Probability Distributions 1

If we want to perform some mathematical computation on top of the outcome from a random exp. we need the outcome in a numerical form.

Random Variables

maps the outcome of a random process / random exp. to a number

(H) Random exp: The toss of a coin

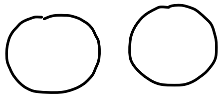
$\Omega = \{H, T\}$

X = # of heads

$X = [1, 0]$

y = # of tails

$y = [0, 1]$



Exp: Toss of two coins

$\Omega = \{\overset{1/4}{HH}, \overset{1/4}{HT}, \overset{1/4}{TH}, \overset{1/4}{TT}\}$ → Equally likely

X = # of heads

$X = [\underset{1/4}{2}, \underset{2/4}{1}, 0 \rightarrow 1/4]$

y = # of tails

$y = [2, 1, 0]$

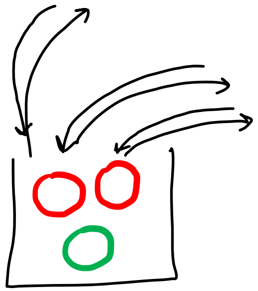
Random exp: Hitting the ball to the goal post (3 shots)

$\Omega = \{HHH, HHM, HMM, HMH, \ldots\}$

X = # of successful hits

$X = [3, 2, 1, 0]$

Random Variables



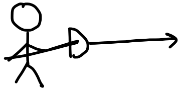
Random exp: Take a ball out of the bag
note down the colour of the ball
put the ball back to the bag } 3

$$\underset{1}{R}\quad \underset{2}{G}\quad \underset{3}{R}$$

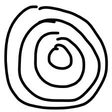
$$\Omega = \{ RRR,\ RRG,\ RGR,\ RGG \ldots \}$$

X = # of Red balls

$$X = [3, 2, 1, 0]$$



5 trails



$$\Omega = \{ HHHHH,\ HHHHM,\ HHHMM,\ \ldots\ldots \}$$

$$(2)^5$$

X = # of bully's eye hit

$$X = [5, 4, 3, 2, 1, 0]$$

Expected Value

The avg value of the Random variable

Random exp: Roll a dice

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$X = [1, 2, 3, 4, 5, 6]$

the outcome of the dice

$E(X) = 3.5$

If we perform this exp. for very long, we are expected to get the outcome as 3.5

$[6, 6, 1, 5, 3, 5, 4, 4, 3, 1]$ 10 times

$\boxed{\bar{X} = 3.8}$

Toss a coin

$\Omega = \{H, T\}$

$X = \#$ of heads

$X = [1, 0]$

$E(X) = \frac{1+0}{2} = 0.5$

$[0, 1, 1, 1, 0, 0, 0, 1, 1, 1]$

$E(X) = 0.6 \longrightarrow$

If we perform the exp. for very long time, then expected value helps us with the avg outcome of the random var.

Expected Value

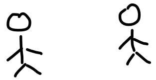
Toss two coins

$\{HH, HT, TH, TT\}$

HH $\longrightarrow$ Sumit will give Jay 20 Rs

HT, TH, TT $\longrightarrow$ Jay have to give sumit 10 Rs

S Jay



x = The amount Jay may win

$x = [+20, -10]$

P(Jay winning 20 Rs) = P(HH) = $1/4 = 0.25$

P(Jay loosing 10 Rs) = P(HT, TH, TT) = $3/4 = 0.75$

$E(x) = (+20 \times 0.25) + (-10 \times 0.75) = -2.5$

on a long run, Jay is expected to loose 2.5 Rs

$$\boxed{E(x) = -2.5}$$

$$E(x) = \sum_{i=1}^{n} x_i \times P(x = x_i)$$

x_i: Random var.

$P(x = x_i)$: Prob. of Random var.

Expected Value

One game that is popular at some carnivals and amusement parks involves selecting a floating plastic duck at random from a pond full of ducks. In most cases, the letter S, M, or L appears on the bottom of the duck, signifying that the winner receives a small, medium, or large prize, respectively. The duck is then returned to the pond for the next game.

Although the prizes are typically toys, crafts, etc., suppose that the monetary values of the prizes are as follows: Small is $\$0.50$, Medium is $\$1.50$, and Large is $\$5.00$.

The probabilities of winning an item on 1 duck selection are as follows: Small 60%, Medium 30%, and Large 10%. What is the expected monetary value of the prizes a player can win?

