1 Mean-fields for central clustering

$$d_{\alpha\gamma} = \frac{1}{2}\left(\underline{\mathbf{x}}^{(\alpha)} - \underline{\mathbf{x}}^{(\gamma)}\right)^2, \text{ approximation to } O_{\left(\frac{1}{p}\right)} \tag{1}$$

$$\begin{aligned}
e_p^{(\alpha)} &= \frac{1}{M\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} \sum_{\delta\neq\alpha} \left\langle m_p^{(\delta)}\right\rangle_Q \left(\underline{\mathbf{x}}^{(\delta)} - \underline{\mathbf{x}}^{(\alpha)}\right)^2 \\
&\quad - \frac{1}{M\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} \sum_{\delta\neq\alpha} \left\langle m_p^{(\delta)}\right\rangle_Q \sum_{\varepsilon\neq\alpha} \left\langle m_p^{(\varepsilon)}\right\rangle_Q \left(\underline{\mathbf{x}}^{(\delta)} - \underline{\mathbf{x}}^{(\varepsilon)}\right)^2 \\
&= \frac{1}{M\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} \sum_{\delta\neq\alpha} \left\langle m_p^{(\delta)}\right\rangle_Q \left\{\underline{\mathbf{x}}^{(\delta)} - 2\left(\underline{\mathbf{x}}^{(\delta)}\right)^T \underline{\mathbf{x}}^{(\alpha)} + \left(\underline{\mathbf{x}}^{(\alpha)}\right)^2\right\} \\
&\quad - \frac{1}{M\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} \sum_{\delta\neq\alpha} \left\langle m_p^{(\delta)}\right\rangle_Q \sum_{\varepsilon\neq\alpha} \left\langle m_p^{(\varepsilon)}\right\rangle_Q \\
&\quad \cdot\left\{\left(\underline{\mathbf{x}}^{(\delta)}\right)^2 - 2\left(\underline{\mathbf{x}}^{(\delta)}\right)^T \underline{\mathbf{x}}^{(\varepsilon)} + \left(\underline{\mathbf{x}}^{(\varepsilon)}\right)^2\right\} \\
&= \frac{1}{M}\left[\frac{\sum\limits_{\delta\neq\alpha}\left\langle m_p^{(\delta)}\right\rangle_Q \left(\underline{\mathbf{x}}^{(\delta)}\right)^2}{\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} - 2\frac{\left(\underline{\mathbf{x}}^{(\alpha)}\right)^T \sum\limits_{\delta\neq\alpha}\left\langle m_p^{(\delta)}\right\rangle_Q \underline{\mathbf{x}}^{(\delta)}}{\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} + \left(\underline{\mathbf{x}}^{(\alpha)}\right)^2\right] \\
&\quad - \frac{1}{M\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} \sum_{\delta\neq\alpha} \left\langle m_p^{(\delta)}\right\rangle_Q \left\{\frac{\sum\limits_{\varepsilon\neq\alpha}\left\langle m_p^{(\varepsilon)}\right\rangle_Q \left(\underline{\mathbf{x}}^{(\varepsilon)}\right)^2}{\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q}\right. \\
&\quad \left. -2\frac{\left(\underline{\mathbf{x}}^{(\delta)}\right)^T \sum\limits_{\varepsilon\neq\alpha}\left\langle m_p^{(\varepsilon)}\right\rangle_Q \underline{\mathbf{x}}^{(\varepsilon)}}{\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} + \left(\underline{\mathbf{x}}^{(\delta)}\right)^2\right\} \\
&\approx \frac{1}{M}\left[\left(\underline{\mathbf{x}}^{(\alpha)}\right)^2 - 2\left(\underline{\mathbf{x}}^{(\alpha)}\right)^T \underline{\mathbf{w}}_p + \underline{\mathbf{w}}_p^2\right]
\end{aligned} \tag{2}$$

with:

$$\underline{\mathbf{w}}_p = \frac{\sum\limits_{p}\left\langle m_p^{(\gamma)}\right\rangle_Q \underline{\mathbf{x}}^{(\gamma)}}{\sum\limits_{\gamma}\left\langle m_p^{(\gamma)}\right\rangle_Q} \tag{3}$$