Date

Unit: 3

* Fourier Transform (5-6 Marks)

(I) Defⁿ of fourier Transform

=) Fourier Transform of f(x) is given by

$$\boxed{F[f(x)] = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx = F(s)}$$

↳ Solⁿ

(II) Inverse of fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$$

Solⁿ

(III) Fourier Cosine Transform

$$\boxed{F_c(s) = \int_{0}^{\infty} f(x) \cos(sx)\,dx}$$

(IV) Fourier Sine Transform

$$\boxed{F_s(s) = \int_{0}^{\infty} f(x) \sin(sx)\,dx}$$

$\boxed{a^{-\infty} = 0}$

Page No.

Date

Q1.) Find Fourier Sine & Cosine Transform.

(1) e^{-ax}

(2) $[e^{-2x} + 4e^{-3x}]$

(3) $\frac{e^{-ax}}{x}$

(1) Let $f(x) = e^{-ax}$

According to the def^n of fourier Cosine Transform

$$F_c(s) = \int_0^\infty f(x) \cdot \cos(sx)\,dx$$

$$= \int_0^\infty e^{-ax} \cdot \cos(sx)\,dx$$

$$\int e^{-ax} \cdot \cos(sx)\,dx = \frac{e^{ax}}{a^2+b^2}[a \cdot \cos bx + b \cdot \sin bx]$$

$$a = -a, \quad b = s$$

$$F_c(s) = \left[\frac{e^{-ax}}{a^2+s^2}[-a \cdot \cos(sx) + (s) \cdot \sin(sx)\, d\right]_{x=0}^{x=\infty}$$

$$= \frac{1}{(a^2+s^2)}[\,0 - 1(-a+0)]$$

$$\boxed{F_c(s) = \frac{a}{s^2+a^2}}$$

Date

2) Let $f(x) = e^{-ax}$

According to the defn of Fourier Sine Transform.

$$f_s(s) = \int_0^\infty f(x).\ \sin(sx)\,dx$$

$$= \int_0^\infty e^{-ax}.\ \sin(sx)\,dx$$

$$\int \frac{e^{ax}}{a^2+b^2} = \frac{e^{ax}}{a^2+b^2}\ \cancel{[-\cos bx - \sin bx]}\ [a.\ \sin bx - b.\ \cos bx]$$

here $a = -a,\quad b = s$

$$= \frac{e^{ax}}{a^2+s^2}\Big[-a.\ \sin(sx) - (s)\cdot\cos(sx)\Big]_0^\infty$$

$$= \frac{1}{s^2+a^2}\big[\,0 - 1.[\,0 - s\,]\,\big]$$

$$= \frac{s}{s^2+a^2}$$

(2) $[e^{-2x} + 4e^{-3x}]$

Soln ⇒ Let $f(x) = [e^{-2x} + 4e^{-3x}]$

According to the defn of Fourier Cosine Transform

$$F_c(s) = \int_0^\infty f(x).\ \cos(sx)\,dx$$

$$= \int_0^\infty (e^{-2x} + 4e^{-3x})\cos(sx)\,dx$$

Page No. Date

$$= \int_0^\infty e^{-2x} \cdot \cos(sx)\,dx + 4\int_0^\infty e^{-3x} \cdot \cos(sx)\,dx$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a \cdot \cos bx + b \cdot \sin bx\right]$$

here, $a = -2$, $b = s$

$$= \frac{e^{-2x}}{4+s^2}\left[-2 \cdot \cos(sx) + (s) \cdot \sin(sx)\right]_0^\infty + 4\int_0^\infty e^{-3x} \cdot \cos(sx)\,dx$$

$$= \frac{e^{-2x}}{s^2+2^2}\left[-2 \cdot \cos(sx) + s \cdot \sin(sx)\right]_0^\infty + 4\int_0^\infty e^{-3x} \cdot \cos(sx)\,dx$$

