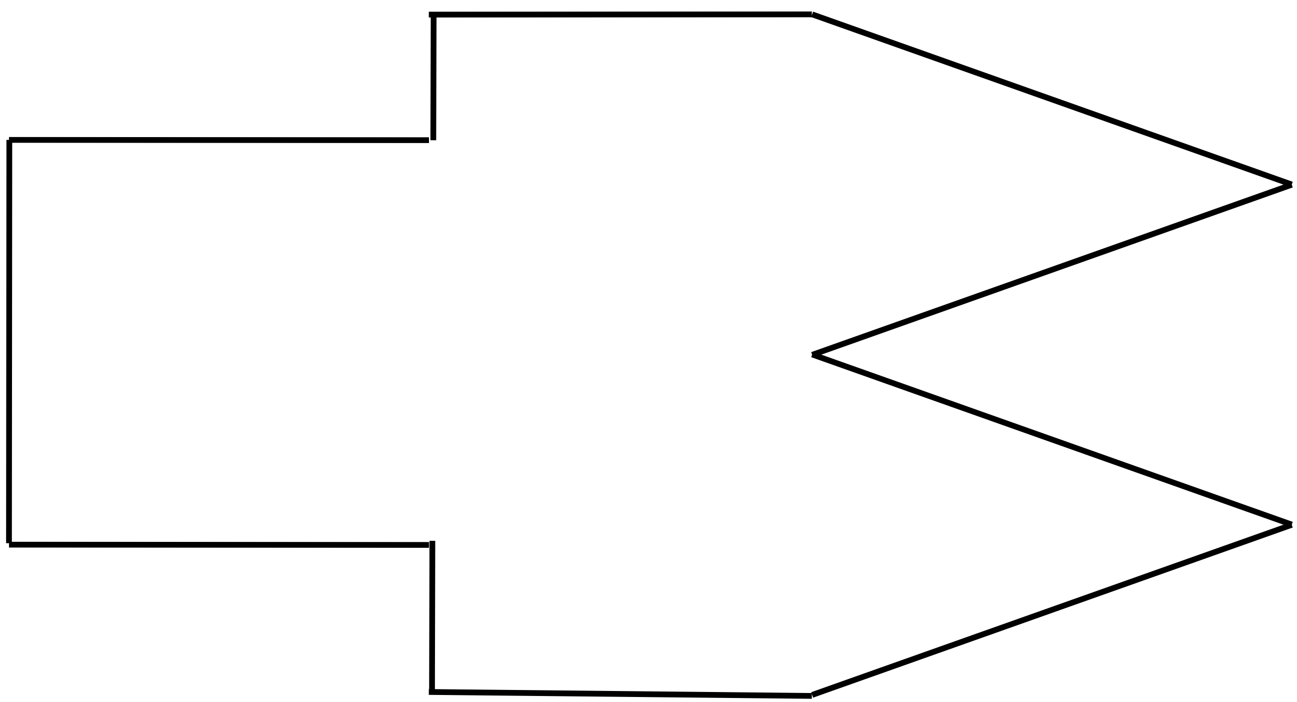
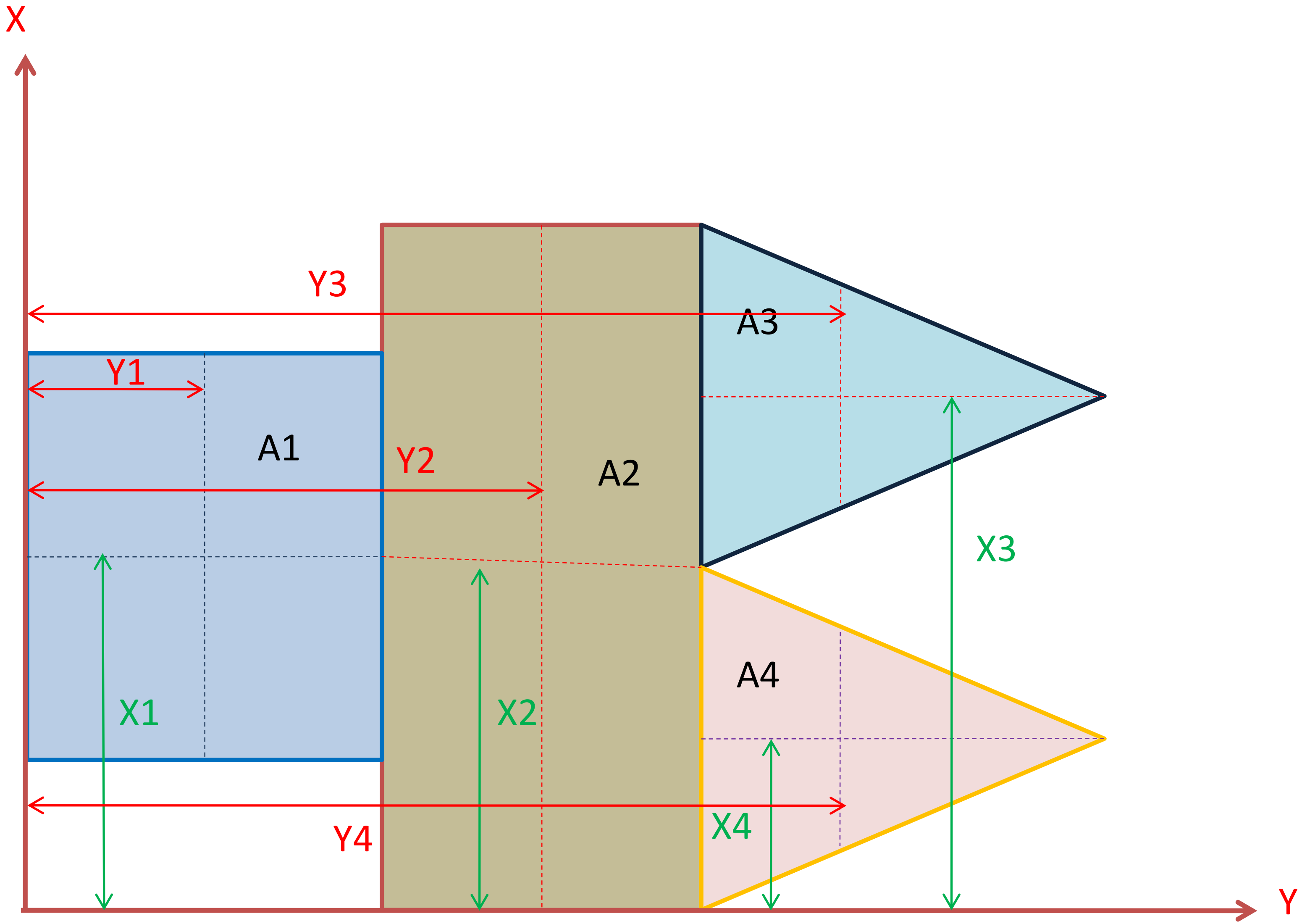
CENTER OF GRAVITY & MOMENT OF INERTIA

Center of Gravity: - The point through which the whole mass of body is supposed to act is known as Center of Gravity.

Centroid : - It is a point through which whole area of body is supposed to act.

To determine the centre of gravity or centroid menace find out the point through which whole mass or whole area of given figure is suppose to act. That meance find out the distance of centroid from X & Y axis. Which is denoted by $\overline{X}$ & $\overline{Y}$




X
Y3
A3
Y1
A1
Y2
A2
X3
A4
X1
X2
X4
Y4
Y

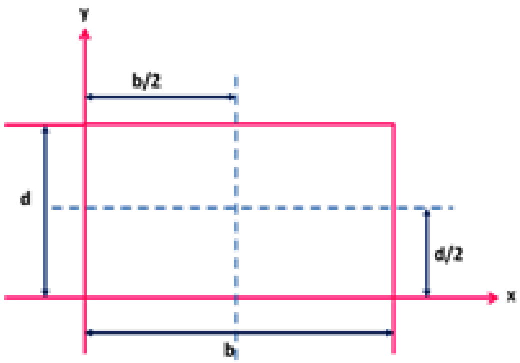
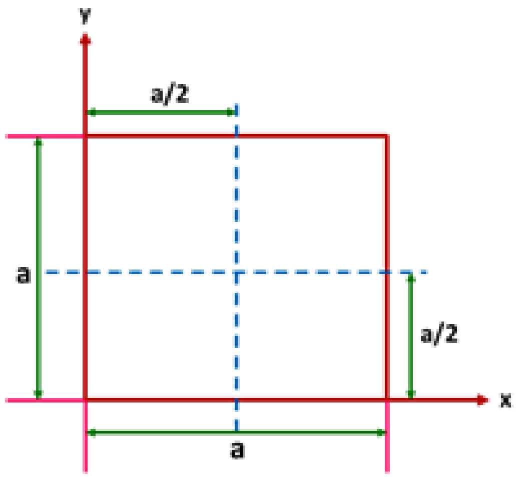
Centroid about X - axis is expressed

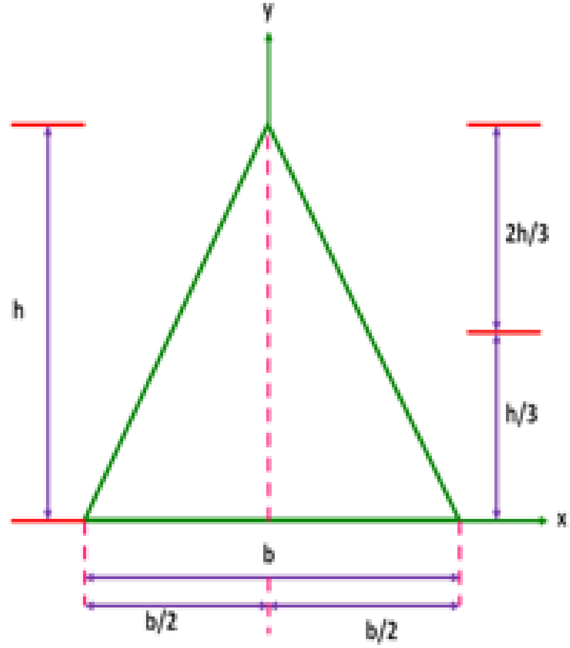
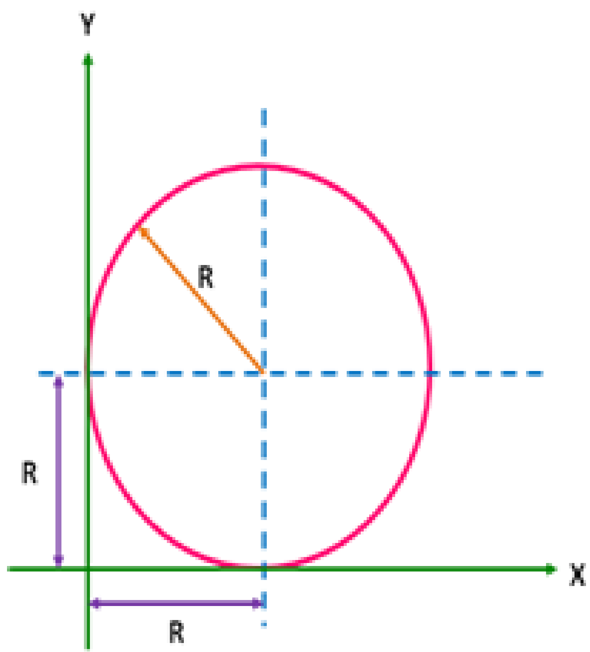
$$\overline{X} = \frac{A1x1 + A2x2 + A3x3 + A4x4}{A1 + A2 + A3 + A4}$$

