# First Mid Term Exam @ MEC

**When:** $08^{th}$ of September 2014

**What:** MA 101 : Calculus and Introduction to Analysis.

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**Problem 5** We have $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$

verifying $f(100) = g(0) = 1$ (1)

and $\forall x, y \in \mathbb{R},\ g(x-y) = g(x)g(y) + f(x)f(y)$. (2)

5.a) Let $x \in \mathbb{R}$. If $y = x \in \mathbb{R}$, the given identity (2) becomes $g(x-x) = g(x)g(x) + f(x)f(x)$

So $g(0) = (g(x))^2 + (f(x))^2$

But $g(0) = 1$ is given, so $1 = (g(x))^2 + (f(x))^2$

Conclusion $\boxed{\forall x \in \mathbb{R},\ 1 = (g(x))^2 + (f(x))^2.}$

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5.b) By applying the previous result (a)

* With $x = 0$ : $1 = (g(0))^2 + (f(0))^2$

or $g(0) = 1$ is given, so $(f(0))^2 = 1 - 1^2 = 1 - 1 = 0$

so $f(0)^2 = 0$, so $\boxed{f(0) = 0}$.

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* And $x = 100$, similarly: $1 = (g(100))^2 + (f(100))^2$

but $f(100) = 1$ is given, so $g(100)^2 = 0$

Hence $\boxed{g(100) = 0}$.

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5.c) Let $x \in \mathbb{R}$.

Thanks to the given identity (2),

$$g(-x) = g(0 - x)$$
$$= g(0)\, g(x) + f(0) f(x)$$

But $g(0) = 1$ was given, and $f(0) = 0$ was proved in 5.b),

So $g(-x) = 1 \cdot g(x) + 0 \cdot f(x)$
$= g(x)$ ie $g(-x) = g(x)$

4/4 Conclusion $\boxed{\text{For any } x \in \mathbb{R},\ g(x) = g(-x).}$

Remark. $g$ is an even function.

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5.d) Let $x \in \mathbb{R}$. Similarly, with (2) we have

$g(100 - x) = g(100) \cdot g(x) + f(100) \cdot f(x)$

But $g(100) = 0$ by 5.b) and $f(100) = 0$ by (1).

So $g(100 - x) = 0 \cdot g(x) + 1 \cdot f(x) = f(x)$

Conclusion: $\boxed{\forall x \in \mathbb{R},\ g(100-x) = f(x).}$

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Last Question 1/1

*) $f$ and $g$ are similar to sine and cosine.

*) We can be more precise, saying that $f$ and $g$ could be : $f(x) = \sin\left(\frac{\pi x}{200}\right)$

$g(x) = \cos\left(\frac{\pi x}{200}\right)$.

+ 1/1 Bonus for cleanliness and presentation.