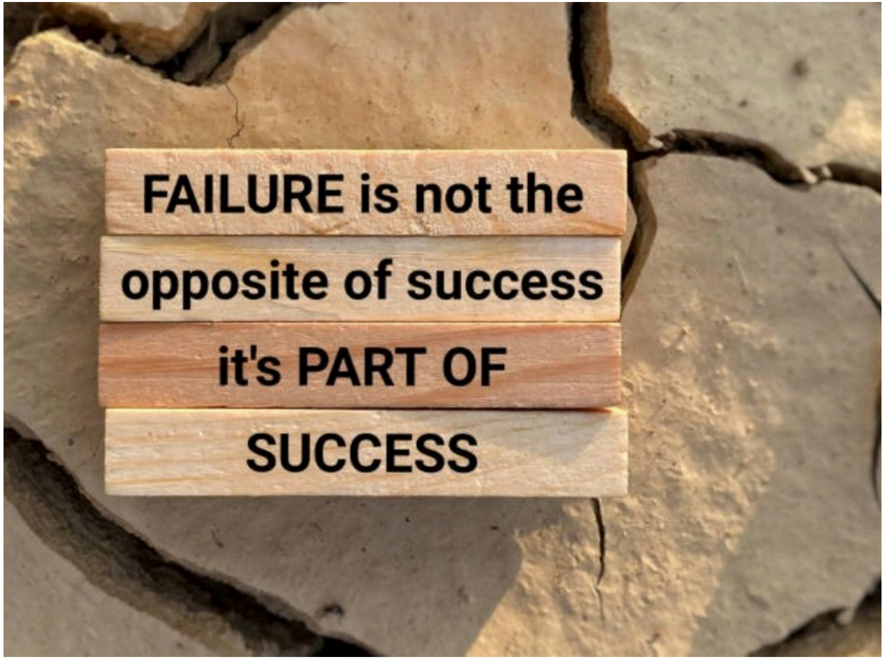
# 1A-3

# Distances



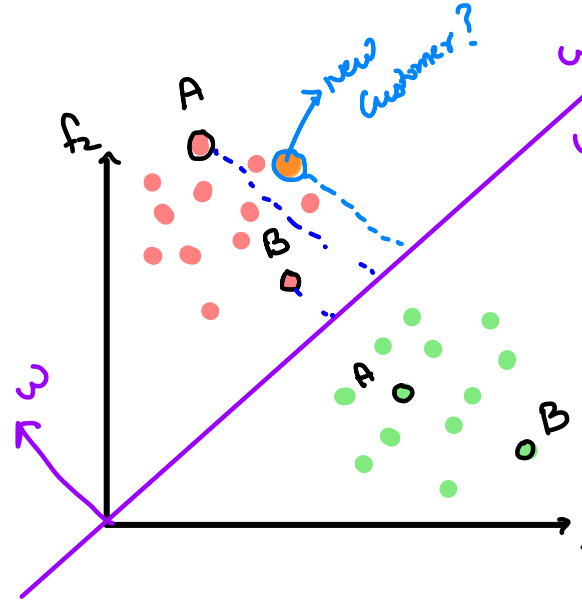
# Agenda :-

1. Recap
2. Weight vector & line relation
3. Distance (origin, line)
4. Distance (point, line)

# Motivation

When does your model is confident about predictions?

Distances?




$w_1 x_1 + w_2 x_2 + w_0 = 0$

$w^T x + w_0 = 0$

$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

## Recap

### Vectors

$$\vec{x}, \vec{y} \in \mathbb{R}^d$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \quad \text{or} \quad \vec{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_d \end{bmatrix}$$

### Norm

↳ length
magnitude

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$$ → L2 Norm

### Dot Product

$$\vec{x} \cdot \vec{y} = x^T y = \|x\| \cdot \|y\| \cos\theta = \sum_{i=1}^{d} x_i \cdot y_i$$