$x =$ The amount a player may win

$x = [0.5, 1.5, 5.0]$

x_i	$P(x = x_i)$
0.5	0.6
1.5	0.3
5.0	0.10

$E(x) = (0.5 \times 0.6) + (1.5 \times 0.3) + (5.0 \times 0.1)$

$\boxed{E(x) = 1.25}$

on a long run, each player is expected to win \$1.25

$(1.25 - 2.0) = -0.75$

Expected Value

Revisiting the charity carnival, recall that when selecting a duck, the average monetary value of the prizes you win per game is $\$1.25$. How can the charity running the carnival make any money if it is paying out $\$1.25$ to each player on average for each game?

To address this, in most cases a player must pay to play a game, and that is where the charity (or any other group running such a game) would earn its money.

Question: Imagine that the cost to play the game is $\$2.00$. What are the expected net earnings for the charity? What are the expected net winnings for a player?

$x =$ The amount a player may win

$$x = [\$0.5, \$1.5, \$5.0]$$

$$x = [\$0.5 - \$2.0, \$1.5 - \$2.0, \$5.0 - \$2.0]$$

$$x = [-1.5, -0.5, 3.0]$$

x_i	$P(x = x_i)$
-1.5	0.6
-0.5	0.3
3.0	0.1

$$E(x) = (-1.5 \times 0.6) + (-0.5 \times 0.3) + (3.0 \times 0.1)$$

$$\boxed{E(x) = -0.075}$$

on along run a player will loose 0.75 per game.

Expected Value

Quiz-1: The world famous gambler from Crime Master Gogo, proposes the following game of chance. You roll a fair die. If you roll a 1, then Gogo pays you $25. If you roll a 2, Crime Master Gogo pays you $5. If you roll a 3, you win nothing. If you roll a 4 or a 5, you must pay Crime Master Gogo $10, and if you roll a 6, you must pay Crime Master Gogo $15. Should you play this game?

amount a player may win

Roll outcome	x_i	$P(x=x_i)$
1	$25	1/6
2	$5	1/6
3	$0	1/6
4	- $10	1/6
5	-$10	1/6
6	- $15	1/6

$$E(x) = \left(25 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(0 \times \frac{1}{6}\right) + \left(-10 \times \frac{1}{6}\right) + \left(-10 \times \frac{1}{6}\right) + \left(-15 \times \frac{1}{6}\right)$$

$$\boxed{E(x) = -0.8}$$

on a long run each player is expected to loose $0.8

way to represent the prob of on exp in graphical form

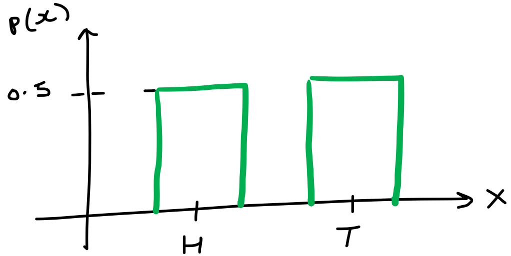
Probability Distribution (PMF, PDF, CDF)

Toss of a coin

$\Omega = \{H, T\}$

$P(H) = 0.5$

$P(T) = 0.5$



PMF (prob. mass function)

Sample space is a discrete sample space we generate prob. mass function

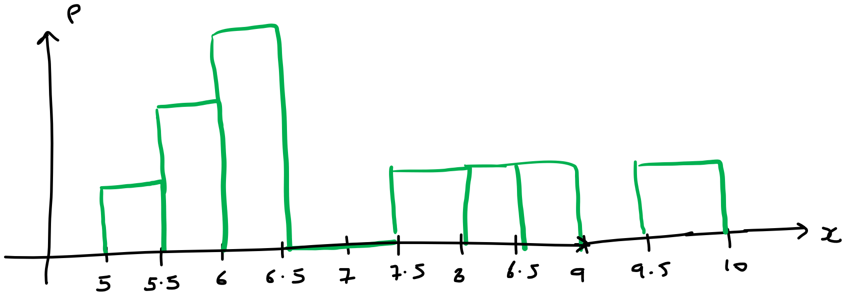
Time taken by Sumit to travel from home to office

$\Omega = \{5 - 10\}$

Any value possible b/w 5 to 10
continuous sample space

$X = [5.5, 7.5, 10, 5.9, 6.2, 8.8, 6.1, 8.9, 5.7, 6.1]$

5 - 5.5	1
5.5 - 6.0	2
6.0 - 6.5	3
6.5 - 7.0	0
7.0 - 7.5	0
7.5 - 8.0	1
8.0 - 8.5	1
8.5 - 9.0	1
9.0 - 9.5	0
9.5 - 10	1

