1 Field Axioms

Let $\mathbb{F}$ denote a set on which two binary operators[1] $\oplus$ and $\odot$ are defined, that is, two functions $\oplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\odot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$.

Let $x, y \in \mathbb{F}$, then we denote the result of the $\oplus$ operator as $x \oplus y$, and the $\odot$ operator as $x \odot y$. Unless it is ambiguous, we denote $x \odot y$ as simply xy.

For the following axioms[2], let $x, y, z \in \mathbb{F}$.

Axiom 1.1 *Commutative laws:* $x \oplus y = y \oplus x$ and $xy = yx$.

Axiom 1.2 *Associative laws:* $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and $x(yz) = (xy)z$.

Axiom 1.3 *Distributive laws:* $x(y \oplus z) = xy \oplus xz$.

Axiom 1.4 *Identity elements:* There exist two (distinct) real numbers 0 and 1, such that for every $x \in \mathbb{F}$, $x \oplus 0 = x$ and $x \odot 1 = x$.

Axiom 1.5 *Negatives:* For every $x \in \mathbb{F}$ there exists $y \in \mathbb{F}$ such that $x \oplus y = 0$.

Axiom 1.6 *Reciprocals:* For every $x \in \mathbb{F} \backslash \{0\}$ (*i.e.* $x \neq 0$), there exists $y \in \mathbb{F} \backslash \{0\}$ such that $xy = 1$. We denote such y by x^{-1}, called the *reciprocal* of x.

Theorem 1. *Let $a, b, c, d \in \mathbb{F}$.*

Property 1.1: If $a \oplus b = a \oplus c$ then $b = c$ *(*i.e. *every element is* cancellable*).*

Property 1.2: There exists a unique $x \in \mathbb{F}$ such that $a \oplus x = b$. (We denote x by $b - a$. Specifically, $0 - a$ is $-a$, the negative *of a).*

Property 1.3: $b - a = b \oplus (-a)$.

Property 1.4: $-(-a) = a$ *(*i.e. *the function $\mathbb{F} \to \mathbb{F}, x \mapsto -x$ is an* involution*).*

Property 1.5: $0 \odot a = a \odot 0 = 0$ *(*i.e. *0 is* absorbent*).*

Property 1.6: $a(b - c) = ab - ac$.

Property 1.7: If $ab = ac$ and $a \neq 0$ then $b = c$.

[1] Also called *internal composition laws*.

[2] An axiom is a fact, assumed as true. You do **not** have to try to prove these axioms!

Property 1.8: *If $a \neq 0$ then there exists a unique $x \in \mathbb{F}$ such that $ax = b$ (we denote this x by b/a or $\frac{b}{a}$. Specifically, $1/a$ is a^{-1}, the* reciprocal *of a.).*

Property 1.9: *If $a \neq 0$ then $b/a = b \odot a^{-1}$.*

Property 1.10: *If $a \neq 0$ then $(a^{-1})^{-1} = a$.*

Property 1.11: *If $ab = 0$ then $a = 0$ or $b = 0$.*

Property 1.12: $(-a)b = -(ab)$ *and* $(-a)(-b) = ab$.

Property 1.13: $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad \oplus bc}{bd}$ *if $b \neq 0$ and $d \neq 0$.*

Property 1.14: $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$ *if $b \neq 0$ and $d \neq 0$.*

Property 1.15: $\frac{a}{b}/\frac{c}{d} = \frac{ac}{bd}$ *if $b \neq 0$, $c \neq 0$, and $d \neq 0$.*

---

2 Problems

Problem 1. *Prove* at least *12 properties among the 15 exposed in Theorem 1.*

Problem 2. *Verify*[3] *whether all the field axioms and all the properties of Theorem 1 hold for:*

1. *$\mathbb{F} = \mathbb{Z}$, the set of integers, $\oplus = +$, and $\odot = \cdot$, the usual addition and multiplication.*
2. *$\mathbb{F} = \mathbb{Q}$, the set of rational numbers, $\oplus = +$, and $\odot = \cdot$, the usual addition and multiplication.*
3. *Let n be a positive integer greater than 1 and let $\mathbb{F} = \{0, 1, 2, \ldots, n-1\}$. For $x, y \in \mathbb{F}$, we define $x \oplus y = (x + y) \mod n$ and $x \odot y = (xy) \mod n$. Furthermore, what is -1 in this case and check if $(-1)(-1) = 1$.*
4. *(**Bonus**) Let S be a nonempty set, and let $\mathbb{F}$ be such that $A \in \mathbb{F}$ if and only if $A \subseteq S$, that is, $\mathbb{F}$ contains all the subsets of S including $\varnothing$ and S ($\mathbb{F}$ is called the* power set of S*, set of all subsets of S). Define $\oplus = \cup$ and $\odot = \cap$, the union and the intersection operators, respectively.*

---

[3] When you think one of 6 axioms is not true for this particular $\mathbb{F}$, you can *try to* prove this with a counter-example.