Remarks on grading

Here is the list of who corrected which groups :

Prof. Lilian: A1 and B1,

Prof. Jai: A2 and B2,

Prof. Satya: A3 and B3,

Prof. Vijay: A4,

Prof. Arya: B4.

Marking scheme And regarding the grading, the following scheme was **suggested**:

- The entire quiz was on 5 points (with grades from 0/5 to 5/5),
- 3 points for question 1 (1 points for justifying differentiability and computing $f'(x)$, 1 point for the condition on a, b, c or δ, γ, α for the discriminant Δ to be non-negative, and 1 point for the good expression of the two *critical points* x_1, x_2),
- 1 point for question 2 (0.5 point for computing $f''(x)$ and its values at x_1, x_2, 0.5 point if conclusion says which is minimum which is maximum),
- 1 point for question 3 (0.5 point for applying question 1, checking that $\Delta > 0$, 0.5 point for applying question 2, saying which is max/min).

Exercise for the B group

Let $a, b, c, d \in \mathbb{R}$ be four *constants*, with $a \neq 0$. We define $f : \mathbb{R} \to \mathbb{R}, x \mapsto ax^3 + bx^2 + cx + d$, a cubic polynomial function.

Question 1) *(3 points)* Use the **First Derivative test** to determine the point(s) where f can have a local extremum (either local maximum or local minimum).
You should find a condition on a, b, c, d for the local extrema to exist, and their (possible) location should also depend on the coefficients a, b, c, d.

Question 2) *(1 points)* Use the **Second Derivative test** to determine which of these extremum is a *local* maximum and which is a *local* minimum.

Question 3) *(1 points)* Apply this to find the local maximum and local minimum for this function $f : x \mapsto 2x^3 + 4x^2 - 5x - 1$ (if they exist).

Solution for A group

Just to check: The function f is indeed a polynomial, of degree exactly 3 because we ask $a \neq 0$ (so it is cubic).

Answer:

1) f is differentiable on $\mathbb{R}$, and $f'(x) = 3ax^2 + 2bx + c$ for every $x \in \mathbb{R}$. We can also compute $f''(x) = 6ax + 2b$.

   Thanks to the *Extreme Value Theorem* (for f continuous and differentiable on $\mathbb{R}$), we know that at one local extremum x_0, we have $f'(x_0) = 0$.

   Assume x to be a local extremum (either maximum or minimum). It satisfies

$$3ax^2 + 2bx + c = 0 \tag{0.1}$$

   But solving such a polynomial equation of degree two is something we can easily do. Here, the discriminant is $\Delta = (2b)^2 - 4(3a)c = 4b^2 - 12ac$. As usual, there is three cases regarding its sign:

   - Assume that $\Delta > 0$. We have two real roots for the equation 0.1 : $x_{1,2} = \frac{-(2b) \pm \sqrt{\Delta}}{2(3a)} = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$. Therefore, there can be two (different) extrema x_1 and x_2.
   - Assume that $\Delta = 0$. We have one unique real root for the equation 0.1 : $x_0 = \frac{-(2b)}{2(3a)} = -\frac{b}{3a}$. Therefore, there can be one extremum x_0.
   - Assume that $\Delta < 0$. We have no real root for the equation 0.1, therefore the function f cannot have any extremum.

2) Thanks to the previous question, we can know use the test with second derivative, because f is continuous, twice differentiable, of first and second derivatives continuous (here, f is said to be of class $\mathcal{C}^1$).

   - If $\Delta > 0$. There can be two (different) extrema $x_1 = \frac{-b - \sqrt{b^2 - 3ac}}{3a}$ and $x_2 = \frac{-b + \sqrt{b^2 - 3ac}}{3a}$.
     - At x_1, the double derivative f'' will take the value $f''(x_1) = 6ax_1 + 2b = 6a\frac{-b - \sqrt{b^2 - 3ac}}{3a} + 2b = -2\sqrt{b^2 - 3ac} < 0$, so we can conclude that the possible extremum x_1 is indeed an extremum, and it is a (*local*) maximum.
     - At x_2, the double derivative f'' will take the value $f''(x_2) = 6ax_2 + 2b = 6a\frac{-b + \sqrt{b^2 - 3ac}}{3a} + 2b = +2\sqrt{b^2 - 3ac} > 0$, so we can conclude that the possible extremum x_2 is indeed an extremum, and it is a (*local*) maximum.
   - If $\Delta = 0$. There can be one extremum x_0. At x_0, the double derivative f'' will take the value $f''(x_0) = 6ax_0 + 2b = 2(3a(-\frac{b}{3a}) + b) = 0$, but in this case we cannot conclude of the nature of this point x_0.
   - If $\Delta < 0$. The function f have no extremum.

