PH101
Lecture12

25.08.14

- Studied motion by approximating a block or big mass by treating it as a collection of point particles and not as an extended body

- In case of planetary motion, it is justified as the distances involved are very large compared to the size of the objects e.g. a planet
- Even in case of motion of elementary particles ($\sim 10^{-15}$ m) in an accelerator of several orders of magnitude larger size, it is justified
- Inapplicable in situations: e.g.

i) landing of a space craft on moon's surface in the gravitational field of moon

Space craft has a structure involving spider like legs, antennas and a lumpy body

ii) Motion of rockets, in which there is a flow of mass!

How to apply ***F*** = *M*.***a*** ?

Newton expressed his 2nd law as

$$\vec{F} = \frac{d}{dt}\left(M\,\vec{v}\right)$$

M is a constant for a Particle in Newtonian mechanics

$$\Rightarrow \frac{d}{dt}\left(\underbrace{M\,\vec{v}}_{\textit{Momentum}}\right) = M\,\frac{d}{dt}\vec{v} = M\,\vec{a}$$

Momentum $\Rightarrow$ 2nd law becomes $\vec{F} = \dfrac{d\,\vec{p}}{dt}$

In the form $$\vec{F} = \frac{d\,\vec{p}}{dt}$$

easier to generalize 2nd law for applying to complex systems

Momentum is more fundamental than *m* or *v* separately

<u>Dynamics of a system of particles</u>

Consider the solar system of sun and its planets:

Due to involved huge distances, these could be approximated as particles

How is solar systems’ dynamics governed?

All planets in this system interact via gravitational influence of sun

+

Interaction between the planets themselves

+ ?

Attraction experienced by the solar system from far-off objects

Consider a billiard ball resting on the board

On a microscopic scale, the ball consists of atoms as its particles

Disregard that atoms themselves have a structure

What would be the primary source of interactions in the ball?

Essentially **<u>internal</u>** inter-atomic electric forces

+

<u>External</u> forces on the ball:

Gravitational force & contact force of the board

Each particle interacts with each other: *internal interaction*

Also with particles external to the system: *external interaction*

Let the system consists of *N* interacting particles having masses

$$m_1, m_2, m_3, m_4, \ldots\ldots, m_N$$

Assume position vector of j^{th} particle as $\mathbf{r}_j$

The force on it as $\mathbf{f}_j$

Its momentum as $\mathbf{p}_j$

Its equation of motion:

$$\vec{f}_j = \frac{d\,\vec{p}_j}{dt};\ j = 1,2,3,\ldots, N$$

$$\downarrow$$

$$\vec{f}_j = \vec{f}_j^{\,\text{int}} + \vec{f}_j^{\,\text{ext}} = \frac{d\,\vec{p}_j}{dt};\ j = 1,2,3,\ldots, N$$

$\vec{f}_j^{\,\text{int}}$: *Interna*l force on j^{th} particle due to all other particles in the system

$\vec{f}_j^{\,\text{ext}}$: *External* force on j^{th} particle due to sources outside the system

$$\vec{f}_1^{\,\text{int}} + \vec{f}_1^{\,\text{ext}} = \frac{d\,\vec{p}_1}{dt}$$

....

$$\vec{f}_1^{\,\text{int}} + \vec{f}_1^{\,\text{ext}} = \frac{d\,\vec{p}_1}{dt}$$

....

$$\vec{f}_j^{\,\text{int}} + \vec{f}_j^{\,\text{ext}} = \frac{d\,\vec{p}_j}{dt}$$

....

$$\vec{f}_N^{\,\text{int}} + \vec{f}_N^{\,\text{ext}} = \frac{d\,\vec{p}_N}{dt}$$

$$\underbrace{\sum_{j=1}^{N} f_j^{\text{int}}}_{\sum \text{ Over all the internal forces } = 0 \text{ because they will cancel in pairs (3rd law)}} + \underbrace{\sum_{j=1}^{N} f_j^{\text{ext}}}_{\equiv \vec{F}_{\text{ext}}} = \sum_{j=1}^{N} \frac{d\,\vec{p}_j}{dt}$$

