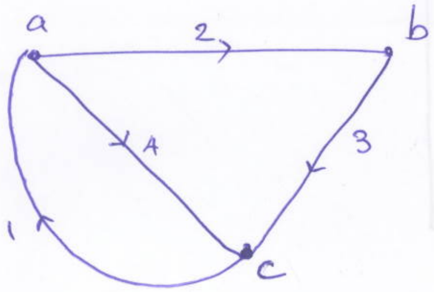
EE101 Assignment - 2.

Solutions

1.(i)



NB: In the graph, do not mark the elements using symbols.

(ii)(a) Conditions for a sub-graph to be a tree of a graph

(i) All nodes should be present in the tree

(ii) There should not be any closed path

(iii) There should be a path from any node to another (ie. All nodes should be connected)

Based on these 12, 23, 34, and 13 are trees.

123 and 234 are not trees.

(b) Condition to be circuit

(i) There should be a closed path between the nodes in the given sub-tree.

Based on this, 123, 234 and 14 are circuits.

124 is not a circuit.

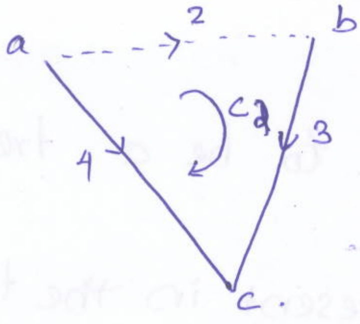
(iii) Complete incidence matrix

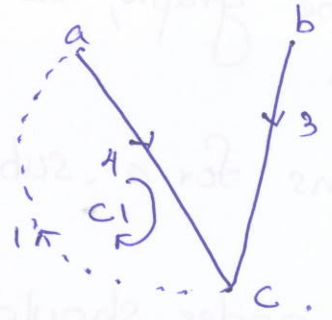
$$A_c = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & +1 & 0 & +1 \\ 0 & -1 & +1 & 0 \\ +1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Reduced incidence matrix with c as reference.

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0. \end{bmatrix}$$

(iv)





$$B_f = \begin{array}{c} \\ c1 \\ c2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

(v) $$AB_f^T = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0+0\;1 & 0+1+0-1 \\ 0+0+0+0 & 0-1+1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence verified.

(vi) KCL

$$A i_b = 0 .$$

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{b_1} \\ i_{b_2} \\ i_{b_3} \\ i_{b_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-i_{b_1} + i_{b_2} + i_{b_4} = 0 .$$

$$\boxed{i_{b_2} + i_{b_4} = i_{b_1}} \quad — (1) .$$

$$-i_{b_2} + i_{b_3} = 0 .$$

$$\boxed{i_{b_2} = i_{b_3}} \quad — (2) .$$

(vii) KVL.

$$B_f V_B = 0 .$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = 0 .$$

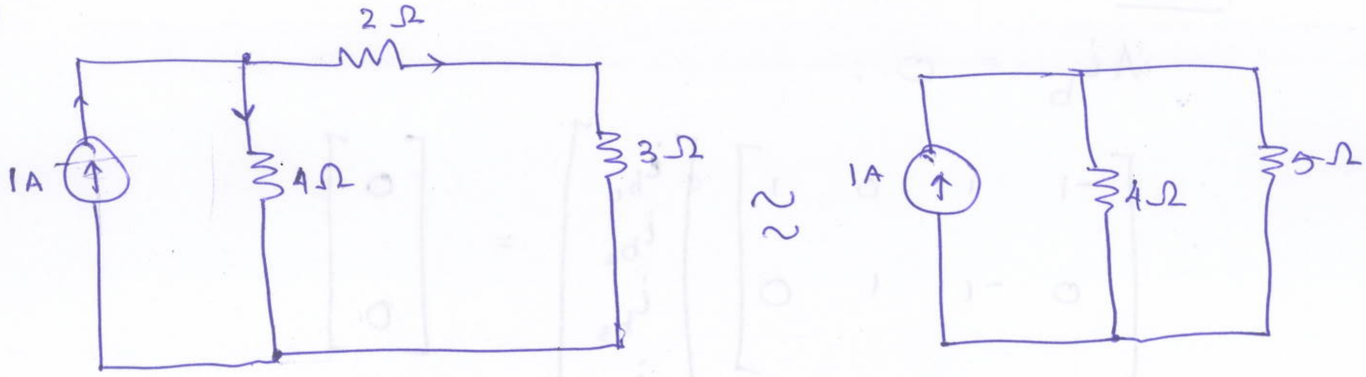
$$V_1 + V_4 = 0$$

$$\boxed{V_1 = -V_4} \quad — (1) .$$

$$V_2 + V_3 - V_4 = 0 .$$

$$\boxed{V_2 + V_3 = V_4} \quad — (2) .$$

(viii)



By current division rule, current through 4Ω resistor $= \dfrac{1 \cdot 5}{4+5} = \dfrac{5}{9}$ A.

