mathbf{P}(0) = \mathbf{A}$, i.e., $s(0) = \frac{1}{2}|\mathbf{A} - \mathbf{B}|$. By the acceleration transformation formula,[16] we have

$$\ddot{\mathbf{P}} = -\frac{f(|\mathbf{P}|)}{|\mathbf{P}|}\mathbf{P} + (\boldsymbol{\Omega} \times \mathbf{P}) \times \boldsymbol{\Omega} + 2\dot{\mathbf{P}} \times \boldsymbol{\Omega} + \mathbf{P} \times \dot{\boldsymbol{\Omega}}, \tag{1}$$

where $f(r)$, $0 \leq r \leq R$ is a spherically symmetric gravitational field of the physical body. The Euler acceleration $\mathbf{P} \times \dot{\boldsymbol{\Omega}}$ is obviously zero. After taking the dot product of both sides of Eq. (1) with $\mathbf{u}$, it is easy to see that $2(\dot{\mathbf{P}} \times \boldsymbol{\Omega}) \cdot \mathbf{u} = 0$, i.e., the Coriolis acceleration is perpendicular to the motion, and

$$((\boldsymbol{\Omega} \times \mathbf{P}) \times \boldsymbol{\Omega}) \cdot \mathbf{u} = |\boldsymbol{\Omega}|^2\mathbf{P} \cdot \mathbf{u} - (\boldsymbol{\Omega} \cdot \mathbf{P})(\boldsymbol{\Omega} \cdot \mathbf{u}) = \left(\frac{2\pi}{Q}\right)^2\left(s(t)\left(1 - u_3^2\right) - cu_3w_3\right).$$

Therefore, the only fictitious acceleration which plays a role here is the centrifugal acceleration. Thus, the function $s(t)$ must satisfy the equation

$$\ddot{s} = \left(-\frac{f\left(\sqrt{c^2 + s^2}\right)}{\sqrt{c^2 + s^2}} + \left(\frac{2\pi}{Q}\right)^2\left(1 - u_3^2\right)\right)s - c\left(\frac{2\pi}{Q}\right)^2 u_3w_3. \tag{2}$$

In Sec. III, we will write Eq. (2) in a dimensionless form.

III. THE MAIN RESULT

In order to write Eq. (2) in a dimensionless form with coefficients expressed in spherical coordinates, take $s(t) = R\chi(\tau)$, $t = Q(2\pi)^{-1}\tau$, $F(\chi) = Q^2R^{-1}(2\pi)^{-2}f(R\chi)$ and $\bar{c} = cR^{-1}$. Observe that $F(\chi)$ is a positive function defined on the interval $[0, 1]$. Using expressions for $\mathbf{u}$ and $\mathbf{w}$ from Sec. II, Eq. (2) becomes

$$\frac{d^2\chi}{d\tau^2} = \left(-\frac{F\left(\sqrt{\chi^2 + \bar{c}^2}\right)}{\sqrt{\chi^2 + \bar{c}^2}} + a\right)\chi + b, \tag{3}$$

with real numbers

$$a = 1 - \frac{(\sin\varphi - \sin\varphi_0)^2}{4(1 - \bar{c}^2)},$$

$$b = \frac{\sin(\varphi + \varphi_0)\sin(\varphi - \varphi_0)}{4\sqrt{1 - \bar{c}^2}}.$$

The formula for b is correct also in the case $c = 0$ since then $\varphi = -\varphi_0$ and b is zero, as it must be due to the last term in Eq. (2).

According to the definitions, we have $0 \leq \bar{c} < 1$ and $0 \leq a \leq 1$. Define $\bar{g} = \omega^2Q^2(2\pi)^{-2}$, where $\omega = \sqrt{g/R}$ and $g = f(R)$ is the gravitational field strength at the surface of the body. For the Earth $\omega_\oplus = 1.241 \times 10^{-3}\ \mathrm{s}^{-1}$ and $Q_\oplus = 8.6164 \times 10^4\ \mathrm{s}$, therefore $\bar{g}_\oplus = 289.5$. As mentioned before, the popular choices for F are the constant gravitational field $F_{\mathrm{con}} = \bar{g}$ and the linear gravitational field $F_{\mathrm{lin}}(\chi) = \bar{g}\chi$. Here, $F_\oplus(\chi)$ denotes the gravitational field predicted by the PREM.