Also follows from conservation of momenta

Thus

$$\sum_{j=1}^{N} f_j^{\text{ext}} = F_{\text{ext}} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}$$

$$F_{\text{ext}} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt} = \left(\frac{d}{dt}\right) \sum_{j} \vec{p}_j$$

$$\sum_{j=1}^{N} \vec{p}_j \equiv P$$

Total momentum of the system

$$\Rightarrow \quad \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$$

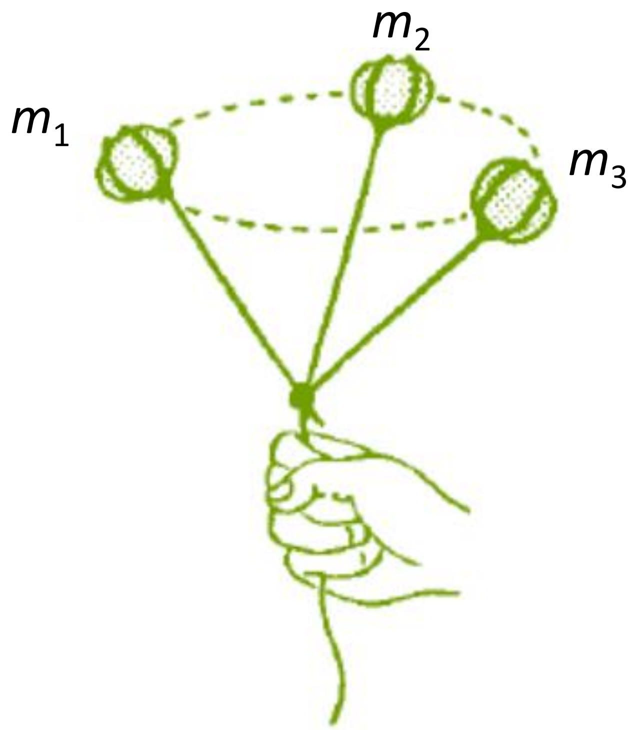
i.e. Total external force on the system equals rate of change of the system's momentum

irrespective of the details of the interactions

Single force $\mathbf{F}_{\text{ext}}$ on a single particle or resultant of many tiny Interactions involving each particle of the system

Consider a Bola, which is a weapon to entangle animals

It consists of three balls of stone or iron (masses m_1, m_2 & m_3) connected by thongs



Gaucho whirls the bola in the air and hurls it at the animal. What can you say about its motion?

In terms of total momentum: $$\frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} = \vec{f}_1^{\,\text{ext}} + \vec{f}_2^{\,\text{ext}} + \vec{f}_3^{\,\text{ext}}$$

$$= m_1\vec{g} + m_2\vec{g} + m_3\vec{g}$$

$$\Rightarrow \frac{d\vec{P}}{dt} = \left(m_1 + m_2 + m_3\right)\vec{g} = M\,\vec{g}$$

single particle of mass *M*

Goucha does not realize complexity of the system!

Let us rewrite
$$\vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$$

$$\vec{F} = \frac{d\vec{P}}{dt}; \vec{F} \equiv \vec{F}_{\text{ext}} \qquad (1)$$

It is identical to the eq of motion of a single particle

This analogy can be further extended as follows:

We write
$$\vec{F} = M\frac{d^2\vec{R}}{dt^2} = M\ddot{\vec{R}} \qquad (2)$$

$\vec{R}$ is a radius vector not yet defined

Since $\vec{P} = \sum_j m_j \dot{\vec{r}}_j \qquad \Rightarrow M\ddot{\vec{R}} = \frac{d\vec{P}}{dt} = \sum_j m_j \ddot{\vec{r}}_j$

This will be true if
$$\vec{R} = \frac{1}{M}\sum_j m_j r_j$$

$$\vec{R} = \frac{1}{M}\sum_j m_j r_j$$

$\vec{R}$: a vector connecting origin to a point, which we call as the *center of mass*

As if the entire mass is concentrated at the center of mass at which all the external forces act

- For rigid bodies like automobiles, one may consider it to be a system of particles fixed relatively to each other by strong internal forces
- w.r.t external forces it behaves as if it were a point particle
- Our earlier assertion that each body was as if a particle is justified if we accept the concept of center of mass

However in practice an extended body e.g. a plank may behaves differently from a compact body like a rock, even if they have same mass

Relation

$$\vec{F} = M\frac{d^2\vec{R}}{dt^2} = M\ddot{\vec{R}}$$

is applicable to only translational motion!

