Week 6 Assignment

For any function denoted as $f(x)$, y(x) *or* $u_i(x)$ (for any $i \in \mathbb{N}^*$) that represents a mapping $\mathbb{R} \to \mathbb{R}$, it will be assumed here and henceforth that the function is continuous and differentiable within our domain of interest, unless explicitly stated otherwise.

Section A: Homework

Different questions are of different levels of depth and intensity and will not take equal time to answer. *Give to each question its due attention*. There are no "marks" to these questions. Mark them yourself on the degree of advance in the level of your understanding (and you if feel, you may share that marking with us).

A.1.(a & b). Figures 1 and 2 each show the graph of a function over a closed interval D (defined within figure). At what points of the domain are the functions:

(a) differentiable?

(b) continuous but not differentiable?

(c) neither continuous nor differentiable?

Give reasons for your answers.

A.2 Work out from the basic definition of a derivative in terms of limits, if there is any point in the domain of the function $y(x) = x^3$ where the slope becomes negative.

A.3 Work out from the basic definition of derivatives (in terms of limits), if there is any point on the quadratic $y(x) = 2x^2 – 13x + 5$ where the tangent acquires a slope of -1? If so, identify that point of tangency and the equation of that tangent.

$$\left\{ \begin{array}{l} \text{Recall that the equation of a line written as y = mx + c, where m is slope and c is} \\ \text{the intercept on the y-axis, can alternately be written in the form } \dfrac{(y-y_1)}{(x-x_1)} = m, \text{ where} \\ (x_1, y_1) \text{ is a point through which the line passes.} \end{array} \right\}$$

A.4 Evaluate the derivative of the function

$$y(x) = (x^2+1)(x+5+\frac{1}{x})$$

by (a) applying product rule, and (b) multiplying the factors and creating a sum of simpler terms.

A.5 Find the derivative of

$$y(x) = \frac{(x+1)(x+2)}{(x-1)(x-2)}$$

A.6 Find the first and second derivatives of

$$p(q) = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right)$$

A.7 Find the first and second derivatives of

$$p(q) = \frac{q^2+3}{(q-1)^3+(q+1)^3}$$

A.8 Find equations for the horizontal tangents to the curve of $y(x) = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

A.9 If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V-nb} - \frac{an^2}{V^2}$$

in which a, b, n and R are constants. Find dP/dV.

A.10 The graphs in fig. 3 show the position *s*, the velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph represents which parameter? Give reasons for your answers.

A.11 Acquiring insight into Simple Harmonic Motion (Discuss in Tutorial Class if unable to make progress). Figure 4 shows a body hanging from a vertical spring and displaced, oscillating above and below its rest position. Let us say it is stretched 5 units beyond its rest position and then released at time t=0. Its position at any later time t is given by the equation

$$s = 5\cos t$$

(implication: at t = 0, i.e. at start, its position is +5 (units), at t= $\pi/2$ is 0 (rest), and at t = π position is -5).

What is its velocity and acceleration at time t?

What is the period (time for one cycle) and amplitude (max. displacement / vel / accl) of the motion?

It is obviously inconvenient to view time 't' as an angle like $\pi/2$, π, etc. Just suppose you replaced the argument of the cosine function 't' by the term '$2\pi t$'. Then, again starting from the maximum stretch position with t=0, at what value of t will the body reach the rest position, maximum compression position, and return to the maximum stretch position where it would exactly complete one cycle?

It is somewhat unreal to think that the period of oscillation (one full cycle or one "wave") of a body would exactly be one second. Usually oscillation is much faster, and the term "frequency" denotes the number of cycles covered in one second. Now suppose the argument of the cosine function '$2\pi t$' is replaced by the term '$2\pi ft$'. What physical parameter would 'f' represent? Suppose the term '$2\pi ft$' is replaced by the term 'ωt'. How would you physically relate 'ω' and '$2\pi f$'?