### Angle $(\vec{v}_1, \vec{v}_2)$

$$\cos\theta = \frac{v_1^T v_2}{\|v_1\| \cdot \|v_2\|}$$

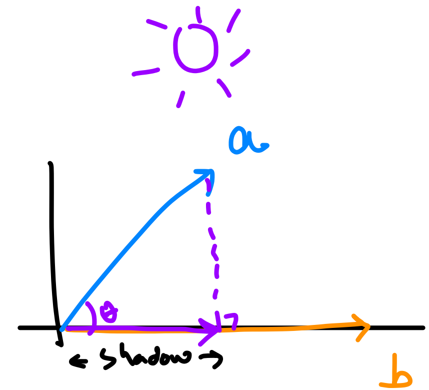
**Unit Vector**

$$\hat{a} = \frac{\vec{a}}{\|a\|}$$

**$Proj_y(x)$**

$$\|P\| = x^T \hat{y}$$

$$= \|x\| \cos\theta$$




$a^T \hat{b}$

Q - $x_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad$ $x_2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

(a) Find Norm !

$\|x_1\| = \sqrt{4 + 9}$
$= \sqrt{13}$

$\|x_2\| = \sqrt{36 + 49}$
$= \sqrt{85}$

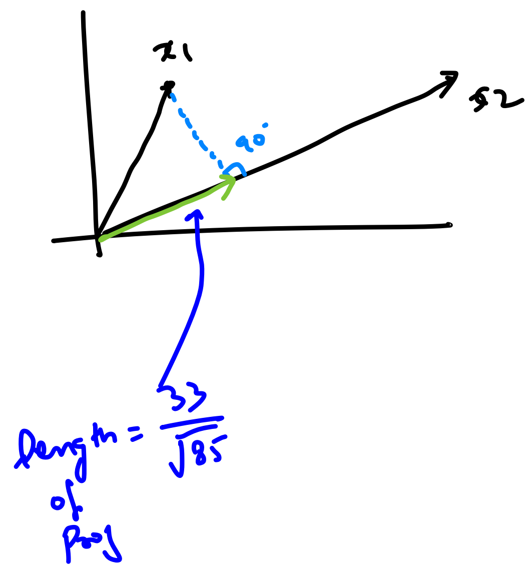
(b) Angle b/w 2 vectors ?

$\cos\theta = \dfrac{x_1^T x_2}{\|x_1\| \|x_2\|} = \dfrac{33}{\sqrt{13} \cdot \sqrt{85}}$

$[2, 3] \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

$\theta = \cos^{-1}\left(\dfrac{33}{\sqrt{13} \cdot \sqrt{85}}\right)$

(c) $Proj_{x_2}(x_1)$ : Proj of $x_1$ on $x_2$




$$Proj_{x_2} x_1 = x_1^T \cdot \hat{x}_2$$

$$= \begin{bmatrix} 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6/\sqrt{85} \\ 7/\sqrt{85} \end{bmatrix}$$

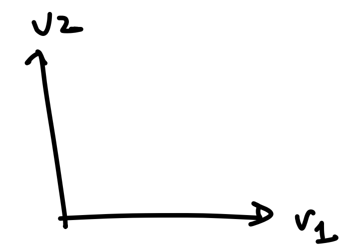
$$= \frac{33}{\sqrt{85}}$$

Q - $v_1^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad v_2^T = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$v_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad v_4^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

Calculate angle b/w them!