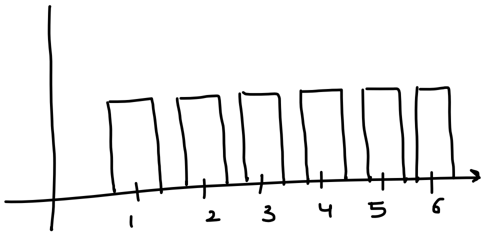


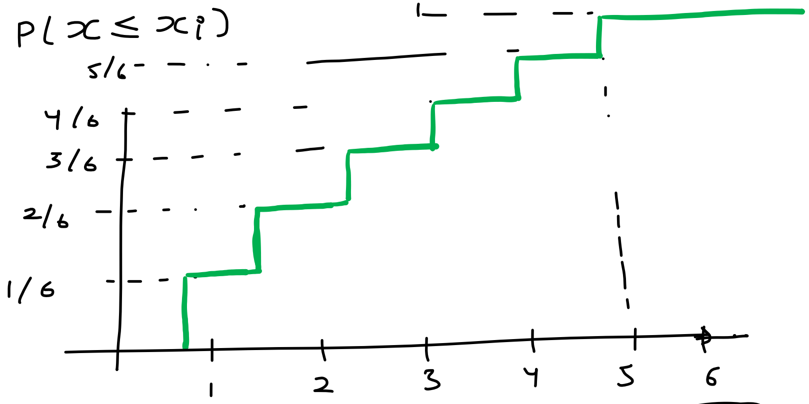
PDF → prob. distribution function

when sample space is continuous in nature.

Probability Distribution (PMF, PDF, CDF)

CDF — cumulative dist. function





CDF

$P(x \leq 1) = 1/6$

$P(x \leq 2) = 2/6$

$P(x \leq 3) = P(x=1) + P(x=2) + P(x=3) = 3/6$

$P(x \leq 4) = P(x=1) + P(x=2) + P(x=3) + P(x=4) = 4/6$

$P(x \leq 5) = 5/6$

$P(x \leq 6) = 6/6$

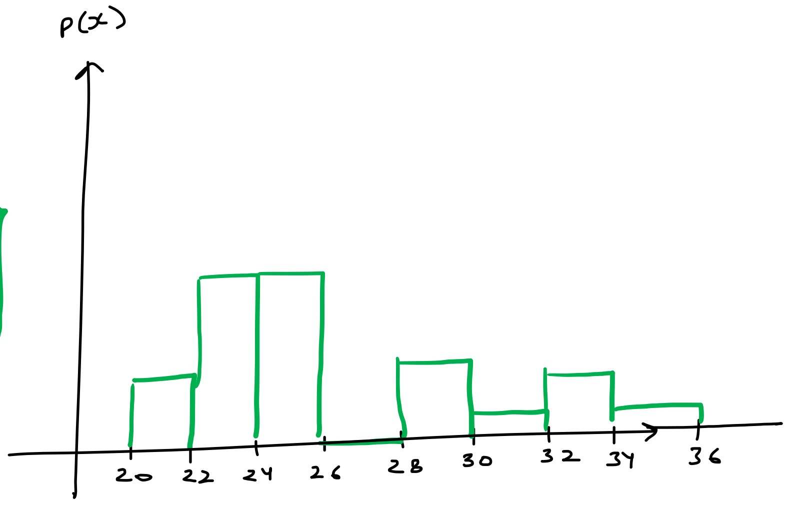
$P(x \leq 5) = 5/6$

Probability Distribution (PMF, PDF, CDF)

