PH101
Lecture15

01.09.14

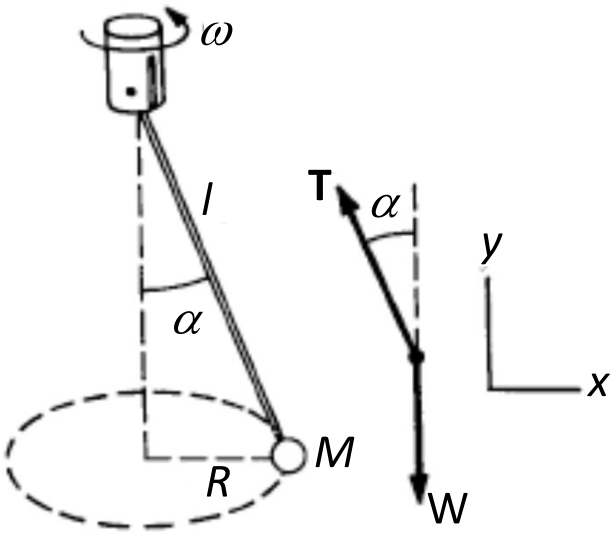
Recap:

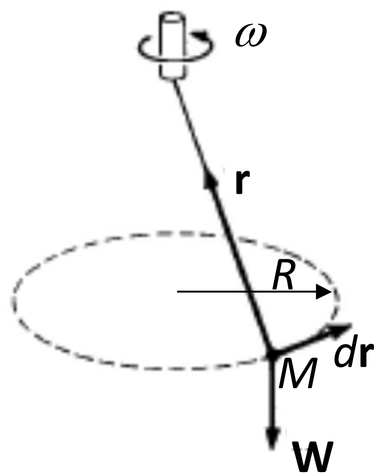
The work-energy theorem for the translational motion of an extended system is written in terms of center of mas as

$$\int_{\vec{R}_a}^{\vec{R}_b} \vec{F} \cdot \underbrace{d\vec{R}}_{d\vec{R} = \vec{V}\, dt} = \frac{1}{2} M V_b^2 - \frac{1}{2} M V_a^2$$

$d\vec{R} = \vec{V}\, dt$: displacement of the center of mass in *dt*

Example of the Conical pendulum:



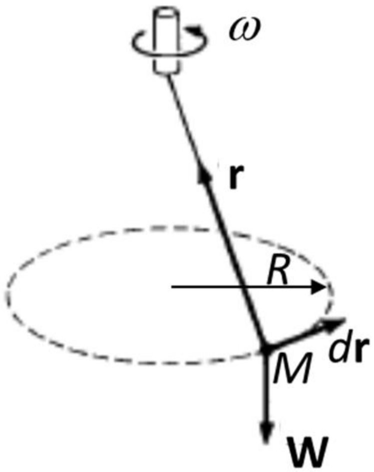


M moves with a const ω and traces out a circular path of const radius *R*

$\Rightarrow$ K.E. of the mass *M*

$$= \frac{1}{2} M R \omega^2$$

is a constant



$\Rightarrow$ From work-energy theorem no work is being done on the mass *M*

$$\text{Since } \vec{v} = \frac{d\vec{r}}{dt} \Rightarrow d\vec{r} \text{ is always parallel to } \vec{v}$$

$$\Downarrow$$

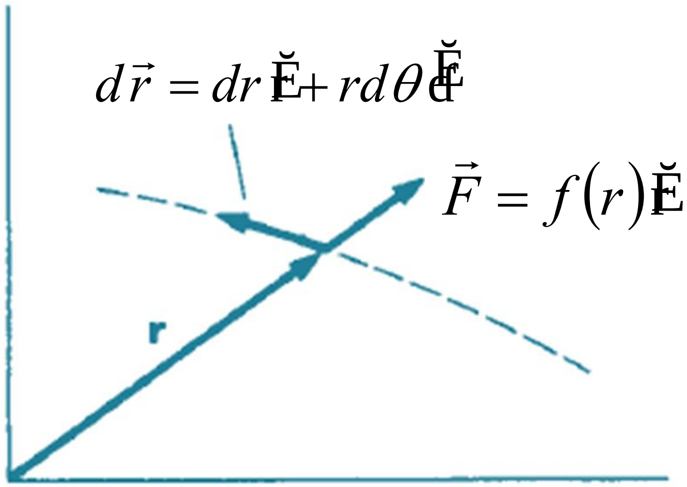
In the work integral: $\int \vec{F}.d\vec{r}$

the vector *d***r** is along the particle's path

Further since each of the forces due to the string and the weight is perpendicular to the path of the particle, the integrand