Assume that the gravity train is at rest at A. It is possible to integrate Eq. (3) to obtain

$$\left(\frac{d\chi}{d\tau}\right)^2 = \int_{\sqrt{\chi^2+\bar{c}^2}}^{1} 2F(u)du - a\left(1 - \chi^2 - \bar{c}^2\right) + 2b\left(\chi - \sqrt{1 - \bar{c}^2}\right). \tag{4}$$

The gravity tunnel is traversable if and only if the right-hand side of Eq. (4) is, as a function of χ, positive for $-\sqrt{1-\bar{c}^2} < \chi < \sqrt{1-\bar{c}^2}$ and equals zero for $\chi = -\sqrt{1-\bar{c}^2}$. While the first condition ensures that the gravity train reaches B, the second condition means that it also stops there. The latter condition which is equivalent to $b=0$ or $|\varphi| = |\varphi_0|$ is crucial for the gravity train to be able to move back to A. We thus have two necessary and sufficient conditions for traversability:

(1) For $0 \le \chi < \sqrt{1-\bar{c}^2}$, it follows that

$$\int_{\sqrt{\chi^2+\bar{c}^2}}^{1} F(u)\mathrm{d}u > \frac{a}{2}\left(1-\chi^2-\bar{c}^2\right);$$

(2) $|\varphi| = |\varphi_0|$.

The first condition is always satisfied in the case of constant and linear gravitational fields until $\bar{g} > 1$. This is taken for granted in what follows. The PREM also satisfies this condition since $F_{\oplus}(\chi)$ is always greater than $F_{\mathrm{lin}}(\chi)$. Therefore, the second condition is crucial. For the sake of simplicity, we say that the gravity tunnel is *horizontal* if $\varphi = \varphi_0$ and *vertical* if $\varphi = -\varphi_0$. Throughout this note, these two types of gravity tunnels are of interest to us since we are only interested in traversable gravity tunnels.

It is not hard to find gravitational fields and gravity tunnels which do not satisfy the first condition. Take $\mu > 1$ and $F(\chi) = \bar{g}\chi^{2\mu-1}$. If $\bar{g} \le (1-\varepsilon^2)\mu$ for some $\varepsilon \in [0,1)$, then gravity tunnels with $\bar{c} = 0$ and $\varphi_0 = 0$ violate the first condition for $0 \le \chi \le \varepsilon$.

From Eq. (4), the general expression for the fall-through time is deduced:

$$T = \frac{2\sqrt{\bar{g}}}{\omega}\int_{\bar{c}}^{1} w\left((w^2-\bar{c}^2)\left(\int_w^1 2F(u)\mathrm{d}u + a(w^2-1)\right)\right)^{-1/2}\mathrm{d}w. \tag{5}$$

This integral depends on φ_0 and $\bar{c}$ because $\varphi = \pm\varphi_0$ and a depends only on φ_0 and $\bar{c}$. If the angle between vectors $\mathbf{B}$ (or $\mathbf{A}$) and $\mathbf{w}$ is denoted by θ_0, then $\bar{c} = \cos\theta_0$. Therefore, $\bar{c}$ has a similar role as θ_0 in Klotz's paper.[4] For the horizontal gravity tunnel, we have $a=1$ and T really depends only on $\bar{c} \in [\sin|\varphi_0|, 1]$. But for the vertical gravity tunnel, we have $a = 1-(1-\bar{c}^2)^{-1}\sin^2\varphi_0$ and φ_0 must be taken into account where $\bar{c} \in [0, \cos\varphi_0]$. Of course, T is an even function in variable φ_0.