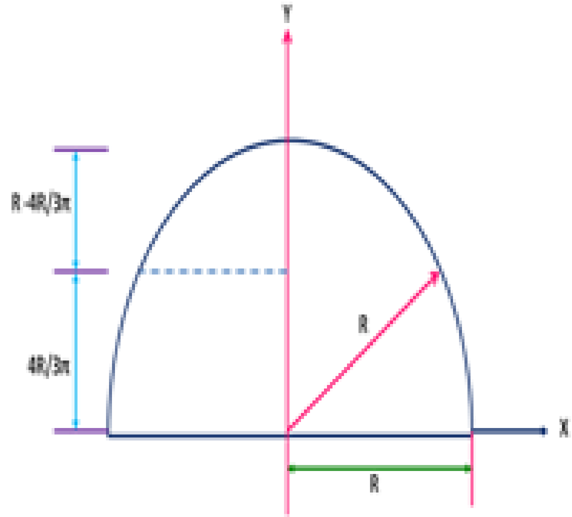
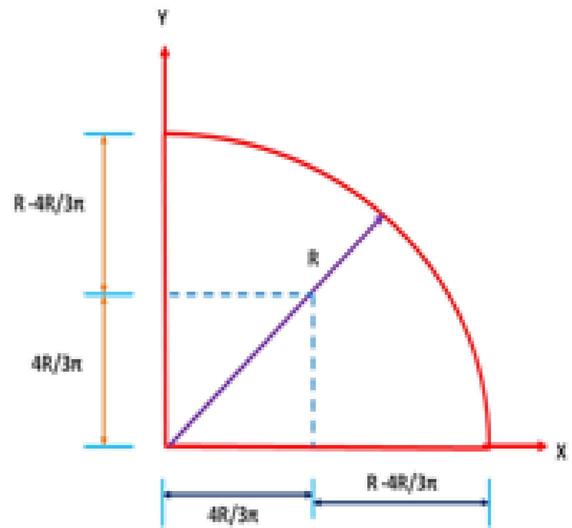
Centroid about Y - axis is

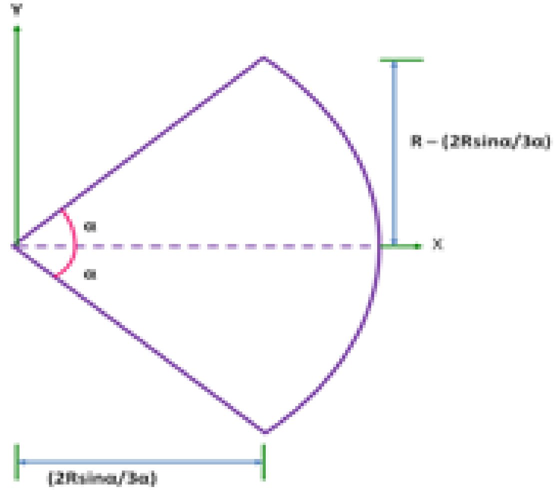
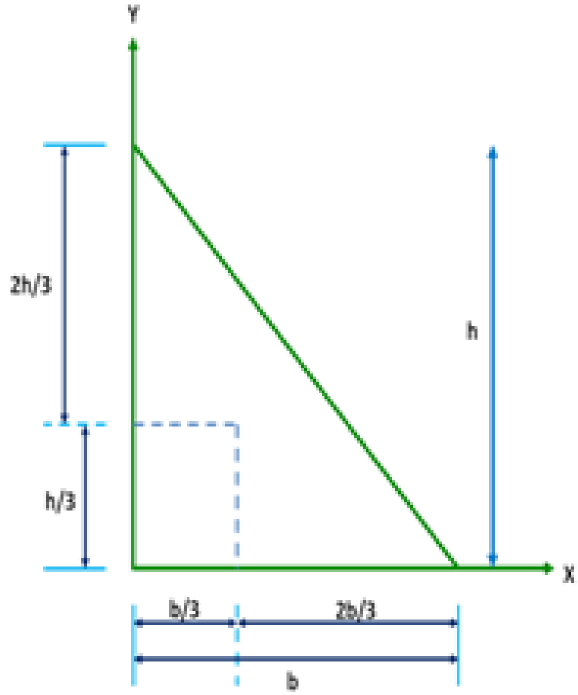
$$\overline{Y} = \frac{A1y1 + A2y2 + A3y3 + A4y4}{A1 + A2 + A3 + A4}$$

Centroid of some common figure

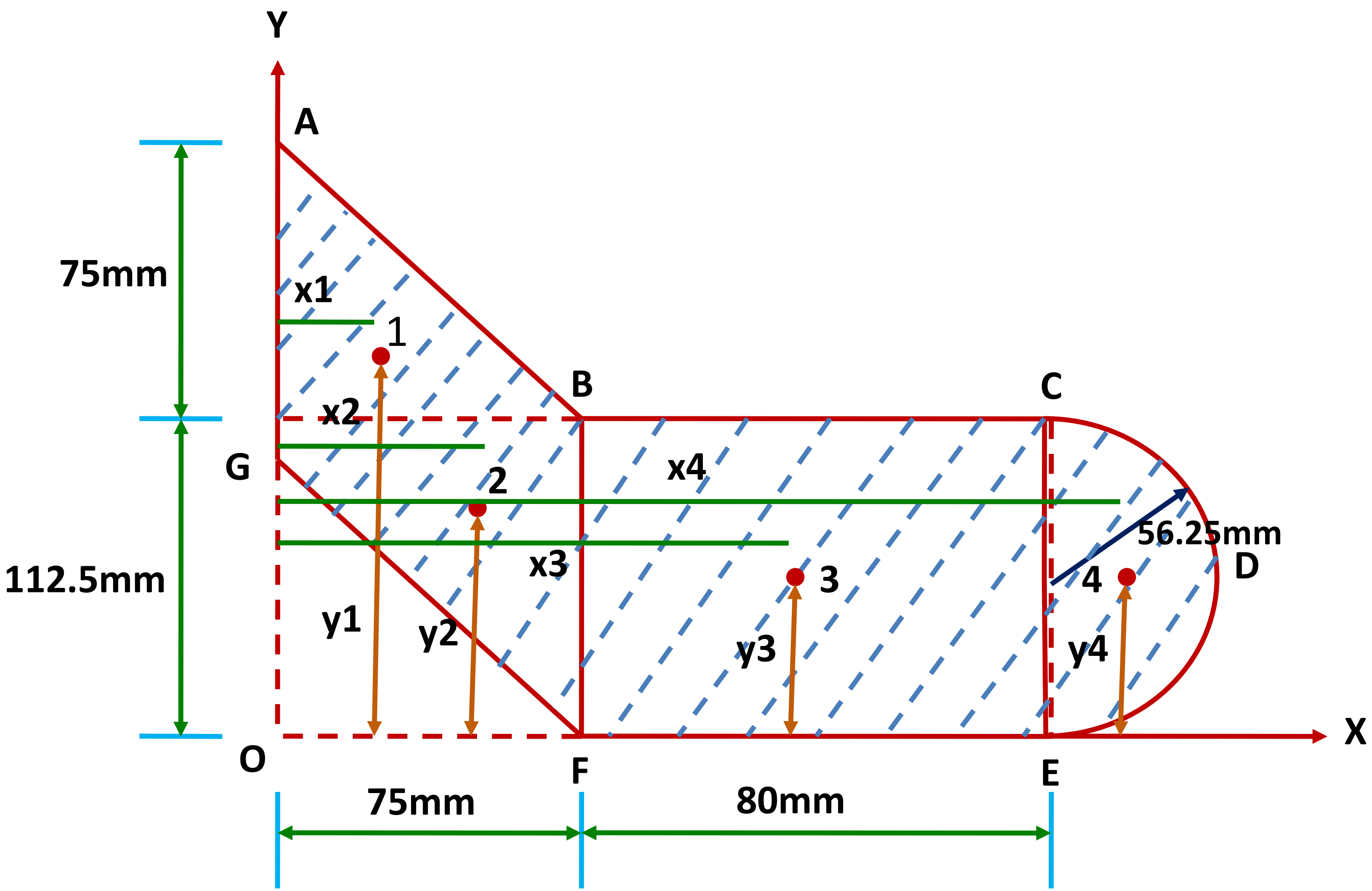
Sr. No	Name	Figure	$\overline{X}$	$\overline{Y}$	Area
01	Rectangle		b/2	d/2	b x d
02	Square		a/2	a/2	a^2

03	Triangle		0	h/3	½ x bxh
04	Circle		R	R	i) πR^2 or ii) $\frac{\pi}{4} D^2$

05	Semicircle	 	0	4R/ 3π	i) $\pi R^2/2$ or ii) $\frac{\pi}{8}D^2$
06	Quarter Circle	 	4R/ 3π	4R/ 3π	i) $\pi R^2/4$ Or ii) $\frac{\pi}{16}D^2$

07	Sector of Circle		2Rsinα/ 3α	0	αR^2
08	Right Angle Triangle		b/3	h/3	½ b.h

1. Find the position of the Centroid of the plane lamina as shown with reference to origin ‘O’.



Area of $\Delta ABG = \frac{1}{2} x\ b\ x\ h$

$A_1 = \frac{1}{2} x\ 75\ x\ 75$

$= 2812.50\ mm^2$

Area of $\Delta BFG = \frac{1}{2} x\ 75\ x\ 112.5$

$A_2 = 4218.75\ mm^2$

Area of $\square BCEF = b\ x\ h$

$A_3 = 80\ x\ 112.5$

$= 9000\ mm^2$

Area of Semicircle CDE

$$A_4 = \frac{\pi R^2}{2}$$

$$= \frac{\pi\, x\, 56.25^2}{2}$$

$$= 4970.10 \text{ mm}^2$$

$$Y_1 = 112.5 + \frac{1}{3} x\ 75\ (h/3)$$

$$= 137.50 \text{ mm}$$

$$Y_2 = \frac{2}{3}\ x\ 112.5\ (h)$$

$$= 75 \text{ mm}$$

$$Y_3 = \frac{h}{2} = \frac{112.5}{2} = 56.25 \text{ mm}$$

$Y_4 = \frac{D}{2} = \frac{112.5}{2} = 56.25$ mm

$X_1 = \frac{1}{3} x$ 75 (b) = 25 mm

$X_2 = \frac{2}{3}$ x 75(b) = 50 mm

$X_3 =$ 75 + 80 (b)/2 = 115 mm

$X_4 = 75 + 80 + \frac{4\ x\ 56.25}{3\pi}$ (4R/3∏) = 178.873 mm

Distance from y – axis i.e. x

$$\bar{X} = \frac{A1x1 + A2x2 + A3x3 + A4x4}{A1 + A2 + A3 + A4}$$

$$= \frac{(2812.50\ x\ 25) + (4218.75\ x\ 50) + (9000\ x\ 115) + (4970.10\ x\ 178.873)}{(2812.50 + 4218.75 + 9000 + 4970.10)}$$

