Final Term Report

Option Pricing Models and Their Accuracy

Nihal H

210070056

Contents

INTRODUCTION:

Options are financial derivatives that give the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price within a specific time period. The underlying asset can be stocks, commodities, indices, currencies, or even other derivative products.

There are two types of options:

- Call Option: A call option gives the holder the right to buy the underlying asset at a specified price, known as the strike price, before the expiration date. If the market price of the underlying asset rises above the strike price, the holder can exercise the option and profit from buying the asset at a lower price.

- Put Option: A put option gives the holder the right to sell the underlying asset at a specified strike price before the expiration date. If the market price of the underlying asset falls below the strike price, the holder can exercise the option and profit from selling the asset at a higher price.

Options provide flexibility and opportunities for investors and traders to manage risk, speculate on price movements, and generate potential profits. They are commonly traded on options exchanges, and their prices are influenced by factors such as the underlying asset price, time to expiration, volatility, interest rates, and market conditions.

Option pricing is significant because it provides a fair value estimation for options, enabling us to make informed decisions regarding our trading strategies. By knowing the theoretical value of an option, we can assess whether it is overpriced or underpriced in the market. This information is crucial for executing profitable trades, managing risk, and maximizing potential returns. Option pricing models allow us to evaluate different scenarios, compare different options contracts, and select the ones that align with their investment goals and risk tolerance. Option pricing also contributes to market efficiency by ensuring that options are traded at prices that reflect market conditions and expectations.

Option pricing models are mathematical models used to estimate the theoretical value of options. These models help us determine the fair price of an option based on various factors, such as the current price of the underlying asset, the strike price, time to expiration, interest rates, and market volatility.

The most widely used option pricing model is the Black-Scholes model, developed by economists Fischer Black and Myron Scholes in 1973. Other than Black-Scholes model, Binomial Option Pricing Model, Monte Carlo Simulation and Heston Model are some of the key option pricing models. Each has its own assumptions and limitations, and the choice of model depends on factors such as the type of option, market conditions, and investor preferences.

GEOMETRIC BROWNIAN MOTION:

Geometric Brownian motion is a mathematical model commonly used to describe the dynamics of stock prices or other financial assets over time. It is based on the assumption that the logarithmic returns of the asset follow a random walk with constant drift and volatility.

The formula for the geometric Brownian motion of a stock price (S) can be expressed as:

$$dS = \mu * S * dt + \sigma * S * dW$$

- ➔ dS represents the infinitesimal change in the stock price over a small time interval (dt).
- ➔ μ is the expected return or drift of the stock price. It represents the average rate of growth in the stock price over time.
- ➔ σ is the volatility of the stock price. It measures the degree of fluctuation or dispersion in the stock price returns.
  - ➔ dW represents the increment of a Wiener process or standard Brownian motion. It represents the random component or "noise" in the model.

The term $\mu * S * dt$ represents the deterministic component, reflecting the expected growth or drift of the stock price over time. The term $\sigma * S * dW$ represents the stochastic component, accounting for the random fluctuations or volatility in the stock price.

Geometric Brownian motion assumes that the logarithmic returns of the stock price are normally distributed and independent over time, which implies that the stock price itself follows a lognormal distribution. This property aligns with empirical observations that stock prices tend to exhibit lognormal behavior.

The geometric Brownian motion model is widely used in option pricing models, as it allows for the simulation of future stock price paths and the estimation of option prices based on these paths.

MONTE CARLO SIMULATION

INTRODUCTION:

Monte Carlo simulation is a computational technique that uses random sampling to model and analyze complex systems or processes. The method involves running repeated simulations to estimate the behavior and outcomes of a system by generating a large number of random inputs. It is particularly valuable when dealing with complex systems that cannot be easily modeled analytically or when there is uncertainty or variability in the inputs.

Monte Carlo simulation is extensively used in option pricing to estimate the value of options and analyze their associated risks. The method allows for the consideration of various factors, such as the underlying asset's price, volatility, interest rates, and time to expiration, which can significantly impact option prices.

KEY STEPS INVOLVED:

Model the Asset Price: Choose an appropriate model (here geometric Brownian motion), to describe the underlying asset's price dynamics.

