Ex. 1 (B group): Corner Circle

A unit circle is placed against a right angle. What is radius of the smaller circle (in `cm`) ?

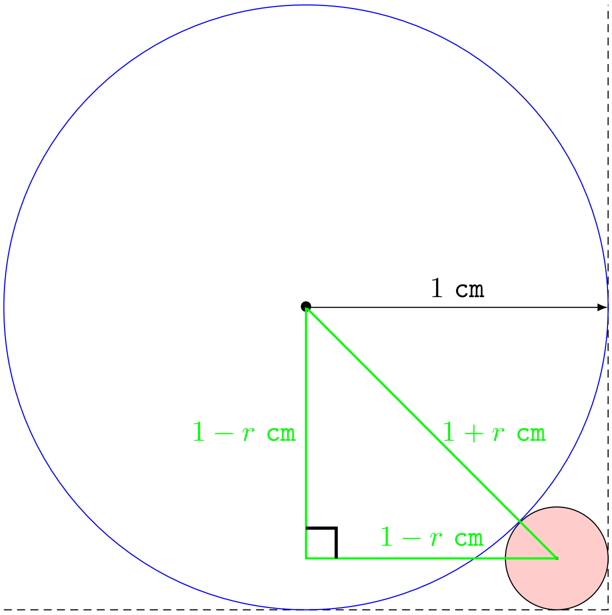

Figure 1: Corner Circle

Solution: We have $(1+r)^2 = 2(1-r)^2$, from the PYTHAGOREAN theorem (in the rectangle triangle in Figure 1). Then r is a solution of the equation $r^2 - 6r + 1 = 0$, so $r = 3 \pm 2\sqrt{2}$. But $r < 1$, then $r = 3 - 2\sqrt{2} \simeq 0.1715$ cm.

Ex. 2 (B group): Cubes and Multiples of 7

Prove that for any number that is not a multiple of seven, its cube is one more or one less than a multiple of seven. Formally, this would be written as: $\forall n \in \mathbb{N}^*, 7 \nmid n \Rightarrow 7 \mid n^3 \pm 1$.

Solution: Let write n as $7a + b$, with $b \in \{1, 2, 3, 4, 5, 6\}$ the remainder of n divided by 7. We have $n^3 = (7a+b)^3 = 7(7^2 \times a^3 + 3 \times 7a^2 \times b + 3 \times a \times b^2) + b^3 = 7c + b^3$ (with some c), therefore $n^3 = b^3 \bmod 7$. Therefore, we just have to compute $b^3 \bmod 7$ for all the 6 possibles values for b:

b	1	2	3	4	5	6
b^3	1	8	27	64	125	216
$b^3 \bmod 7$	1	1	6	1	6	6

And $6 = -1 \bmod 7$, so $n^3 = \pm 1 \bmod 7$

Ex. 3 (B group): About continuity

Let f be a function from an interval $[a, b]$ to $\mathbb{R}$ (with $a, b \in \mathbb{R}, a < b$), and $x_0 \in [a, b]$.

1. Recall the *formal* definition of f being continuous at this point x_0 (as a pure formula with variables, well introduced with quantifiers $\forall$ and $\exists$).
2. If f is continuous at x_0, does this mean f is differentiable at x_0?

   If yes, (try to) prove it. If not, (try to) give a counter-example.

Solution:

1. f is continuous at x_0 **iff** $\forall \varepsilon > 0, (\exists \eta_{\varepsilon,x_0} > 0, (\forall y \in [a, b], (|x - y| < \eta_{\varepsilon,x_0} \Rightarrow |f(x_0) - f(y)| < \varepsilon)))$,
2. No, with the simple example of the absolute value $|.| : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$, continuous on $\mathbb{R}$, but not differentiable on $x_0 \stackrel{\text{def}}{=} 0$.

Ex. 1 (A group): 3 prime numbers?

Given that $a, b, c \in \mathbb{N}^*$ are three non-negative integers, if we have $abc+ab+bc+ca+a+b+c+1 = 1001$, what is the value of $a + b + c$? Hint: $1001 = 7 \times 11 \times 13$.

