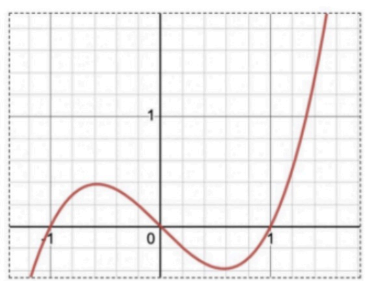
Differentiation

Real Analysis Student

YOU NEED THAT FOR $f: A \to \mathbb{R}$, $c \in A$, THE FUNCTION IS CONTINUOUS AT C IF AND ONLY IF $\forall\ \varepsilon > 0\ \exists\ \delta > 0 \ni |x-c| < \delta$ and $x \in A$ implies $|f(x)-f(c)| < \varepsilon$!!! OTHERWISE IT'S NOT SUFFICIENTLY RIGOROUS!!!!

Precalculus Student



If I can draw it without picking my pen up, it's continuous.

When your friend asks what the normal vector to a plane looks like

Agenda:

- → Recap
- → Differentiation – Geometric Intuition
- → Derivatives
- → Differentiability
- → Common Derivatives
- → Rules
- → Maxima & Minima – Optimization

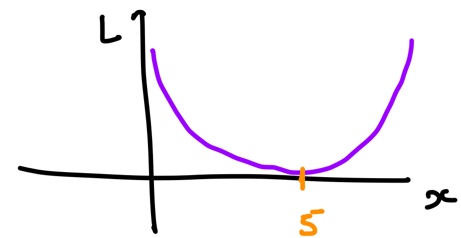
Recap

$$w^*, w_0^* = \underset{w, w_0}{\arg\max}\ G(D, w, w_0)$$

$$\Updownarrow$$

$$w^*, w_0^* = \underset{w, w_0}{\arg\min}\ \text{Loss}(D, w, w_0)$$

$$L = (x-5)^2$$



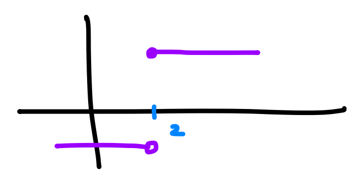
minima @ $x=5$

① Left Hand limit

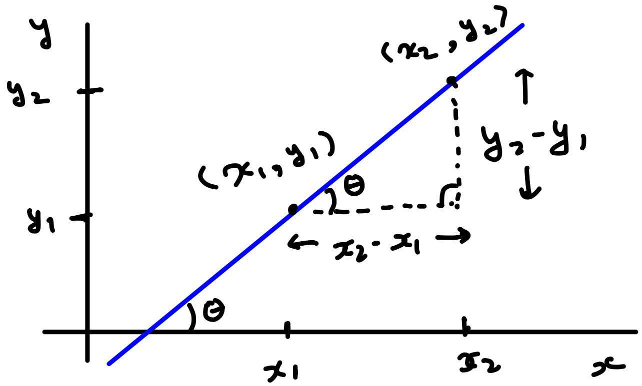
$$\lim_{x \to a^-} f(x)$$

② Right Hand limit

$$\lim_{x \to a^+} f(x)$$



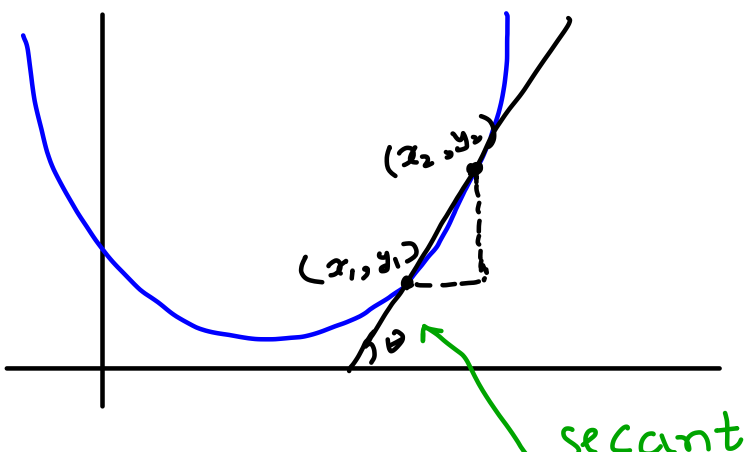
Geometric Intuition - Differentiation