- This involves considering factors like the current asset price (S), volatility (σ), and risk-free interest rate (r).

$$dS = rS\,dt + \sigma S\,dW$$

- Generate Random Price Paths: To generate random price paths, we discretize the time interval into small increments (dt) and simulate the asset price at each time step. Starting with the initial asset price (S[0]), we can calculate the asset price at each subsequent time step using the following formula:

$$S[t + dt] = S[t] * \exp((r - 0.5 * \sigma^2) * dt + \sigma * \sqrt{dt} * Z)$$

Here, Z is a random variable drawn from a standard normal distribution. It captures the element of randomness in the asset price, allowing for the simulation of various potential future paths.

- Calculate Option Payoffs: For each simulated price path, we calculate the option payoff at its expiration based on the option type (call or put) and the difference between the asset price and the strike price.

  For example, if we have a call option, the payoff at expiration is max(S[T] - K, 0), where S[T] is the asset price at expiration and K is the strike price. For a put option, the payoff is max(K - S[T], 0).

- Discount and Average Payoffs: Discount the calculated payoffs back to present value using the risk-free interest rate. Then, take the average of all the discounted payoffs obtained from the simulated price paths.
- Repeat and Refine: Repeat the entire process by generating a larger number of random price paths and recalculating the option payoffs and their averages. The more simulations performed, the more accurate the option pricing estimation becomes.

Determine Option Value: The average of the discounted payoffs obtained from the

Monte Carlo simulations provides an estimate of

- the option's value. This value represents the fair price at which the option can be bought or sold in the market.

By following these steps using the geometric Brownian motion model, we can conduct a Monte Carlo simulation to estimate the value of an option and gain insights into its potential risks and profitability.

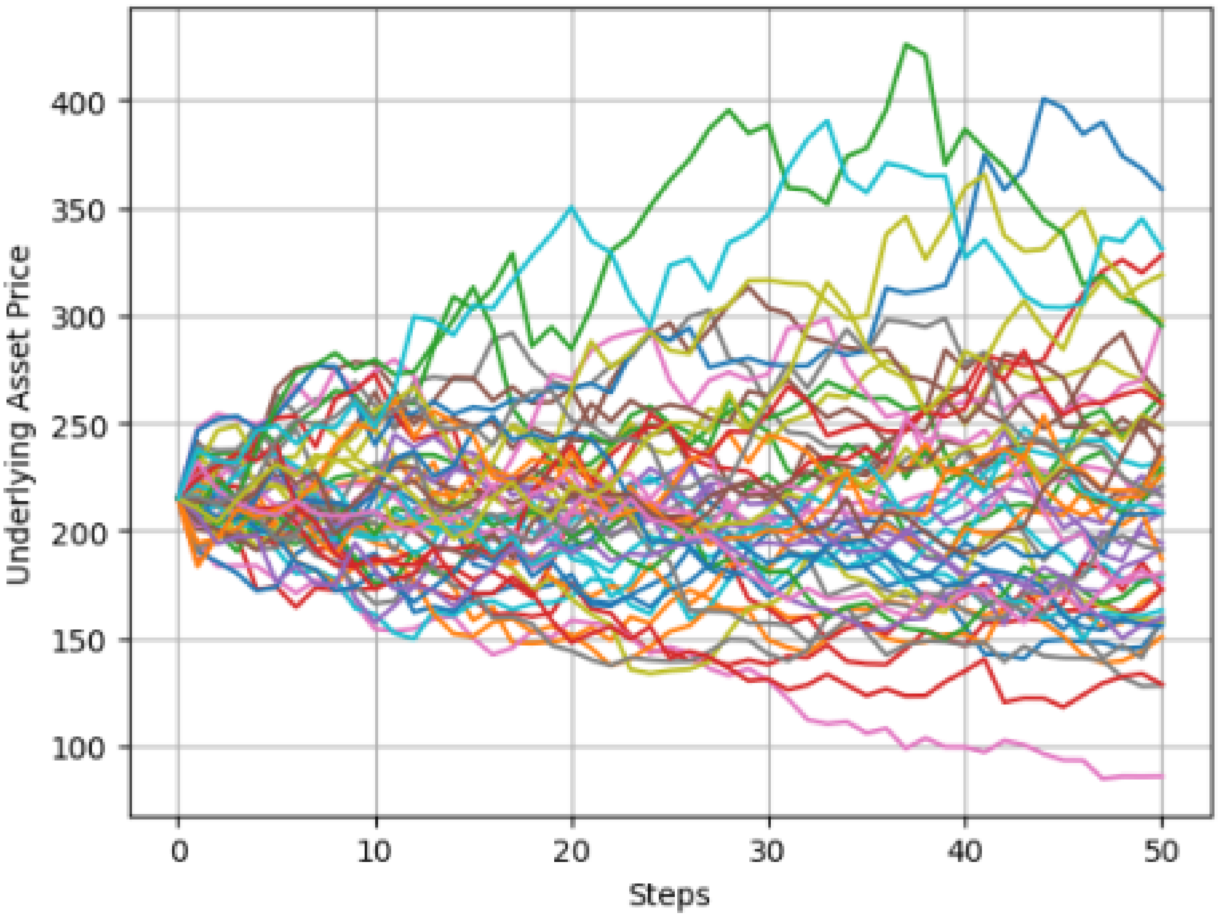
CODE & OUTPUT:

```
import math
from numpy import *
from time import time
import matplotlib.pyplot as plt
random.seed(123)  #random seed used for the reproducibility of the values
S0 = 215.         #Current value of the underlying
K = 230.          #Strike price
T = 1.0           #Time period
r = 0.05          #risk-free rate
sigma = 0.3       #volatility of the asset
steps = 50        #no. of steps
dt = T / steps
simulations = 100000 #Total no. of simulations
S = S0*exp(cumsum((r - 0.5 * sigma ** 2)*dt + sigma * math.sqrt(dt) * random.standard_normal((steps+1,simulations)),axis=0))
S[0] = S0

C0 = math.exp(-r * T) * sum(maximum(S[-1] - K, 0)) / simulations
print('The European Call Option Value is: ${:.2f}'.format(C0))

P0 = math.exp(-r * T) * sum(maximum(K - S[-1], 0)) / simulations
print('The European Put Option Value is: ${:.2f}'.format(P0))

plt.plot(S[:,:50]) #plotting the first 50 simulations
plt.grid(True)
plt.xlabel('Steps')
plt.ylabel('Underlying Asset Price')
plt.show()
```

