MA101: Midterm I Solution to Problem 2. Date 08.09.2014

2.a: Note that

$$z = \frac{1+i}{1-i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{2} = i$$

By Euler's identity: $i = e^{i(\pi/2 + 2n\pi)}$, for all $n \in \mathbb{Z}$.

Hence, $\boxed{\text{modulus of } z = 1}$

and $\boxed{\text{principal argument} = \pi/2}$

2.b. Method 1:

Let $z = x+iy$ where $x, y \in \mathbb{R}$. Hence, $z^2 = (x^2-y^2) + i\cdot 2xy$.

Then $z^2 = 2i$ is equivalent to

$$x^2-y^2 = 0 \quad \text{and} \quad xy = 1.$$

It is clear that only real solutions to these equations are $x=1, y=1$ and $x=-1$, $y=-1$.

Hence, $\boxed{z = 1+i \text{ or } -(1+i)}$.

Method 2

From Euler's identity $2i = 2e^{i(\pi/2 + 2n\pi)}$, $n \in \mathbb{Z}$.

Let $z = re^{i\theta}$ where $r>0$ and $\theta \in \mathbb{R}$

(such r and θ exists because of Thm).

Hence, $r^2 e^{i2\theta} = 2e^{i(\pi/2 + 2n\pi)}$ ——— (1)

Equating modulus and argument in (1) we get

$$r^2 = 2 \quad \text{and} \quad 2\theta = \frac{\pi}{2} + 2n\pi, \; n \in \mathbb{Z}.$$

$\Rightarrow$ $r = \sqrt{2}$ (since $r>0$)

and $\theta = \frac{\pi}{4} + n\pi$, $n \in \mathbb{Z}$.

General solution: $\boxed{z = \sqrt{2}\, e^{i(\frac{\pi}{4} + n\pi)}}$, $n \in \mathbb{Z}$.

Distinct solutions: $\boxed{z_1 = \sqrt{2}e^{i\pi/4} = 1+i}$ and $\boxed{z_2 = \sqrt{2}\, e^{-i(\frac{3\pi}{4})} = -1-i}$

2.C $\boxed{\text{Method 1}}$

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Let $r > 0$ and $\phi \in \mathbb{R}$ be such that $u = re^{i\phi}$ (existence of r and ϕ is guaranteed from Thrm).

Let $z = x + iy$ where $x, y \in \mathbb{R}$.

Hence, $e^z = e^{x+iy} = e^x \cdot e^{iy}$

$\Rightarrow |e^z| = e^x$ and $\arg(e^z) = y + 2n\pi,\ n \in \mathbb{Z}$.

Since $e^z = u$ it follows that

$$r = e^x \quad \text{and} \quad \phi = y + 2n\pi,\ n \in \mathbb{Z},$$

or, equivalently,

$x = \ln r$ (since $r>0$ this is valid).

and $y = \phi + 2n\pi,\ n \in \mathbb{Z}$.

Hence,

$$\boxed{\begin{aligned} z &= \ln r + i(\phi + 2n\pi),\ n \in \mathbb{Z} \\ &= \ln|u| + i\arg(u) \end{aligned}}$$

$\boxed{\text{Method 2}}$ Let $a, b \in \mathbb{R}$ be such that $u = a + ib$ and let $z = x + iy$ where $x, y \in \mathbb{R}$. It is easy to show that (using Euler's identity) $e^z = e^x\cos y + ie^x\sin y$.

Hence, $e^x \cos y = a$ and $e^x \sin y = b$.

$\Rightarrow e^x = \sqrt{a^2+b^2} \Rightarrow \boxed{x = \ln\left(\sqrt{a^2+b^2}\right) = \ln|u|}$.

and y is the solution to $\sin y = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos y = \dfrac{a}{\sqrt{a^2+b^2}}$

which implies that $\boxed{y = \arg(u)}$.

Hence, $\boxed{z = \ln|u| + i\arg(u)}$

Since $\arg(u)$ is not unique z is not unique.