$$= \frac{1}{s^2+2^2}\left[0 \cancel{- 1(-2)}\right.$$

$$= \left[\frac{e^{-2x}}{4+s^2}(-2 \cdot \cos(sx) + s \cdot \sin sx]\right] + 4\left[\frac{e^{-3x}}{9+s^2}(-3 \cdot \cos(sx) + s \cdot \sin sx]\right]$$

$$= \frac{1}{s^2+4}\left[0 - 1(-2)\right) + \frac{4}{s^2+9}\left[0 - 1[-3+0]\right]$$

$$\boxed{F(s) = \frac{2}{s^2+4} + \frac{12}{s^2+9}}$$

Date

(ii) According to the defn of fourier sine transform

$$F_s(s) = \int_0^\infty f(x) . \sin(sx)\, dx$$

$$= \int_0^\infty [\cancel{e^{x}} e^{-2x} + 4 . e^{-3x}] \sin(sx)\, dx$$

$$= \int_0^\infty e^{-2x} . \sin(sx)\, dx + 4\int_0^\infty e^{-3x} . \sin(sx)\, dx$$

$$= \quad a = -2 , b = s$$

$$= \left. \frac{e^{-2x}}{4+s^2} [-2 . \sin(sx) - (s) . \cos(sx)] \right]_0^\infty$$

$$+ 4 \left[\frac{e^{-3x}}{9+s^2} (-3 . \sin(sx) - s . \cos(sx) \right]_0^\infty$$

$$= \frac{1}{s^2+4} [0 - 1(0-s) + \frac{4}{s^2+9} (0 - 1(0-s))]$$

$$F_s(s) = \frac{s}{s^2+4} + \frac{4s}{s^2+9}$$

Find Fourier Sine & Cosine transform.

$$f(x) = x; \quad 0 < x < 1$$
$$= (2-x); \quad 1 < x < 2$$
$$= 0; \quad x > 2$$

(i) According to Fourier Cosine Transform

$$F_c(s) = \int_0^\infty f(x)\cos(sx)\,dx$$

$$= \left[\int_0^1 x\cdot\cos(sx)\,dx + \int_1^2 (2-x)\cdot\cos(sx)\,dx + \int_2^\infty 0\,dx\right]$$

$$= \left[x\left(\frac{\sin sx}{s}\right) - 1\left(-\frac{\cos sx}{s^2}\right)\right]_0^1 + \left[(2-x)\left[\frac{\sin x}{s}\right] - (-1)\left(-\frac{\cos(sx)}{s^2}\right)\right]_1^2$$

$$= \left(x\cdot\frac{\sin(sx)}{s} + \frac{\cos(sx)}{s^2}\right)_0^1 + \left[(2-x)\frac{\sin(sx)}{s} - \frac{\cos(sx)}{s^2}\right]_1^2$$

$$= \left[\frac{\sin s}{s} + \frac{\cos s}{s^2}\right] - \left[0 + \frac{1}{s^2}\right] + \left[0 - \frac{\cos 2s}{s^2}\right] - \left[\frac{\sin s}{s} - \frac{\cos s}{s^2}\right]$$

$$= \left[\cancel{\frac{\sin s}{s}} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right] + \left[-\frac{\cos 2s}{s^2} - \cancel{\frac{\sin s}{s}} + \frac{\cos s}{s^2}\right]$$

$$= \frac{2\cos(s)}{s^2} - \frac{\cos(2s)}{s^2} - \frac{1}{s^2}$$

$$\boxed{F_c(s) = \frac{2\cos(sx) - \cos(2s) - 1}{s^2}}$$

Q2] Given function is

$$f(x) = x; \quad 0 < x < 1$$
$$= (2-x); \quad 1 < x < 2$$
$$= 0; \quad x > 2$$

(i) According to fourier sine Transform.

