Problem 4(a): Let $c \in \mathbb{R}$. show that

$$\lim_{x \to c} x^2 = c^2, \text{ using } \epsilon - \delta \text{ definition.}$$

Proof:

Let $h(x) := x^2, \ \forall x \in \mathbb{R}$

we want to make the difference

$|h(x) - c^2| = |x^2 - c^2| < \epsilon$, by taking x sufficiently close to c, i.e., $|x - c| < \delta$ (where $\epsilon > 0$)

Note: $x^2 - c^2 = (x+c)(x-c)$

If $|x - c| < 1 \Rightarrow |x| \leq |c| + 1$

$\therefore \ |x + c| \leq |x| + |c| \leq |c| + 1 + |c| = 2|c| + 1.$

So, if $|x - c| < 1$, we have

$$|x^2 - c^2| = |x + c||x - c| \leq (2|c| + 1)|x - c| \quad -- ①$$

Moreover this last term will be less than ϵ provided we take

$$|x - c| < \frac{\epsilon}{2|c| + 1}.$$

consequently, if we choose

$$\delta(\epsilon) := \inf\left\{1, \frac{\epsilon}{2|c|+1}\right\}$$

then if $0 < |x - c| < \delta(\epsilon)$, it will follow first that $|x - c| < 1$ so that (1) is valid, and therefore, since

$$|x - c| < \frac{\epsilon}{(2|c| + 1)}$$

$$|x^2 - c^2| \leq (2|c| + 1)|x - c| < \epsilon$$

Since we have a way of choosing $\delta(\epsilon) > 0$ for any arbitrary choice of $\epsilon > 0$, we infer that

$$\lim_{x \to c} h(x) = \lim_{x \to c} x^2 = c^2.$$

Problem 4 (a): Let $c \in \mathbb{R}$. show that

$$\lim_{x \to c} x^2 = c^2 \text{ using } \epsilon - \delta \text{ definition.}$$

$[x \in \mathbb{R}]$

Proof: Let $f(x) = x$ and $g(x) = x$

Note: For a given $a \geq 0$, $b > 0$

$$|f(x) - f(c)| < b, \ \forall \ |x - c| < a$$

and $$|g(x) - g(c)| < b, \ \forall \ |x - c| < a$$

Next,

$$|f(x) g(x) - f(c) g(c)|$$
$$= |f(x) g(x) - f(x) g(c) + f(x) g(c) - f(c) g(c)|$$
$$\leq |f(x)| |g(x) - g(c)| + |g(c)| |f(x) - f(c)|$$
$$= |x| |x - c| + |c| |x - c|$$
$$= (|x| + |c|)(|x - c|) \quad - - \quad (1)$$

we know that, $|x - c| < a \Rightarrow c - a < x < c + a$

So, $|x| = |x - c + c| \leq |x - c| + |c| \leq a + |c|$

So, $|f(x) g(x) - f(c) g(c)|$
$$\leq (a + 2|c|) a, \quad \forall \ |x - c| < a$$

Let $b > 0$, then

$$(a + 2|c|) a < b \Leftrightarrow a^2 + 2|c| a - b < 0$$
$$\Rightarrow \quad a < \frac{-2|c| + \sqrt{4|c|^2 + 4b}}{2}$$

Hence, we can find a positive a such that

$$(a + 2|c|) a < b.$$

Let $\epsilon > 0$ and $\delta > 0$ such that

$$(\delta + 2|c|) \delta < \epsilon$$

Then for all $|x - c| < \delta$,

$$|x^2 - c^2| < (\delta + 2|c|) \delta < \epsilon$$

Hence proved..

Problem 4(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2 - x^2, & |x| \le 2 \\ 2, & |x| > 2 \end{cases}$$

Determine the formula for $h(x) = f(g(x))$.

Are f, g, h continuous?

Proof:

For $x \in \mathbb{R}$, $h(x) = (f \circ g)(x) = f(g(x))$

$$\text{So, } h(x) = \begin{cases} 1, & \text{if } |g(x)| \le 1 \\ 0, & \text{if } |g(x)| > 1 \end{cases}$$

$$= \begin{cases} 1, & \text{if } |2 - x^2| \le 1 \\ 0, & \cancel{\text{if}}\ \text{otherwise} \end{cases} \quad [\text{By the def. of } g(x)]$$

So, $|2 - x^2| \le 1 \iff -1 \le 2 - x^2 \le 1$

$\iff 1 \le x^2 \le 3$

$\therefore x \in [-\sqrt{3}, -1]$ and $[1, \sqrt{3}]$

$$\therefore h(x) = \begin{cases} 1, & \text{if } x \in [-\sqrt{3}, -1] \cup [1, \sqrt{3}] \\ 0, & \text{otherwise} \end{cases}$$

Checking the continuity of $f(x)$ at $x = 1$ and $x = -1$

At $x = 1$: $L^+ = 0$, $L^- = 1$.