In this case we have taken the underlying asset price as $215 and the strike price as $230, over a period of one year, with 30% volatility and 5% risk-free rate. We plan to do 100000 simulations each having 50 steps.

After the completion of the Monte Carlo simulation, we get the Call option value as $**24.54** and the Put option value as $**27.80**.

First 50 of the 100000 simulations have been plotted, and is shown below,



ADVANTAGES:

- Flexibility: Monte Carlo Simulation can handle a wide range of complex derivatives with non-linear payoffs and multiple sources of uncertainty. It allows for realistic modeling by incorporating factors such as time-varying volatility, discrete trading, transaction costs, and dividend payments.
- Accuracy: With a sufficiently large number of simulations, Monte Carlo Simulation provides accurate estimates of option prices, especially for options with irregular payoffs.
- Versatility: Monte Carlo Simulation can be applied to value both European and American-style options, as well as other types of derivatives.

DISADVANTAGES:

- Computational Intensity: Monte Carlo Simulation requires generating a large number of random paths, which can be computationally demanding and time-consuming. The computational requirements increase with higher precision requirements or complex derivatives.
- Convergence Issues: The accuracy of Monte Carlo estimates depends on the number of simulations performed. Achieving convergence, where the estimated option price stabilizes, may require a substantial number of simulations.
- Potential Biases: Monte Carlo Simulation is subject to sampling errors and potential biases, which can impact the accuracy of the estimated option prices.

CONCLUSION:

Monte Carlo simulation is a valuable technique for option pricing, especially when dealing with complex derivatives or situations where closed-form solutions are unavailable. By simulating numerous random price paths and calculating the corresponding option payoffs, Monte Carlo simulation provides a quantitative estimate of an option's value. This approach allows us to assess risks, make informed trading decisions, and gain insights into the potential profitability of different options strategies. However, it's important to note that Monte Carlo simulation is a computational approximation and its accuracy depends on the number of iterations used!

