Week 7 Assignment

For any function denoted as $f(x)$, y(x) *or* $u_i(x)$ (for any $i \in \mathbb{N}^*$) that represents a mapping $\mathbb{R} \to \mathbb{R}$, it will be assumed here and henceforth that the function is continuous and differentiable within our domain of interest, unless stated or implied otherwise.

Section A: Homework

Different questions are of different levels of depth and intensity and will not take equal time to answer. *Give to each question its due attention*. There are no "marks" to these questions. Mark them yourself on the degree of advance in the level of your understanding (and you if feel, you may share that marking with us).

A.1. The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When l = 12 cm and w = 5 cm, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

A.2 A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. See fig. 1.

a. How fast is the top of the ladder sliding down the wall then?

b. At what rate is the area of the triangle formed by the ladder, wall and ground changing then?

c. At what rate is the angle θ between the ladder and the ground changing then?

A.3 Show that the linearization of f(x) = $(1 + x)^k$ at x = 0 is L(x) = (1 + kx).

Using this approximation, estimate:

(a) $(1.0002)^{50}$, and

(b) $(1.009)^{1/3}$.

A.4 Evaluate the differential dy in terms of the differential dx for the relation

$$xy^2 - 4x^{3/2} - y = 0$$

A.5 The function f(x) expressed below changes value when x changes from x_0 to x_0 + dx.

$f(x) = 2x^2 + 4x - 3$, x_0 = -1, dx = 0.1

Find

a) the change $\Delta f = f(x_0 + dx) - f(x_0)$;

b) the value of the estimate $df = f'(x_0)dx$; and

c) the approximation error $|\Delta f - df|$.

A.6 Workout the proof of the first derivative theorem for a minimum, i.e. if a function f(x) is defined and continuous within a domain [a, b] $\subset \mathbb{R}$ and differentiable within (a, b), and acquires a local minimum at an interior point 'c' of that domain, then f'(c) = 0.

A.7 Find the extreme values of the function below, and their locations:

$$y(x) = \frac{x}{x^2+1}$$

A.8 Find the derivative at each critical point and determine the local extreme values of the function below:

$$y(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$

First determine using limits from left and right if the derivative exists at 1.

A.9 Write out logically the proof of Rolle's Theorem assuming the Extreme Value Theorem and the First Derivative Theorem as starting premises. *Do not copy it out from a textbook*, however, you may first read your notes or a text book proof before writing out yourself.

A.10 Find the value or values of 'c' in the conclusion relation of the Mean Value Theorem

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$, where *a, b* and *c* have their usual meanings,

for the functions and intervals below:

a) $f(x) = x + \frac{1}{x}, \text{ interval } \left[\frac{1}{2}, 2\right]$

b) $f(x) = \sqrt{x-1}, \text{ interval } [1,3]$

A.11 The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at x = 0 and x = 1 and differentiable on (0, 1), but its derivative on (0, 1) is never zero. How can this be? Does it not seem to contradict Rolle's Theorem? Give reasons for your answer.

A.12 For what values of a, m and b does the function

$$f(x)=\begin{cases}3, & x = 0\\ -x^2+3x+a, & 0 < x < 1\\ mx+b, & 1 \le x \le 2\end{cases}$$

satisfy the hypothesis of the Mean Value Theorem on the interval [0, 2]?

A.13 Plot the zeros of the given polynomial on a line (approximately) together with the zeros of its first derivative:

$$y(x)=x^3-33x^2+216x=x(x-9)(x-24)$$

Do you see any pattern?

A.14 Use Rolle's Theorem to prove that between every two zeros of

$$x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0$$

there lies a zero of

$$nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots\ldots + a_1.$$

A.15 The geometric mean of two positive numbers a and b is $(ab)^{1/2}$. Show that the value of 'c' in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = (ab)^{1/2}$.

A.16 We have seen that if the first derivatives of any two functions are identical, then they differ by a constant. What can you say about two functions whose second derivatives are identical? Logically expound the reasons for your conclusion. Demonstrate with a simple example.

