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The acceleration of gravity in Earth's interior is determined by the density distribution in Earth. A remarkable result is that the acceleration is approximately constant all over the mantle, which amounts to about 84% of Earth's volume. This result can be explained by a simple two-layer model of Earth, showing that the constancy of the acceleration in the mantle is a consequence of the particular size and density of Earth's core with respect to the size and density of the whole Earth. In other planets, with different mass distributions, the dependence of acceleration on depth could be very different.

Knowing the gravity field in Earth's interior is important because gravity is one of the main forces controlling the inner movements of Earth, such as convective motions occurring in the core and in the mantle. These motions are the origin of fundamental manifestations of the planet's activity, such as the magnetic field and the surface dynamics (plate tectonics, volcanism, earthquakes). In particular, the adiabatic temperature gradient, that is the minimum temperature gradient allowing convection in a compressible liquid, is proportional to the acceleration of gravity.[1]

Calculation of the acceleration of gravity requires the knowledge of the distribution of mass density within Earth. As a first approximation, it can be assumed that Earth has spherical symmetry, with density depending only on the distance r from the center of the planet. Under this assumption, the density distribution can be obtained from the Adams-Williamson equation,[2] relating the radial density gradient to the elastic properties in a planet with spherical symmetry under hydrostatic conditions. The solution of the equation requires the knowledge of the seismic wave velocities as functions of r, that can be derived from seismometric data.

Graphs of the acceleration of gravity as a function of r within Earth show a surprising behavior.[3,4] Gravity is approximately constant and equal to the surface value in the whole mantle, a shell having a thickness of about 3000 km, representing 84% of the total volume of Earth. This fact can be demonstrated on the basis of a simple two-layer Earth model.[5,6]

The chief role of models is to reproduce the observed phenomena and to disclose the relationships existing between the quantities describing a physical system. A model is always a simplification of the system: only a few aspects are considered, while many others are neglected purposefully, because they are not considered to be relevant for the problem.

In the present paper, we use the two-layer model in order to show that the constancy of the acceleration of gravity in the mantle comes from the particular values of the ratios between the radius of the core and the radius of Earth and between the average density of the core and the average density of the Earth. In order to make clear this statement, some simple models of planets with spherical symmetry are considered and the acceleration of gravity in their interiors is calculated for different density distributions on the basis of the Newtonian theory.

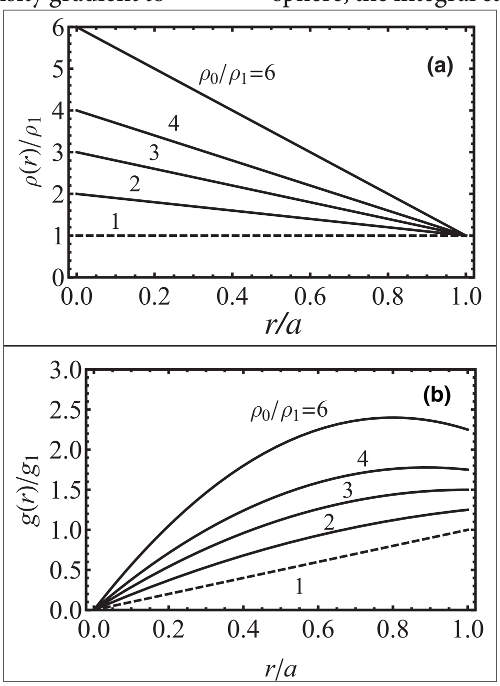

Fig. 1. (a) Density distributions $\rho(r)$ within a planet of radius a, for different values of the ratio ρ_0/ρ_1; (b) acceleration of gravity $g(r)$ for the density distributions shown in (a).

Acceleration of gravity

Let a be the radius of the planet and ρ be its density. The acceleration of gravity g in the planet's interior can easily be calculated remembering that Newton's law of gravitation implies Gauss' law, according to which the acceleration of gravity on any spherical surface of radius r is

$$g(r) = -\frac{Gm(r)}{r^2}, \tag{1}$$

where G is the gravitational constant and $m(r)$ is the mass contained in the sphere of radius r and volume V:

$$m(r) = \int_V \rho \, dV. \tag{2}$$

If the mass distribution has spherical symmetry, that is, density ρ depends only on the distance r' from the center of the sphere, the integral can be calculated by summing the contributions of infinitesimal spherical shells as

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'. \tag{3}$$

Then Eq. (1) yields

$$g(r) = -\frac{4\pi G}{r^2} \int_0^r \rho(r') r'^2 dr', \tag{4}$$

where the negative sign indicates that the acceleration is directed toward the planet's center.

