Stochastic Energy Dynamics:
From Mean Reversion to Merton Jump-Diffusion

1 Introduction

Modeling energy commodities like Brent Crude requires accounting for two distinct market behaviors: long-term equilibrium (mean reversion) and sudden, discontinuous price shocks (jumps). This paper provides the mathematical derivation for transitioning from a standard Ornstein-Uhlenbeck (OU) process to a Merton Jump-Diffusion (MJD) framework.

2 Phase 1: The Ornstein-Uhlenbeck Process

The foundation of our model assumes that prices S_t tend to revert to a long-term mean θ. The continuous dynamics are described by the SDE:

$$dS_t = \kappa(\theta - S_t)dt + \sigma S_t dW_t \tag{1}$$

Where dW_t is a standard Wiener process. Under this model, the returns are assumed to be normally distributed, represented by the probability density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(x-\mu)^2}{2\sigma^2 t}} \tag{2}$$

Technical Reasoning: While the OU model captures the equilibrium of the oil market, it fails to account for the "Fat-Tails" (high kurtosis) observed during geopolitical crises.

3 Phase 2: Merton Jump-Diffusion Expansion

To address the inadequacy of the normal distribution in energy markets, we introduce a Poisson-driven jump component.

3.1 The Expanded SDE

We define the new price process by adding a jump term dJ_t:

$$dS_t = \kappa(\theta - S_t)dt + \sigma S_t dW_t + S_t dJ_t \tag{3}$$

The jump term is defined as $dJ_t = (Y - 1)dN_t$, where:

- N_t is a Poisson process with intensity λ, such that $P(dN_t = 1) = \lambda dt$.
- Y is the random jump magnitude, where $\ln(Y) \sim N(\mu_j, \sigma_j^2)$.

3.2 Log-Price Transformation

For numerical stability and to prevent negative prices in our simulation engine, we apply Itô's Lemma to the log-price $x_t = \ln S_t$. The resulting dynamics are:

$$dx_t = \left[\kappa(\ln\theta - x_t) - \frac{1}{2}\sigma^2\right] dt + \sigma dW_t + \ln(Y) dN_t \tag{4}$$

4 Conclusion: Technical Justification

The transition to Jump-Diffusion is necessitated by the empirical observation of "Black Swan" events in Brent Crude data.

- **Volatility Separation**: By isolating jumps via the intensity λ, we obtain a "Clean σ" for the diffusion component, preventing the overestimation of daily market noise.
- **Risk Management**: The MJD model provides a more accurate Value-at-Risk (VaR) by accounting for the discontinuous price "spikes" that a smooth OU process ignores.