$\overline{X}$ = 105 mm

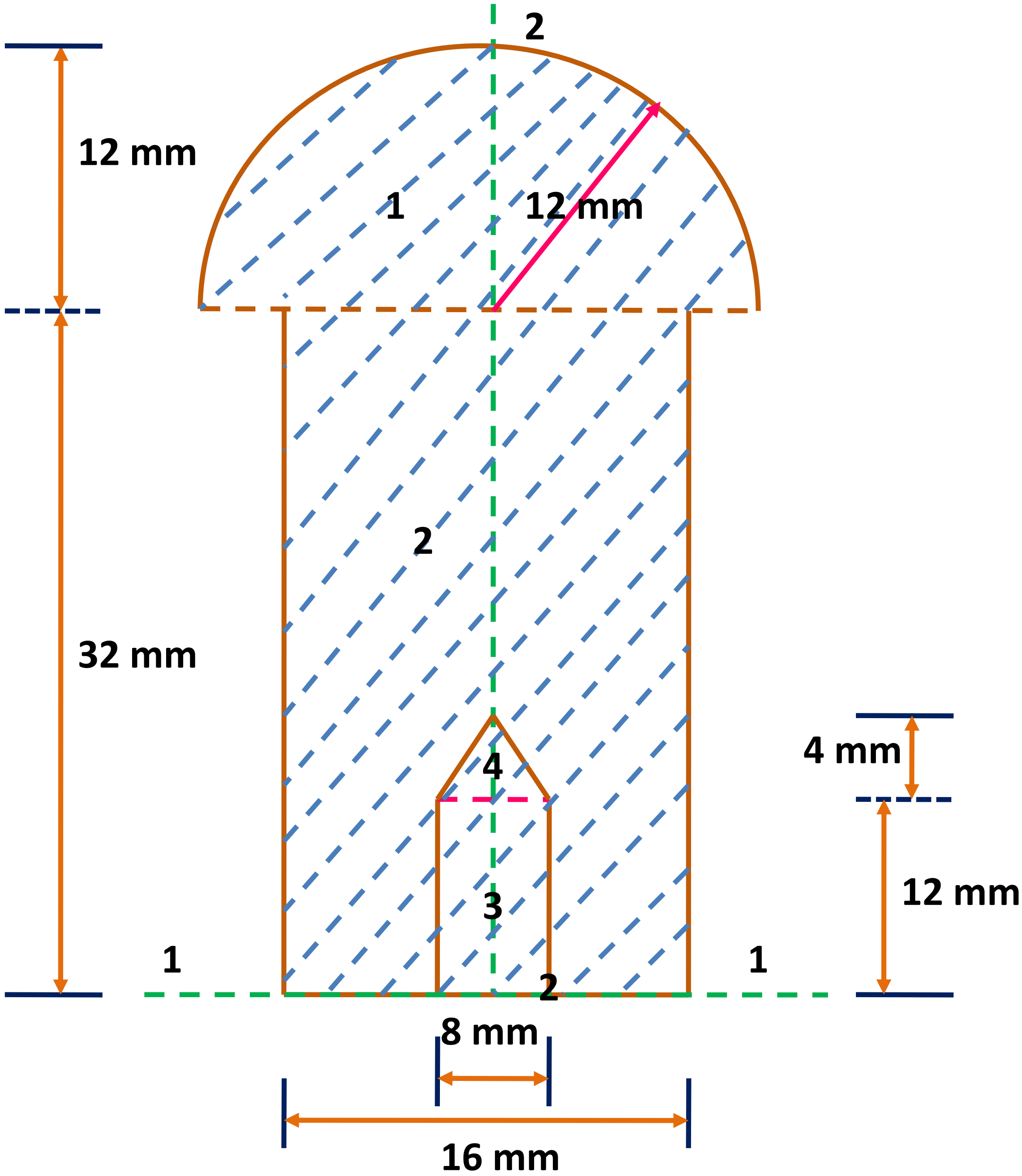
Distance from x axis i.e. y

$$\overline{Y} = \frac{A1y1 + A2y2 + A3y3 + A4y4}{A1 + A2 + A3 + A4}$$

$$= \frac{(2812.50\ x\ 137.50) + (4218.75\ x\ 75) + (9000\ x\ 56.25) + (4970.10\ x\ 56.25)}{2812.50 + 4218.75 + 9000 + 4970.10}$$

$\overline{Y}$ = 70.9 mm

2.Find the centroid of lamina as shown in figure about given axes 1-1 & 2-2



As the given lamina is symmetrical about 2-2 axes the distance of centre of gravity i.e. X is zero.

Therefore this case we want to find the value of Y only.

Area of semicircle

$$A_1 = \frac{\pi R^2}{2}$$

$$A_1 = \frac{\pi \text{ x } 12^2}{2}$$

A_1 = 226.19 mm^2

Area of Rectangle

$$A_2 = 32 \text{ x } 16$$

$$A_2 = 512 \text{ mm}^2$$

Area of Rectangle (small)

$$A_3 = 8 \text{ x } 12$$

$$A_3 = 96 \text{ mm}^2$$

Area of Triangle

$$A_4 = \frac{1}{2} \text{ x } 8 \text{ x } 4$$

$$A_4 = 16 \text{ mm}^2$$

$$y_1 = 32 + \frac{4 \text{ x } 12}{3\pi}$$

$$y_1 = 37.09 \text{ mm}$$

$$y_2 = \frac{32}{2} = 16 \text{ mm}$$

$$y_3 = \frac{12}{2} = 6 \text{ mm}$$

$$y_4 = \frac{1}{3} x \ 4{+}12 = 13.33 \text{ mm}$$

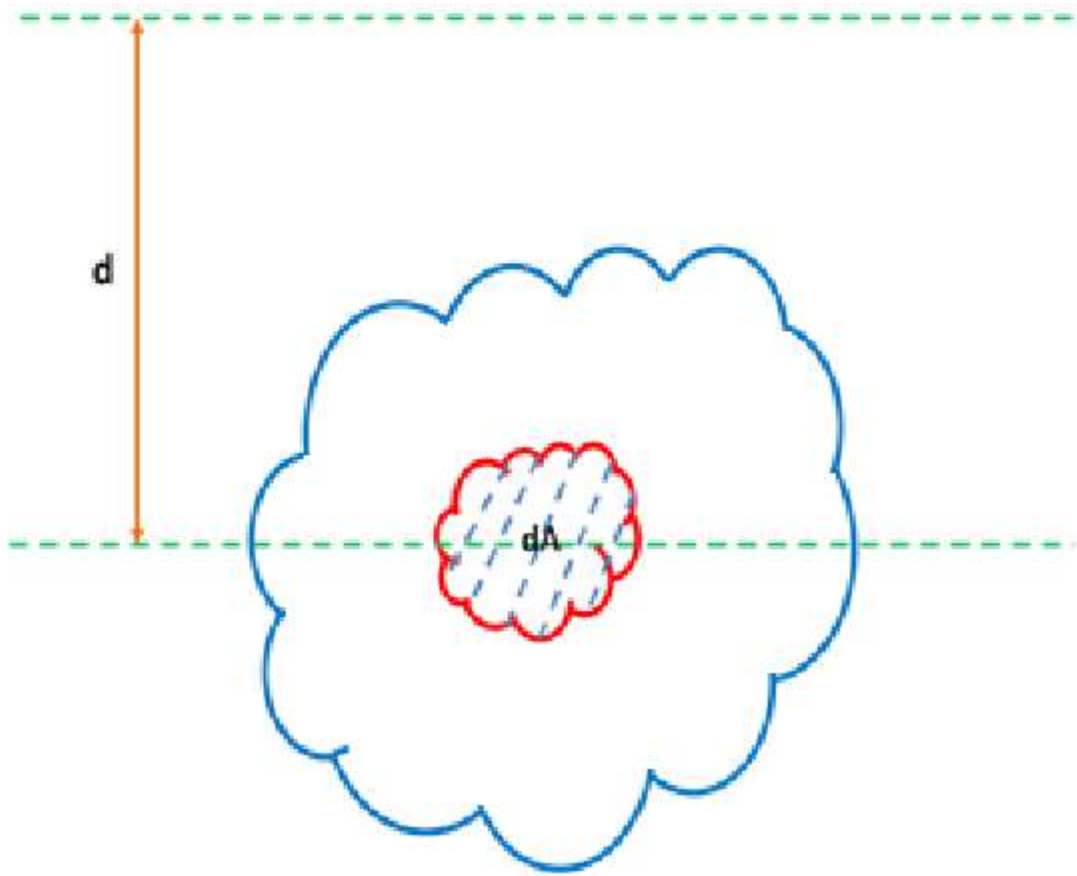
$$Y = \frac{A1y1{+}A2y2{-}A3y3{-}A4y4}{A}$$

$$Y = \frac{(226.19 \text{ x } 37.09){+}(512 \text{ x } 16){-}(96 \text{ x } 6){-}(16 \text{ x } 13.33)}{(226.19{+}512{-}96{-}16)}$$

$$Y = 22.22 \text{ mm}$$

Moment of Inertia

Defination :-The moment of inertia of any plane area about any axis is the second moment of area about that axis.