Planet with linearly variable density

In planets, density typically increases with depth. A continuous increase of density from the surface to the center of Earth was generally assumed until the end of the 19th century.[7] Suppose that the density at the center of the planet is ρ_0 and decreases linearly to the surface value ρ_1, that is,

$$\rho(r) = \rho_0 - \frac{\rho_0 - \rho_1}{a} r. \tag{5}$$

A graph of the function $\rho(r)/\rho_1$ is plotted in Fig. 1(a) for different values of the ratio ρ_0/ρ_1. From Eq. (4), the acceleration of gravity within the planet is

$$g(r) = -\frac{4}{3}\pi G\rho_0 r + \pi G\frac{\rho_0 - \rho_1}{a}r^2. \tag{6}$$

Setting $\rho_0 = \rho_1$ and $r = a$, we obtain the acceleration at the surface of a planet with uniform density ρ_1:

$$g_1 = -\frac{4}{3}\pi G\rho_1 a. \tag{7}$$

A graph of the function $g(r)/g_1$ is plotted in Fig. 1(b) for different values of the ratio ρ_0/ρ_1. It can be seen that the function $g(r)$ is not monotonic, in general. Only if $\rho_0/\rho_1 \le 3$, the function is always increasing, with a maximum at $r = a$. If $\rho_0/\rho_1 > 3$, the acceleration has a maximum at $r = r_0$, with

$$\frac{r_0}{a} = \frac{2}{3}\frac{\rho_0}{\rho_0 - \rho_1}. \tag{8}$$

As the ratio ρ_0/ρ_1 increases, the spherical surface $r = r_0$ moves downward, with a minimum value $r_0 = 2a/3$, that is approached for $\rho_0 \gg \rho_1$. Therefore, a very simple density distribution as given by Eq. (5) can produce different profiles of $g(r)$, according to the value of the ratio ρ_0/ρ_1.

In the particular case when the density ρ is uniform, shown with dashed lines in Fig. 1, Eq. (6) gives

$$g(r) = -\frac{4}{3}\pi G\rho r. \tag{9}$$

This is the well-known result that the acceleration of gravity decreases linearly with r. In fact, according to Eq. (4), the acceleration at any depth is proportional to the ratio $m(r)/r^2$, where m is the mass responsible for the attraction. As one goes deeper into a planet with uniform density, m decreases proportionally to volume, that is, to r^3. Therefore, the ratio $m(r)/r^2$ decreases proportionally to r.

Two-layer planet

The German physicist Emil Wiechert was the first to propose in 1896 an Earth model with an iron core and a stony shell, with a jump in density at the boundary between the two parts.[7] As an approximate model for Earth, let us consider a two-layer planet with a piecewise constant density

$$\rho(r) = \begin{cases} \rho_1, & 0 \le r \le r_1 \\ \rho_2, & r_1 < r \le a, \end{cases} \tag{10}$$

with $\rho_1 > \rho_2$. We can assume that the inner sphere $0 \le r \le r_1$ is the core and the outer shell $r_1 < r \le a$ is the mantle. The average density of the planet is

$$\rho^* = \frac{(\rho_1 - \rho_2)r_1^3 + \rho_2 a^3}{a^3}. \tag{11}$$

A graph of the function $\rho(r)$ is shown in Fig. 2(a), where values appropriate to Earth have been used for the parameters[8]: $r_1 = 3485$ km, $a = 6371$ km, $\rho_1 = 10{,}800\ \text{kg}\cdot\text{m}^{-3}$, and $\rho_2 = 4500\ \text{kg}\cdot\text{m}^{-3}$, yielding $\rho^* \simeq 5530\ \text{kg}\cdot\text{m}^{-3}$. The actual density distribution in Earth[4] is also plotted in Fig. 2(a).

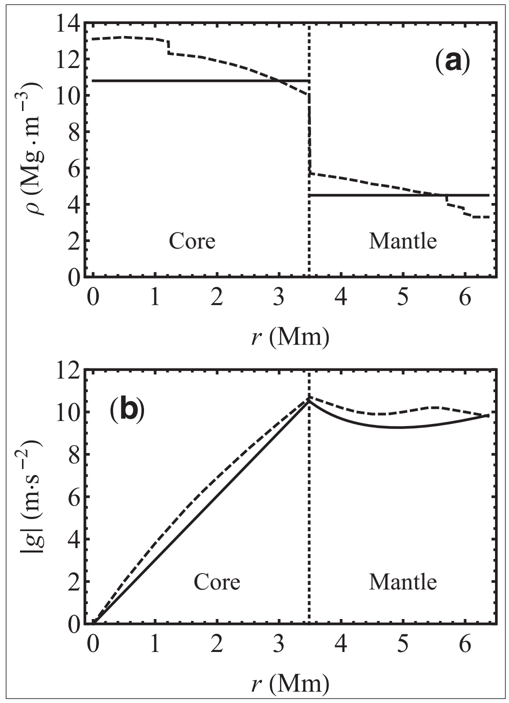

Fig. 2. (a) Density distribution $\rho(r)$ in a two-layer model of Earth (solid lines) and the actual density within Earth[4] (dashed line); (b) acceleration of gravity resulting from the model (solid line) and the actual acceleration in Earth[3] (dashed line).

