ESE 2180 Project One: Solving Linear Circuits

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1 Project Information

This project will provide familiarity with techniques for solving matrix equations by exploring a case study in linear circuits. The project is due on **September 24**. You may work in groups of up to **two**. You will turn in a lab report and source code as described in Section 6.

2 Learning Outcomes

The learning outcomes of this project are:

- Formulate a real-world physical problem as a system of linear equations
- Write code to solve a linear system using Gaussian elimination
- Prepare a report and well-commented code

3 Resistor Networks

The electric voltage at a particular point refers to the electrical potential energy at that point, typically relative to a ground state that is assumed to have zero voltage. Ohm's Law gives a linear relationship between the voltage at two points A and B, denoted V_A and V_B respectively, and the current I_{AB} from A to B. That relationship is given by $V_A - V_B = I_{AB}R_{AB}$, where R_{AB} is the electrical resistance of the connector between A and B (Fig. 1). An equivalent relationship is given by $I_{AB} = C_{AB}(V_A - V_B)$, where $C_{AB} = 1/R_{AB}$ is the conductance between A and B.

A resistor network is a collection of points (or nodes) $n_1, \ldots, n_M$ that are connected by resistors (Fig. 2), where M is the number of nodes. Some of these nodes may be voltage sources, i.e., they are fixed to a particular voltage. By convention, we choose one node as a *ground* that has zero voltage. The voltages of the remaining nodes are determined by the Ohm's Law and the Kirchhoff Voltage and Current Laws [1] (KVL and KCL), which are defined by

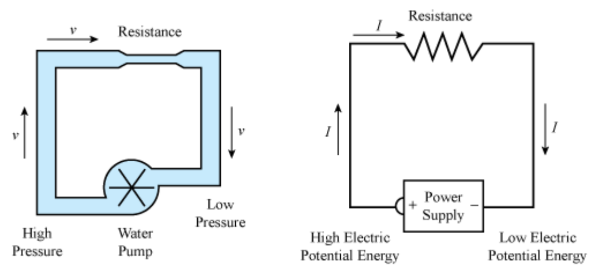

Figure 1: Ohm's Law illustration

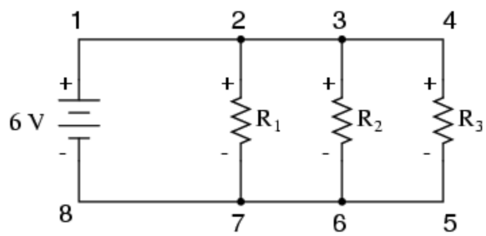

Figure 2: Resistor Networks

- Kirchhoff Voltage Law (KVL) – The sum of the changes in voltage around any loop in the network is equal to zero. In other words, if (A_1, A_2), $(A_2, A_3), \ldots, (A_r, A_1)$ is a sequence of links in the network, then

$$\sum_{i=1}^{r-1} (V_{i+1} - V_i) + (V_1 - V_r) = 0. \tag{1}$$

- Kirchhoff Current Law (KCL) – The sum of the net currents entering any junction is equal to zero. In other words, if $A_1, \ldots, A_r$ is the set of nodes that are connected to node B, then

$$\sum_{j=1}^{r} I_{A_j B} = 0 \tag{2}$$

4 Solving Circuits Using Node Voltages

The main idea of node voltages is to determine the voltages at each node in the graph using KCL (Eq. (2)). Each node that is not a voltage source results in one linear equation. The following procedure is used to solve linear circuits using node voltages:

1. Identify the nodes in the graph. For each node i that acts as a voltage source, add an equation $V_i = u_i$, where u_i is the voltage assigned to i.

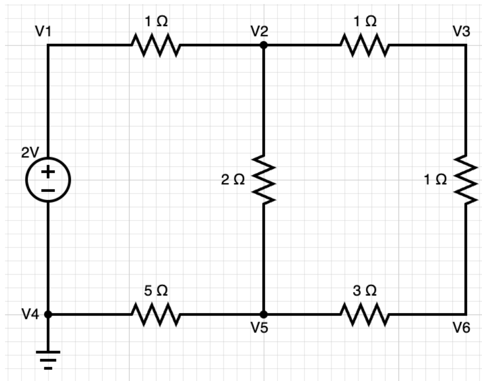

Figure 3: Example of node voltage analysis.