The first moment of Area = Area x Distance

= dA x d

And

The second moment of area = Area x $(\text{Distance})^2$

The second moment of area = dA x d^2

OR $\quad I = \int d^2 . \, dA$

$I = A. \, d^2$

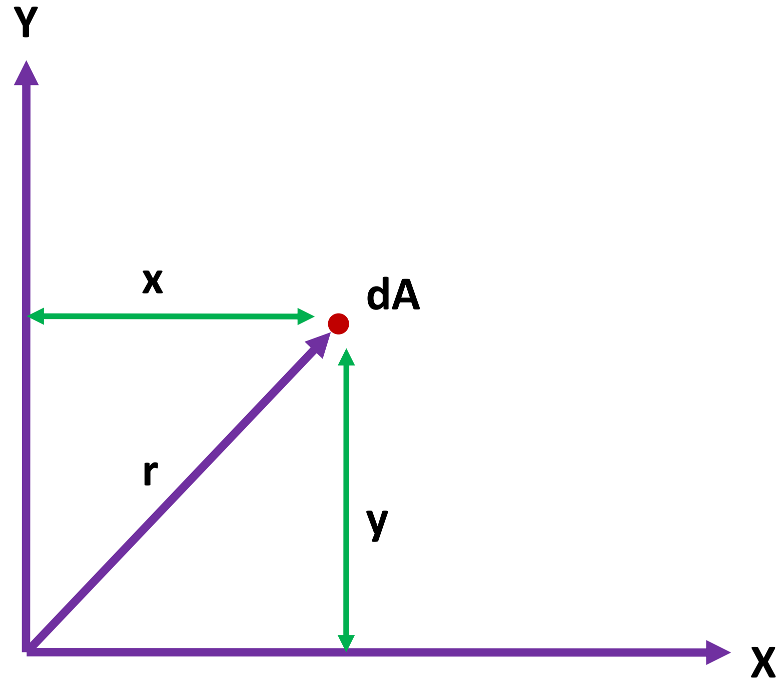
Where d = Centroidal distance from axis

The unit is mm^4, cm^4 or m^4

If moment of area is taken about X-axis, Y-axis or Z-axis respectively. The Moment of Inertia is denoted by Ixx, Iyy and Izz

<u>Polar Moment of Inertia</u>:-

Moment of inertia about an axis perpendicular to the plane of an area is known as polar moment of Inertia.



It may be denoted by J or Izz, thus the moment of inertia about an axis perpendicular to the plane area at O as shown in figure is called as polar moment of inertia at point O is given by

$$I_{zz} = \sum r^2 \, dA$$

$$I_{zz} = I_{xx} + I_{yy}$$

Radius of Gyration:-

Radius of gyration is a mathematical turn defined by the relation

$$K = \sqrt{\frac{I}{A}}$$

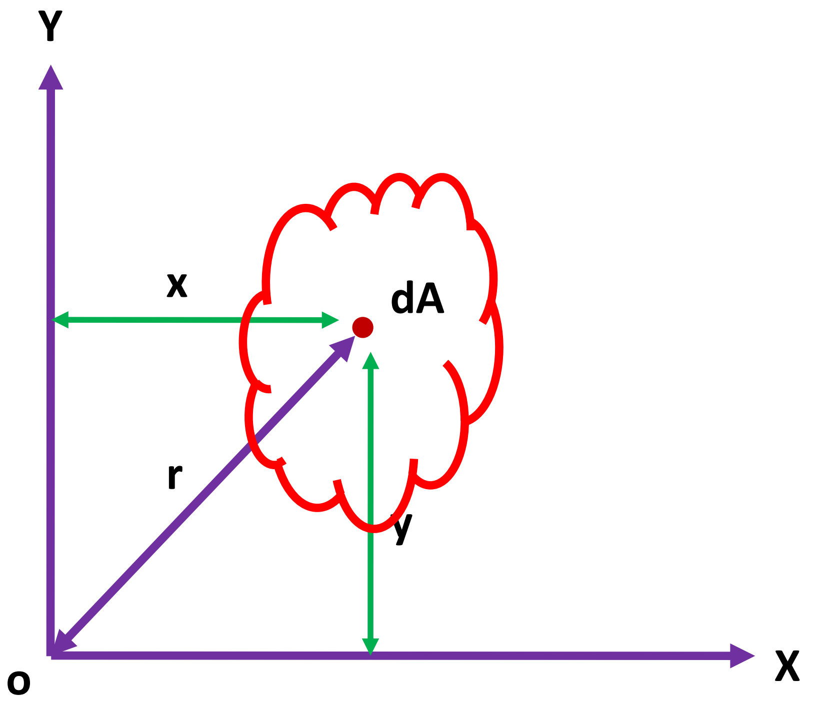
Where, K = Radius of Gyration

I = Moment of Inertia

A = Cross section Area

Theorem of Moment of Inertia

Perpendicular Axis Theorem:-



The Moment of Inertia of an area about an axis perpendicular to its plane (polar Moment of Inertia) at any point O is equal to the sum of Moment of Inertia about any two mutually perpendicular axis through the same point O and lying in the plane of the area.

In figure if Z-Z is the axis normal to the of paper passing through point O.

So, as per this theorem

$$I_{zz} = I_{xx} + I_{yy}$$

As the above theorem can be easily proved, as below

Consider an elementary areas dA at a distance r from o.

Let the co-ordinate of dA be x & y, Then from definition.

$$I_{zz} = \sum r^2 \, dA$$

$$I_{zz} = \sum (x^2 + y^2) \, dA$$

$$I_{zz} = \sum x^2 \, dA + \sum y^2 \, dA$$

$$I_{zz} = I_{xx} + I_{yy}$$

Where $\sum x^2 \, dA = I_{xx}$ and $\sum y^2 \, dA = I_{yy}$

Parallel Axis Theorem:-

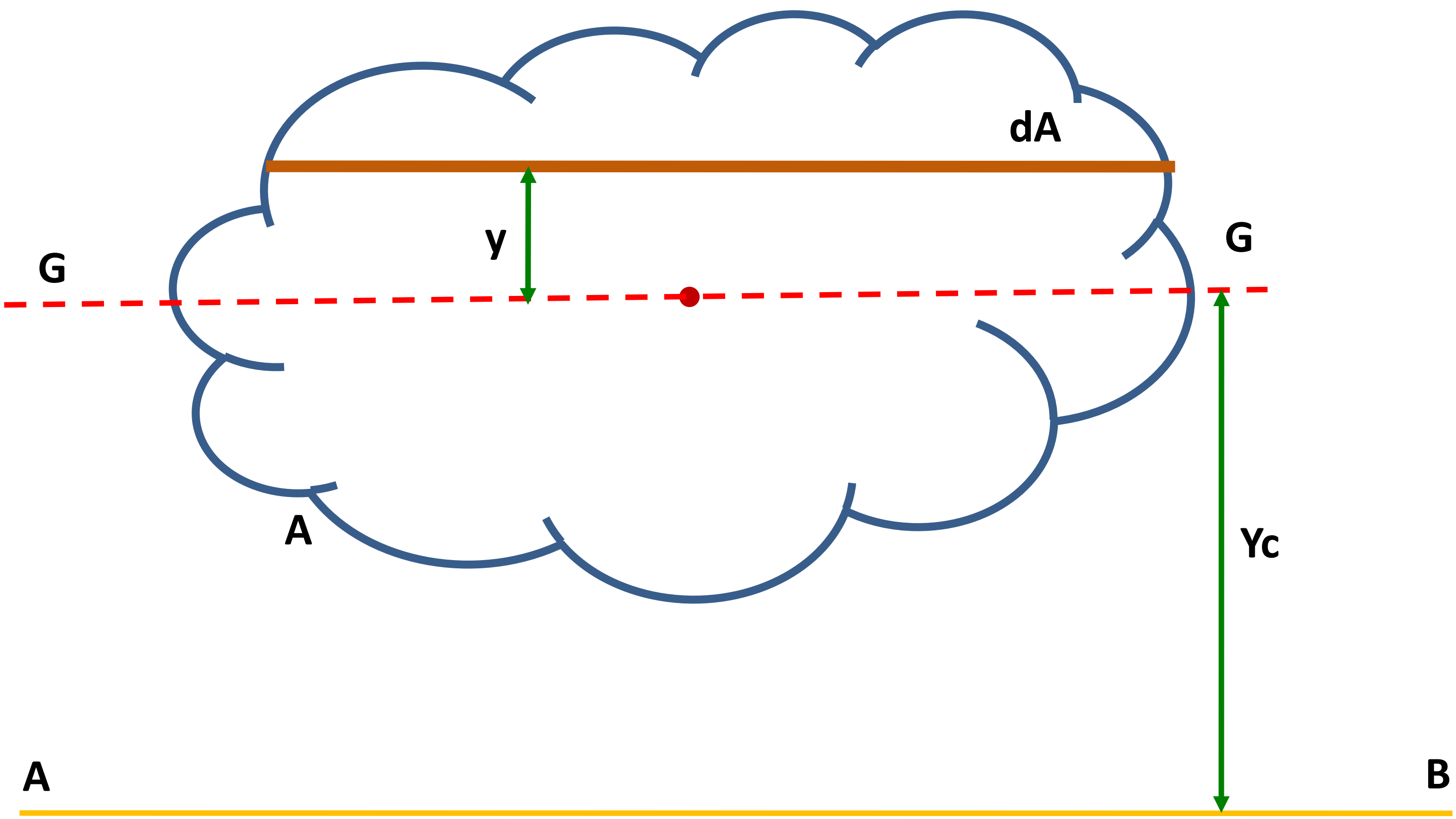
Moment of Inertia about any axis in the plane of an area is equal to the sum of Moment of Inertia about a parallel Centroidal axis the product of area & square of the distance between the two parallel axes i.e. $I_{AB} = I_{GG} + AYc^2$

Where, IAB = Moment of Inertia about axis AB.

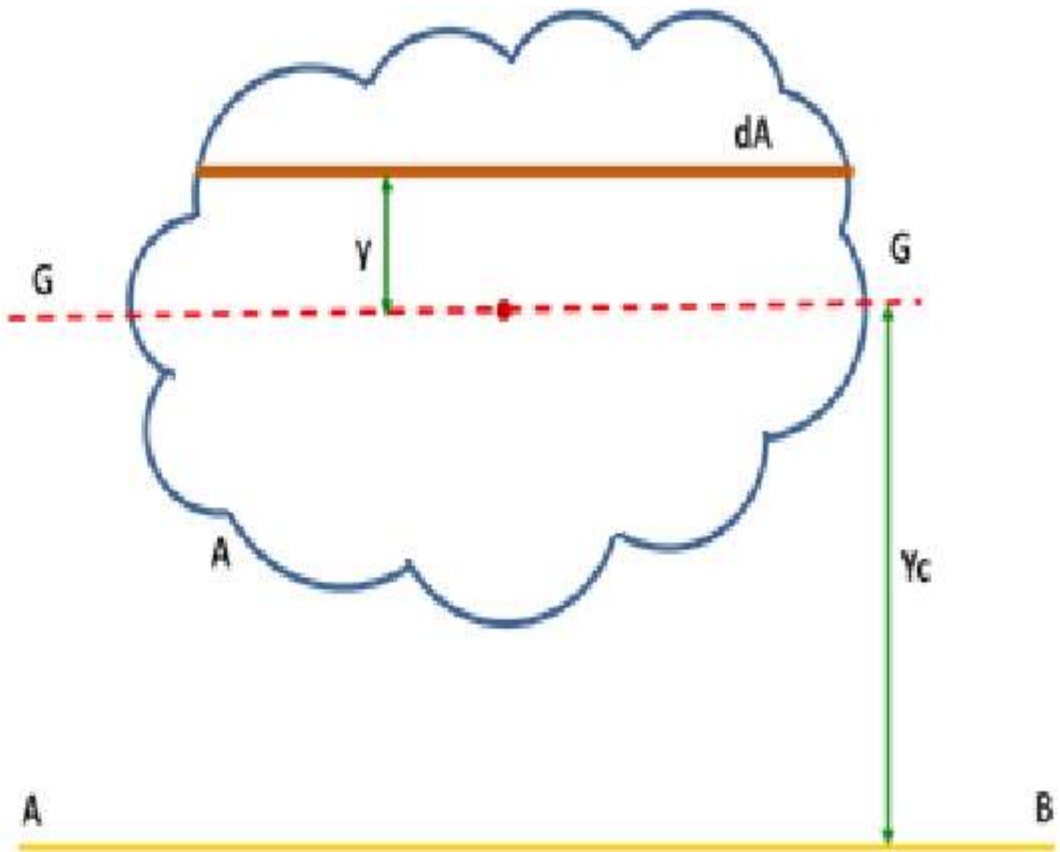
IGG = Moment of Inertia about centroid axis GG parallel to AB.