$$\cos\theta = \frac{v_1^T v_2}{\|v_1\| \cdot \|v_2\|} = 0$$




$$\theta = 90^\circ$$

# Shifting lines

$$w_1 \boxed{x_1} + w_2 x_2 + w_0 = 0$$

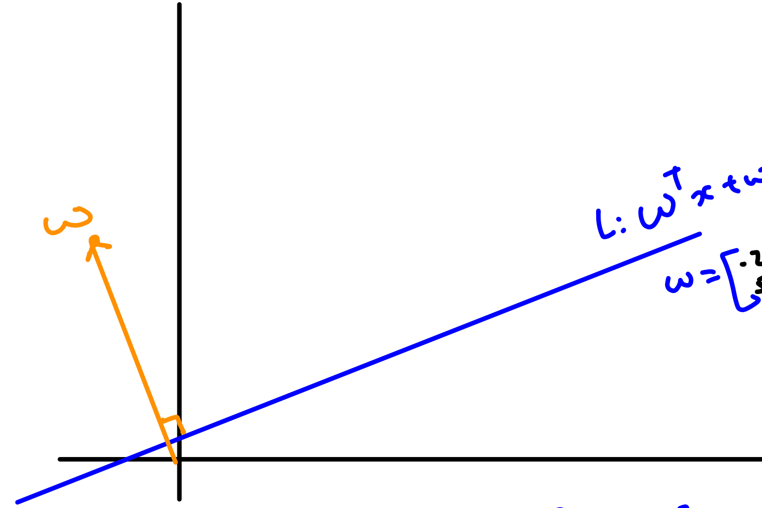
$$w_1 (x_1 + a) + w_2 x_2 + w_0 = 0$$

$$w_1 x_1 + w_1 a + w_2 x_2 + w_0 = 0$$

$$w_1 x_1 + w_2 x_2 + \underbrace{(w_0 + w_1 a)}_{w_0'} = 0$$

$$[w_1 x_1 + w_2 x_2 + w_0' = 0] \rightarrow \text{shifted line}$$

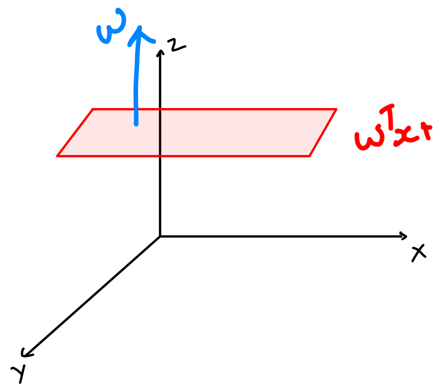
## Weight Vector Orthogonality:




$l: w^T x + w_0 = 0$

$w = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$-2x_1 + 5x_2 + w_0 = 0$




$w^T x + w_0 = 0$

$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

$w_1 x_1 + w_2 x_2 + w_3 x_3 + w_0 = 0$

$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$$ax + by + c = 0$$

$$w_1x_1 + w_2x_2 + w_0 = 0$$

Suppose, $\boxed{w_0 = 0}$

$$w_1x_1 + w_2x_2 = 0$$

$$\boxed{w^Tx = 0}$$

why dot$(w, x) = 0$ ?

$$w = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad x = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

which lies exactly on line

Both $w$ & $x$ are $90°$.

Note.

if $w_0 = 0$

line passes through origin.

$$w \cdot x = 0$$

angle b/w $w$ & $x$ = $90°$

Shortest distance (line, origin)