A.17 Figure 2 shows the displacement s of a body moving as a function of time, $s = f(t)$, back and forth on a coordinate line. (a) When is the body moving away from the origin, and towards the origin? Approximately when are: (b) velocity equal to zero? (c) acceleration equal to zero? (d) when are the accelerations positive and negative? Give reasons.

A.18 Suppose the derivative of the function $y = f(x)$ is given by

$$y'(x) = (x-1)^2(x-2)(x-4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

A.19 Refer fig. 3. The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

A.20 Figure 4 shows the flight path of an aircraft flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the aircraft. Assume that the landing trajectory of the aircraft is provided by the graph of the cubic polynomial function

$$y = ax^3 + bx^2 + cx + d$$

where $y(-L) = H$ and $y(0) = 0$.

a) What is dy/dx at $x = 0$?

b) What is dy/dx at x = -L?

Interpret answers to (a) and (b) from the figure.

c) Use the above two values from (a) and (b) together with y(0) = 0 and y(-L) = H, to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right]$$

A.21 You operate a tour service that offers the following rates:

Rs. 2000 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by Rs. 20.

It costs Rs. 60,000 (a fixed cost) plus Rs. 320 per person to conduct the tour. How many people does it take to maximize your profit?

A.22 For the expression below, evaluate the limit using L'Hopital's rule. Then also evaluate using an alternate method you know and cross-check.

$$\lim_{x\to\infty}\frac{2x^2+3x}{x^3+x+1}$$

A.23 Use L'Hopital's rule to find the limit of

$$\lim_{r\to 1}\frac{a(r^n-1)}{r-1},\quad n\in\mathbb{N}^*$$

A.24 Explore evaluation of the expression below using both L'Hopital's rule as well as alternative methods

$$\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{\sin x}}$$

A.25 Find the Taylor series generated by f(x) at x = a = -2, when

$f(x) = x^4 + x^2 + 1$

A.26 Take the Maclaurin series expansion of e^x and the value of e as 2.732. Evaluate the error for estimating e^x at x = 1 by considering upto the 3rd term, i.e. 2nd derivative, using hand calculation. Now find the error upper bound by Remainder Formula (RF). How does the actual error compare with upper bound? Next, estimate e^x at x = -1 at the 4th term, i.e. 3rd derivative, by hand. Find upper bound using RF. How do they compare?

A.27 a) Use Taylor's formula with n = 2 to find the quadratic approximation of $f(x) = (1 + x)^k$ at x = 0 (k a constant)

b) If k = 3, for approximately what values of x in the interval [0, 1] will the error in the quadratic approximation be less than 1/100?

c) Again, for k = 3, for approximately what values of x in the interval [0, 1] will the error in the quadratic expansion be less than 1/10?

d) Do you see any relation between proximity of the point of interest to the central point in a Taylor expansion (where derivatives are known), and the error in an approximation (order of approximation remaining unchanged)?

Section B: Tutorial for class workout

B.1 Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate. [Hint: By "picks up moisture" is meant that its mass, interpreted as volume, increases].

B.2 Show that the normal line at any point of the circle

$$x^2 + y^2 = a^2$$

passes through the origin (Hint: try to express the line equation in the form y = mx + c, where c is the intercept on the y-axis).

B.3 Supertankers off-load oil at a docking facility 4 km. offshore. The nearest refinery is located on the coast, 9 km down from the coastal point closest to the docking facility (point A, see fig. 5). A pipeline must be constructed connecting the docking facility with the refinery. The pipeline costs Rs. 3 Crores per mile if constructed underwater, and Rs. 2 Crores per mile overland.

a) Find an optimal point B on the coast, between the closest point and the refinery, that minimizes the cost of construction

b) The cost of underwater construction is expected to increase, while the overland costs are unlikely to change. Then at what cost of the former does it become optimal to construct the underwater pipeline directly to point A?

B.4 Suppose that f'' is continuous on *[a, b]* and that f has three zeros in that interval. Show that f'' has at least one zero in *(a, b)*. *Generalize this result*. Also use this to prove that a cubic polynomial cannot have more than three real zeroes.

B.5 Use the equation

$$f(x) = f(a) + f'(\mathrm{a})(\mathrm{x}-\mathrm{a}) + \frac{f''(c_2)}{2}(x-a)^2$$

to establish the following test.