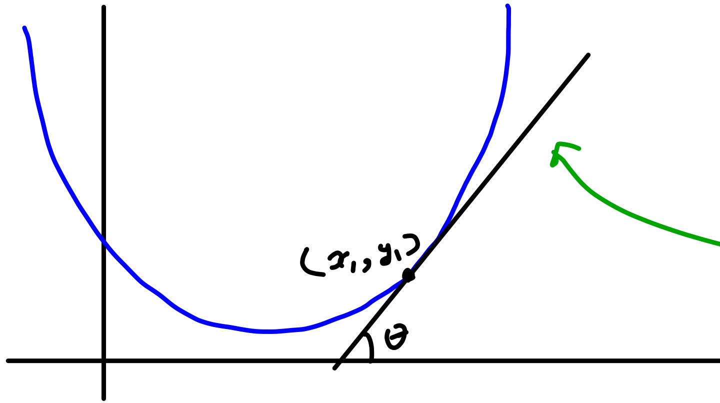
$$m = \tan\theta = \frac{p}{b} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

$$y_1 = f(x_1) \quad , \quad y_2 = f(x_2)$$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



$$m = \tan\theta = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



$x_2 = x_1 + 0.00001$

tangent

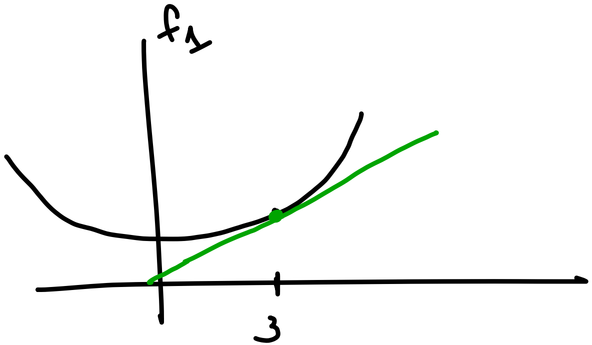
(x_1, y_1)

$x_2 = x_1 + \underbrace{0.00001}_{\Delta x}$

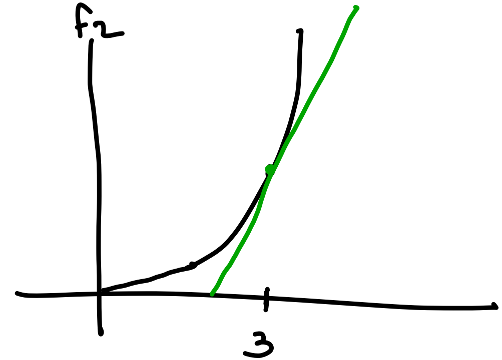
$$m = \frac{f(x_1 + \Delta x) - f(x_1)}{\cancel{x_1} + \Delta x - \cancel{x_1}}$$

V. Imp.

$$\boxed{m = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}}$$



3.001



3.001

Derivative = Slope of tangent (line touching at a)

= rate of change of fn w.r.t x

given $\rightarrow f(x)$

$$\frac{df(x)}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\boxed{\text{III}}$ How do I find derivative of a function

Example : $y = f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x^2} + \Delta x^2 + 2x\Delta x - \cancel{x^2}}{\Delta x}$$

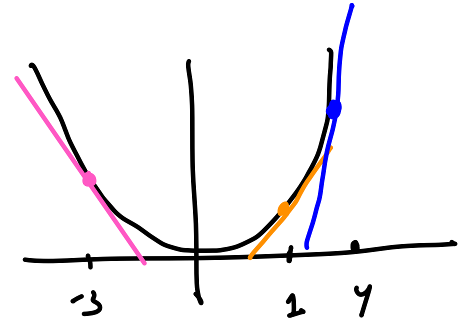
$$\lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x \cdot \Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \Delta x + 2x$$