$$F_S(s) = \int_0^\infty f(x) \cdot \sin(sx)\, dx$$

$$= \left[\int_0^1 x \cdot \sin(sx)\, dx + \int_1^2 (2-x) \cdot \sin(sx)\, dx + \int_2^\infty 0\, dx\right.$$

$$= \left[x \cdot \left(-\frac{\cos(sx)}{s}\right) - 1 \cdot \left(-\frac{\sin(sx)}{s^2}\right)\right]_0^1 + \left[(2-x) \cdot \left(-\frac{\cos(sx)}{s}\right) - (-1)\left(-\frac{\sin(sx)}{s^2}\right)\right]_1^2$$

$$= \left[-\left(\frac{x \cos(sx)}{s} + \frac{\sin(sx)}{s^2}\right)\right]_0^1 + \left[-\frac{(2-x) \cdot \cos(sx)}{s} - \frac{\sin(sx)}{s^2}\right]_1^2$$

$-2+1$

$$= \cancel{[0 - 0] + [0 -}$$

$$= \left[\frac{-1\cos s}{s} + \frac{\sin(s)}{s^2}\right] - [0+0] + \cancel{\left[0 - \left(-\frac{\cos s}{s}\right) - \frac{\sin(s)}{s^2}\right]}$$

$$\left[0 - \frac{\sin 2s}{s^2}\right] - \left[-\frac{\cos s}{s} - \frac{\sin s}{s^2}\right]$$

$$= \cancel{\frac{-x1\cdot\cos s}{s}} + \cancel{\frac{\sin(s)}{s^2}} + \frac{\cancel{\cos s}}{s} - \frac{\sin(s)}{s^2}$$

$$\cancel{= 0}$$

$$= -\cancel{\frac{\cos s}{s}} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \cancel{\frac{\cos s}{s}} + \frac{\sin s}{s^2}$$

$$= 2\frac{\sin s}{s^2} - \frac{\sin 2s}{s^2}$$

$$= \frac{2.\sin s - \sin 2s}{s^2}$$

Note: ①
$|x| > 1 \Rightarrow 1$ to ∞

Page No.
Date

** 11. Imp

Find Fourier Transform of

$$f(x) = 1 \;;\; |x| \le 1$$
$$= 0 \;;\; |x| > 1$$

& hence Evaluate $\int_0^{\infty} \frac{\sin x}{x}\,dx$

(2) Note: If in the problem of Fourier Transform Evaluate is given it means we have to apply Inverse Fourier Transform

Soln:

Given function is

$$f(x) = 1 \;;\; |x| \le 1 \;;\; -1 \le x \le 1$$
$$= 0 \;;\; |x| > 1 \;;\; 1 \le x \le \infty$$

According to the defn of Fourier Transform

$$F(s) = \int_{-\infty}^{\infty} e^{isx}\,1.dx + \int_1^{\infty} 0\,dx$$

$$= \int_{-\infty}^{\infty} e^{isx}\,dx$$

$$= \left[e^{isx}\right]$$

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \int_{-1}^{1} e^{isx}.1\,dx + \int_1^{\infty} 0\,dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{x=-1}^{x=+1}$$

$$= \frac{1}{is}\left[e^{is} - e^{-is}\right]$$

$$= \frac{1}{is}\left[2i \sin s\right]$$

$$\boxed{F(s) = \frac{2 \sin(s)}{s}}$$

According to the defn of Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) \cdot e^{-isx} dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin s}{s} e^{-isx} ds$$

$$f(x) = \frac{1}{2\pi} 2\int_{0}^{\infty} \frac{2 . \sin(s)}{s} e^{-isx} dx$$

Put $x=0$

$$f(0) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(s)}{s} ds$$

$$1 = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(s)}{s} ds$$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin(s)}{s} ds$$

Replace 's' by 'x'

$$\boxed{\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}}$$

Note:

$$\cosh x = \frac{e^{x} + e^{-x}}{2},$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$2i \sin x = e^{ix} - e^{-ix}$$

$$e^{i\theta} = (\cos\theta + i \sin\theta)$$

$$e^{-i\theta} = (\cos\theta - i \sin\theta)$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

$$e^{i\theta} - e^{-i\theta} = 2 \sin\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Date

VV Imp.