$\vec{F}.d\vec{r}$ will be zero

Work done by a uniform force e.g. a central force



$$\oint \vec{F}.d\vec{r} = W_{ba} = \oint_a^b f(r)\,\hat{r}.\left(dr\,\hat{r} + r\,d\theta\,\hat{\theta}\right) = \int_a^b f(r)\,dr$$

Pl note correction: $\cancel{\oint} \longrightarrow \int$

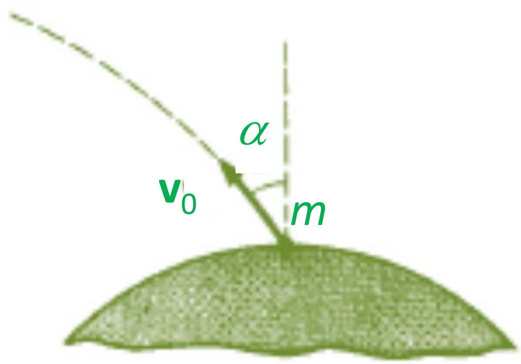
$\downarrow$

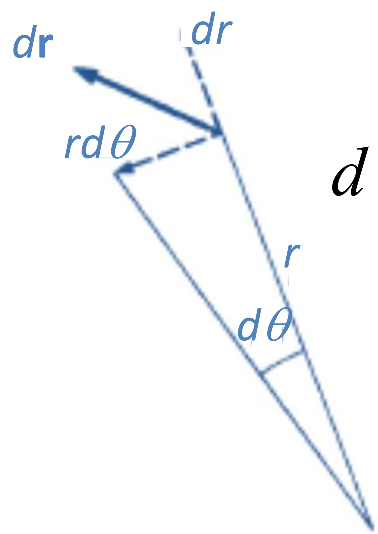
Work done depends only on the initial and final positions besides the functional dependence of *f*(*r*) and not on the particular path

This is in contrast to force of sliding friction, where work done would be different for different paths between the initial and final points

Example:

Escape velocity of a mass projected at an angle α to the vertical





$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} \Rightarrow$$

Force experienced by the mass m is central: $\vec{F} = -\frac{GMm_e}{r^2}\hat{r}$

Acceleration g on earth's surface: $g = \frac{GM_e}{R_e^{\,2}} \Rightarrow \vec{F} = -mg\left(\frac{R_e}{r}\right)^2 \hat{r}$

$$\vec{F}.d\vec{r} = -mg\frac{R_e^2}{r^2}\hat{r}.\left(dr\,\hat{r} + r\,d\theta\,\hat{\theta}\right) = -mg\frac{R_e^2}{r^2}dr$$

$$\vec{F}.d\vec{r} = -mg\frac{R_e^2}{r^2}.dr$$

Accordingly, work-energy theorem becomes

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -mgR_e^2\int_{R_e}^{r}\frac{dr}{r^2} = -mgR_e^2.\left(-\frac{1}{r}\right)\Bigg|_{R_e}^{r}$$

$$= -mgR_e^2\left(\frac{1}{R_e} - \frac{1}{r}\right)$$

Escape vel (v_0^{esc}) $\Rightarrow$ $v = 0$ & $r = \infty$

Thus

$$\cancel{-}\frac{1}{2}\cancel{m}v_0^2 = \cancel{-}\cancel{m}gR_e^{\cancel{2}}\left(\frac{1}{\cancel{R_e}} - \frac{1}{\infty}\right) \quad \Rightarrow \quad v_0 = \sqrt{2g\,R_e}$$

Escape vel is independent of the launch direction!

Some comments:

1. Work-energy theorem requires evaluation of the line integral

$$\underbrace{\int_{\substack{\text{Path}\\ (a\rightarrow b)}} \vec{F}.d\vec{r}} = W_{ab}$$

$\oint$: the integral is to be evaluated along a particular path

$\Rightarrow$ the integral depends on what particular path the particle actually follows !

thereby $\Rightarrow$ that we need to know the path, which requires solving the problem completely beforehand!!

$\Rightarrow$ As if work-energy theorem is of no benefit to us, which is merely a mathematical consequence of Newton's 2nd law, no new physics emerged!!