∴ Current through 2Ω and 3Ω resistors

$$= 1 - \frac{5}{9} = \frac{4}{9} \text{ A.}$$

Voltage across 2Ω resistor $= \dfrac{4}{9} \times 2 = \dfrac{8}{9}$ V.

Voltage across 3Ω resistor $= \dfrac{4}{9} \times 3 = \dfrac{4}{3}$ V.

Voltage across 4Ω resistor $= \dfrac{5}{9} \times 4 = \dfrac{20}{9}$ V.

Votage across current source = − Voltage across 4Ω resisor $= -\dfrac{20}{9}$ V.

(ix) <u>Tellegens theorem.</u>

According to Tellegen's theorem $\sum_{k=1}^{b} V_k i_k = 0$.

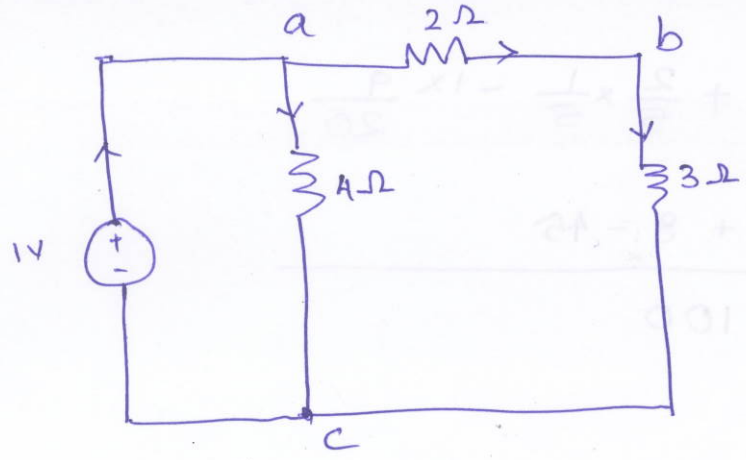
b ≈ Total number of branches

$$\frac{20}{9} \times \frac{5}{9} + \frac{4}{9} \times \frac{8}{9} + \frac{4}{9} \times \frac{4}{3} - 1 \times \frac{20}{9}$$

$$= \frac{100 + 32 + 48 - 20 \times 9}{9 \times 9} = 0$$

Hence verified. (4).

(x)



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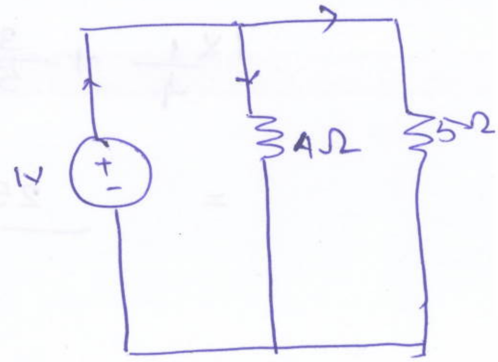


Voltage across 4Ω resistor = voltage from voltage source = 1V.

Voltage across the voltage source in the assumed current direction = -1V.

Voltage across 5V resistor = voltage from voltage source = 1V.

Now applying voltage division rule, voltage across 2Ω resistor = $\frac{1 \times 2}{5} = \frac{2}{5}$ V.

Voltage across 3Ω resistor = $1 \times \frac{3}{5} = \frac{3}{5}$ V.

Current through 4Ω resistor = $\frac{V}{R} = \frac{1}{4}$ A.

Current through 2Ω and 3Ω resistor = $\frac{1}{5}$ A.

Current through the voltage source,

Apply KCL at node a, $\frac{1}{5} + \frac{1}{4}$

$= \frac{9}{20}$ A.