From Eq. (4), the acceleration of gravity in the core is a linear function

$$g(r) = -\frac{4}{3}\pi G\rho_1 r, \quad 0 \le r \le r_1, \tag{12}$$

while in the mantle it is given by

$$g(r) = -\frac{4}{3}\pi G\frac{(\rho_1 - \rho_2)r_1^3 + \rho_2 r^3}{r^2}, \quad r_1 \le r \le a. \tag{13}$$

A homogeneous planet with density ρ^* would produce a surface acceleration

$$g^* = -\frac{4}{3}\pi G\rho^* a. \tag{14}$$

In Fig. 2(b) the magnitude $|g|$ of the acceleration is plotted as a function of r for the density distribution given in Fig. 2(a). It is remarkable that g is approximately constant with r in the outer layer. The actual acceleration of gravity in Earth[3] is also plotted for comparison.

Discussion

As shown in Fig. 2(a), the density distribution within Earth is not a piecewise constant function of r, but it increases with depth both in the core and in the mantle. However, the functions (12) and (13) are a good approximation for our aim, as shown in Fig. 2(b).

The remarkable fact that g is almost constant in the mantle can easily be understood if we consider that, according to Eq. (14), the acceleration of gravity at the surface of a spherical mass is proportional to the product of the average density times the radius of the sphere. In fact, expression (13) for the acceleration in the mantle can be written as

$$g(r) = -\frac{4}{3}\pi G r \bar{\rho}(r)\,, \tag{15}$$

where

$$\bar{\rho}(r) = \frac{(\rho_1 - \rho_2) r_1^3 + \rho_2 r^3}{r^3}. \tag{16}$$

This is the average density in a sphere with radius r, with $r_1 \le r \le a$, so that $\bar{\rho}(a) = \rho^*$.

In the case of Earth, it happens that the product $r\,\bar{\rho}(r)$ has about the same value for the whole Earth (or for spherical portions of Earth including the core) as for the sole core. In other words, the product $r\,\bar{\rho}(r)$ has a constant value all over the mantle. In fact

$$r\,\bar{\rho}(r) \approx r_1 \rho_1 \qquad r_1 \le r \le a \tag{17}$$

so that

$$g(r) \approx g(r_1) \qquad r_1 \le r \le a, \tag{18}$$

meaning that the acceleration of gravity at any point in the mantle is about equal to that at the surface of the core. In particular,

$$r_1 \rho_1 \approx a \rho^*. \tag{19}$$

The core is smaller than the entire Earth, but has a higher average density: the two factors approximately compensate, so that

$$\frac{r_1}{a} \approx \frac{\rho^*}{\rho_1}. \tag{20}$$

With the values appropriate to the Earth, it results $r_1/a \simeq 0.55$ and $\rho^*/\rho_1 \simeq 0.51$, with a relative difference equal to about 7%. Therefore the acceleration $g(a)$ at Earth's surface is about the same as the acceleration $g(r_1)$ at the surface of the core, and the effect of the mantle is to keep this value virtually unchanged across 3000 km.

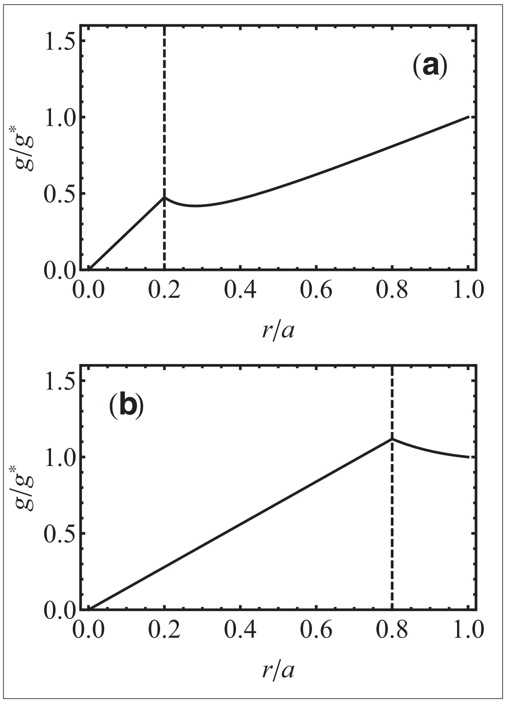

Fig. 3. Acceleration of gravity in the interior of two-layer planets with different values of the ratio r_1/a: (a) 0.2, (b) 0.8. Density values ρ_1 and ρ_2 are the same as in Earth.

