Mathematical Derivation: Ito's Lemma & GBM

This document provides a rigorous derivation of the log-price dynamics for a stock following Geometric Brownian Motion (GBM). This transformation is the foundation of the Black-Scholes-Merton model.

1. Initial Stochastic Process

We assume the asset price S_t follows the standard GBM SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

2. Applying the Lemma

To find the dynamics of the log-price, we define the transformation $f(S_t) = \ln(S_t)$. According to **Ito's Lemma**, the differential df is given by:

$$df = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW_t$$

3. Calculation of Derivatives

We calculate the necessary partial derivatives of $f(S) = \ln(S)$:

- **First Derivative:** $\frac{\partial f}{\partial S} = \frac{1}{S}$
- **Second Derivative:** $\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$
- **Time Derivative:** $\frac{\partial f}{\partial t} = 0$

4. Final Result

Substituting these derivatives back into the Ito formula:

$$d(\ln S_t) = \left(\mu S \left(\frac{1}{S} \right) + \frac{1}{2}\sigma^2 S^2 \left(-\frac{1}{S^2} \right) \right) dt + \sigma S \left(\frac{1}{S} \right) dW_t$$

Simplifying the terms, we arrive at the SDE for the log-return:

$$d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t$$