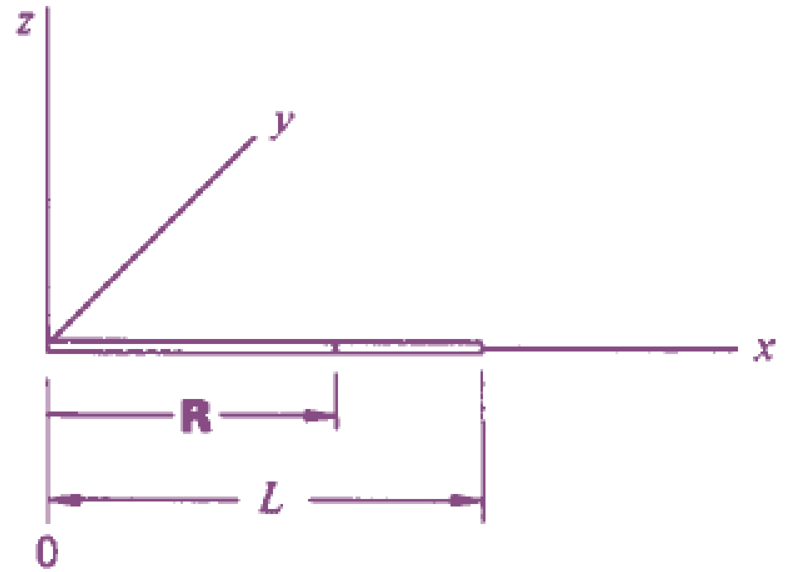
In case of rotational motion of an extended body, it depends on

shape as well as the point where the forces are applied

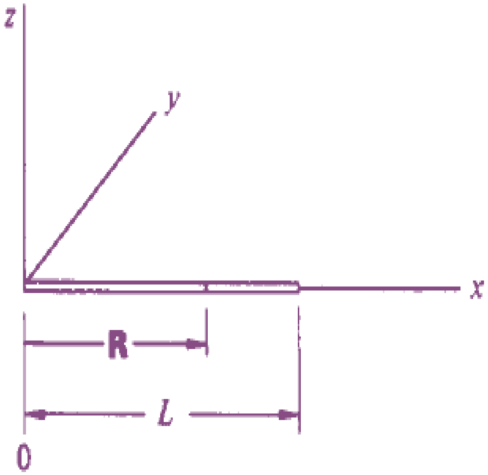
#1. Find the center of mass of a rod of length *L*, which has non-Uniform density; mass/length represented through λ varies as

$$\lambda = \lambda_0\left(\frac{s}{L}\right)$$

where λ_0 is a constant and *s* is the distance from the end marked 0.



Set up a coordinate system whose origin coincides with 0 and the rod's lay is along the *x*-axis $\Rightarrow s = x$



Elemental mass *dm* in an elemental length *dx:*

$$dm = \lambda dx = \lambda_0 \left(\frac{x}{L} \right) dx$$

Thus

$$M = \int dm = \int_0^L \lambda_0 \left(\frac{x}{L} \right) dx$$

$$= \frac{\lambda_0}{\cancel{L}} \frac{L^{\cancel{2}}}{2} = \frac{1}{2} \lambda_0 L$$

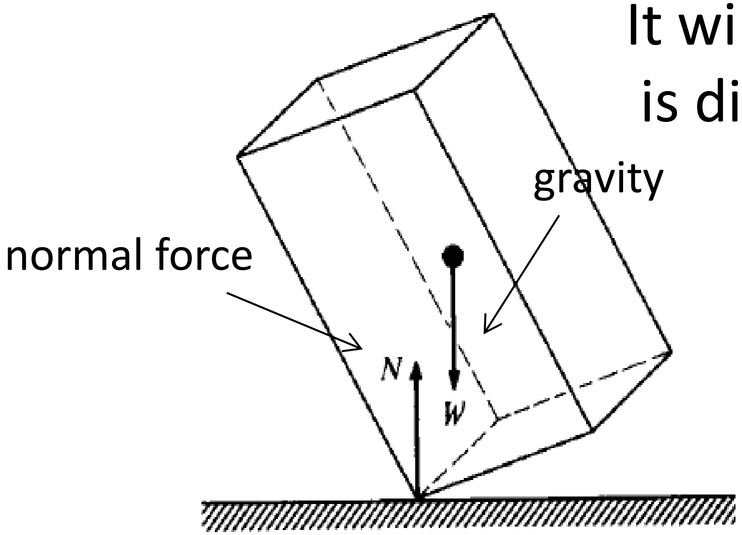
Thus, center of mass will be at

$$\vec{R} = \frac{1}{M} \int \vec{r} \lambda dM$$

$$= \frac{2}{\lambda_0 L} \int_0^L \left(x\hat{\mathrm{i}} + \underbrace{y\hat{\mathrm{j}} + z\hat{\mathrm{k}}}_{0} \right) \frac{\lambda_0 x d x}{L}$$

$$\vec{R} = \frac{2}{L^2} \frac{\hat{\mathrm{i}}}{3} x^3 \Big|_0^L = \frac{2}{3} L \hat{\mathrm{i}}$$

#2. A rectangular box is held with one corner resting on a frictionless table is gently released. Find the trajectory of its center of mass.