Put $\Delta x \approx 0$

$$\boxed{f'(x) = 2x}$$

$$\frac{d\,x^2}{dx} = 2x$$

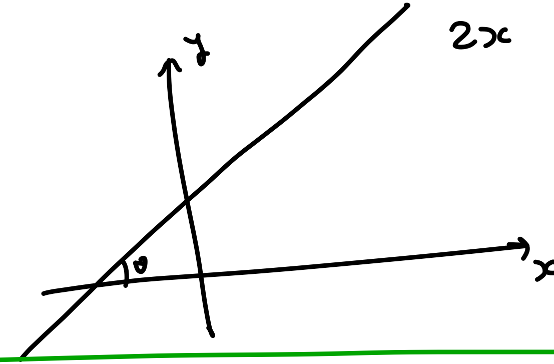


$$\left.\frac{dx^2}{dx}\right|_{x=1} = 2x$$
$$2 \cdot 1 = 2$$

$$\left.\frac{dx^2}{dx}\right|_{x=-3} = 2 \cdot x$$
$$2 \cdot (-3) = -6$$

$$\left.\frac{dx^2}{dx}\right|_{x=4} = 2 \cdot x$$
$$2 \cdot 4 = 8$$

$y = f(x) = 2x$



$f'(x) = 2$

$$\frac{d\,2x}{dx}\Big|_{x=1} = 2$$

$$\frac{d\,2x}{dx}\Big|_{x=-4} = 2$$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

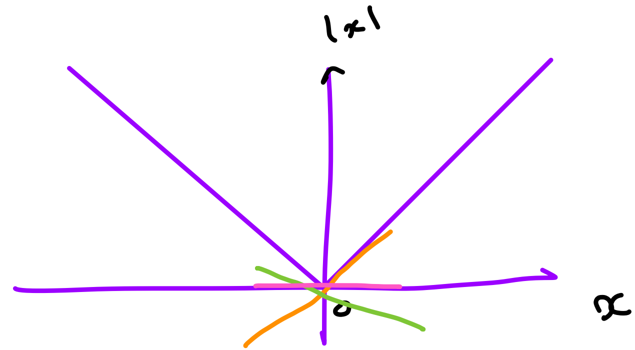
$$\frac{2(x + \Delta x) - 2x}{\Delta x}$$

$$\frac{2\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= 2$$

$$\frac{d\,x^n}{dx} = n.x^{n-1}$$

$f(x) = |x|$



$$f(x) = |x| = \begin{cases} x & : x > 0 \\ -x & : x < 0 \\ 0 & : x = 0 \end{cases}$$

Case I : $x > 0$

$f(x) = x$

$f'(x) = 1$

Case II : $x < 0$

$f(x) = -x$

$f'(x) = -1$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= -(x+\Delta x) - (-x)$$

$$= \frac{-\cancel{x} - \Delta x + \cancel{x}}{\Delta x}$$

$$= -1$$

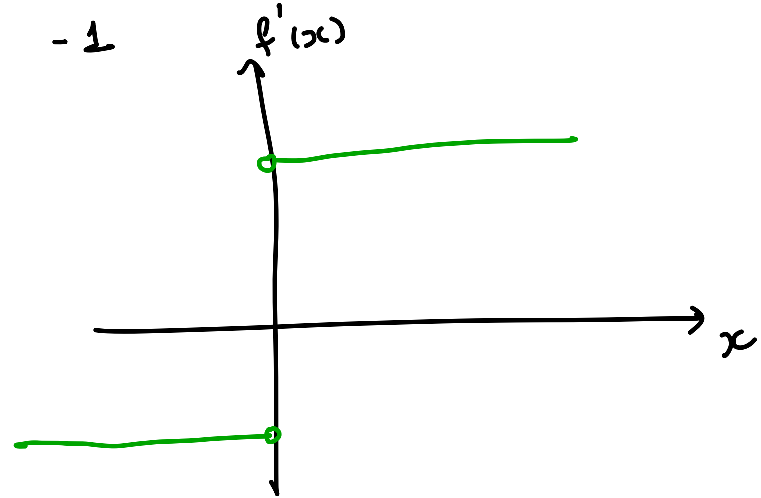
Case III : $x = 0$

$$\lim_{x \to 0^+} f'(x) \neq \lim_{x \to 0^-} f'(x)$$

Not Differentiable.