It should be evident that the constancy of g in Earth's mantle is accidental, coming from the particular values of the ratios r_1/a and ρ^*/ρ_1. In another planet, with different values of these ratios, the gravity profile within the planet could be very different.

For instance, if the core were smaller (say, $r_1/a = 0.2$), with the same values of ρ_1 and ρ_2, the acceleration of gravity $g(r_1)$ at the surface of the core would be only 47% of that at the surface of the planet [Fig. 3(a)]. If the core were greater ($r_1/a = 0.8$), $g(r_1)$ would be 12% greater than $g(a)$ [Fig. 3(b)]. Incidentally, the two values of r_1/a are close to the values inferred for the Moon and Mercury, respectively.[4] In both cases, there would be a remarkable radial gradient of g in the mantle, as in the core. This would affect the adiabatic temperature gradient and the buoyancy force and might alter the conditions for the occurrence of thermal convection.

Free fall along a diameter

In recent years, the motion of a body falling in a hypothetical tunnel along a diameter of a model Earth has been considered by several authors.[9-11] A model with an acceleration of gravity that is linearly increasing in the core and constant in the mantle provides a simple analytical solution of the problem. The equation of motion is

$$\frac{d^2 r}{dt^2} = g(r), \tag{21}$$

where t is time and

$$g(r) = \begin{cases} -\omega^2 r, & 0 \le r \le r_1 \\ -\omega^2 r_1, & r_1 < r \le a \end{cases} \tag{22}$$

with

$$\omega^2 = \frac{4}{3}\pi G \rho_1. \tag{23}$$

If the motion starts at $t = 0$, the solution for the mantle is

$$r(t) = a - \frac{1}{2}\omega^2 r_1 t^2. \tag{24}$$

The body reaches the mantle-core boundary at time

$$t_1 = -\frac{v_1}{\omega^2 r_1} \tag{25}$$

with a velocity

$$v_1 = -\omega\sqrt{2 r_1 (a - r_1)}. \tag{26}$$

For $t \ge t_1$, the motion in the core is

$$r(t) = r_1 \cos\omega(t - t_1) + \frac{v_1}{\omega}\sin\omega(t - t_1) \tag{27}$$

up to time

$$t_2 = t_1 + \frac{1}{\omega}\arctan\,\frac{1}{\omega t_1} \tag{28}$$

when the body reaches the center of Earth with a velocity

$$v_2 = v_1\sqrt{1+\frac{1}{\omega^2 t_1^2}}. \tag{29}$$

The motion continues specularly in the other hemisphere and the time required for crossing the whole Earth is $2t_2$: it is equal to 37.4 min if calculated with the values of ρ_1, r_1, and a assumed above, close to the 38.2 min given by PREM.[12] Position and velocity as functions of time are shown in Fig. 4.

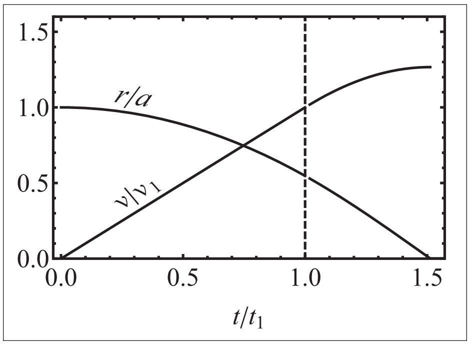

Fig. 4. Position $r(t)$ and velocity $v(t)$ of a body falling along a diameter of a two-layer Earth model. The dashed line indicates the crossing of the mantle-core boundary. With the assumed values of r_1 and a, it results $t_2 \simeq 1.5\ t_1$.

Conclusions

In Earth, the particular values of the ratios between the radius of the core and the radius of the planet and between the average density of the core and the average density of the planet produce an approximately constant acceleration of gravity all over the mantle. This is a remarkable fact, due to the role of gravity in controlling the inner dynamics of Earth. In another planet, with different values of the two ratios, the gravity dependence on depth would be different and would alter the conditions for the occurrence of thermal convection in the mantle. A two-layer Earth model, in the approximation of constant field throughout the mantle, also provides a simple analytical solution for the free fall of a body in a tunnel along a diameter of the planet.

The models presented above can be used in introductory physics or astronomy classrooms in addition to the case of a simple uniform-density model. The models show that gravity in a planet's interior can be a non-intuitive function of depth and that differences in density distributions can produce different gravity fields, with consequences on the inner dynamics of the planet. Therefore they can be used in a lesson of comparative planetology. Finally, the free fall of a body in a tunnel across the two-layer Earth is an exercise of Newtonian mechanics in the presence of a force that is a non-smooth function of position.

Acknowledgments

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