Binomial Option Pricing Model

The Binomial Options Pricing model is an options valuation method that provides a generalized and iterative numerical method for the pricing of path-dependent options (generally, American options that are exercisable at any time before the expiration of the options contract). It is a relatively simpler mathematical model based on the concept of no-arbitrage that traces the evolution of the option's key underlying variables in discrete time using a binomial lattice or tree. As this model is based on the description of an underlying instrument over a period of time rather than a single point, it can handle a variety of conditions that are difficult to predict with other models. The binomial tree is a diagram that represents different possible paths that might be followed by the stock price over the option's life. The underlying assumption is that the stock price follows a random walk. In each time step, it has a certain probability of moving up by a certain percentage amount and a certain probability of moving down by a certain percentage amount. The underlying security follows a path of evolution so that it either increases or decreases by a fixed percentage during each period of time- hence the binomial part of the model. Using a binomial tree, all potential values for the underlying asset are projected at different points of time(nodes in the binomial tree) till the expiration of the options contract and from those values, all possible final values for the option are obtained. This model is used by investors to estimate how likely they are to buy or sell at a given price in the future.

In the limit, as the time step becomes smaller, this model is the same as the Black–Scholes model. The European option price given by the binomial tree converges to that given by Black–Scholes as the time step becomes infinitesimally small.

Assumptions of the Binomial Option Pricing Model:

The binomial option pricing model makes several assumptions to simplify the pricing process. The following are the key assumptions used

1. **Discrete Time:** Time is divided into a series of discrete intervals, typically referred to as periods or steps. Price changes in these discrete time steps. Times between nodes of an event tree can be different at each link, allowing the method to be adapted the times to your specific problem if required.

2. **Risk-Neutral Valuation:** The model is based on the concept that investments (which is not the case in the real world) are risk-free. Such risk-free investments give a return equal to the risk-free ratio, or the investors are compensated for the time value only, not for the associated risk. It turns out that assuming a risk-neutral world gives us the right option price for the real world as well as for a risk-neutral world.

3. **Constant Risk-Free Rate:** Risk-Neutral Valuation implies the discount rate used for the expected payoff on an option (or any other instrument) is the risk-free rate. This risk-free rate is assumed to be constant throughout the option contract

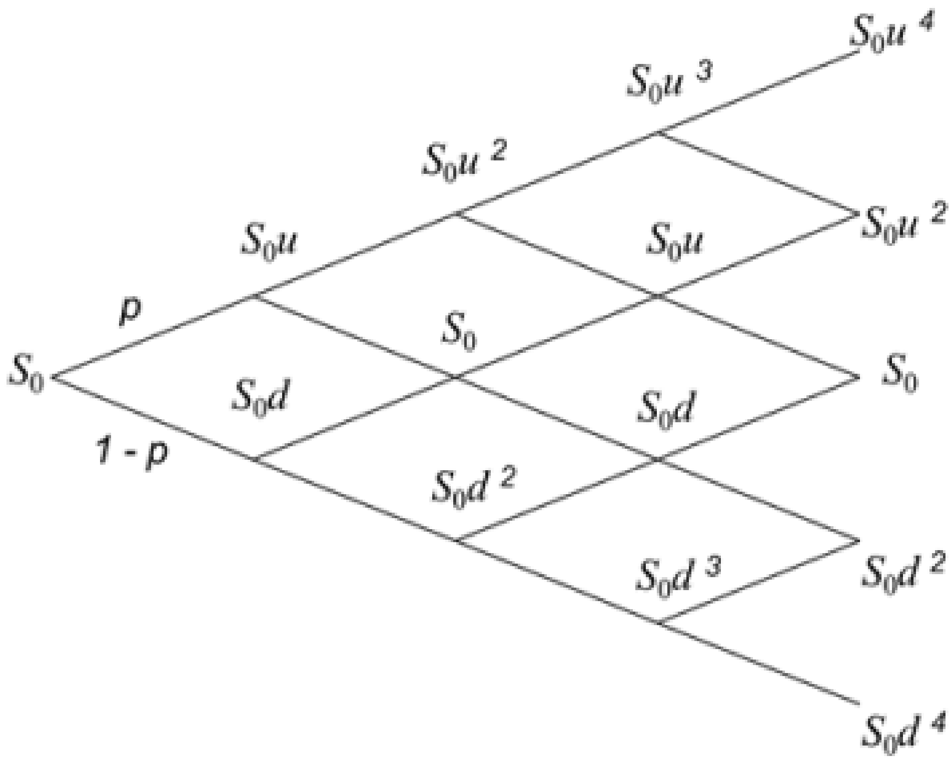
period.

