1 Mercer's Theorem

Preliminaries:

χ : compact subset of $\mathbb{R}^N$

K : $\chi \times \chi \to \mathbb{R}, k \in L_\alpha,$ symmetric function ("kernel")

$$\left.\begin{array}{ll} T_k: & L_{2(\chi)} \to L_{2(\chi)} \\ (T_k f)_{(\underline{\mathbf{x}})} := & \int_\chi K_{(\underline{\mathbf{x}},\underline{\mathbf{x}}')} f_{(\underline{\mathbf{x}}')} d\underline{\mathbf{x}}' \end{array}\right\} \text{corresponding integral operator}$$

$$\left.\begin{array}{ll} \lambda_j: & \text{eigenvalues} \\ \psi_j \in L_{2(\chi)}: & \text{normalized eigenfunction} \end{array}\right\} \text{of } T_k$$

essential condition: T_k positive definite

$$\int_{\chi\times\chi} K_{(\underline{\mathbf{x}},\underline{\mathbf{x}}')} f_{(\underline{\mathbf{x}})} f_{(\underline{\mathbf{x}}')} d\underline{\mathbf{x}} d\underline{\mathbf{x}}' > 0 \qquad \forall f \in L2(\chi) \tag{1}$$

then:

$$K_{(\underline{\mathbf{x}},\underline{\mathbf{x}}')} = \sum_{j=1}^{n} \lambda_j \underbrace{\psi_{j(\underline{\mathbf{x}})} \psi_{j(\underline{\mathbf{x}}')}}_{\substack{\text{eigenvalue} \\ \text{decomposition}}} \tag{2}$$

$n \to \infty$: absolute and uniform convergence (non-trivial part)

consequences of Mercer's theorem:

$$\begin{aligned} \underline{\phi} : \underline{\mathbf{x}} &\to \left(\sqrt{\lambda_1}\psi_{1(\underline{\mathbf{x}})}, \sqrt{\lambda_2}\psi_{2(\underline{\mathbf{x}})}, \ldots, \sqrt{\lambda_n}\psi_{n(\underline{\mathbf{x}})}\right)^T \\ K_{(\underline{\mathbf{x}},\underline{\mathbf{x}}')} &= \underline{\phi}^T_{(\underline{\mathbf{x}})} \underline{\phi}_{(\underline{\mathbf{x}}')} \end{aligned} \tag{3}$$