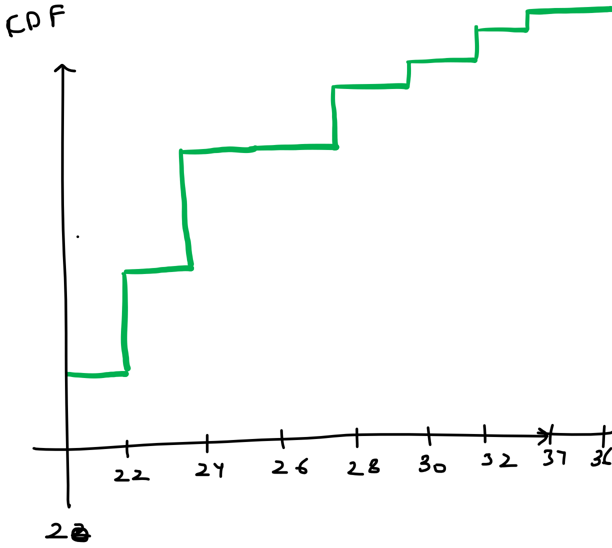
Age

$X = [20, 21, 25, 35, 29, 32, 33, 29, 28, 23, 25, 25, 27, 31, 22, 20, 22, 25, 23, 22]$

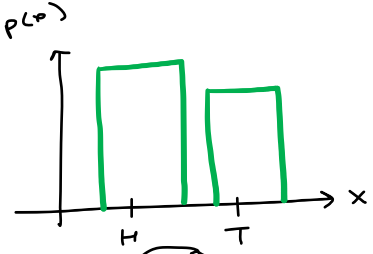
Age Buck.	P(x)	
20-22	3	3/20
22-24	4	4/20
24-26	4	4/20
26-28	0	0/20
28-30	3	3/20
30-32	1	1/20
32-34	2	2/20
34-36	1	1/20



PDF



Empirical vs Theoretical Probability



pmf

After performing the exp.

Standard prob. before performing the exp.

Exp: Toss of a coin {H, T}

P(H) = 0.5
P(T) = 0.5 } Theoretical

Exp: { T, H, H, H, T, T, T, H, H, H }

P(H) = 0.6
P(T) = 0.4 } empirical

Empirical prob ⟶ Theoretical

[when exp is performed for very long time

(Law of Large numbers)

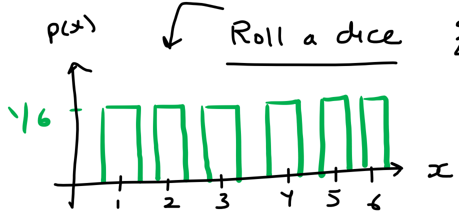
$\frac{100}{600} - 1$

10

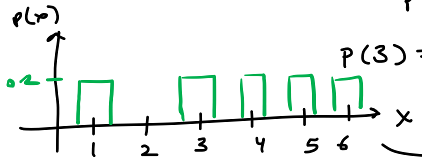
[6, 6, 1, 5, 3, 5, 4, 4, 3, 1]

pmf

Roll a dice { 1, 2, 3, 4, 5, 6 }



P(1) = 1/6
P(2) = 1/6
P(3) = P(4) = P(5) = P(6) = 1/6

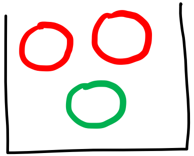


pmf empirical

P(1) = 2/10
P(2) = 0/10
P(3) = 2/10
P(4) = 2/10
P(5) = 2/10

P(6) = 2/10

Binomial Probability



$\Omega = \{ RRR, RRG, GRR, GRG \ldots GGG \}$

$P(RRG) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}$

① The outcome of the exp. should be binary (success failure)

② fix no. of trails (n)

③ prob. of success should be same across all n trails

Random Exp:

Take a ball out of the bag
note down the colour of the ball
put the ball back to the bag

3

P(Two red and one green ball) =

$P(RRG) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)$

$P(RGR) = \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} = \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)$

$P(GRR) = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)$

$= 3 \times \left[\frac{2}{3}\right]^2 \times \left(\frac{1}{3}\right)$

Prob. of green

no. of combination

Prob. of Red

Assume: getting a red ball is success

$n = 3$

$x = 2$

$p = 2/3$

$q = 1/3$

$3_{C_2} (2/3)^2 (\frac{1}{3})^1$

$\frac{3!}{2!\,1!} = 3$

$3 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^1$

$$\boxed{= n_{C_x} (p)^x (q)^{n-x}}$$

n = # of trails

x = # of success in n trails

p = prob. of success

q = prob. of failure

Binomial Probability

$^nC_x \, p^x q^{n-x}$

n = # of trails
x = no. of success in n trials
p = prob. of succ
q = prob. of failure

Quiz-2: The probability of a man hitting the target at a shooting range is 1/4. If he shoots 10 times, what is the probability that he hits the target exactly three times?

r

$n = 10$

{ HHHHHHHHHH, HHHHHHHHHM, ... }

X = # of correct hits = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0]

$$120 \times \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^7 = 0.25$$

(HHHMMMMMMM)

(HMHMHMMMMM)

120

^nC_x

$= {}^{10}C_3$

$$= \frac{10!}{3! \; 7!} = \frac{\overset{5}{\cancel{10}} \times \overset{3}{\cancel{9}} \times \cancel{8} \times \cancel{7!}}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{7!}} = 120$$

Binomial Probability

The probability of a man hitting the target at a shooting range is 1/4. If he shoots 10 times, what is the probability that he hits the target at most 3 times?

$P(X \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

cdf till 3