IV. FALL-THROUGH TIMES

In this section, exact formulas for fall-through times in linear and constant gravitational fields are provided. In general, these equations depend on variables a and $\bar{c}$ and on parameters $\bar{g}$ and ω. Observe that the limiting case $\bar{g} \to \infty$ corresponds to the non-rotating scenario. This means that we must obtain all non-rotating formulas after taking this limit in the equations below. At the end, we compare both models with the PREM.

A. Linear gravitational field

If we take $F(\chi) = F_{\mathrm{lin}}(\chi)$, then Eq. (5) gives

$$T = \frac{\pi}{\omega}\sqrt{\frac{1}{1-a/\bar{g}}}.$$

In this case, the fall-through time is constant for horizontal gravity tunnels only but changes a little if $\bar{g}$ is sufficiently large. For the Earth, this value is 42.26 min, slightly more than for a non-rotating Earth.

B. Constant gravitational field

If we take $F = F_{\mathrm{con}}$, then it is possible to find an equation for T in terms of complete elliptic integrals. From Eq. (5), it follows that

$$T = \frac{\sqrt{\bar{g}}}{\omega}\int_{\bar{c}}^{1}\frac{2w\mathrm{d}w}{\sqrt{(a(w+1)-2\bar{g})(w-1)(w-\bar{c})(w+\bar{c})}}.$$

In order to write this integral in a closed form, we need to consider three separate cases: $a > 0$ and $\bar{c} > 0$, $a > 0$ and $\bar{c} = 0$, and $a=0$. Accordingly, we have

$$T = \frac{4\bar{c}}{\omega\sqrt{(1+\bar{c})(2-a(1+\bar{c})/\bar{g})}} \times \left(2\mathbf{\Pi}\left(\frac{1-\bar{c}}{1+\bar{c}}, \sqrt{\frac{(1-\bar{c})(2-a(1-\bar{c})/\bar{g})}{(1+\bar{c})(2-a(1+\bar{c})/\bar{g})}}\right) - \mathbf{K}\left(\sqrt{\frac{(1-\bar{c})(2-a(1-\bar{c})/\bar{g})}{(1+\bar{c})(2-a(1+\bar{c})/\bar{g})}}\right)\right); \tag{6}$$

$$T = \frac{2}{\omega}\sqrt{\frac{\bar{g}}{a}}\log\frac{\bar{g}-a}{\bar{g}-\sqrt{a(2\bar{g}-a)}}; \tag{7}$$

$$T = \frac{2\sqrt{2}}{\omega\sqrt{1+\bar{c}}}\left((1+\bar{c})\mathbf{E}\left(\sqrt{\frac{1-\bar{c}}{1+\bar{c}}}\right) - \bar{c}\mathbf{K}\left(\sqrt{\frac{1-\bar{c}}{1+\bar{c}}}\right)\right), \tag{8}$$

respectively. Here, $\mathbf{K}$, $\mathbf{E}$, and $\mathbf{\Pi}$ are the first, second, and third complete elliptic integrals.[17] Equations (7) and (8) can be obtained from Eq. (6) after taking limits $\bar{c} \to 0$ and $a \to 0$, respectively. Equation (8) is the limiting case of Eq. (6) after $\bar{g} \to \infty$ and thus also presents the non-rotating fall-through time. This equation is the same as Klotz's[4] Eq. (A10) except that in his equation, a minus must be used between elliptic integrals, the same remark made also by

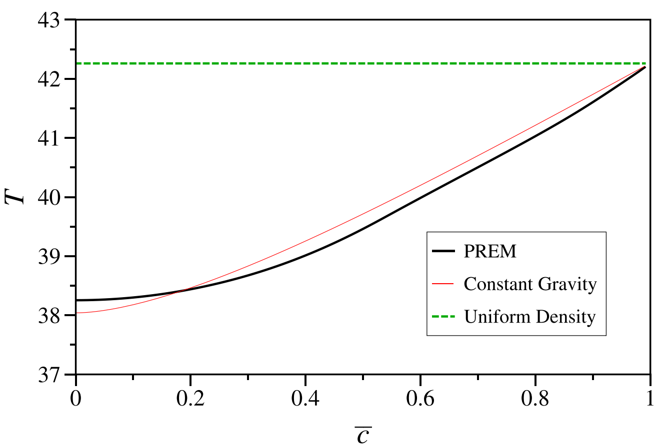

Fig. 2. Fall-through time T (in minutes) as a function of distance of the horizontal gravity tunnel from the Earth's center per the Earth's radius $\bar{c}$ according to the PREM, constant gravity, and uniform density models.