Now return to the original equation with argument of cosine as 't'. At t=0, we know that the displacement is maximum. At what points of displacement, are the velocities maximum and minimum? At what points of displacements, are the accelerations maximum and minimum? At the extreme values of velocity, what are the accelerations? At the extreme points of displacement, what are the velocities? The accelerations? Hint: Just take x-axis varying from 0 to 4π, and sketch on the same graph the approximate curves of displacement, velocity and acceleration (as values of y). Do you see any link between extreme values of a function and its derivatives? Its second derivatives?

A.12 At what order of derivative of the function y(x)= sin(x) would we get back the original function?

A.13 Find derivatives of the two functions below. For each, use two alternate approaches and cross-check the answers:

$$y(x) = (\sec x + \tan x)(\sec x - \tan x)$$

$$y(x) = (\sin x + \cos x)\sec x$$

A.14 Find $dr/d\theta$ for $r(\theta) = \sec\theta \csc\theta$

A.15 Using composite functions, find the derivative of

$$y(x) = (4x+3)^4 (x+1)^{-3}$$

A.16 Using composite functions, find the derivative of $f(\theta) = \left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$

A.17 Find dy/dt for $y(t) = 4\sin\left(\sqrt{1+\sqrt{t}}\right)$

A.18 Find y'' for $y(x) = \left(1-\sqrt{x}\right)^{-1}$

A.19 Let *g* be a composite function of *f* and *u*, i.e. $g = f \circ u$,

where $f(u) = \left(\dfrac{u-1}{u+1}\right)^2$, and $u(x) = \dfrac{1}{x^2} - 1$.

Find g'(x) at x = -1.

A.20 Suppose, at x= 2, the functions f(x), g(x), f'(x) and g'(x) have the values 8, 2, 1/3 and -3, respectively. Then find the value of the derivative of $\sqrt{f^2(x)+g^2(x)}$ at that point.

A.21 Find dy/dx if $y(x)= x^{3/2}$ by using the chain rule where the composite functions are taken as

$y(u)= u^3$ and $u(x) = \sqrt{x}$

or as $y(u) = \sqrt{u}$ and $u(x) = x^3$.

Compare results from the two decompositions.

A.22 Suppose that u= g(x) is differentiable at x= 1, and y= f(u) is differentiable at u= g(1). If the graph of y= f[g(x)] against x has a horizontal tangent at x= 1, can we conclude anything about the tangent to the graph of g against x at x= 1, or the tangent to f against u at u = g(1)? Give reasons for your answer.

A.23 Which of the following could be true if $f''(x) = x^{-1/3}$?

a. $f(x) = \frac{3}{2}x^{2/3} - 3$

b. $f(x) = \frac{9}{10}x^{5/3} - 7$

c. $f'''(x) = -\frac{1}{3}x^{-4/3}$

d. $f'(x) = \frac{3}{2}x^{2/3} + 6$

Section B: Tutorial for class workout

B.1 Consider the product rule for derivatives, i.e.

$$\frac{d(u_1 \cdot u_2)}{dx} = \frac{du_1}{dx} \cdot u_2 + u_1 \cdot \frac{du_2}{dx}$$ expressed simply as $(u_1.u_2)' = u_1'.u_2 + u_1u_2'$.

Prove by the method of induction that the product rule holds for any 'n' number of functions, i.e. $(u_1.u_2. \; ..... \; .u_n)' = u_1'. \, u_2. \; .. \; .u_n + u_1. \, u_2'. \; .. \; .u_n + ... + u_1. \, u_2. \; .. \; .u_n'$.

Recall that the method of induction says that if an identity holds for the first term, and then if assumed to hold for any 'k' number of terms, can be proven to hold for 'k+1' number of terms, then it will hold for any arbitrary number of terms.

B.2 Find a, b and c if the curves representing $y(x) = x^2 + ax + b$, and $y(x) = cx - x^2$, have to have a common tangent line at the point (1, 0).

