Technical Analysis: Mean-Reverting Oil Price Dynamics

1. Model Selection: Why Ornstein-Uhlenbeck?

Standard equity models typically utilize Geometric Brownian Motion (GBM), which assumes a constant upward drift. However, commodity markets such as Brent Crude are fundamentally different due to supply and demand equilibrium. We select the **Ornstein-Uhlenbeck (OU) process** because it accounts for **Mean Reversion**—the tendency of price levels to return to a long-term equilibrium (θ) after a shock.

2. Stochastic Differential Equation (SDE)

The spot price X_t is modeled by the following SDE:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t \tag{1}$$

Where:

- κ: Reversion speed (calculated via regression or observation).
- θ: The long-term mean price, empirically derived as **$79.75**.
- σ: Annualized volatility, empirically derived as **33.93%**.
- dW_t: Increment of a standard Wiener process.

3. Mathematical Proof using Ito's Lemma

To solve for the price at time t, we apply **Ito's Lemma** to the transformation $f(X_t, t) = X_t e^{\kappa t}$. Applying the Lemma:

$$d(X_t e^{\kappa t}) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX_t)^2 \tag{2}$$

Calculating partial derivatives:

- $\frac{\partial f}{\partial t} = \kappa X_t e^{\kappa t}$
- $\frac{\partial f}{\partial X} = e^{\kappa t}$
- $\frac{\partial^2 f}{\partial X^2} = 0$

Substituting Eq. (1) into the expansion:

$$d(X_t e^{\kappa t}) = \kappa X_t e^{\kappa t}dt + e^{\kappa t}[\kappa(\theta - X_t)dt + \sigma dW_t] \tag{3}$$

Simplifying the drift terms:

$$d(X_t e^{\kappa t}) = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dW_t \tag{4}$$

4. Numerical Discretization

For the Python implementation in `src/ou_model.py`, we utilize the **Euler-Maruyama method**. The discrete update rule is:

$$X_{t+\Delta t} = X_t + \kappa(\theta - X_t)\Delta t + \sigma\sqrt{\Delta t}\epsilon \tag{5}$$

Where $\epsilon \sim N(0, 1)$. This allows for the high-resolution simulation of paths that oscillate around the equilibrium mean.