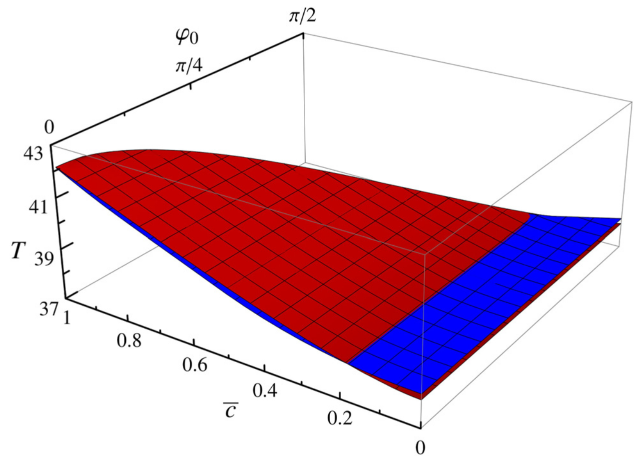

Fig. 3. Fall-through time T (in minutes) as a function of distance of the vertical gravity tunnel from the Earth's center per the Earth's radius $\bar{c}$ and departure latitude φ_0 (in radians) according to the PREM (the blue surface) and constant gravity (the red, mostly upper surface) models.

Isermann[3] who obtained analytic expressions for fall-through times in the case of a non-rotating Earth with piecewise linear approximation of the PREM.

C. The PREM model

Klotz[4] used the PREM to get fall-through times depending on θ_0. Interestingly, the PREM results are much closer to the constant gravitational field model than the linear gravitational field model although the latter is more physically realistic concerning gravity around the origin. Our results for a rotating Earth are, unsurprisingly, essentially the same, see Figs. 2 and 3.

Isermann[13] discusses a hypothetical free fall gravity tunnel between Asia and South America. Inspired by "The Fall" from *Total Recall*, conditions on the latitudes of the tunnel's endpoints exclude the possibility of a hypothetical traversable straight gravity tunnel between Great Britain and Australia. However, we can connect Europe and Australia, e.g., Heraklion on Crete in Greece (35.28° N, 25° E) with Canberra (35.28° S, 149.2° E). According to the PREM, for such a vertical gravity tunnel the center of the Earth is passed at a distance of 2434 km, which is twice the radius of Earth's inner core, with velocity $8.8\,\mathrm{km}/s$, comparable to the re-entry speeds of spacecrafts. The fall-through time is about 38.9 min.

V. CONCLUSION

The point of this note is that we must have traversable gravity tunnels in order to be able to speak about fall-through times. While this is irrelevant for non-rotating physical bodies, it is crucial in other cases. In this note, we have investigated the necessary conditions for straight gravity tunnels to be traversable. One of them greatly narrows the choice of allowed positions to the tunnel's endpoints on a massive body; the absolute values of endpoints' latitudes must be the same. This condition could be explained by a physical argument. Because the energy, which is the sum of kinetic energy, the gravitational potential, and the centrifugal potential, is conserved, the centrifugal potential, which at the surface depends only on the latitude, should be the same at the endpoints.

We compare the theoretically obtained fall-through times in linear and constant gravitational fields for the Earth with the numerics of the PREM. The results are close to the non-rotating times because the Earth rotates relatively slowly. If we want the rotation to have a great impact, then $\bar{g}$ must be close to 1. This means that the Earth should rotate close to the period 1.4 h.

Equation (4) also implies that if $|\varphi| > |\varphi_0|$, then the gravity train will not reach its destination, and if $|\varphi| < |\varphi_0|$, then it will not stop there. It may be interesting to provide some numerics in such cases on depths and ejecting speeds of gravity trains under different gravitational fields.

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