$$1 \neq -1$$

$$f'(x) = \begin{cases} +1 & ; x > 0 \\ -1 & ; x < 0 \\ \text{N.D} & ; x = 0 \end{cases}$$



Differentiability

Step 1   get $f'(x)$

If $f'(x)$ is discontinous, we can say $f(x)$ is not differentiable

Differentiability = Continous function + Smooth fun.

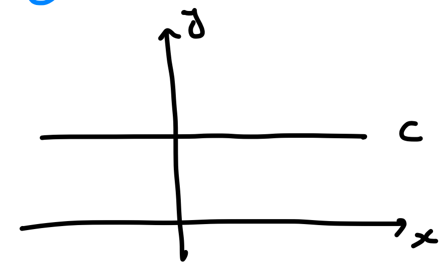
Commonly used derivatives

1. $\frac{d}{dx} x^n = n \cdot x^{n-1}$

2. $\frac{d}{dx} \log x = \frac{1}{x}$

3. $\frac{d}{dx} e^x = e^x$

4. $\frac{d}{dx} c = 0$



5. $\frac{d}{dx} \sin x = \cos x$

6. $\frac{d}{dx} \cos x = -\sin x$

7. $\frac{d}{dx} \tan x = \sec^2 x$

III Rules of derivatives

1. Linearity Rule.

$$h(x) = f(x) \pm g(x)$$

$$h'(x) = f'(x) \pm g'(x)$$

$$h(x) = \sin x - x^2$$

$$h'(x) = \cos x - 2x$$

$$f(x) = 3x^2 + 4x$$

$$f'(x) = 6x + 4$$

2 Product Rule

$$h(x) = f(x) \cdot g(x)$$

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$f(x) = \underbrace{x}\cdot\underbrace{\log x}$$

$$f'(x) = 1 \cdot \log x + x \cdot \frac{1}{x}$$

$$f'(x) = 1 + \log x$$

$$f(x) = x^2 = \underbrace{x}\cdot\underbrace{x}$$

$$f'(x) = 1 \cdot x + x \cdot 1$$

$$= 2x$$

$$f(x) = \frac{\log x}{x}$$

$$f(x) = (\log x) \cdot \left(\frac{1}{x}\right)$$

$$f(x) = \log x \cdot x^{-1}$$

$$f'(x) = \frac{1}{x} \cdot x^{-1} + \log x \cdot (-1)\, x^{-2}$$

$$\Rightarrow \frac{1}{x^2} - \frac{\log x}{x^2} \Rightarrow \frac{1 - \log x}{x^2}$$

3. Quotient Rule

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

1. $f(x) = \dfrac{e^x}{x^2}$

$$= \frac{e^x \cdot x^2 - e^x \cdot 2 \cdot x}{x^4}$$

$$= \frac{e^x \cdot x - 2e^x}{x^3}$$

4. Chain Rule.

$$h(x) = f(g(x))$$

v. Imp

$$\boxed{h'(x) = f'(g(x)) \cdot g'(x)}$$

$$h(x) = \log(x^2)$$

$f = \log$

$g = x^2$

$$= \frac{d \log x^2}{dx^2} \cdot \frac{dx^2}{dx}$$

$$= \frac{1}{x^2} \cdot 2x = \frac{2}{x}$$

$$\frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{d}{dx} \log x^2 = \cancel{\ }$$