A = Area of plane figure.

Yc = The distance between the axis AB and parallel Centroidal axis G-G.


dA
y
G
G
A
Yc
A
B

Proof:- Consider an elemental parallel strip dA at a distance y from the Centroid axis as shown.



$$I_{AB} = \Sigma(y+y_c)^2 dA$$

$$I_{AB} = \sum (y^2 + 2yy_c + y^2)\, dA$$

$$I_{AB} = \Sigma y^2\, dA + \Sigma\, 2yy_c\, dA + \Sigma\, y_c^2\, dA$$

$\Sigma\, y^2 dA$ = Moment of Inertia about axis GG=I_{GG}

$$\Sigma\, 2yy_c dA = 2y_c \Sigma\, ydA = 2y_c A \frac{\Sigma\, ydA}{A}$$

In above term $2y_c A$ is constant and $\frac{\Sigma ydA}{A}$ is a distance of centroid from the reference Axis GG.

Since GG is passing through the centroid itself $\frac{ydA}{A}$ is zero and hence the term $\sum 2yy_c\mathrm{dA}$ is zero.

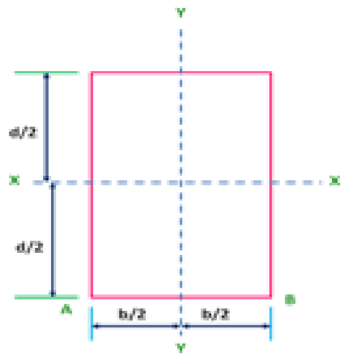
Now the third term

$$\sum y_c^2\mathrm{dA} = y_c^2 \sum dA$$

$$\sum y_c^2\mathrm{dA} = \mathrm{A}y_c^2$$

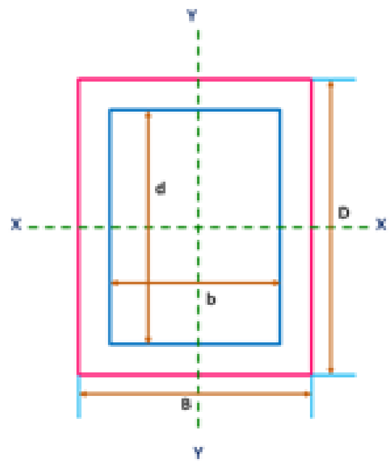
$$\therefore I_{AB} = I_{GG} + \mathrm{A}y_c^2$$

The Common moment of Inertia of Std Section

<table>
<tr><th>Sr. No.</th><th>Shape</th><th>Axis</th><th>Moment of Inertia</th></tr>
<tr><td>1</td><td>Rectangle

</td><td>X-X

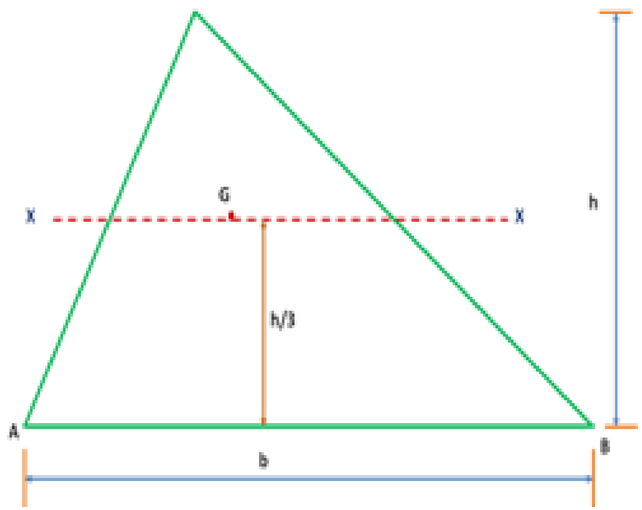
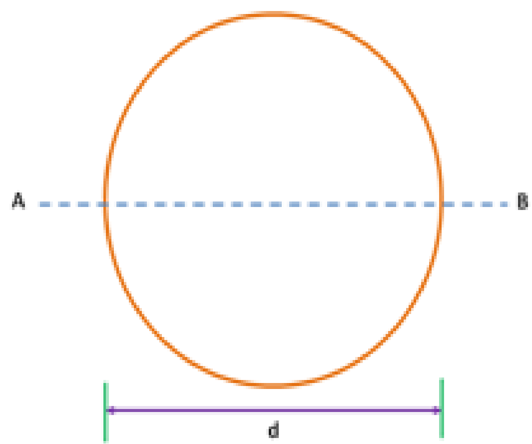
Y-Y

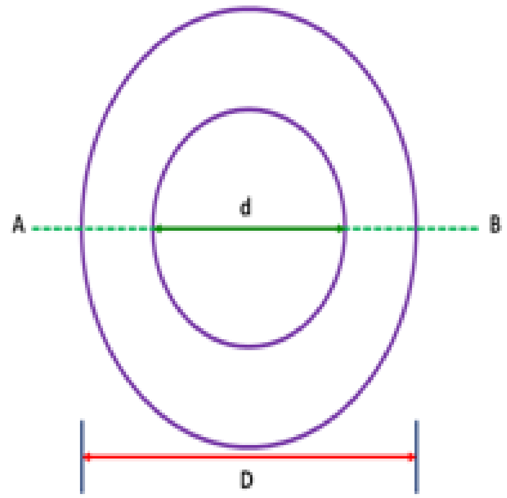
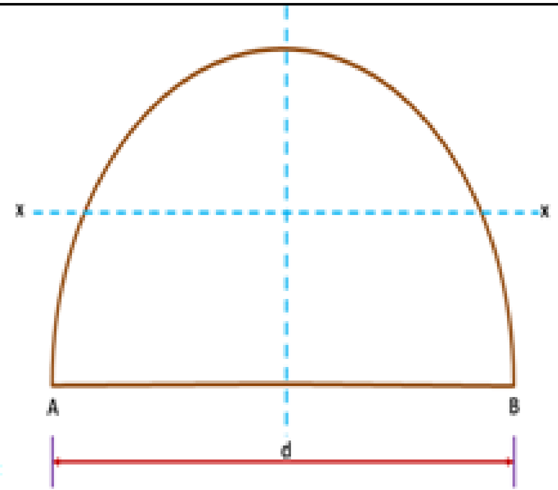
A-B</td><td>$I_{xx}=\frac{bd^3}{12}$
$I_{yy}=\frac{db^3}{12}$

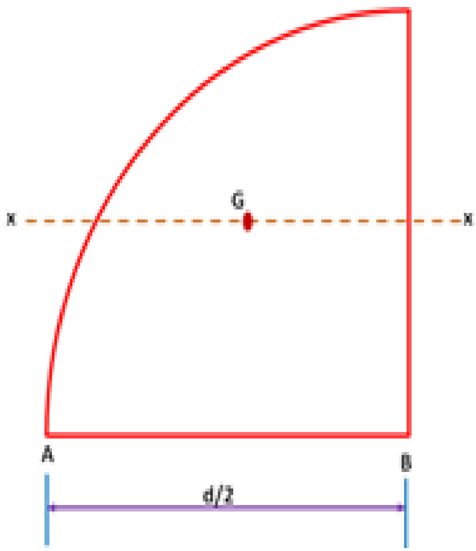
$I_{AB}=\frac{bd^3}{3}$</td></tr>
<tr><td>2</td><td>Hollow Rectangle

</td><td>X-X

Y-Y</td><td>$I_{xx}=\frac{BD^3-bd^3}{12}$

$I_{yy}=\frac{DB^3-db^3}{12}$</td></tr>
</table>

3	Triangle  	X-X A-B	$I_{xx}=\frac{bh^3}{36}$ $I_{AB}=\frac{bh^3}{12}$
4	Circle  	Diameteral Axis i.e. A-B	$I_{AB}=\frac{\pi d^4}{64}$

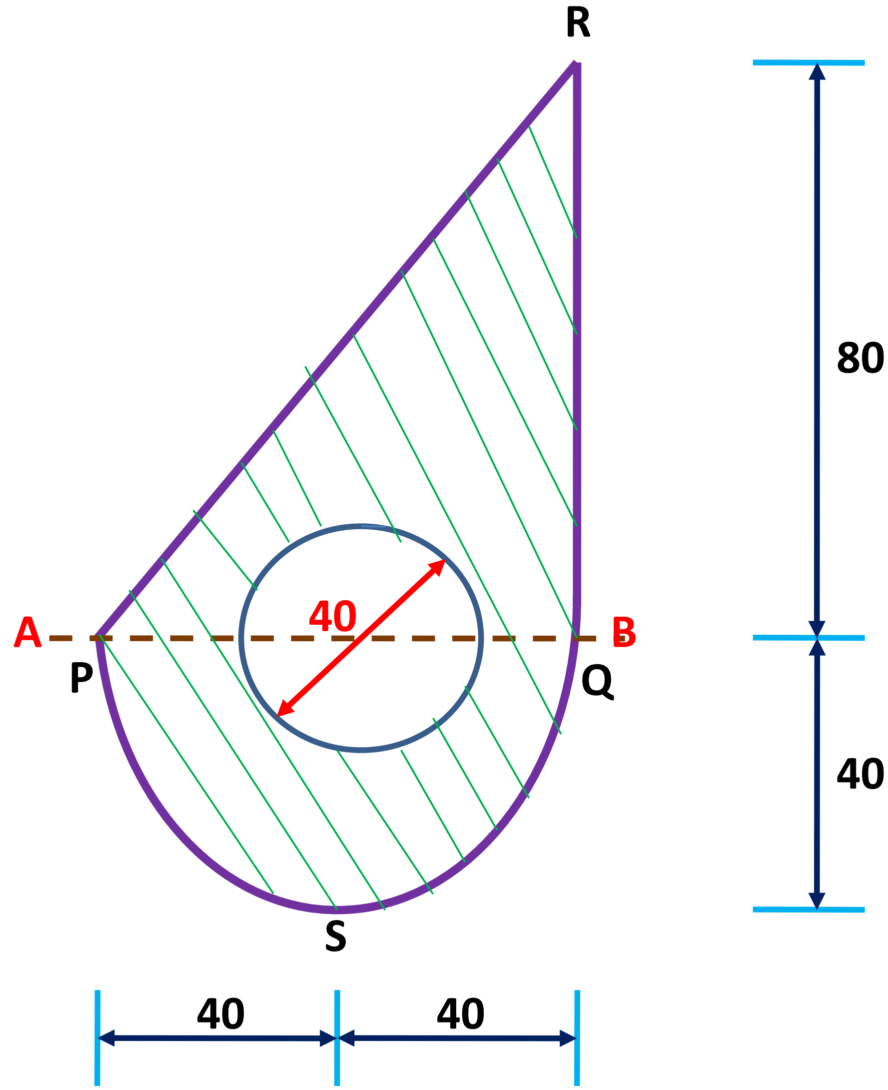
5	Hollow Circle 	A-B	$I_{AB}=\frac{\pi}{64}(D^4-d^4)$
6	 Semicircle	A-B centroidal i.e. X-X	$I_{AB}=\frac{\pi d^4}{128}$ $I_{xx}=0.0068598d^4$