★ Find fourier Transform of

$$f(x) = (1-x^2) \; ; \quad |x| \leq 1$$
$$= 0 \; ; \quad |x| > 1$$

Sol^n:- Given Fr Function is

$$f(x) = (1-x^2) \; ; \quad |x| \leq 1 \; ; \quad -1 \leq x \leq 1$$
$$= 0 \; ; \quad |x| > 1 \; ; \quad 1 \leq x \leq \infty$$

According to the def^n of fourier Transform

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \int_{-1}^{1} e^{isx} (1-x^2)\, dx + \int_{1}^{\infty} e^{isx} \cdot 0\, dx$$

$$= \left[(1-x^2) \cdot \left(\frac{e^{isx}}{is} \right) - (-2x) \left(\frac{e^{isx}}{i^2 s^2} \right) + (-2) \left(\frac{e^{isx}}{i^3 s^3} \right) \right]_{-1}^{1}$$

$$= \left[\frac{(1-x^2) \cdot e^{isx}}{is} - \frac{2x \cdot e^{isx}}{s^2} + \frac{2 \cdot e^{isx}}{is^3} \right]_{-1}^{1}$$

$$= \left[\left[0 - \frac{2e^{is}}{s^2} + \frac{2e^{is}}{is^3} \right] - \left[0 + \frac{2e^{-is}}{s^2} + \frac{2 \cdot e^{-is}}{is^3} \right] \right]$$

$$= \left[-\frac{2e^{is}}{s^2} + \frac{2e^{is}}{is^3} - \frac{2e^{-is}}{s^2} - \frac{2 \cdot e^{-is}}{is^3} \right]$$

$$= -\frac{2}{s^2} \left[e^{is} - e^{-is} \right] + \frac{2}{is^3} \left[e^{is} - e^{-is} \right]$$

Page No. Date

$$= \frac{-2}{s^2}[2\cos(s)] + \frac{2}{\cancel{i}s^3}[2\cancel{i}\sin(s)]$$

$$\boxed{F(s) = \frac{4\sin(s)}{s^3} - \frac{4\cos(s)}{s^2}}$$

v.Imp Find Fourier Sine & Cosine Transform of $\frac{e^{-ax}}{x}$

Soln; Given function is

$$f(x) = \frac{e^{-ax}}{x}$$

According to defn of fourier Cosine Transform,

$$F_c(s) = \int_0^\infty f(x).\cos \cancel{(s)x} (sx)\, dx$$

$$F_c(s) = \int_0^\infty \frac{e^{-ax}}{x}.\cos(sx)\, dx$$

here 's' is an parameter.

Diffn paritially w.r.t. parameter 's' by using Rule of DUIS

$$F'(s) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}}[-\cancel{x}\sin(sx)\, d\cancel{x}]\, dx$$

$$= \cancel{\int_0^\infty \frac{e^{-ax}}{x}}.$$

$$= -\int_0^\infty e^{-ax}.\sin(sx)\, dx$$

$$\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a.\sin bx - b\cos bx]$$

$a = -a$

$b = s$

Page No.
Date

$$= -\left[\frac{e^{-ax}}{a^2+s^2} [-a \cdot \sin(sx) - s \cdot \cos(sx)] \right]_0^\infty$$

$$= \frac{-1}{s^2+a^2} [0 - 1[0-s]]$$

$$F_c'(s) = \frac{-s}{s^2+a^2}$$

Integrating w.r.t. parameter's

$$F_c(s) = -\frac{1}{2} \int \frac{2s}{s^2+a^2} \, ds$$

$$\boxed{F_c(s) = -\frac{1}{2} \log(s^2+a^2) + c}$$