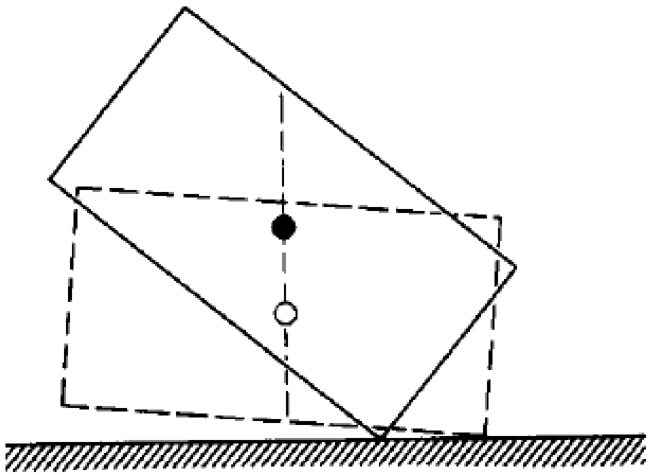


It will fall in a complex tumbling motion, which is difficult to obtain!

External forces acting on it are gravity and normal force

None of these forces have any horizontal component

$\Rightarrow$Thus the center of mass must accelerate vertically



For a uniform body, its geometrical center will be the center of mass

When released from rest, its center will fall straight down

Conservation of momentum

What are the implications of the relation

$$\vec{F} = \frac{d\,\vec{p}}{dt}$$

for an isolated system ? Does not interact with its surrounding

Naturally, for an isolated system

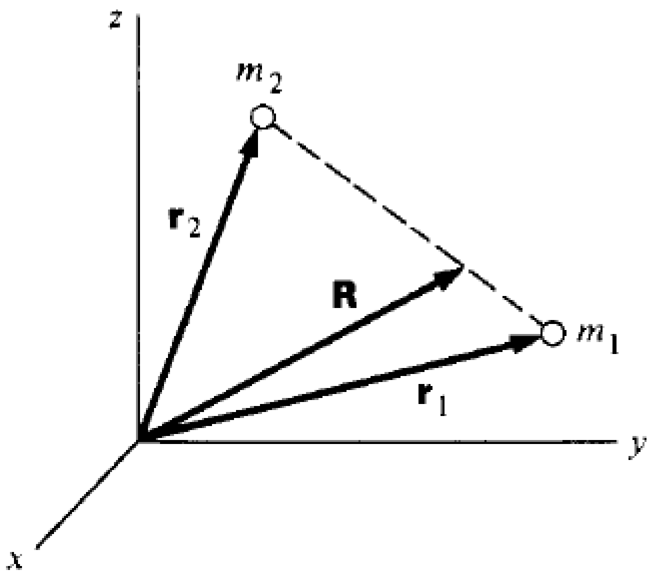
$$\vec{F} = 0 \Rightarrow \frac{d\,\vec{P}}{d\,t} = 0$$

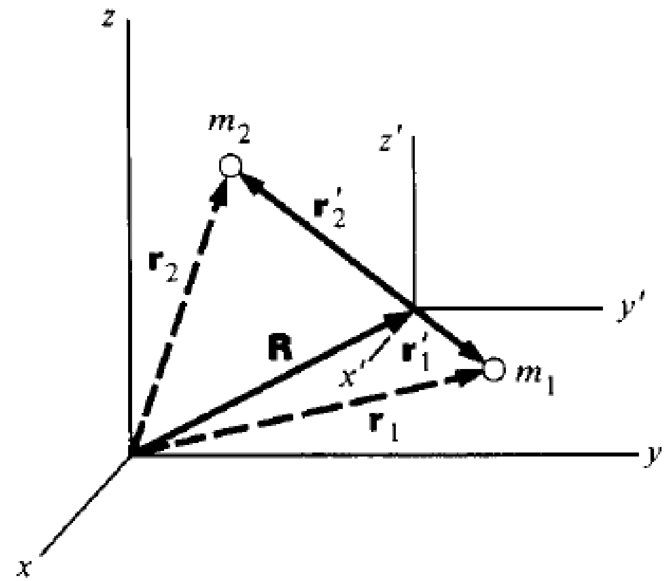
$\Rightarrow$ Total momentum is constant

This is law of conservation of momentum

It can provide powerful insights to complicated systems

Center of mass coordinates





Origins are displaced by **R**

Center of mass is at

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Center of mass coordinates :

$$\vec{r}_1^{\,/} = \vec{r}_1 - \vec{R}$$

$$\vec{r}_2^{\,/} = \vec{r}_2 - \vec{R}$$

Since $m_1 r_1 + m_2 r_2 = 0$

Center of mass coordinates are Natural coordinates for an isolated body

If motion of particle 1 is known the other's will follow directly