<table>
<tr>
<td>7</td>
<td>Quarter Circle

</td>
<td>A-B

X-X</td>
<td>$I_{AB}=\frac{\pi d^4}{256}$

$I_{xx}=0.00343d^4$</td>
</tr>
</table>

1. Find the Moment of Inertia of the area shaded as shown in figure about the axis AB



The section is divided into a ΔPQR, a semicircle PSQ having base on axis AB, and circle having its center point on axis AB.

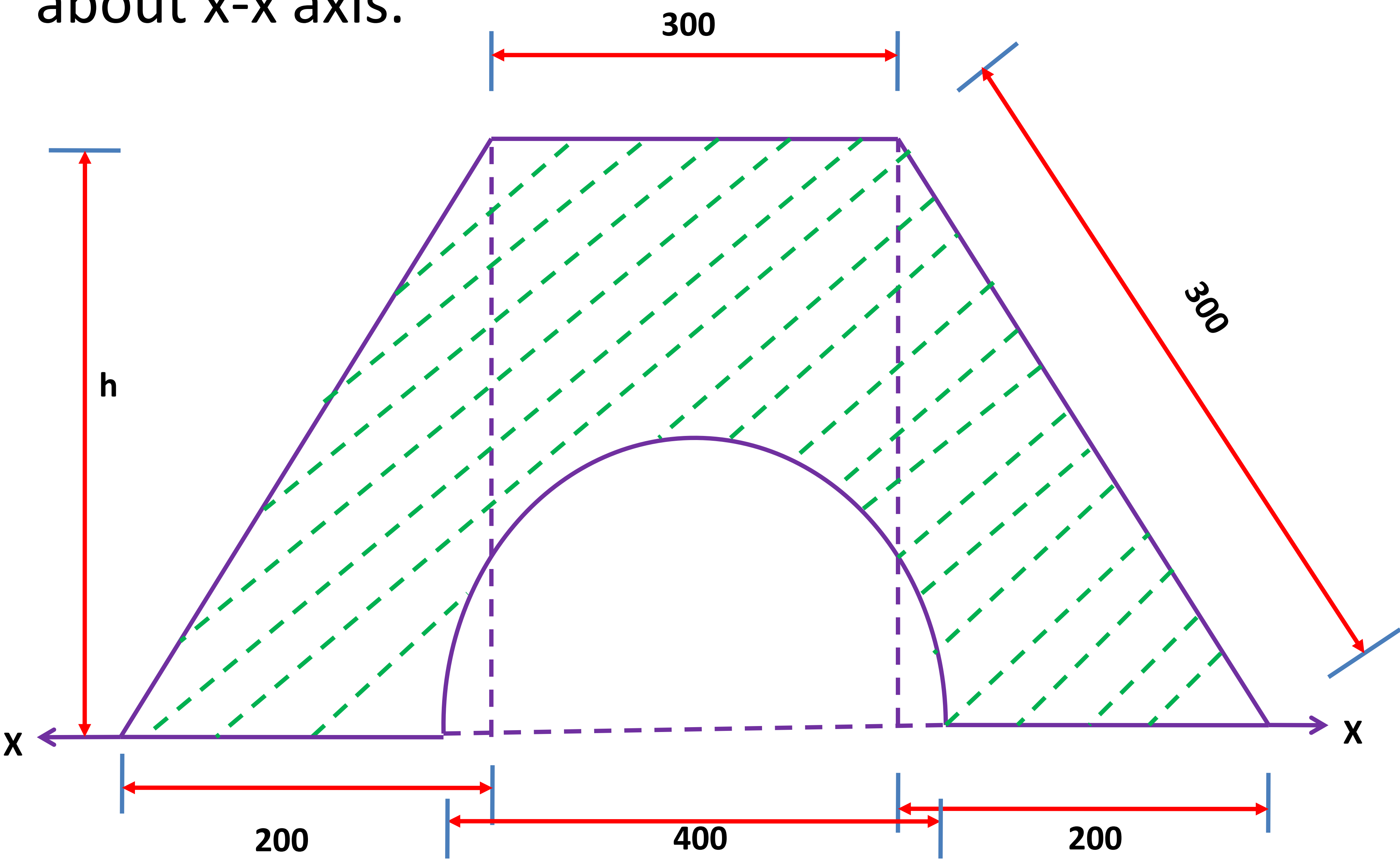
Moment of Inertia of the section about axis AB

= MI of ΔPQR about AB+MI of semicircle @AB-MI of circle @AB

$$= \frac{bh^3}{12} + \frac{\pi d^4}{128} - \frac{\pi d^4}{64} = \frac{80 \text{ x } 80^3}{12} + \frac{\pi \text{ x } 80^4}{128} - \frac{\pi \text{ x } 40^4}{64}$$

$$= 4292979.00 \text{ mm}^4$$

2. Compute the second moment of shaded area about x-x axis.



From geometry,

By using Pythagoras theorem

$$300^2 = 200^2 + h^2$$

$$h^2 = 300^2 - 200^2$$

$$h = \sqrt{300^2 - 200^2}$$

$$h = 223.606$$

Ixx of shaded Area

Ixx = Moment of Inertia rectangular portion + 2 X Moment of Inertia of triangular portion – Moment of Inertia of semicircle

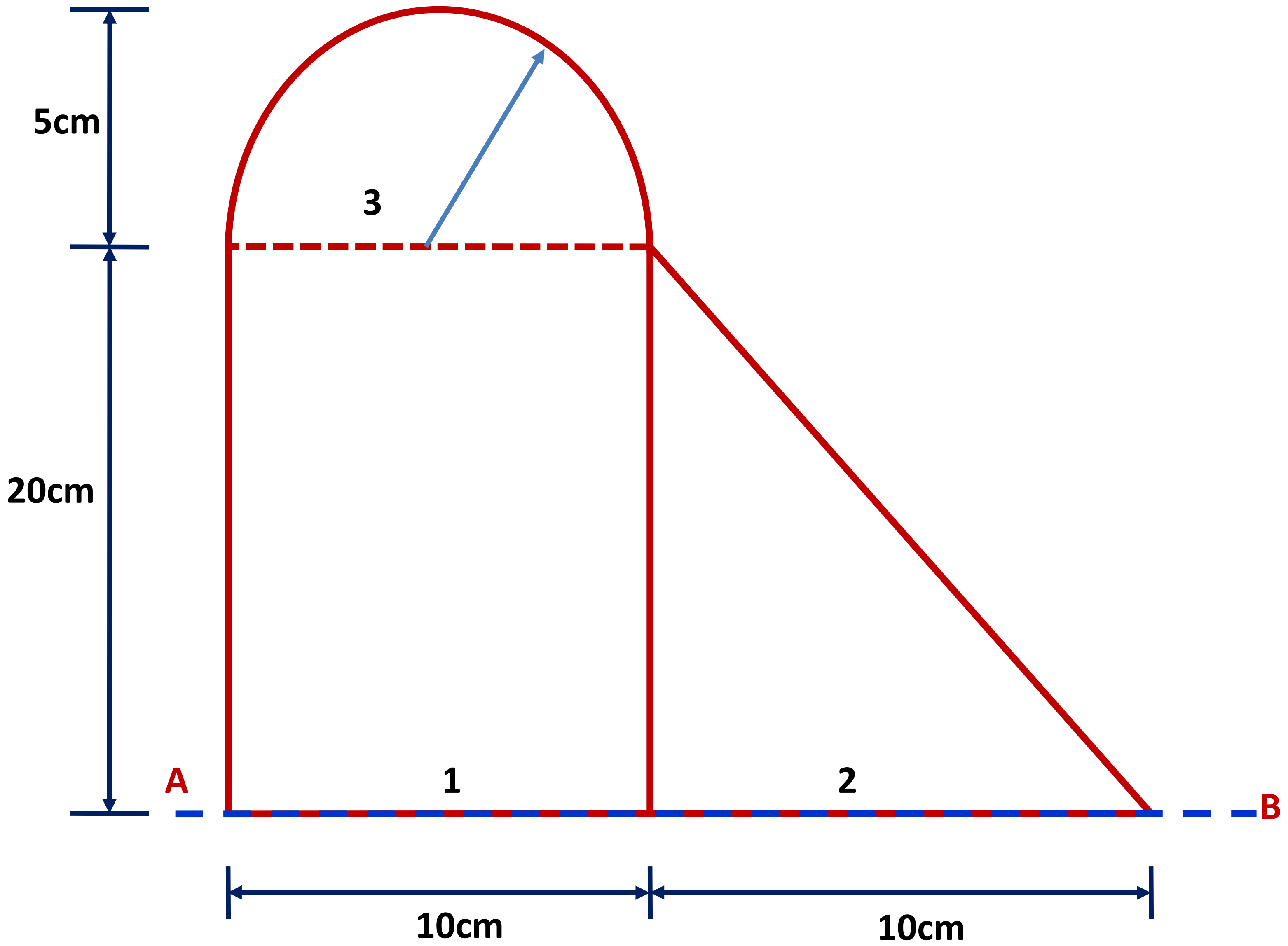
$$= \frac{bh^3}{3} + 2 \text{ x } \left(\frac{bh^3}{12}\right) - \frac{\pi d^4}{128}$$

$$= \frac{300 \text{ x } (223.606)^3}{3} + 2 \text{ x } \left(\frac{200 \text{ x } (223.606)^3}{12}\right) - \frac{\pi \text{ x } (400)^4}{128}$$

$$= 1118022023 + 1666654.775 - 628318530.7$$

$$= 1056857147 \text{ mm}^4$$

3.Compute the second Moment of Inertia of composite area with respect to AB axis.