Å Fortuitously, special cases exist in which the integral depends only on the end points and not on particular path

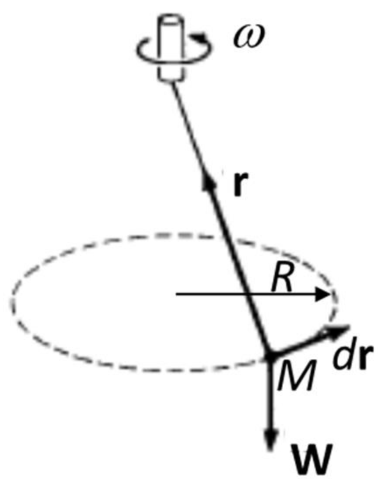
Å Such forces are known as *conservative* forces, for which this theorem can be expressed in a very simple form

Å Another special case is one for which path of an object is known because of its constrained motion because external constraints act on it to keep it on a predetermined trajectory

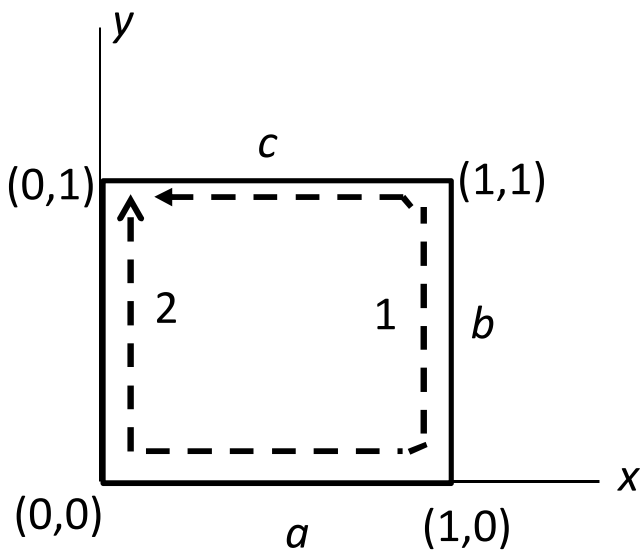
Å e.g. in a roller coaster. which is held on wheels below and above the track and hence follows that laid out track

- Another example is Conical pendulum, in which the constraint is that length of the string is fixed

- In all these examples, the constraining force does no work
- It only ensures that the direction of velocity (**v**) is always tangential to the predetermined path and does no work



Illustrative example of a path dependent line integral:



Let us consider an arbitrary force represented through

$$\vec{F} = A\left(xy\,\hat{i} + y^2\,\hat{j}\right)$$

Calculate the integral

$$\oint_1 \vec{F}.d\vec{r}$$

along path 1 and then along path 2

Along path 1

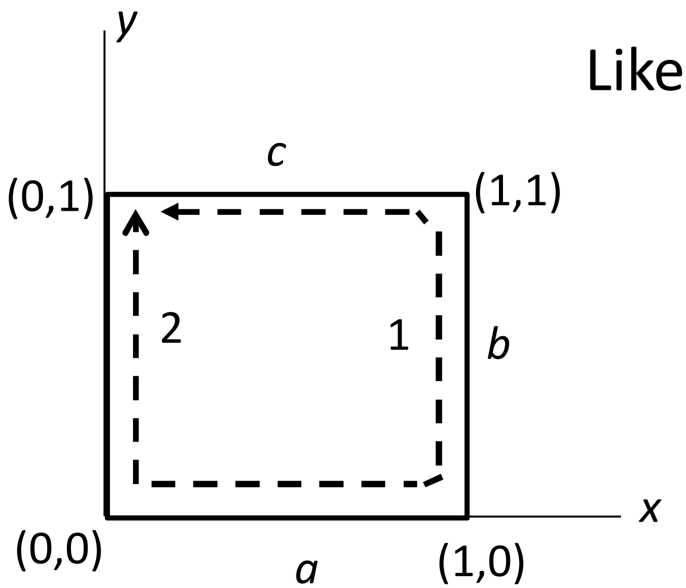
$$\oint_1 \vec{F}.d\vec{r} = \int_a \vec{F}.d\vec{r} + \int_b \vec{F}.d\vec{r} + \int_c \vec{F}.d\vec{r}$$