(a) Right limit: For given $\epsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - L^+| < \epsilon, \quad \forall\ 1 < x < 1 + \delta$

$\Rightarrow |0 - 0| < \epsilon, \quad \forall\ 1 < x < 1 + \delta$

$\therefore \lim_{x \to 1^+} f(x) = 0$

(b) Left limit: For given $\epsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - L^-| < \epsilon, \quad \forall\ 1 - \delta < x < 1$

$\cancel{\Rightarrow |1 - 1|}$

$|1 - 1| < \epsilon, \quad \forall\ 1 - \delta < x < 1$

$\Rightarrow 0 < \epsilon, \quad \forall\ 1 - \delta < x < 1$

$\therefore \lim_{x \to (+1)^-} f(x) = 1$

Since

$$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$$

$\therefore$ $f(x)$ is not continuous at $x = 1$.

Similarly,

$$\lim_{x\to(-1)^-} f(x) \neq \lim_{x\to(-1)^+} f(x)$$

$\therefore$ $f(x)$ is not continuous at $x = -1$.

Checking ~~the continuity~~ ~~the~~ continuity of $g(x)$ at $x = 2$ and $x = -2$

At $x = 2$: $L^+ = 2$: $L^- = -2$.

ⓐ Right limit: For given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|g(x) - L^+| < \epsilon, \quad \forall \; 2 < x < 2 + \delta$$

$$\Rightarrow |2 - 2| < \epsilon, \quad \forall \; 2 < x < 2 + \delta$$

$$\Rightarrow 0 < \epsilon, \quad \forall \; 2 < x < 2 + \delta$$

$$\therefore \lim_{x\to 2^+} g(x) = 2$$

ⓑ Left limit: For given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|g(x) - L^-| < \epsilon, \quad \forall \; 2 - \delta < x < 2$$

$$\Rightarrow |2 - x^2 - (-2)| < \epsilon, \quad \forall \; 2 - \delta < x < 2$$

$$\Rightarrow |4 - x^2| < \epsilon$$

$$\Rightarrow |x - 2||x + 2| < \epsilon \quad \text{①}$$

$|x + 2| \le |x| + 2 \le 2 + 2 = 4$

$\Rightarrow |x + 2| \le 4$

The last term will be less than ϵ, provided we take $\delta(\epsilon) = \inf\{1, \frac{\epsilon}{4}\}$ {Refer Problem 4 (a) for details}.

$$\therefore |4 - x^2| = |x + 2||x - 2| \le 4|x - 2| < \epsilon$$

So, $\lim_{x\to 2^-} g(x) = -2$.

Since

$$\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$$

$\therefore h(x)$ is not continuous at $x = 2$.

Similarly,

$$\lim_{x\to (-2)^+} g(x) \neq \lim_{x\to (-2)^-} g(x)$$

$\therefore$ $g(x)$ is not continuous at $= -2$.

Checking the continuity of $h(x)$ at $x = 1$, $x = -1$, $x = \sqrt{3}$ and $x = -\sqrt{3}$

At $x = 1$: $L^+ = +1$, $L^- = \cancel{1}\ 0$

(a) Right limit: For given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|h(x) - L^+| < \epsilon, \ \forall \ 1 < x < 1 + \delta$$

So, $|1 - 1| < \epsilon, \ \forall \ 1 < x < 1 + \delta$

$\Rightarrow$ $0 < \epsilon, \ \forall \ 1 < x < 1 + \delta$

$$\therefore \lim_{x\to 1^+} h(x) = 1.$$

(b) Left limit: For given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|h(x) - L^-| < \epsilon, \ \forall \ 1 - \delta < x < 1$$

So, $|0 - 0| < \epsilon, \ \forall \ 1 - \delta < x < 1$

$\Rightarrow$ $0 < \epsilon, \ \forall \ 1 - \delta < x < 1$

$$\therefore \lim_{x\to 1^-} h(x) = 0$$

Since $\lim_{x\to 1^+} h(x) \neq \lim_{x\to 1^-} h(x)$

$\therefore$ $h(x)$ is not continuous at $x = 1$.

Similarly,

$$\lim_{x \to (\sqrt{3})^+} h(x) \neq \lim_{x \to (\sqrt{3})^-} h(x)$$

$$\lim_{x \to (-1)^+} h(x) \neq \lim_{x \to (-1)^-} h(x)$$

$$\lim_{x \to (-\sqrt{3})^+} h(x) \neq \lim_{x \to (-\sqrt{3})^-} h(x).$$

So, $h(x)$ is not continuous at $x = \sqrt{3}$, $x = -1$ and $x = -\sqrt{3}$

Conclusions

① $f(x)$ is continuous except at $x = 1$ and $x = +1$

② $g(x)$ is continuous except at $x = 2$ and $x = -2$

③ $h(x)$ is continuous except at $x = 1$, $x = -1$, $x = \sqrt{3}$ and $x = -\sqrt{3}$.