Let f(x) have continuous first and second derivatives and suppose that f′(a) = 0. Then

a. *f(x)* has a local maximum at *a* if $f'' \le 0$ throughout an interval whose interior contains *a*

b. *f(x)* has a local minimum at *a* if $f'' \ge 0$ throughout an interval whose interior contains *a*

B.6 Use the Taylor expansions of e^x, sin x and cos x to derive the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Section C: Additional Exercises

C.1 The positions of two particles oscillating on the s-axis are given as functions of t by

$s_1 = \sin(t)$ and $s_2 = \sin(t + \pi/3)$, with s_1, s_2 in meters and t in seconds.

a. At what time/s in the interval $0 \leq t \leq 2\pi$ do the particles meet?

b. What is the farthest apart that the particles ever get?

c. When in the interval $0 \leq t \leq 2\pi$ is the distance between the particles changing the fastest?

C.2 Obtain the Taylor expansion of $(1 + x)^m$. This is the Binomial series. Show that it converges for $|x| < 1$.

Figures follow in next page.

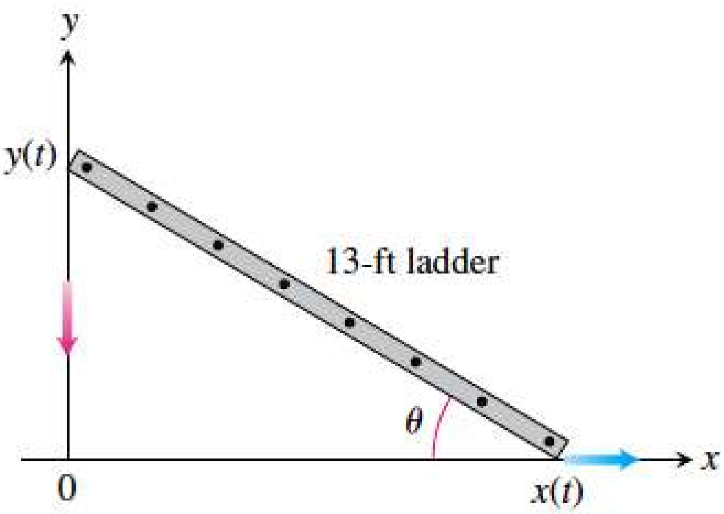

Fig.1 – Relates to problem A.2.

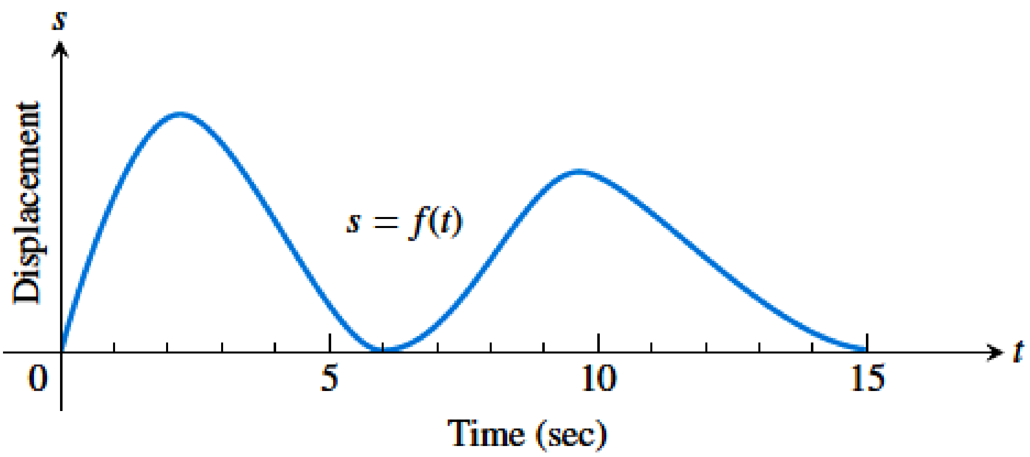

Fig. 2. – Problem A.17.

