1 Exercises to (try to) do before tutorials.

1. Find[1] a necessary and sufficient condition (*NSC*) for $z \in \mathbb{C}$ to satisfy the identity: $|z+i| = |z-i|$.

2. Resolve (in $\mathbb{C}$) the following equations:

   (1) $z^2 + (1-i)z - 1 + \dfrac{1}{4} = 0$,

   (2) $z^2 + (2-3i)z - 6i = 0$,

   (3) $3z^2 + (1+i)z + 2 = 0$.

3. Write on *polar form* the following complex numbers (*i.e.* find the modulus and **one** argument): $1+i$, $\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}$, $1 + e^{i\theta}$ for $\theta \in \mathbb{R}$, and $e^{ia} + e^{ib}$ for $a, b \in \mathbb{R}$.

4. Compute the product of all the n^{th} roots of unity in $\mathbb{C}$ (for n a natural integer, bigger than 2), *i.e.* the product $\displaystyle\prod_{z \in \mathbb{U}_n} z$, when $\mathbb{U}_n = \{z \in \mathbb{C} : z^n = 1\}$.

5. Let $z, z' \in \mathbb{C}$.

   (1) Prove the following identity: $|z+z'|^2 + |z-z'|^2 = 2(|z|^2 + |z'|^2)$ (which is called the *parallelogram identity*).
   It could be a good idea to illustrate the situation with a drawing, and try to justify graphically the name of this identity.

   (2) Conclude that one could have: $|z| + |z'| = |a + \dfrac{z+z'}{2}| + |a - \dfrac{z+z'}{2}|$ with a being **one** square root of zz'.

2 Exercises to (try to) do during or after tutorials.

1. Let $u \in \mathbb{C}\backslash\{1\}$ and $z \in \mathbb{C}\backslash\mathbb{R}$. Prove the following equivalence: $|u| = 1 \Leftrightarrow \dfrac{z - u\overline{z}}{1-u} \in \mathbb{R}$.

2. Simplify the following expressions: $\tan(3\pi + \theta)$, $\cos\left(\dfrac{\pi}{2} - \theta\right)$, where θ is a real such that these expressions make sense.

3. Simplify the expressions $\cos\left(\dfrac{\pi}{8}\right)$ and $\sin\left(\dfrac{\pi}{8}\right)$, first by using a trigonometric method and then an algebric[2] one.

[1] One could try to raise to the square and use the formula $|z|^2 = z\overline{z}$.

[2] For the second method, if $z = e^{i\frac{\pi}{8}}$ then $z^2 = e^{i\frac{\pi}{4}}$ and then we know a method to compute **one** square root of a complex number.

4. Resolve in $\mathbb{C}$ the following equation (z being the unknown): $z^4 + z^3 + z^2 + z + 1 = 0$. One could introduce the new variable $Z := z + \frac{1}{z}$. Then show that the solutions are all 5^{th} roots of unity (*i.e.* in the set $\mathbb{U}_5$), and use this to compute the value of $\cos\left(\frac{2\pi}{5}\right)$.

5. Resolve in $\mathbb{C}$ the following equation (in z): $(z+1)^6 + (z-1)^6 = 0$.

6. Let a be a complex number of modulus 1 (*i.e.* $a \in \mathbb{U}$) and $z_1, \dots, z_n$ the complex solutions of the equation $z^n = a$, for some $n \in \mathbb{N}^*$ (*i.e.* the n^{th} roots of a). Show that the points of complex affixes $(1+z_1)^n, \dots, (1+z_n)^n$ are aligned in an euclidian plan[3].

3 Bonus exercises.

1. Simplify and compute the following integrals: $\int_0^{\frac{\pi}{2}} \sin^4(x)\mathrm{d}x$, and $\int_0^{\frac{\pi}{2}} \cos^5(x)\mathrm{d}x$.

2. Resolve (in $\mathbb{R}$) the following equations:

   (1) $\cos(x) + \sin(x) = 0$,

   (2) $\sin(4x) - \sqrt{3}\sin(3x) + \sin(2x) = 0$.

3. Let $\xi = \mathrm{e}^{\frac{2i\pi}{n}}$ with $n \in \mathbb{N}^*$. Compute the sum $\sum_{k=0}^{n} |\xi^k - 1|$.

4. Let us consider a recursive sequence, defined by its first two terms u_0 and u_1, two complex numbers, and by its recurrence equation: $u_{n+2} - u_{n+1} = a(u_{n+1} - u_n)$, for **one** $a \in \mathbb{C}$ and for any $n \in \mathbb{N}$. Find a necessary **and** sufficient condition on this parameter a for the sequence $(u_n)_{n\in\mathbb{N}}$ to be *periodical*[4].

5. Let a and b be two real numbers (*i.e.* $a, b \in \mathbb{R}$). Resolve, in $\mathbb{C}$, the following equation:

$$z^4 - 4\cos(a)\cos(b)z^3 + 2(1 + \cos(2a) + \cos(2b))z^2 - 4\cos(a)\cos(b)z + 1 = 0.$$

[3] One could try to see if there is a line, coming from the origin, and passing by each of these points. This could be expressed with a criteria over their arguments.

[4] A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be *periodical* when there is an integer $T \geqslant 1$, called a *period*, such that $\forall k \in \mathbb{N}, u_{k+T} = u_k$