B.3 Using the basic definition of a derivative in terms of limits, and common trigonometric identities, evaluate the derivative of cos x.

B.4 This deals with your insight on the slopes of sine curves:

(a) find equations for the tangents to the curves representing y(x) = sin 2x and y(x) = -sin (x/2) at the origin. Is there anything special about how the tangents are related? Give reasons.

(b) the function y(x)= sin x completes one period on the interval $[0, 2\pi]$, the function y(x) = sin 2x completes two periods, y(x)= sin x/2 completes half a period, etc.. Is there any relation between the number of periods y(x)= sin mx completes on [0, 2π] and the slope of the curve representing y(x) = sin mx at the origin? Give reasons for your answer.

B.5 The graph in fig. 5 shows the average Fahrenheit temperatures in Fairbanks, Alaska, during a typical year of 365 days. The equation that approximates the temperature on day x is

$$y(x) = 37\sin\left[\frac{2\pi(x-101)}{365}\right] + 25$$

a) On what day is the temperature increasing the fastest?

b) About how many degrees per day is the temperature increasing when it is increasing the fastest? For both this and the previous part, use calculus and then confirm from figure.

c) Why exactly one full cycle in a year (draw on your basic knowledge of geography – earth's revolution and axis tilt)?

c) Do you see any relations between the temperature curves and the lengths of days and nights?

d) Any relations to spring and autumn equinoxes (Mar 21 & Sep 23), summer and winter solstices (June 21 & Dec 22)?

e) Any patterns and extremes match with your analysis and insight into the spring-mass displacement and velocities problem? Can you extract some generic insights into the laws of nature and their relations with Mathematics and Calculus?

B.6 For oscillations of small amplitude (small swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is the local acceleration due to gravity. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. If we represent temperature by u, then this variation can be represented as

$$\frac{dL}{du} = kL$$

where k is a constant of proportionality.

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is kT/2.

Section C: Additional Exercises

C.1 a) The empirical equation discussed in Prob. B.6 was created by engineers / scientists with a good foundation in Mathematics. Restating the original equation

$$y(x) = 37\sin\left[\frac{2\pi(x-101)}{365}\right] + 25$$

(1)

this expresses the temperature in Fahrenheit. Given the conversion between Fahrenheit (F) and Centigrade (C) as

$$F = \frac{9}{5}C + 32 \qquad (2)$$

convert equation (1) in terms of Centigrades.

(b) Again, draw from your elementary knowledge of geography, relating to differences in summer and winter periods between northern and southern hemispheres. The annual temperature profile shown in eq (1) is valid for *Fairbanks, Alaska*. *Prime Head*, the northernmost point in Antartica, is at the same latitude as Fairbanks but in the southern hemisphere. Transform eq (1) for *Prime Head*. Identify which aspect of the transformation may not turn out to be exactly valid, and which should turn out to be precise.

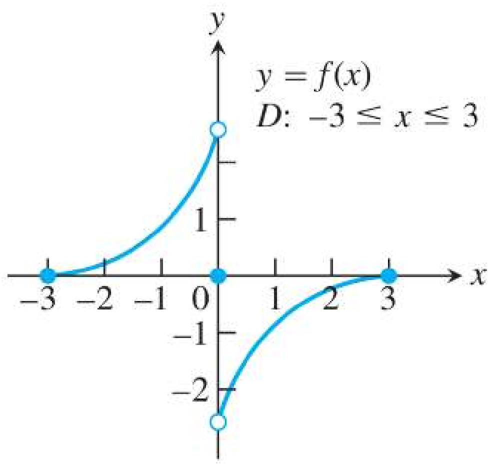

Fig. 1. Relates to problem A.1.a.