4. **Only Two Possible Values** (in a given period of time): The model assumes that starting from a node, the underlying can take only two possible values, either by moving up or by moving down in a given period of discrete time step.

5. **Constant Probability:** The same probability each period for an up or down movement of the underlying asset's price is used for evaluation until the option expires. This assumption allows for the creation of a binomial tree with a fixed branching structure. However, different probabilities for each period can be used based on new information obtained as time passes.

6. **No Arbitrage:** The model assumes the absence of arbitrage opportunities.

7. **No Transaction Costs:** There are no transaction costs, such as commissions or fees, associated with buying or selling the underlying asset or the option. This assumption simplifies the model by disregarding these additional costs.



Binomial tree for an option with the security priced at S_o

Determining the Option Pricing using the Binomial Model:

After a Binomial Tree is constructed for a number of time steps(n) between valuation and expiration of the option, the evolution of the key variables for the underlying asset is traced. Each node in the lattice represents a possible price of the underlying at a given point in time. Starting at each of the final nodes and working backwards through the tree towards the first node (valuation date), values for different variables are found iteratively. The value computed at each stage is the value of the option at that point in time. Following are the steps for evaluating an option by this method

1. Constructing the binomial tree- u and d are assumed as price multipliers for option up and option down, respectively. So if Sis the current price of the underlying, then after the time step t, the price will be either S_{up}=Su or S_{down}=Sd. These are then weighed by their respective probabilities- p of an up move of the underlying and probability (1-p) of a down move. Under the risk neutrality assumption, the present value of a financial derivative equals the expected value of its future payoff discounted by the risk free rate. The expected value is then discounted at r, the risk free rate corresponding to the option.

2. Finding the expected option value- At each final node of the tree—i.e., at the expiration of the option—the option value is simply its intrinsic, which is equal to

   max [$(S_n - K)$, 0], for a call option and max [$(K - S_n)$, 0], for a put option, where K is the strike price and $S_n$is the spot price of the underlying asset at the n^{th} period. The expected stock price $E(S_t)$ at time t is given by

   $E(S_t) = pS_ou+ (1\text{-}p)S_od,$

   which is equal to $E(S_t) = S_oe^{rt}$ under the riskless assumption.

   This gives the value of $p= (e^{rt}- u)/(u\text{-}d).$

3. Finding the option value at each time step

Binomial Option Value = $\{p \times (\text{Option up}) + (1-p) \times (\text{Option Down})\} \times e^{-rt}$

Where $p = (e^{rt} - u)/(u-d)$.

This value represents the fair price of the option at a particular point in time (i.e., at each node), given the evolution in the price of the underlying to that point. It is the value of the option if it were to be held—as opposed to exercised at that point.

4. Depending on the style of the option, evaluate the possibility of early exercise at each node: if the option can be exercised, and the exercise value exceeds the Binomial Value, then the value at the node is the exercise value.

- European option: there is no early exercise option, and the binomial value applies at all nodes.
  - American option: since the option may either be held or exercised prior to expiry, the value at each node is: Max (Binomial Value, Exercise Value).

Advantages and Disadvantages of the Binomial Pricing Model:

Despite its limitations, it also offers several advantages over other methods. Some of these advantages are:

1. **Simplicity and Intuitiveness:** The binomial model is relatively easy to understand and implement. It is based on straightforward probability calculations, making it accessible to both professionals and individuals with basic financial knowledge.

2. **Flexibility:** The binomial model can accommodate various assumptions and market conditions for the whole period of an option's life or for some discrete time periods. It can handle options with different exercise styles (e.g., American or European), as well as options on assets that pay dividends or have discrete cash flows.

3. **Discrete Time Steps:** The model breaks down the time period until option expiration into discrete steps, which simplifies the valuation process and computational requirements. This characteristic makes it more suitable for manual calculations or for educational purposes.

4. **Accuracy for Short-Term Options:** The binomial model can be more accurate for valuing short-term options or options on assets with significant price volatility. In these cases, the discrete steps and ability to model price movements more accurately may outperform other methods.