Along the segment *a*

$$d\vec{r} = dx\,\hat{i}$$

$\int_a \vec{F}.d\vec{r} = \int F_x\,dx = \int Axy\,dx$; but *y* = 0 along the line of integration

$$\Rightarrow \int_a \vec{F}.d\vec{r} = 0$$



Likewise for

$$\int_b \vec{F}.d\vec{r} = \int_{x=1,y=0}^{x=1,y=1} Ay^2\,dy = \frac{A}{3}$$

For the path *c*:

$$\int_c \vec{F}.d\vec{r} = \int_{x=1,y=1}^{x=0,y=1} Axy\,dx = A\int_1^0 x\,dx = -\frac{A}{2}$$

Thus

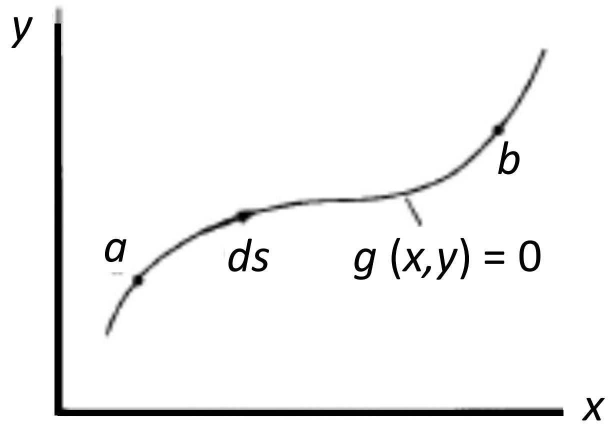
$$\oint_1 \vec{F}.d\vec{r} = \int_a \vec{F}.d\vec{r} + \int_b \vec{F}.d\vec{r} + \int_c \vec{F}.d\vec{r} = 0 + \frac{A}{3} - \frac{A}{2} = -\frac{A}{6}$$

Likewise for

$$\oint_2 \vec{F}.d\vec{r} \quad = A\int_{x=0.y=0}^{x=0,y=1} y^2 dy = \frac{A}{3} \neq \oint_1 \vec{F}.d\vec{r}$$

$\Rightarrow$ Work done is path dependent!

For a general path given by $g(x,y) = 0$



For a change in s by ds

Corresponding change in x & y are:

$$dx = \left(\frac{dx}{ds}\right) ds \quad dy = \left(\frac{dy}{ds}\right) ds$$

$$\Rightarrow \vec{F} = F_x(s)\hat{i} + F_y(s)\hat{j}$$

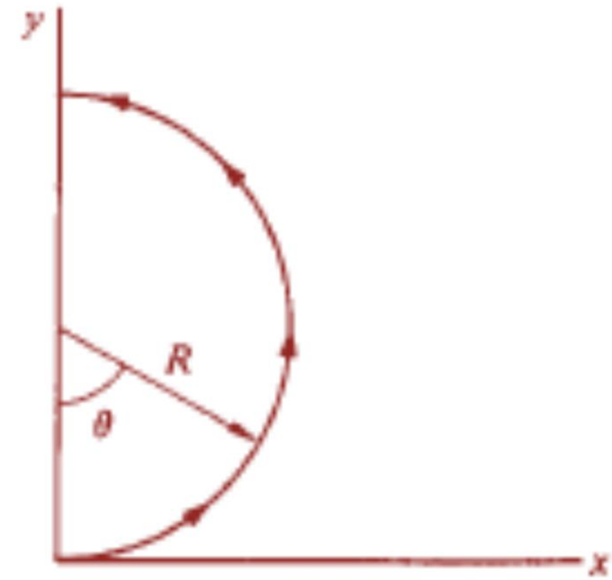
$$\oint \vec{F} \cdot d\vec{r} = \int_a^b \left(F_x dx + F_y dy\right) = \int_a^b \left[F_x(s)\frac{dx}{ds} + F_y(s)\frac{dy}{ds}\right] ds$$

$\Rightarrow$ Effectively reduced the problem to evaluating a 1-D integral

Evaluate the integral of (**F**. *d***r**) along the semi-circle for a force:

$$\vec{F} = A\left(x^3\,\hat{\mathrm{i}} + x y^2\,\hat{\mathrm{j}}\right)$$

from $x = 0$, $y = 0$ to $x = 0$, $y = 2R$



Radius vector R sweeps out a semi-circle as θ varies from 0 to π

$$x = R\sin\theta \Rightarrow dx = R\cos\theta\, d\theta$$

$$y = R - R\cos\theta \Rightarrow dy = R\sin\theta\, d\theta$$

$$F_x = Ax^3 = AR^3\sin^3\theta$$

$$F_y = Axy^2 = AR^3\sin\theta(1-\cos\theta)^2$$

$$\oint \vec{F}.d\vec{r} = A\int_0^{\pi}\left[(R\sin\theta)^3 R\cos\theta + R^3\sin\theta(1-\cos\theta)^2 R\sin\theta\right]d\theta$$