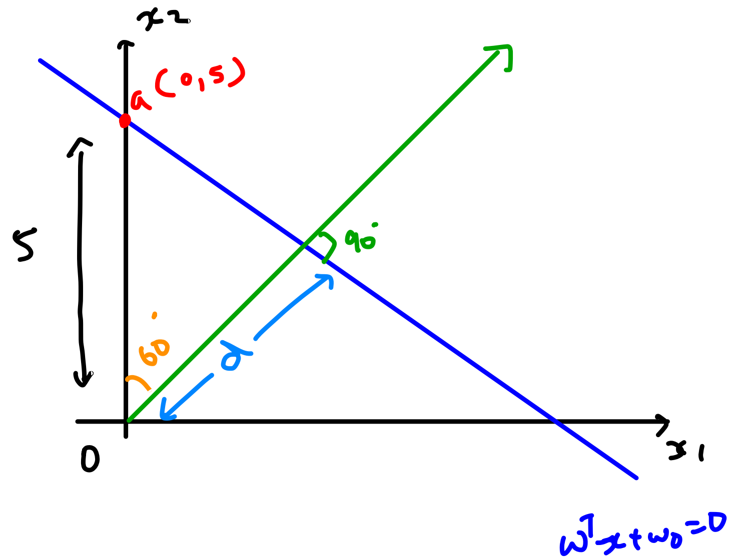
$d =$

$$\cos 60^\circ = \frac{b}{h} = \frac{d}{5}$$

$$d = \cos 60^\circ \times 5$$

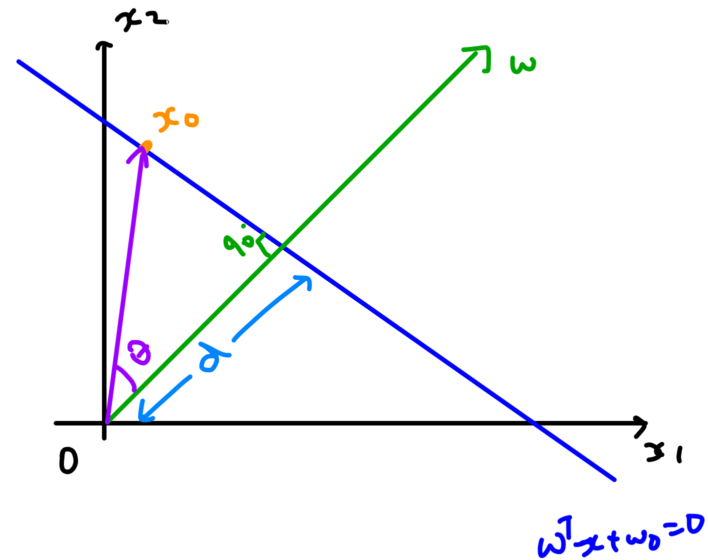
$$d = 5 \times \frac{1}{2}$$

$$\boxed{d = 2.5}$$



Suppose $x_0$ lies on line.

$$w^T x_0 + w_0 = 0$$




$$\cos\theta = \frac{d}{\|x_0\|}$$

$$d = \|x_0\| . \cos\theta$$

angle$(w, x_0)$

$$\cos\theta = \boxed{\frac{w^T x_0}{\|w\| \|x_0\|}}$$

$$d = \cancel{\|x_0\|} \frac{w^T x_0}{\|w\| \cancel{\|x_0\|}}$$

$$d = \frac{w^T x_0}{\|w\|}$$

$$w^T x_0 + w_0 = 0$$

$$\boxed{w^T x_0 = -w_0}$$

$$\boxed{d = \frac{-w_0}{\|w\|}}$$

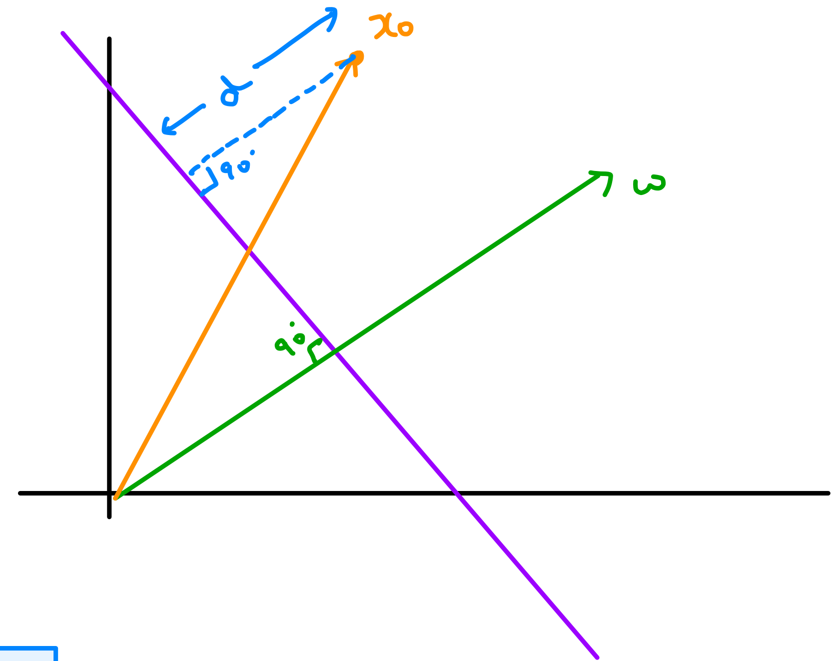
# Distance (Point, line)

Given $x_0 = [\ ]$ $\quad w = [\ ]$, $w_0$

To find: 'd'

