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Terrestrial Brachistochrone

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NOTES AND DISCUSSION

An Example of the Need for Adequate References

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ON occasion an article in a technical journal receives unwarranted and sensational public attention because of a misunderstanding journalist's review.

This happened recently[1] after the note "Through the Earth in Forty Minutes" appeared in the January issue of the *American Journal of Physics*,[2] probably, at least partly, because suitable references to previous work were not included, although required by the *Statement of Editorial Policy* which appears in the same issue.

It is usually difficult to trace the origins of ideas, but many of those advanced in this note are old and reasonably well known.

The fact that a particle moving in a straight tunnel between any two points on the surface of the earth traverses the distance in about 42 minutes was known prior to 1898, as it appears in example 124, page 65, of *A Treatise on Dynamics of a Particle*, by Edward J. Routh (Cambridge University Press, 1898), and most likely before this on some Tripos examination.

This fact and the mode of transportation suggested in the note were known to many undergraduate engineering students at the University of Minnesota in the 1930's and 40's once they had worked problem 8, page 229, of *Engineering Mechanics*, by William E. Brooke and Hugh B. Wilcox (Ginn and Company, Boston, 1929), which deals with a train in a tunnel between Minneapolis and Chicago.

Contemporary physicists and mathematicians are not the applied mechanicians they once were, but certainly the work of Routh, especially since it is now more accessible as a Dover reprint than it ever has been, should be known among them.

Perhaps undue sensationalism would have been avoided in this instance if references dating back to 1898 had been appended to the note.

[1] Time **87**, No. 6, 42 (11 Feb. 1966).
[2] P. W. Cooper, Am. J. Phys. **34**, 68 (1966).

Terrestrial Brachistochrone

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IN a recent note in this JOURNAL, Cooper[1] discussed the motion of a particle through tunnels of various shapes inside a homogeneous gravitating sphere. Among the paths he considers is that of least time, the brachistochrone, and for this path Cooper derives the differential equation and exhibits numerical solutions. The object of this note is to point out that the problem is simple enough so that a solution can be given in terms of elementary functions.

The potential inside the sphere is $\frac{1}{2}\,mgr^2/R$, so that for a particle that starts from rest at the surface the velocity is obtained from the equation for the conservation of energy,

$$v^2 = gR - gr^2/R. \tag{1}$$

The element of time along a path $r(\theta)$ is thus

$$\begin{aligned} dt &= ds/v, \\ &= (R/g)^{\frac{1}{2}}(r'^2+r^2)^{\frac{1}{2}}(R^2-r^2)^{-\frac{1}{2}}d\theta, \end{aligned} \tag{2}$$

where r' stands for $dr/d\theta$.

The problem thus reduces to solving the Euler–Lagrange equation for an effective Lagrangian

$$L = (r^2+r'^2)^{\frac{1}{2}}(R^2-r^2)^{-\frac{1}{2}}. \tag{3}$$

Since L does not contain θ explicitly, a first integral of Lagrange's equation is

$$r'\partial L/\partial r' - L = c,$$

where c is a constant, which can be evaluated at the minimum radius, ρ, where r' is zero. The resulting integral is

$$r^2(r^2+r'^2)^{-\frac{1}{2}}(R^2-r^2)^{-\frac{1}{2}} = \rho(R^2-\rho^2)^{-\frac{1}{2}}, \tag{4}$$

which finally gives

$$\begin{aligned} \theta &= \frac{\rho}{R}\int\left(\frac{R^2-r^2}{r^2-\rho^2}\right)^{\frac{1}{2}}\frac{dr}{r}, \\ &= \arctan\left[\frac{R}{\rho}\left(\frac{r^2-\rho^2}{R^2-r^2}\right)^{\frac{1}{2}}\right] - \frac{\rho}{R}\arctan\left(\frac{r^2-\rho^2}{R^2-r^2}\right)^{\frac{1}{2}}, \end{aligned} \tag{5}$$

where θ is measured from the axis of symmetry. The total angle subtended at the center by the end points of the path is

$$2\phi = \pi(1-\rho/R).$$

The time can also be integrated explicitly, since dt can now be written in terms of r and dr

$$\begin{aligned} t &= (Rg)^{-\frac{1}{2}}(R^2-\rho^2)^{\frac{1}{2}}\int r(R^2-r^2)^{-\frac{1}{2}}(r^2-\rho^2)^{-\frac{1}{2}}dr, \\ &= \tfrac{1}{2}(Rg)^{-\frac{1}{2}}(R^2-\rho^2)^{\frac{1}{2}}\arccos[(R^2+\rho^2-2r^2)/(R^2-\rho^2)], \end{aligned} \tag{6}$$

where again the time origin has been taken at the lowest point in the path. The total travel time between two points on the surface is therefore

$$T = \pi(R^2-\rho^2)^{\frac{1}{2}}(Rg)^{-\frac{1}{2}}. \tag{7}$$

The parametric equations of the trajectory can be obtained directly from Eqs. (5) and (6). They are

$$\theta = \arctan[(R/\rho)\tan\Omega t] - (\rho/R)\Omega t, \tag{8}$$

and

$$r^2 = \tfrac{1}{2}(R^2+\rho^2) - \tfrac{1}{2}(R^2-\rho^2)\cos 2\Omega t, \tag{9}$$

where $\Omega = \pi/T$.

Equations (8) and (9) represent a hypocycloid generated by a circle of radius $\frac{1}{2}(R-\rho)$ rolling at a constant speed inside a circle of radius R. This is analogous to Bernoulli's brachistochrone problem in which the trajectory is a cycloid.[2]

[1] P. W. Cooper, Am. J. Phys. **34**, 68 (1966).
[2] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publ. Co., Inc. Reading, Mass., 1959), p. 56.