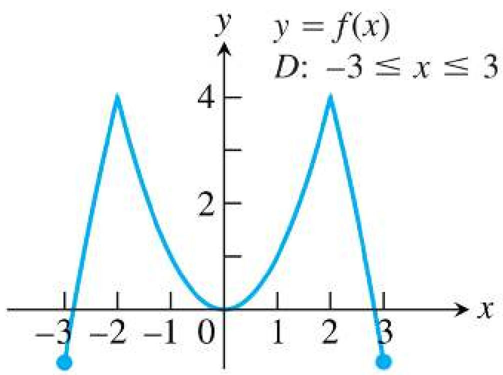

Fig. 2. Problem A.1.b.

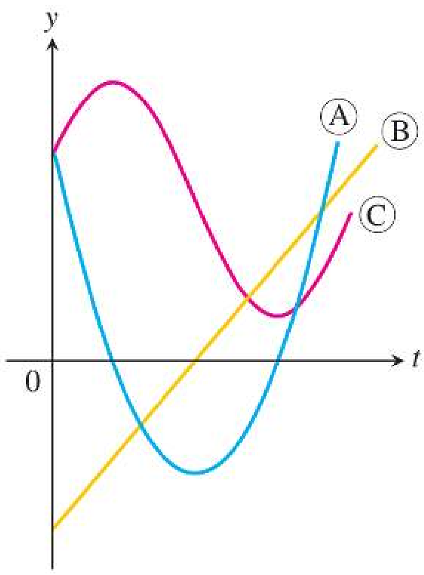

Fig. 3. Problem A.10.

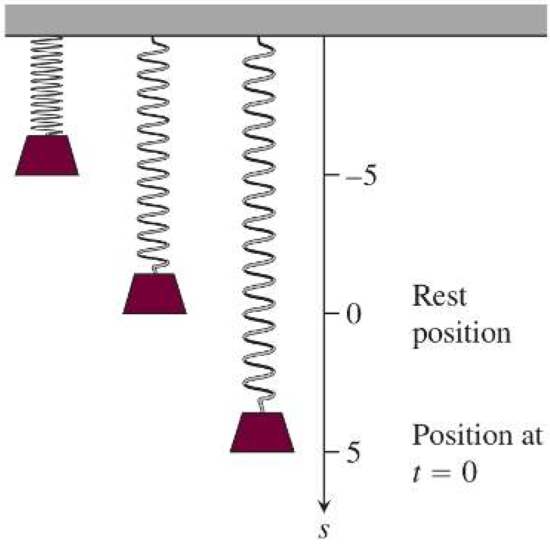

Fig. 4. Problem A.11.

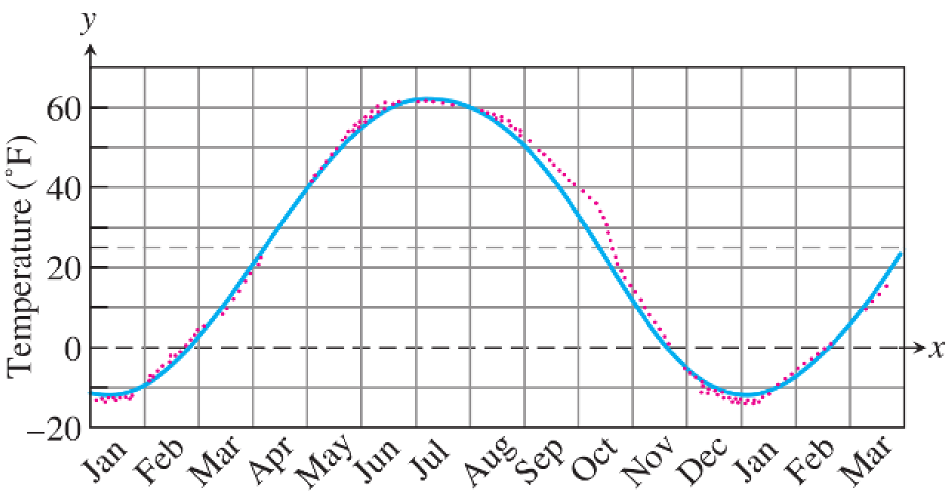

Fig. 5. Problem B.6 and problem C.1.