$$\boxed{d = \frac{w^T x_0 + w_0}{\|w\|}}$$

$$\boxed{d = \frac{w_1 x_{0_1} + w_2 x_{0_2} + w_0}{\sqrt{w_1^2 + w_2^2}}}$$

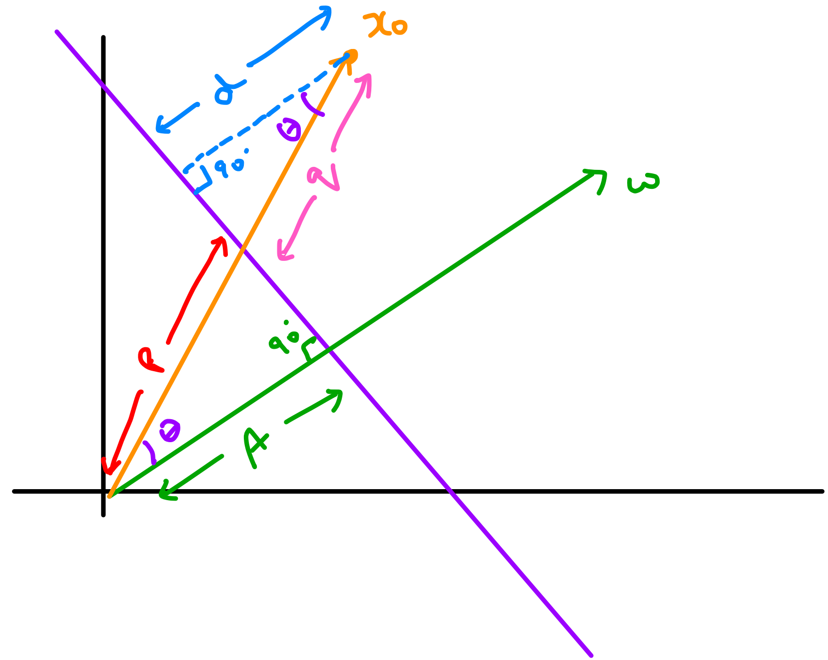


$$\|x_0\| = p + q$$

$$A = \frac{-w_0}{\|w\|}$$

$$\cos\theta = \frac{b}{h} = \frac{A}{p}$$

$$\boxed{p = \frac{A}{\cos\theta}}$$




$$q = \|x_0\| - p$$

$$\boxed{q = \|x_0\| - \frac{A}{\cos\theta}} \quad \text{— } 1$$

$$\cos\theta = \frac{d}{q}$$

$$\boxed{q = \frac{d}{\cos\theta}} \quad \text{— } 2$$

$$\frac{d}{\cos\theta} = \left( \|x_0\| - \frac{A}{\cos\theta} \right)$$

$$d = \left[ \|x_0\| \cos\theta - A \right]$$

$$d = \left[ \|x_0\| \cos\theta + \frac{w_0}{\|w\|} \right]$$

angle b/w w & $x_0$

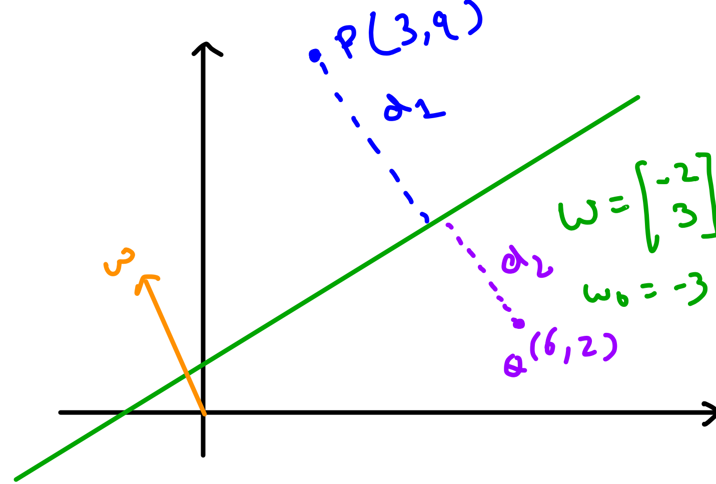
$$d = \left[ \cancel{\|x_0\|} \cdot \frac{w^T x_0}{\|w\| \cancel{\|x_0\|}} + \frac{w_0}{\|w\|} \right]$$

Imp:

$$d = \frac{w^T x_0 + w_0}{\|w\|}$$

$w = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad w_0 = -3$

$P = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$




$$d_1 = \frac{w^T P + w_0}{\|w\|}$$

$$= \frac{21 - 3}{\sqrt{13}}$$

$$d_1 = \frac{18}{\sqrt{13}} = 5.83 \ (+ve)$$

$\because$ weight vector & datapoint are in (same dir$^n$)

$$d_2 = \frac{w^T Q + w_0}{\|w\|}$$

$$= \frac{-6 - 3}{\sqrt{13}}$$

$$= -9/\sqrt{13} = -2.49 \ (-ve)$$