Area of Rectangle A1 = b X d

A1 = 20 x 10 = 200 cm^2

Area of Triangle A2 = ½ *x* b *x* h

A2 = ½ *x* 10 *x* 20 = 100 cm^2

Area of Semicircle $A_3 = \frac{\pi}{2} \text{ x } R^2$

$$A_3 = \frac{\pi}{2} \text{ x } 5^2$$

$$A_3 = 39.27 \text{ cm2}$$

Distance of Centre of Gravity for Rectangle

$$Y1 = Yc1 = 20/2 = 10 \text{ cm}$$

Distance of Centre of Gravity for Triangle

$$Y2 = Yc2 = 20/3 = 6.67 \text{ cm}$$

Distance of Centre of Gravity for Semicircle

$$Y3 = Yc3 = 20 + \frac{4 \text{ x } 5}{3\pi} = 22.12 \text{ cm}$$

Moment of Inertia about base is AB.

$$I_{AB} = I_{G1} + A_1 \text{ x } y^2{}_{C1} + I_{G2} + A_2 \text{ x } y^2{}_{C2} + I_{G3} + A_3 \text{ x } Y^2{}_{C3}$$

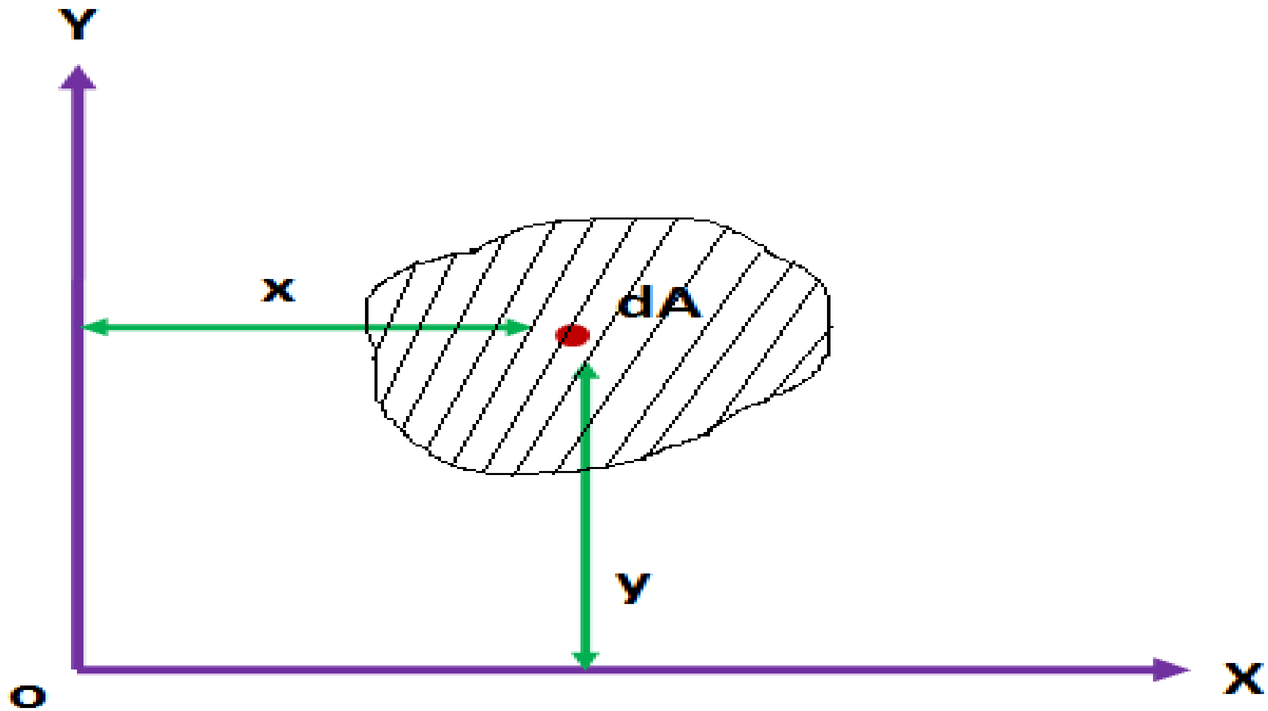
$$I_{AB} = \frac{10\ x\ 20^3}{12} + 200x\ 10^2 + \frac{10\ x 20^3}{36} + 100x6.67^2$$

$$+0.0068598\ \mathrm{x}10^4 + 39.27x\ 22.12^2$$

$$I_{AB} = 26666.67 + 6671.11 + 19283.19$$

$$I_{AB} = 52620.97\ cm^4$$

Product of inertia



The fig shows a body of area A. consider a small area dA. The moment of this area about x-axis is y.dA. Now the moment of y.dA about y- axis is x-y dA. Then x-y dA is known as the product of inertia dA with respect to x-axis and y-axis.

The integral ∫ xy dA is known as the product of inertia of area A with respect to x and y-axis. This product of inertia is represented by I_{xy}.

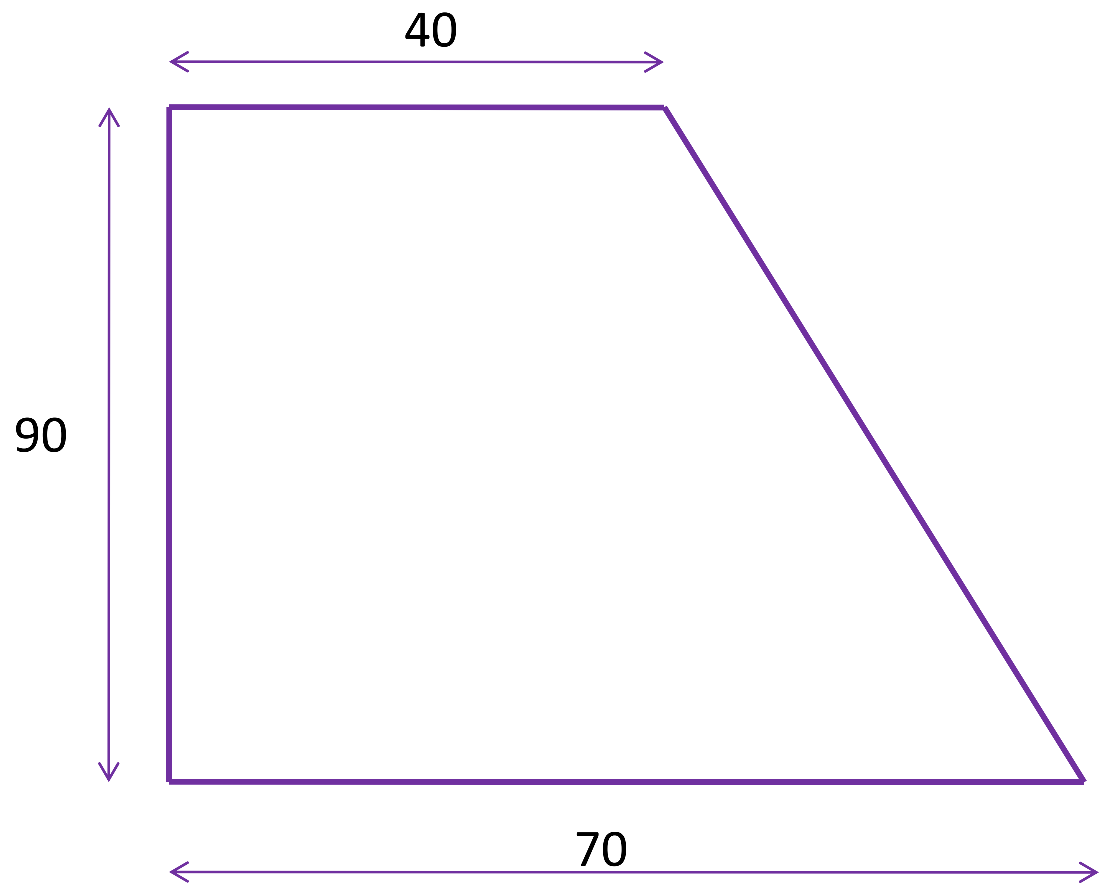
$$I_{xy} = \int xy \, dA$$

The product of inertia I_{xy} can also be written mathematically as

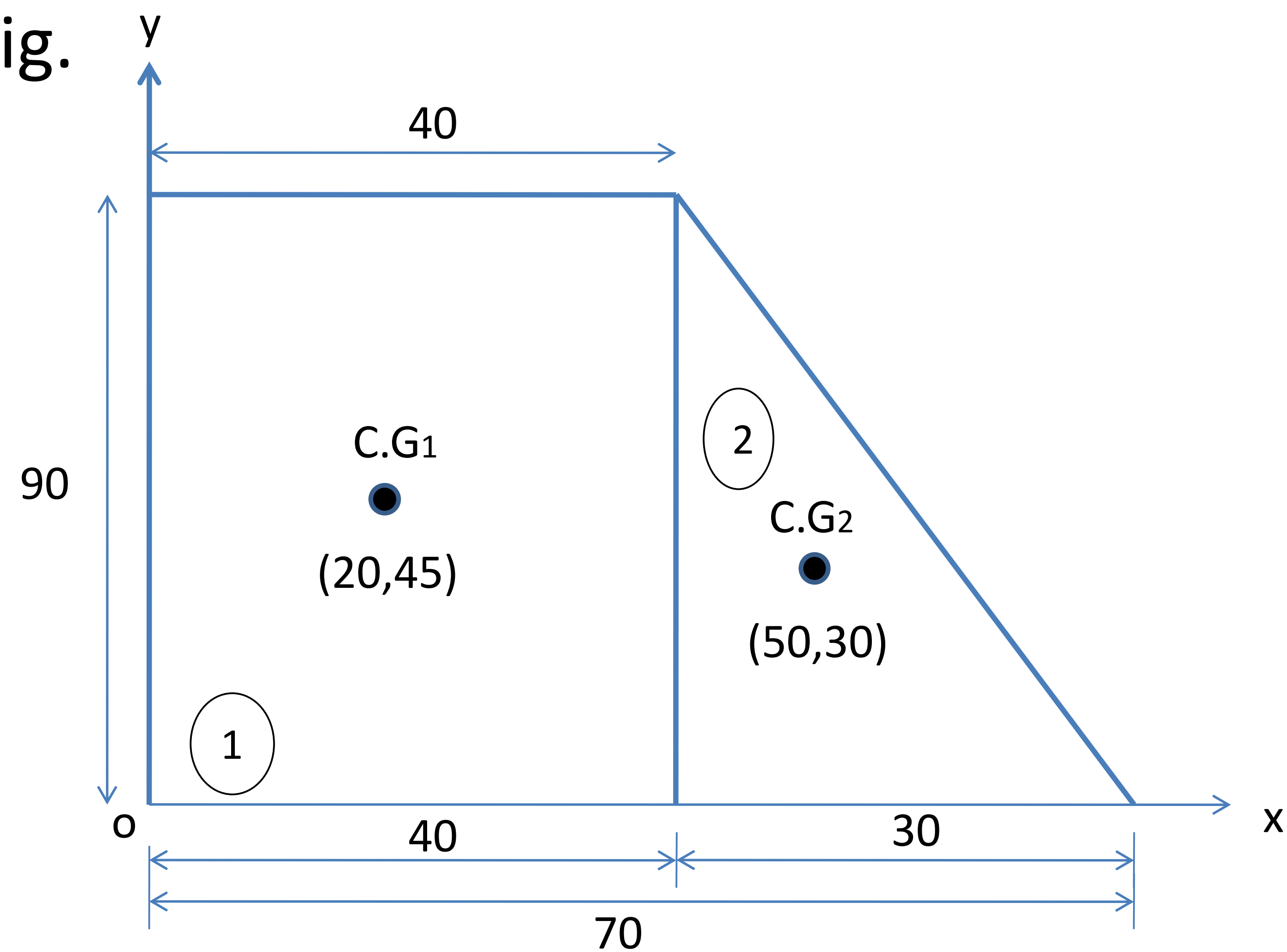
$$I_{xy} = \sum x_i y_i A_i = x_1 y_1 A_1 + x_2 y_2 A_2 + \dots\dots$$