Tellegens theorem.

$$\sum_{k=1}^{b} v_k i_k = 0.$$

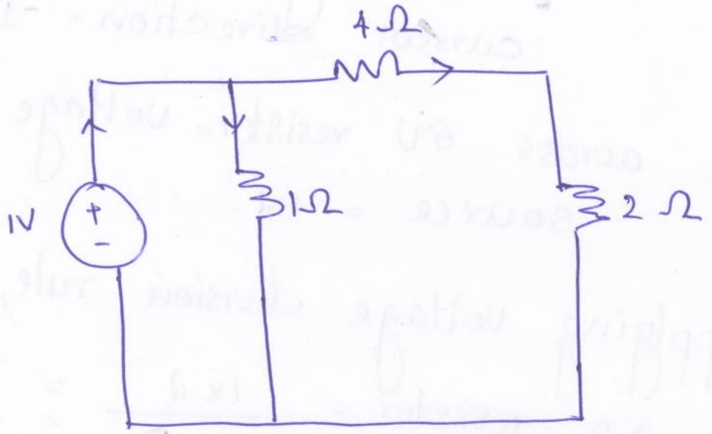
$$1\times\frac{1}{4} + \frac{3}{5}\times\frac{1}{5} + \frac{2}{5}\times\frac{1}{5} - 1\times\frac{9}{20}.$$

$$= \frac{25+12+8-45}{100}$$

$$= \underline{0}$$

Hence verified.

(xi)



Voltage across 1Ω = 1V

Current through 1Ω = $1/1 = 1$ A.

Voltage across 4Ω = $\frac{4}{6}$ V.

Current through 4Ω = $\frac{1}{6}$ A.

Voltage across 2Ω = $\frac{1\cdot 2}{6} = \frac{2}{6}$ V.

current through 2Ω = $\frac{1}{6}$ A.

Voltage across voltage source is the assumed current direction = -1 V.

Current through voltage source = $1 + \frac{1}{6} = \frac{7}{6}$ A

Tellegen's theorem

$$\sum^{b} V_k i_k = 0$$

$$1\times1 + \frac{4}{6}\times\frac{1}{6} + \frac{2}{6}\times\frac{1}{6} - 1\times\frac{7}{6}$$

$$= \frac{36+4+2-42}{36}$$

$$= \underline{\underline{0}}$$

Hence verified.

(xiii) Let's take voltage set from (viii)

$V_2 = \frac{8}{9}V$, $V_3 = \frac{4}{3}V$, $V_1 = -\frac{20}{9}V$, $V_4 = \frac{20}{9}V$.

Let's take current set from (x)

$i_4 = \frac{1}{4}A$, $i_2 = \frac{1}{5}A$, $i_3 = \frac{1}{5}A$, $i_1 = \frac{9}{20}A$.

Now applying Tellegen's theorem

$$\sum_{k=1}^{b} v_k i_k$$

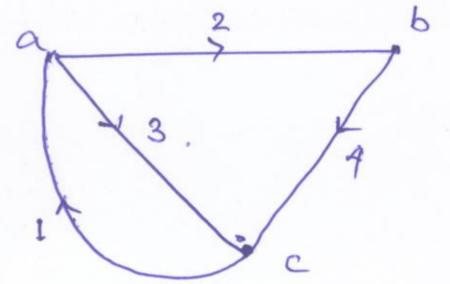
$$= -\frac{20}{9}\cdot\frac{9}{20} + \frac{8}{9}\cdot\frac{1}{5} + \frac{4}{3}\cdot\frac{1}{5} + \frac{20}{9}\cdot\frac{1}{4}$$

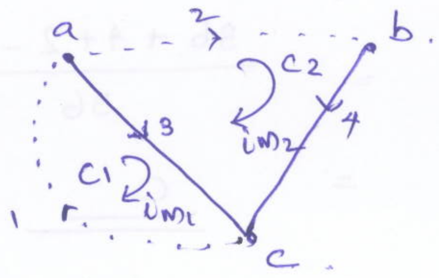
$$= -1 + \frac{8}{45} + \frac{4}{15} + \frac{20}{36}.$$

$$= -1+1 = \underline{\underline{0}}$$

Hence verified.

Q2. Oriented graph. Selected tree.





$$A_c = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & +1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ +1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Considering c as the reference node,

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$B_f = \begin{array}{c} \\ c_1 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

Node voltages : V_a, V_b, V_c.

Branch voltages : V_1, V_2, V_3, V_4.

Branch currents : i_1, i_2, i_3, i_4

Mesh currents : i_{m_1}, i_{m_2}.

Node Analysis.

