1 Exercises to (try to) do before tutorials.

1.1 Prove the following classical inequalities:

(1) $\forall a, b \in \mathbb{R}, (a+b)^2 \leqslant 2(a^2+b^2)$,

(2) $\forall a, b \in \mathbb{R}, |ab| \leqslant \dfrac{a^2+b^2}{2}$,

(3) $\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1+x)^n \geqslant 1+nx$,

(4) $\forall x \in \mathbb{R}, \forall a > 1, \exists n \in \mathbb{N}, a^n \geqslant x$.

1.2 Set upper and lower bounds:

(1) Bound, for $x \in [0,3]$, the value of this expression[1]: $\dfrac{x-1}{\mathrm{e}^x+1}$,

(2) Show that $\left\{\dfrac{\sin(x) - 2\cos(x)}{\mathrm{e}^{\sin x}} : x \in \mathbb{R}\right\}$ is a bounded set,

(3) Show[2] that $\left\{\sqrt{n^3+n+2} - \sqrt{n^3+1} : n \in \mathbb{N}\right\}$ is a bounded set,

1.3 Determine the biggest (resp. smallest) element, and an upper bound (resp. lower) of the following sets E (whenever they exist):

(1) $E = \left\{\dfrac{1}{n}, n \in \mathbb{N}^*\right\}$ (where $\mathbb{N} \stackrel{\text{def}}{=} \mathbb{Z} \cap \mathbb{R}^+$ is the set of non-negative integers number, and $\mathbb{N}^* \stackrel{\text{def}}{=} \mathbb{N}\backslash\{0\} = \{n : n \geqslant 1\}$),

(2) $E = A + B$, with[3] A and B two close intervals of $\mathbb{R}$ (where $A + B \stackrel{\text{def}}{=} \{x+y, (x,y) \in A \times B\}$),

(3) $E = A - B$, with A and B two close intervals of $\mathbb{R}$ (where $A - B \stackrel{\text{def}}{=} \{x-y, (x,y) \in A \times B\}$),

(4) (**Bonus**) What can we say if A and B are both non-empty, bounded and open intervals? And what if they are just bounded and non-empty subsets of $\mathbb{R}$?

(5) $E = A \cup B$, with A and B two bounded sets of $\mathbb{R}$?

(6) (**Bonus**) $E = \left\{(-1)^n a + \dfrac{b}{n}, n \in \mathbb{N}^*\right\}$ with two parameters $a, b \in \mathbb{R}_+^*$ (where $\mathbb{R}_+^* \stackrel{\text{def}}{=} (\mathbb{R}\backslash\{0\}) \cap [0, +\infty] = \mathbb{R} \cap (0, +\infty]$ is the set of positive real numbers).

1.4 Let A be a non-empty and upper bounded subset of $\mathbb{R}$, with $\sup A > 0$. Show that A contains a non-negative element.

1.5 Let $a, b \in \mathbb{R}^+$, with $a \geqslant b$. Simplify the expression: $\sqrt{a + 2\sqrt{a-b}\sqrt{b}} + \sqrt{a - 2\sqrt{a-b}\sqrt{b}}$.

1.6 Let $a, b \in \mathbb{R}$, here and now on, $\mathtt{E}$ will be the **integer part function**[4] (also written $\mathtt{E}(x) = \lfloor x \rfloor$). Show the following statements:

(1) $a \leqslant b \Rightarrow \mathtt{E}(a) \leqslant \mathtt{E}(b)$ ($\mathtt{E}$ is a non-decreasing function),

(2) $\mathtt{E}(a) + \mathtt{E}(b) \leqslant \mathtt{E}(a+b) \leqslant \mathtt{E}(a) + \mathtt{E}(b) + 1$.

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[1] Be sure to verify that it is well defined before using it!

[2] Hint: *conjugate quantities* like $x - y = \dfrac{(x-y)(x+y)}{x+y} = \dfrac{x^2-y^2}{x+y}$, and $x + y = \dfrac{x^2-y^2}{x-y}$ (if $x \pm y \neq 0$).

[3] One could show that, if $A = [a,b]$ and $B = [c,d]$ are two close intervals, then $A + B = [a+c, b+d]$.

[4] $\forall x \in \mathbb{R}, \mathtt{E}(x)$ is given by $\mathtt{E}(x) \in \mathbb{Z}$ and $\mathtt{E}(x) \leqslant x < \mathtt{E}(x) + 1$.

2 Exercises to (try to) do during or after tutorials.

2.1 Let B a bounded and non-empty subset of $\mathbb{R}$, and $A \subset B$ is non-empty. Show that $\sup A \leqslant \sup B$.

2.2 Solve, in $\mathbb{R}$, the following equation: $\sqrt{2-2x} + \sqrt{3+x} = 1$.

2.3 Let f, g be bounded[5] functions from $\mathbb{R}$ to $\mathbb{R}$. Prove the following inequality:

$$\sup_{x \in \mathbb{R}} |(f+g)(x)| \leqslant \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |g(x)|.$$

2.4 For $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$, show this property: $\mathtt{E}(\frac{\mathtt{E}(nx)}{n}) = \mathtt{E}(x)$.

3 Bonus exercises.

You can try to solve them after the tutorials, or before exams to practice.

3.1 Prove these classical inequalities:

(1) $\forall x \in \mathbb{R}, |\sin(x)| \leqslant |x|$,

(2) $\forall x \in [0, \frac{\pi}{2}], \frac{2}{\pi} x \leqslant \sin(x) \leqslant x$,

3.2 Caracterize the set of the functions $f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \stackrel{\text{def}}{=} \mathbb{R}^{\mathbb{R}}$ (*i.e.* the set of all functions $f : \mathbb{R} \to \mathbb{R}$) verifying: $\forall x, y \in \mathbb{R}, |f(x) - f(y)| = |x - y|$ (such function is called an *isometry*).

3.3 Let $n \in \mathbb{N}^*$, and $(x_i)_{i \in [1,\dots,n]}$ be n non-negative real numbers. Prove the following:

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \geqslant n^2.$$

3.4 (1) Prove: $\forall n \in \mathbb{N}^*, \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}$,

(2) Compute the integer part (*i.e.* $\mathtt{E}$ applied to) of $\frac{1}{2} \sum_{k=1}^{n^2} \frac{1}{\sqrt{k}}$, pour $n \in \mathbb{N}^*$.

[5] $f : \mathbb{R} \to \mathbb{R}$ is *bounded* if there is a $M \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, |h(x)| \leqslant M$.