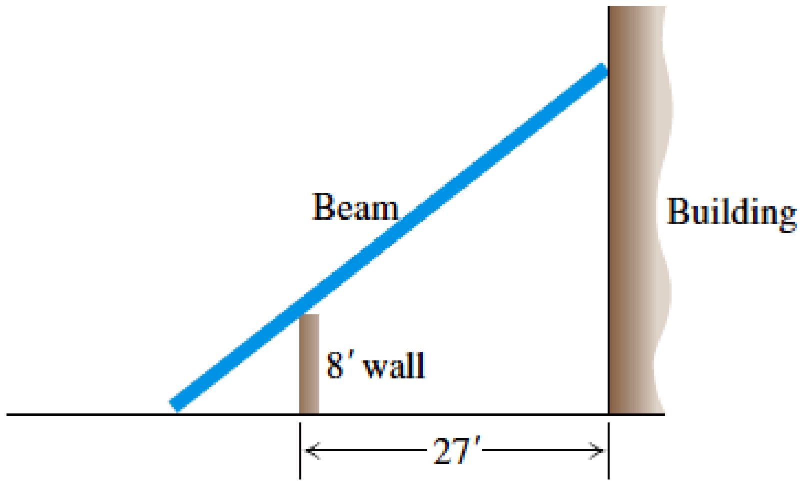

Fig. 3. – Problem A.19.

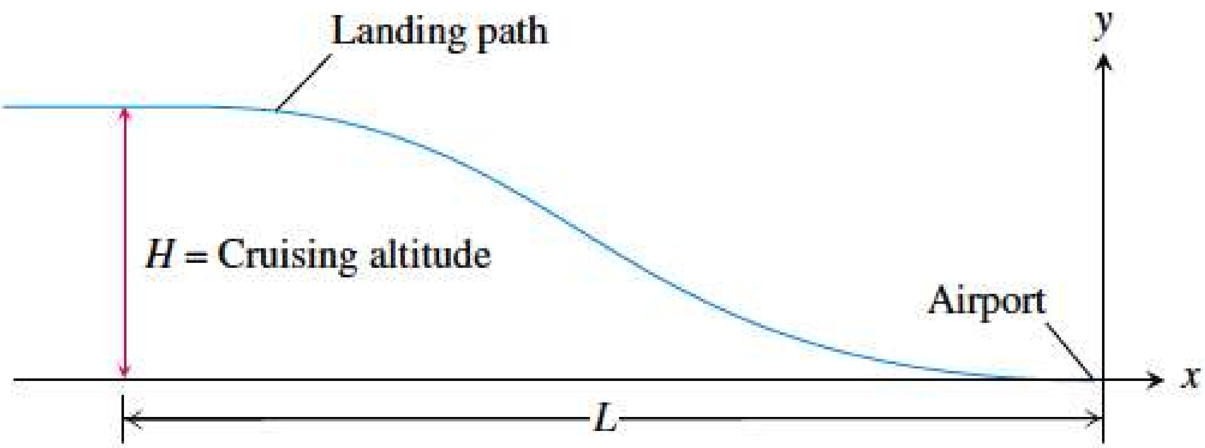

Fig. 4. – Relates to problem A.20

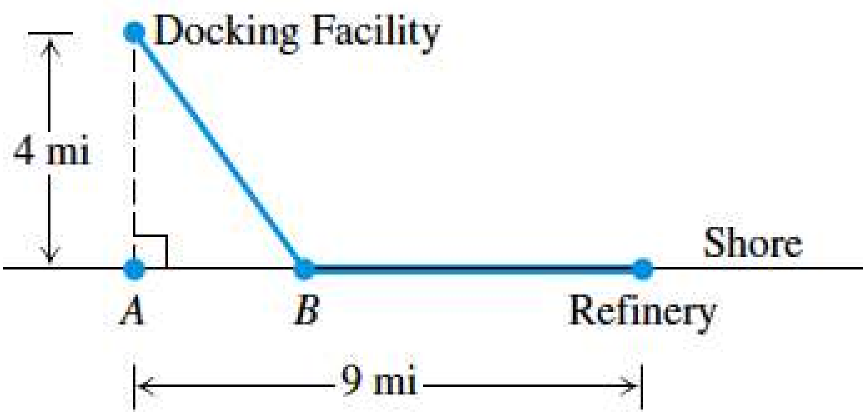

Fig. 5. – Problem B.3.