3) With $f(x) = 2x^3 + 4x^2 - 5x - 1$, we have $a = 2, b = 4, c = -5, d = -1$. Here, $\Delta = 4b^2 - 12ac = 4 \times 16 + 12 \times 2 \times 5 = 64 + 120 = 184 = 4 \times 46 > 0$. So, there is two real solutions for $f'(x) = 0$: $x_{1,2} = \frac{-2b \pm \sqrt{\Delta}}{6a} = \frac{-4 \pm \sqrt{46}}{6}$, ie $x_1 = \frac{-4 - \sqrt{46}}{6} \simeq -1.79705$ and $x_2 = \frac{-4 + \sqrt{46}}{6} \simeq 0.46372$. The *local* minimum is at x_2 and the *local* maximum is at x_1. This can clearly be seen on the Figure 2.

4) (**Bonus**) Here is a quick graphic of this function f.

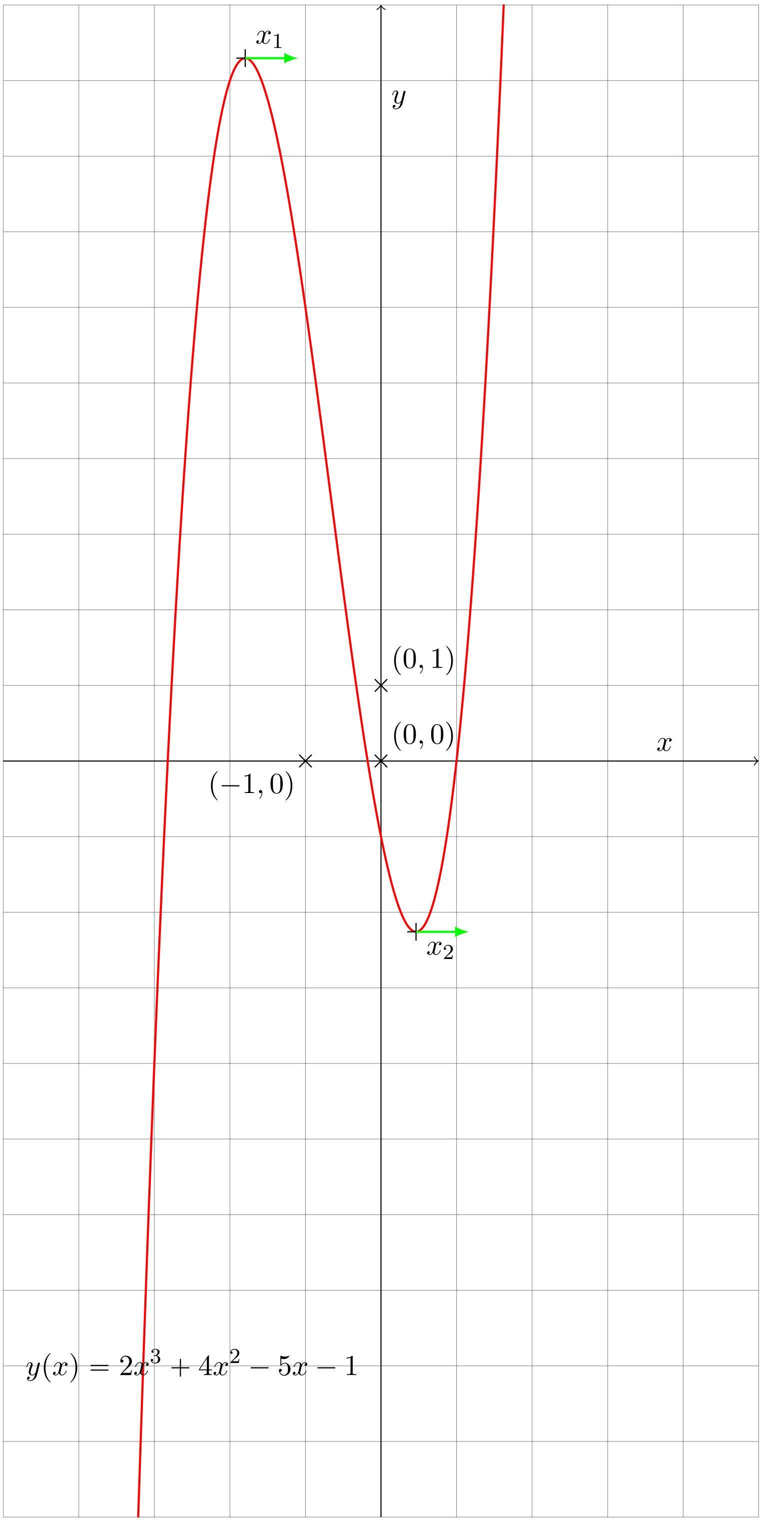

Figure 1: The function $f : x \mapsto 2x^3 + 4x^2 - 5x - 1$ on $[-5, 5]$

Exercise for the A group

Let $\gamma, \delta, \alpha, \beta \in \mathbb{R}$ be four *constants*, with $\gamma \neq 0$. And we define $f : \mathbb{R} \to \mathbb{R}, x \mapsto \gamma x^3 + \delta x^2 + \alpha x + \beta$, a cubic polynomial function.