2. For each remaining node, write the KCL equation. For a node B, this will be of the form (2).

3. Convert the currents in the KCL equation to their corresponding voltages using Ohm's Law, so that (2) becomes

$$\sum_{j=1}^{r} \frac{1}{R_{A_j B}} (V_{A_j} - V_B) \tag{3}$$

Consider the example shown in Fig. 3. There are six nodes. Solving for the node voltages, we obtain the following system of equations:

For **Node** V_2, we apply KCL:

$$\frac{V_2 - 2}{1} + \frac{V_2 - V_3}{1} + \frac{V_2 - V_5}{2} = 0$$

Multiplying through by 2 to clear the fraction and simplifying, we get:

$$7V_2 - 2V_3 - V_5 = 4$$

For **Node** V_3, the KCL equation is:

$$\frac{V_3 - V_2}{1} + \frac{V_3 - V_6}{1} = 0$$

Simplifying, this gives:

$$-V_2 + 2V_3 - V_6 = 0$$

For **Node** V_5, we have:

$$\frac{V_5 - V_2}{2} + \frac{V_5}{5} + \frac{V_5 - V_6}{3} = 0$$

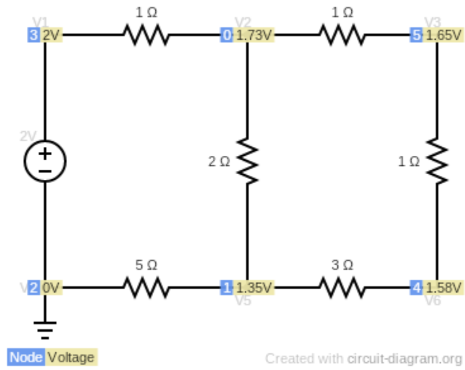

Figure 4: Simulation result of the example circuit.

Multiplying through by 30 and simplifying:

$$31V_5 - 15V_2 - 10V_6 = 0$$

Finally, for **Node** V_6, the KCL equation is:

$$\frac{V_6 - V_3}{1} + \frac{V_6 - V_5}{3} = 0$$

Simplifying, we get:

$$4V_6 - 3V_3 - V_5 = 0$$

Next, we write these equations in matrix form:

$$\begin{pmatrix} 7 & -2 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -15 & 0 & 31 & -10 \\ 0 & -3 & -1 & 4 \end{pmatrix} \begin{pmatrix} V_2 \\ V_3 \\ V_5 \\ V_6 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this system, the node voltages are found to be:

$$V_2 = 1.73 \text{ V}, \quad V_3 = 1.65 \text{ V}, \quad V_5 = 1.35 \text{ V}, \quad \text{and} \quad V_6 = 1.58 \text{ V}$$

Let's test it out in the circuit simulation website [1]. The simulation result is shown in Fig. 4. These values match the simulation results, confirming the accuracy of our calculations.

[1]Circuit Diagram A free, user-friendly program for making electronic circuit diagrams. https://www.circuit-diagram.org/

5 Your Assignment

Please complete the following steps.

1. Consider a 5x5 grid. Write a system of equations $Ax = b$ to solve for the node voltages. Provide your answer in the form $A_{ij} = *$, $i = 1, \ldots, 25$, $j = 1, \ldots, 25$, $b_i = *$, $i = 1, \ldots, 25$.

2. Write a function that reads in a file to obtain the resistances on each link. The file will contain one line for each link of the form

   ```
   NODE1  NODE2 RESISTANCE
   ```

   For example, if there is a resistance of 5 between nodes 2 and 3, then the corresponding line would be

   ```
   2 3 5
   ```

3. Write a function that reads in a file to obtain the set of voltages that are at fixed points. For example, if the voltage at node 3 is equal to 6.2, the corresponding line would be

   ```
   3 6.2
   ```

4. Compute the A matrix using the data read in by the previous two files (two .json files provided).

5. Compute the LU factorization of the matrix.

6. Compute and output the node voltages and the currents through each link.

7. Write the output of the previous three steps to a file.

8. Repeat the above but with a tree network (draw graph). Comment on the relative difficulty of solving for a tree as opposed to a grid.

6 Deliverables

Please submit the following via Canvas:

- A project report (one per group) as described in Section 5
- Your Python source code and your understanding of the solution.

References

[1] C. K. Alexander, M. N. Sadiku, and M. Sadiku, *Fundamentals of electric circuits*. McGraw-Hill Higher Education Boston, MA, USA, 2007.