Solution: We write $abc+ab+bc+ca+a+b+c+1$ as $(a+1)(b+1)(c+1)$, and then we compute the prime decomposition of this number 1001 (given as a hint). The three integers are such that $a > 0$, $b > 0$, $c > 0$, we know that $(a+1) \geqslant 2$, $(b+1) \geqslant 2$, $(c+1) \geqslant 2$. As we have $1001 = 7 \times 11 \times 13 = (a+1)(b+1)(c+1)$, then by uniqueness of the prime decomposition, we can say that $\{a+1, b+1, c+1\} = \{7, 11, 13\}$ (as a non-ordered set). Finally, $a+b+c = (a+1)+(b+1)+(c+1)-3 = 7+11+13-3 = 6+10+12 = 28$.

Ex. 2 (A group): Cables

A telephone company places round cables in round ducts.

Assuming the diameter of a cable is 2 cm, what would the diameter of the duct be for two cables, three cables and four cables?

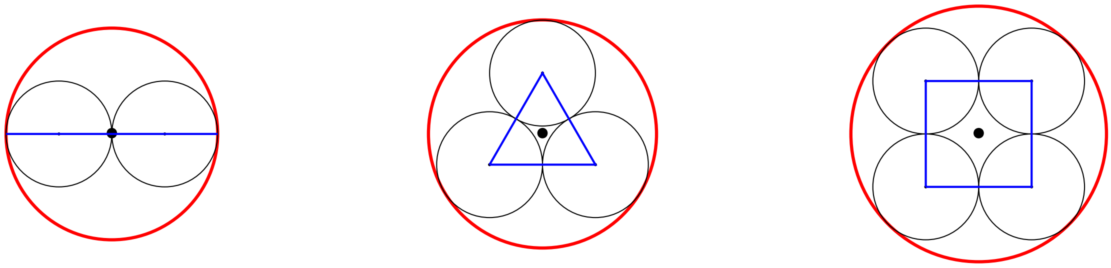

Figure 2: Cables

Solution: Quick version: 2 cm for two cables, $2 + 4\frac{\sqrt{3}}{3} \simeq 4.31$ cm for three cables, and $\sqrt{8} + 2$ cm for four cables.

Clearly the duct for two cables must have a diameter of 4 cm, however, the duct for three and four cables requires a little more effort.

Let us first consider the three cable duct. The center of each circle form a equilateral triangle, of side of length 2 cm and of height h. First of all we use the PYTHAGOREAN theorem to find the height of the triangle, $2^2 = h^2 + 1^2$, therefore $h = \sqrt{3} = 1.73$ cm. Using the geometric result that states the centre of an equilateral triangle is 1/3 the height of the triangle, we deduce that the distance from the centre of the triangle to its apex is $(2/3)h = 2\sqrt{3}/3 \simeq 1.15$ cm. So the duct radius will be $2\sqrt{3}/3 + 1$, hence the diameter will be $4\sqrt{3}/3 + 2$ cm.

Now we consider the duct for four cables. The center of each circle form a square, of side of length 2 cm, and of diagonal d. Using the PYTHAGOREAN theorem, $d^2 = 2^2 + 2^2$, giving $d = \sqrt{8} \simeq 2.83$ cm. Therefore, the duct diameter will be $r = 2 + \sqrt{8} \simeq 4.83$ cm.

Ex. 3 (A group): About differentiability

Let f be a function from an interval $[a, b]$ to $\mathbb{R}$ (with $a, b \in \mathbb{R}, a < b$), and $x_0 \in [a, b]$.

1. Recall the *formal* definition of f being differentiable at this point x_0 (as a pure formula with variables, well introduced with quantifiers $\forall$ and $\exists$).
2. If f is differentiable at x_0, does this mean f is continuous at x_0? If yes, (try to) prove it. If not, (try to) give a counter-example.

Solution:

1. f is differentiable at x_0 **iff** $\exists l \in \mathbb{R}, (\forall \varepsilon > 0, (\exists \eta_{\varepsilon,x_0} > 0, (\forall y \in [a, b]\backslash\{x_0\}, (|x_0 - y| < \eta_{\varepsilon,x_0} \Rightarrow \left|\frac{f(x_0) - f(y)}{x_0 - y} - l\right| < \varepsilon))))$,
2. Yes, by multiplying by $x_0 - y$ the previous inequality, we have $f(y) \underset{y \to x_0}{\longrightarrow} f(x_0)$.