$$\frac{d \log z}{dz} = \frac{1}{z}$$

$$f(x) = \sin(e^x)$$

$$\frac{df}{de^x} = f'(e^x) = \cos e^x$$

$$\frac{df}{dx} = f'(x) = \frac{d \sin e^x}{de^x} \cdot \frac{de^x}{dx}$$

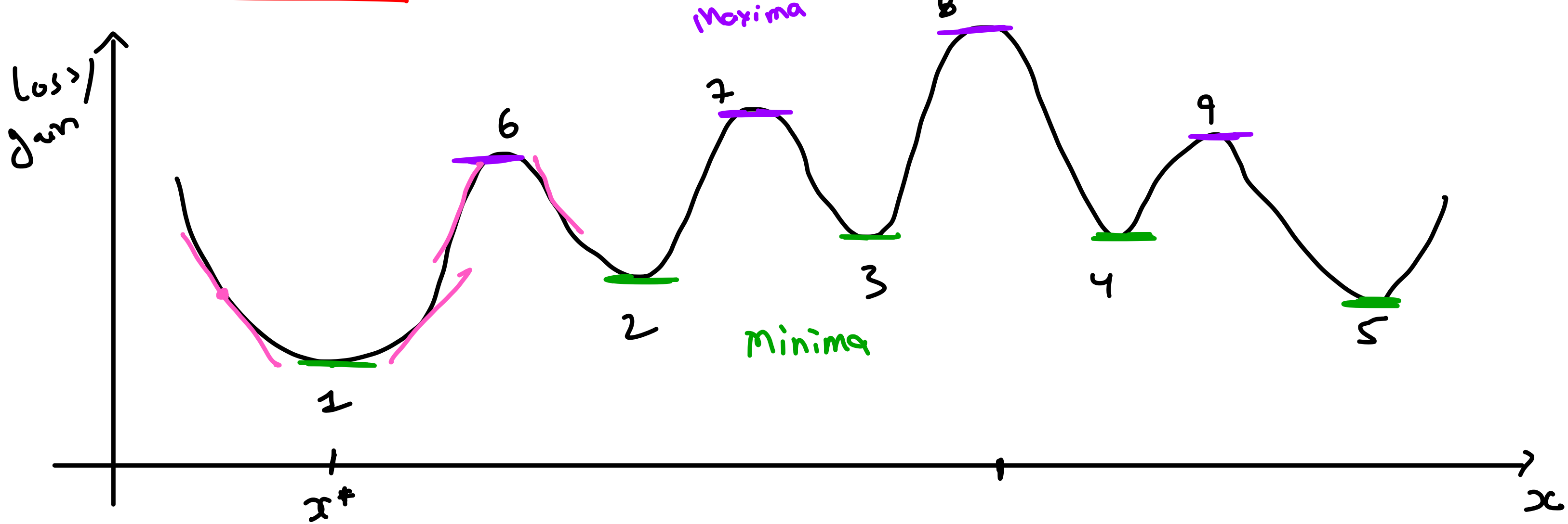
$$= \cos e^x \cdot e^x$$

$$\log(\underbrace{e^{\overbrace{x^2+4x}^{z}}}_{y})$$

$$\log(y)$$

$$\frac{d \log y}{dy} \cdot \frac{dy}{dz} \cdot \frac{dz}{dx}$$

VII How to use derivatives for optimization?



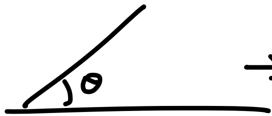
local minima

local maxima

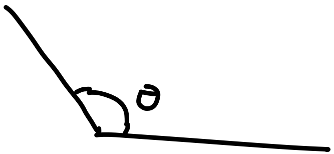
→ global minima/maxima

1 → global minima

8 → global maxima



$\to$ +ve slope



$\to$ -ve slope

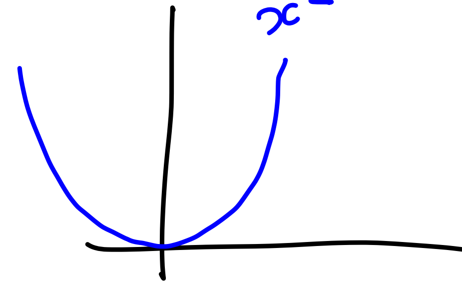
$\boxed{f'(x) = 0} \to$ Critical Point

either maxima / minima.