5. **Early Exercise:** Unlike some other option pricing models, the binomial model explicitly considers the possibility of early exercise for American-style options. This makes it more suitable for valuing such options.

While the binomial options pricing method has some advantages, it also has several disadvantages. Some of the disadvantages of using the binomial model include:

1. **Computationally Intensive for Long Time Periods:** As the number of time steps increases, the complexity and computation time of the binomial model grow significantly. This makes it less efficient for valuing options with longer time horizons, especially when compared to continuous-time models like the Black-Scholes model.

2. **Convergence Issues:** In order to achieve accurate results, the binomial model requires a large number of time steps, which can be impractical in some cases. However, using a small number of steps may lead to convergence issues and less accurate valuations.

3. **Limited Precision:** The binomial model is subject to discrete time and price steps, which can introduce inherent limitations in its precision when compared to continuous models. This discretization can lead to small errors in the valuation of options, especially for complex derivatives.

4. **Impractical for Multiple Factors:** The binomial model is primarily designed for single-factor scenarios, meaning it may not be suitable for valuing options on assets influenced by multiple factors (e.g., interest rates, dividends, and volatility changes simultaneously).

5. **Less Suitable for Complex Options:** For options with several sources of uncertainty (e.g., real options) and for options with complicated features (e.g., Asian options), binomial methods are less practical due to several difficulties.

Due to these disadvantages, the binomial options pricing method is often used as an introductory tool for understanding option valuation concepts. Financial

professionals usually rely on more advanced models, such as the Black-Scholes model or to price options accurately in real-world scenarios.

Black- Scholes Model

The Black-Scholes model, also known as the Black-Scholes-Merton (BSM) model, is one of the most important concepts in modern financial theory. This mathematical equation estimates the theoretical value of derivatives based on other investment instruments, considering the impact of time and other risk factors. The Black-Scholes model is a differential equation widely used to price options contracts. The Black-Scholes model requires five input variables: the strike price of an option, the current stock price, the time to expiration, the risk-free rate, and the volatility. The model is used to find the current value of a call option whose ultimate value depends on the stock's price at the expiration date. Though usually accurate, the Black-Scholes model makes certain assumptions that can lead to predictions that deviate from real-world results. The standard BSM model is only used to price European options, as it does not take into account that American options could be exercised before the expiration date. The assumptions are as follows

1. Prices exhibit Brownian motion
2. Infinite volatility
3. Constant risk-free
4. Efficient markets
5. Perfect liquidity
6. Log-normal distribution of stock prices
7. Frictionless market- there are no transaction costs
8. Neglect dividend payouts throughout the option period.

The Black-Scholes formula estimates the value of a call option by multiplying the current stock prices by a probability factor (D_1) and then subtracting from it the product of discounted exercise payment time and a second probability factor (D_2). D_1 is the cumulative standard normal probability distribution function. However, D_1 and D_2 are hard to explain due to the absence of adequate explanations in original and subsequent research papers about the Black Scholes model. $D_2$is defined as the risk-adjusted probability of exercising an option. D_1 is the difference between the present value of the contingent receipt of stock and the current stock price.

Below is the mathematical notation of the Black-Scholes Formula:

$C= S_tN(d_1) - Ke^{-rt}N(d_2)$

Here,

- $d_1=ln(S_t/K)+(r+\sigma^2/t^2)$
- $d_2= d_1 - t$
- C= *call option price*
- S_t= *current stock price or price of the underlying security*
- K= *strike price*
- r = *risk-free interest rate*
- t = *time to maturity*

N= *normal distribution*

Limitations of the Black-Scholes Model:

While the Black Scholes option pricing model is widely used, it has some limitations too. Below are the limitations of the model:

1. **Works only for European Options**

The model works accurately to determine the prices of European options but is not an accurate measure of the stock options in the U.S (American options). This is because it is based on the assumption that options can be exercised only on the maturity date.

2. **Risk-free Interest rates are not a Reality**

While risk-free interest rates are an assumption behind the BSM model, they do not exist in the real world.

3. **Ignores Transaction Costs**

Brokerage fees, commissions, and other costs mostly accompany trades. The Black Scholes model ignores these costs and assumes that there is no friction in the market. This is far from reality. Ignoring these costs may lead to inaccurate valuations.