Applying KCL; $Ai_b = 0$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Current in Branch 1 is same as the current from the current source.

$\therefore i_1 = I$.

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since now we know i_1 is a constant, let's separate from unknowns and bring to right hand side

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} I.$$

Now write branch currents in terms of branch voltages using the V-I relationship for components.

$$\begin{bmatrix} i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{L}D^{-1} & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & CD \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

$$\boxed{D^{-1} = \int_0^t dt}$$

$$\boxed{D = \frac{d}{dt}}$$

Now write branch voltages in terms node voltages u
g indirect KVL.

$$V_b = A^T V_n.$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b . \end{bmatrix}$$

$$V_1 = -V_a.$$

$$\begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b . \end{bmatrix}$$

Now substing these

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{L}D^{-1} & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & CD \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{L}D^{-1} & G & 0 \\ -\frac{1}{L}D^{-1} & 0 & CD \end{bmatrix} \begin{bmatrix} V_a - V_b \\ V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} I.$$

$$= \begin{bmatrix} \frac{1}{L}D^{-1}(V_a - V_b) + G V_a \\ -\frac{1}{L}D^{-1}(V_a - V_b) + CD V_b . \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} I$$

$$\boxed{\begin{aligned} &\frac{1}{L}D^{-1}(V_a - V_b) + G V_a = I. && \text{—(1)} \\ &+\frac{1}{L}D^{-1}(V_a - V_b) = CD V_b. && \text{—(2)} \end{aligned}}$$

Mesh Analysis.

Applying KVL.

$B_f V_b = 0$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to eliminate v_1 since it corresponds to current source. From the matrix

$v_1 + V_3 = 0$

$V_1 = -V_3$.

$$\Rightarrow \begin{bmatrix} \cancel{0 \quad 1 \quad 0} \\ 1 \quad -1 \quad 1 \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = 0$$

Now write v-I relationships for the branches

$$\begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} LD & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & \frac{1}{C} D^{-1} \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Now write branch currents in terms of mesh currents using indirect KCL.

$i_b = B^T i_m$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{m_1} \\ i_{m_2} \end{bmatrix}$$

$i_1 = I$ for current source.

$$\begin{bmatrix} i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{m_1} \\ i_{m_2} \end{bmatrix}$$

$i_{m_1} = I$.

$$\begin{bmatrix} i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} I + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} i_{m_2}$$

Substituting in

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} LD & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & \frac{1}{C}D^{-1} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} I + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} i_{m_2} \right) = 0$$

$$= \begin{bmatrix} LD & -R & \frac{1}{C}D^{-1} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} I + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} i_{m_2} \right) = 0$$

$$\Rightarrow -RI + LDi_{m_2} + Ri_{m_2} + \frac{1}{C}D^{-1}i_{m_2} = 0$$

$$\Rightarrow \boxed{(i_{m_2} - I)R + LDi_{m_2} + \frac{1}{C}D^{-1}i_{m_2} = 0}$$

Q3 Current through capacitor.

From figure current through capacitor, $i_c = i_4$

current through the inductor, $i_L = i_2$

From $Ai_b = 0$, we have.

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} I.$$

$\Rightarrow i_2 = i_4$.

$$\boxed{\therefore \ i_c = i_L}$$

Voltage across inductor.

From the (2) node analysis.

$$\frac{1}{L} D^{-1}(V_a - V_b) = CDV_b.$$

$V_a - V_b$ is nothing but voltage across inductor, V_L

V_b is nothing but voltage across the capacitor,

$$\boxed{\therefore \ \frac{1}{L} D^{-1} V_L = CDV_c}$$

or.

We already have $i_c = i_L$.

Using the V-I relationships for capacitor and inductor

$$\boxed{C\frac{dV_c}{dt} = \frac{1}{L}\int_0^t V_L \, dt.}$$

current through resistor.

From Mesh analysis.

$$\left(i_{m_2} - I\right) R + LDi_{m_2} + \frac{1}{C} D^{-1} i_{m_2} = 0.$$

$\left(I_{m_2} - I\right)$ is nothing but current through resistor

I_R in the opposite direction of assumed current dir.

$\frac{1}{C} D^{-1} i_{m_2}$ is nothing but voltage across the capacitor

V_c.

Again $i_{m_2} = i_L$, current through the inductor

$$\therefore -I_R R + LDi_L + V_c = 0.$$

$$\boxed{I_R = \frac{1}{R}\left[LDi_L + V_c\right]}$$