Question 1) *(3 points)* Use the **First Derivative test** to determine the point(s) where f can have a local extremum (either local maximum or local minimum).
You should find a condition on γ, δ, α, β for the local extrema to exist, and their (possible) location should also depend on the coefficients γ, δ, α, β.

Question 2) *(1 points)* Use the **Second Derivative test** to determine which of these extremum is a *local* maximum and which is a *local* minimum.

Question 3) *(1 points)* Apply this to find the local maximum and local minimum for this function $f : x \mapsto x^3 + 2x^2 + x + 1$ (if they exist).

Solution for A group

Answer:

1) The first two questions were *exactly* the same, with the only difference being in the notation : for B group, $f(x) = ax^3 + bx^2 + cx + d$ and for A group it was $f(x) = \gamma x^3 + \delta x^2 + \alpha x + \beta$.

2) So, read the solution for B group, with $a = \gamma, b = \delta, c = \alpha, d = \beta$.

3) The example is different from the one for group B. We have $f(x) = x^3 + 2x^2 + x + 1$, so it defines $a = 1, b = 2, c = 1, d = 1$. Here, $\Delta = 4b^2 - 12ac = 4 \times 4 - 12 \times 1 \times 1 = 4 = 2^2 > 0$. So, there is two real solutions for $f'(x) = 0 : x_{1,2} = \dfrac{-2b \pm \sqrt{\Delta}}{6a} = \dfrac{-2 \pm 1}{3}$, ie $x_1 = -1$ and $x_2 = -\dfrac{1}{3}$. The *local* minimum is at x_2 and the *local* maximum is at x_1. This can clearly be seen on the Figure 2.

4) (**Bonus**) Here is a quick graphic of this function f.

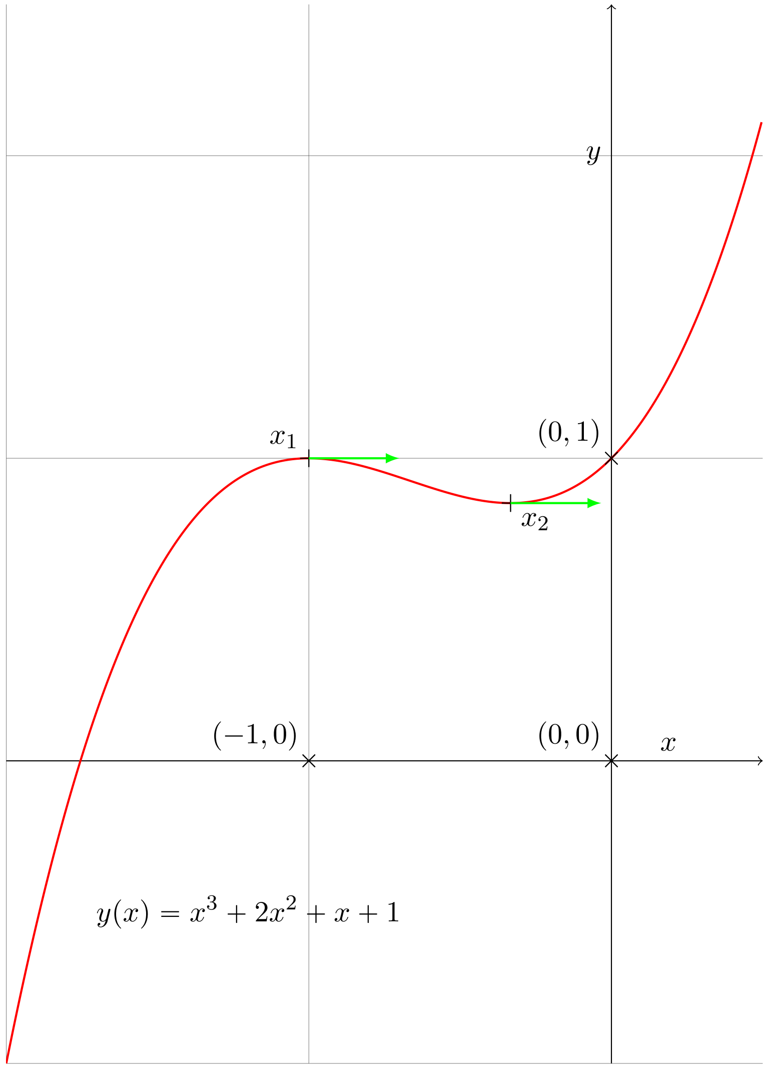

Figure 2: The function $f : x \mapsto x^3 + 2x^2 + x + 1$ on $[-2, 0.5]$