4. **Assumption of No Returns**

The model assumes that there are no dividends or interest earnings from stock options. However, the trading market relies primarily on returns.

Black-Scholes Model vs. Binomial Model

The Black-Scholes model and the binomial option pricing model are mathematical approaches used to value options, but they have different underlying assumptions and computational methods. Some of the differences between these two models are

1. **Assumptions:**

    a. Black-Scholes Model: The Black-Scholes model assumes that the underlying asset price follows geometric Brownian motion, meaning that the price changes are continuous and follow a log-normal distribution. It also assumes constant volatility, no dividends, efficient markets, and the ability to trade continuously.

b. Binomial Option Pricing Model: The binomial model assumes that the underlying asset price can move up or down over discrete time steps. It assumes a risk-neutral framework, meaning that the expected return on the underlying asset is the risk-free rate. This model also assumes that the probability of an up or down movement remains constant over time.

2. **Time and Price Discretization:**

a. Black-Scholes Model: The Black-Scholes model is a continuous-time model. It assumes that the underlying asset price can change at any point in time. The model uses differential equations to calculate the option price and hedging parameters.

b. Binomial Option Pricing Model: The binomial model is a discrete-time model. It divides the time to expiration into a series of discrete intervals or steps. At each step, the underlying asset price can move up or down by a predetermined factor. The model calculates option prices by constructing a binomial tree and computing probabilities at each node.

3. **Volatility:**

a. Black-Scholes Model: The Black-Scholes model assumes constant volatility throughout the option's life. This assumption can be restrictive, especially in situations where volatility changes significantly over time.

b. Binomial Option Pricing Model: The binomial model can handle variable volatility. Traders can adjust the volatility assumption at each time step in the binomial tree to account for changing market conditions.

4. **Complexity and Accuracy:**

a. Black-Scholes Model: The Black-Scholes model provides a closed-form solution for pricing European-style options. It is relatively straightforward to implement and computationally efficient. However, it may not accurately capture certain market conditions, such as severe price jumps or extreme volatility changes.

b. Binomial Option Pricing Model: The binomial model is more flexible and can handle a wider range of option types, including American-style options. It can better capture situations where the underlying asset price is subject to discrete events or changes in volatility. However, it requires constructing and iterating through a binomial tree, making it more computationally intensive compared to the Black-Scholes model.

5. **Option Types:**

a. Black-Scholes Model: The Black-Scholes model is primarily designed for valuing European-style options, which can only be exercised at expiration.

b. Binomial Option Pricing Model: The binomial model can value both European-style and American-style options. American-style options can be exercised at any time before expiration, which introduces additional complexity in the pricing model.

Conclusion

In summary, both methods are essential in valuing financial derivatives, particularly options, but each possesses distinct characteristics that influence their applicability and accuracy in different scenarios.

The Black-Scholes model assumes continuous price movements and constant volatility, providing a closed-form solution for European-style options. On the other hand, the binomial option pricing model including American-style options. The binomial model is more flexible and more accurate, particularly for longer-dated options on securities with dividend payments but computationally more intensive than the Black-Scholes model.

The binomial options pricing model offers simplicity and flexibility, making it a valuable tool for understanding option valuation concepts. It discretizes time and price movements, allowing for variable volatility and handling a broader range of option types allowing for easy implementation and accommodating various assumptions and market conditions., Furthermore, the model explicitly accounts for early exercise possibilities in American-style options, providing an advantage over some other pricing methods.

On the other hand, the Black-Scholes model is renowned for its closed-form solution, allowing for quick and efficient option valuations, particularly for European-style options. It assumes continuous trading and constant volatility, making it more suitable for options with longer time horizons and in markets with relatively stable volatility. The Black-Scholes model is particularly effective when dealing with vanilla options and under risk-neutral assumptions.

In conclusion, the choice between the binomial and Black-Scholes pricing models depends on the specific characteristics of the options being valued and the desired level of accuracy and computational efficiency. For straightforward options, the binomial model remains a valuable tool. However, in complex financial markets, where continuous trading and precise valuations are crucial, the Black-Scholes model stands as a more practical and widely adopted option pricing method. Ultimately, understanding the strengths and limitations of each model allows a trader to select the appropriate approach for their unique valuation requirements.