Where x_i y_i = co-ordinates of the C.G of area A_i.

1.Fig shows a plane area. Determine the product moment of inertia of the given area. All dimensions are in mm.



Solution:- Divide the given area into two parts. The first part is a rectangle and second part is a right angled triangle. Take x-axis and y-axis as shown in fig.



The areas and location of their C.G are given below:

Area of rectangle, $A_1 = 90 \times 40 = 3600\ mm^2$

The co-ordinator of C.G of rectangle are :

x_1=20mm, y_1=45mm.

Area of triangle $A_2 = \dfrac{90 \times 30}{2}$

$= 1350\ mm^2$

the co=ordinates of C.G of triangle are :

$x_2 = 40+1/3 \times 30 = 40 + 10 = 50$ mm,

$y_2 = 1/3 + 90 = 30$mm.

the product of inertia of given area is given by

$$I_{xy} = x_1 y_1 A_1 + x_2 y_2 A_2$$

$$= 3600 \times 20 \times 45 + 1350 \times 50 \times 30$$

$$= 3240000 + 2025000 = 5265000 \text{ mm}^4$$

Principal moments of inertia

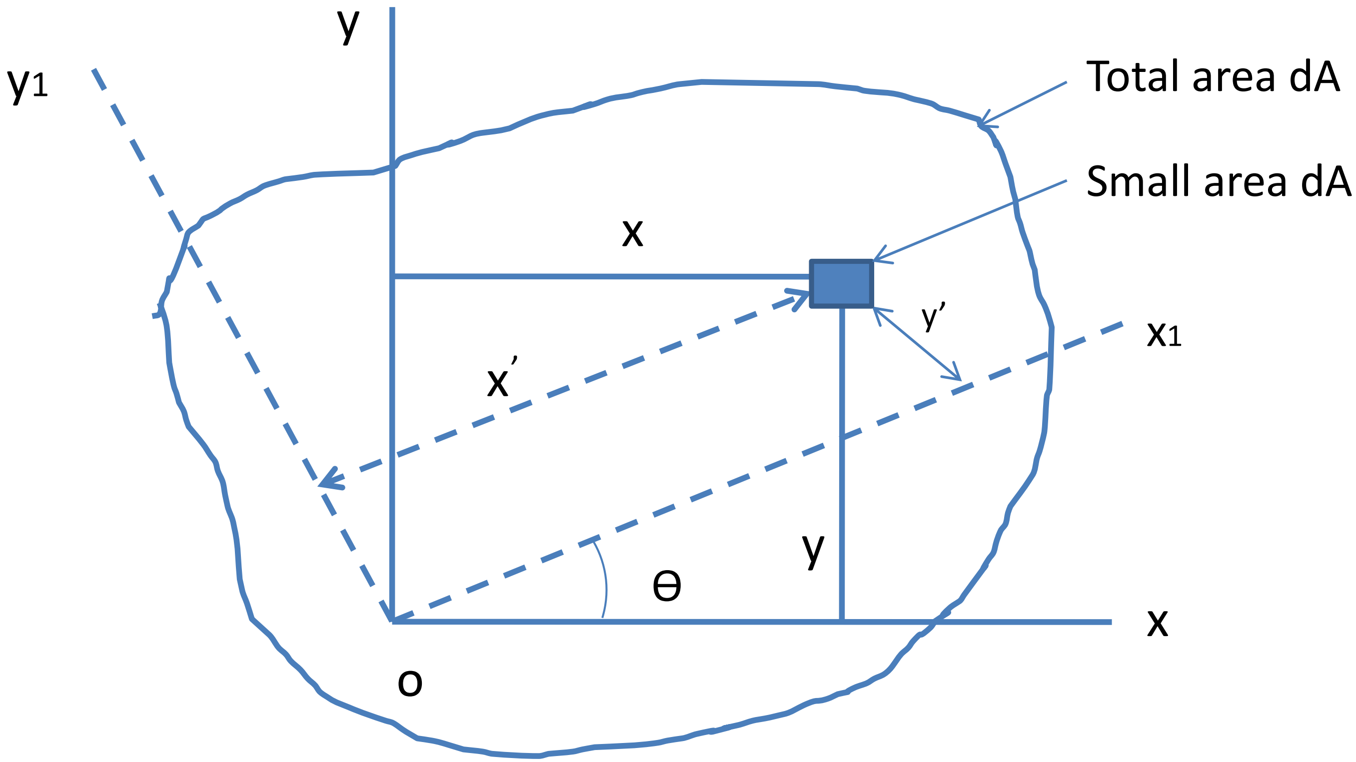


Fig shows a body of area A with respect to old axes (x,y) and new axes (x1,y1). The new axes x1 and y1 have been rotated through an angle ϴ in anticlockwise direction. consider a small area dA.

The co-ordinates of the small area with respect to old axes is (x,y) whereas with respect to new axes, the co-ordinates are x' and y'.

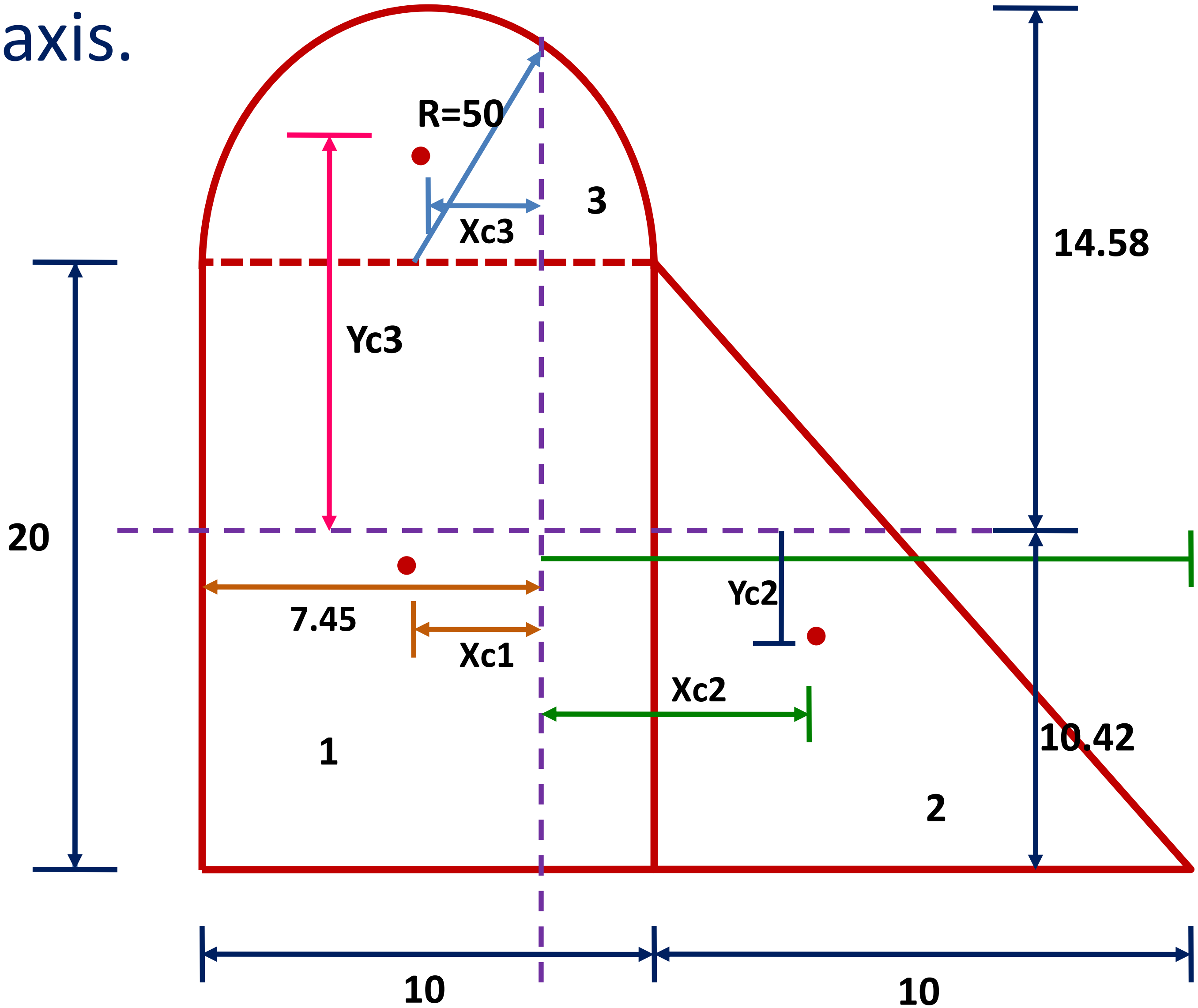
Principal moment of inertia is expressed as

$$Ix_1x_1 = \frac{(Ixx+Iyy)}{2} + \frac{(Ixx-Iyy)}{2}\cos 2\theta - Ixy\sin 2\theta$$

$$Iy_1y_1 = \frac{(Ixx+Iyy)}{2} - \frac{(Ixx-Iyy)}{2}\cos 2\theta + Ixy\sin 2\theta$$

Tutorials

1. Calculate Second Moment of Inertia about centroid axis.



2.Determine the Moment of Inertia of ‘L’ section as shown about its centroidal axis parallel to its Leg. Also find polar Moment of Inertia.