$f''(x) > 0 \to$ minima

$< 0 \to$ Maxima

$= 0 \to$ saddle Point



$f'(x) = 2x$

$2x = 0$

$\boxed{x = 0}$

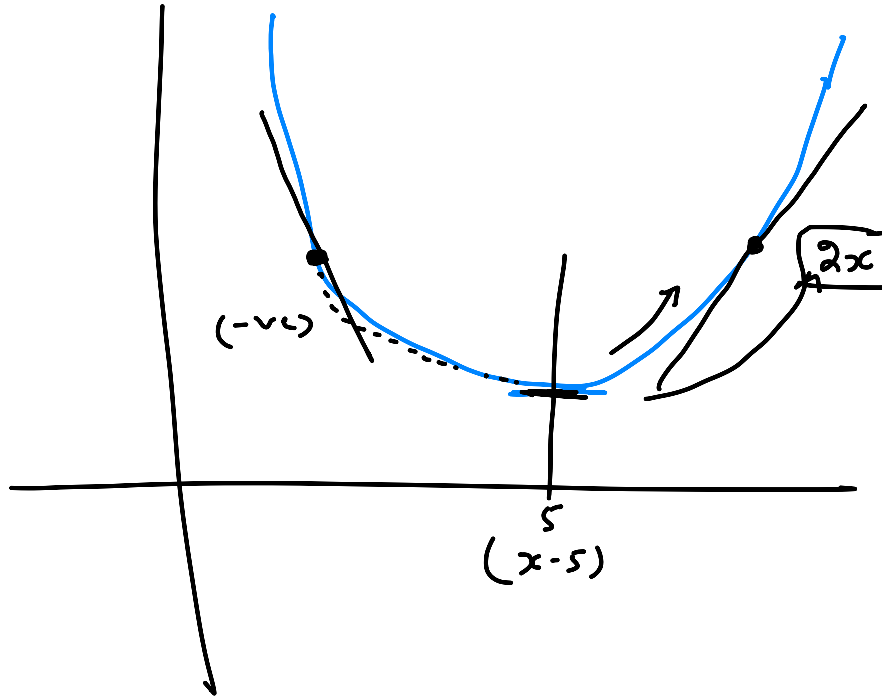
At $x = 0$ either minima or maxima.

$f''(x) = 2 > 0$

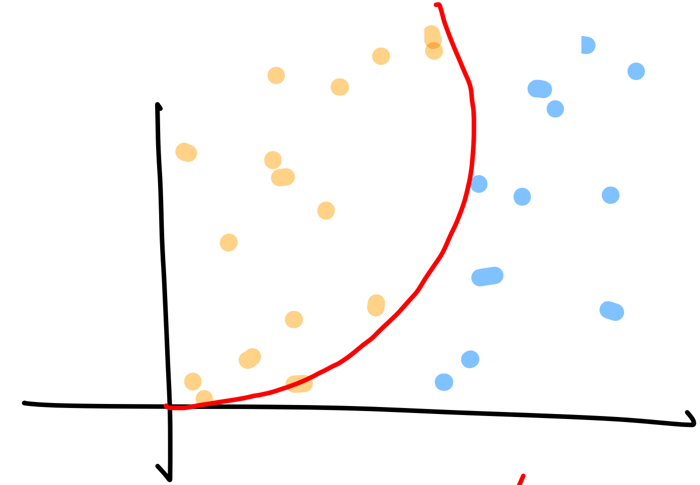
$\downarrow$

minima.

$(x-5)^2$



$$\boxed{f(x)} = x^2$$



$f(x) = \underline{\text{line func}}$

$$y = mx + c$$

$$w_1 x_1 + w_2 x_2 + w_0 = 0$$