Potential energy

For a conservative force, since work done depends only on the end points

Thus for a conservative force one can write

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}.d\vec{r} = -U(\vec{r}_b) + U(\vec{r}_a)$$

The function $U(\vec{r})$ is called *potential energy* function

Though not yet shown that the function $U(\vec{r})$ exists

For the conservative forces, we can assume that $U(\vec{r})$ exists

Because work is path independent

Work-energy theorem takes the form

$$W_{ba} = -U_b + U_a = K_b - K_a$$

$$\Rightarrow$$

$$K_a + U_a = K_b + U_b$$

$$K_a + U_a = K_b + U_b$$

LHS depends on speed of the particle & its P.E. at $\mathbf{r}_a$ without any reference to $\mathbf{r}_b$

So does RHS depends on speed of the particle & its P.E. at $\mathbf{r}_b$ without any reference to $\mathbf{r}_a$

Possible only if each of these equals a constant, say *E*

$$\Rightarrow K_a + U_a = K_b + U_b = E$$

Called total mechanical energy or just total energy

For a conservative force, total work done/energy is independent of the position of the particle $\Rightarrow$ it remains constant can be restated as

The energy is *conserved*

To an extent *E* is arbitrary

Only changes in *E* have physical significance because it was defined through

$$U_b - U_a = -\int_a^b \vec{F}.d\vec{r}$$

Example: PE of an inverse square force

$$\downarrow$$

e.g. central force: $\vec{F} = f(r)\hat{r}$

e.g. Coulomb electrostatic force and gravitational force are central force

PE of a particle in a central force will follow

$$U_b - U_a = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}.d\vec{r} = -\int_{r_a}^{r_b} f(r)\,dr$$

$$\Rightarrow U_b - U_a = -\int_{r_a}^{r_b} \frac{A}{r^2}\, dr = \frac{A}{r_b} - \frac{A}{r_a}$$

In order to write a general expression for P.E., $r_b \rightarrow r$

$$U(r) = \frac{A}{r} + \underbrace{\left(U_a - \frac{A}{r_a} \right)}_{C} = \frac{A}{r} + C$$

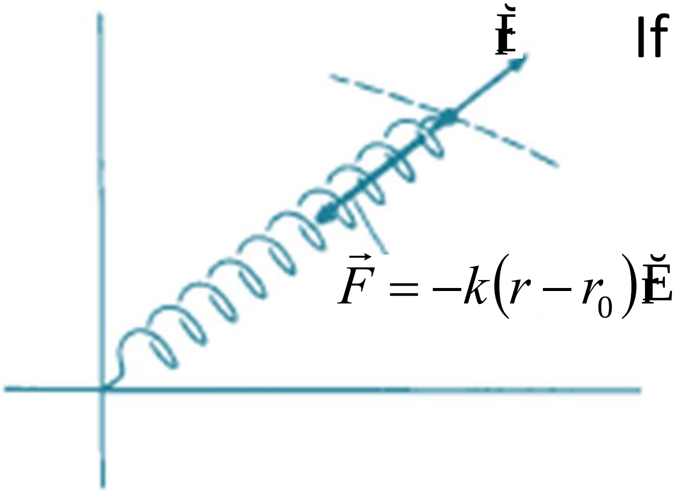
Since only ΔU has significance, C has no physical significance ; it is completely arbitrary. Thus we may put $C = 0 \Rightarrow U(\infty) = 0$

$$\Rightarrow$$

$$U(r) = \frac{A}{r}$$

This is P.E. for an inverse square force

For a spring force



If the spring is stretched to a length *r* along $\hat{r}$

It exerts a force

$$\vec{F} = -k(r - r_0)\hat{r}$$

Since it is central it is a conservative force

Its P.E. is given by

$$= \frac{1}{2}k(r - r_0)^2 \Big|_a^r$$

$$U(r) = \frac{1}{2}k(r - r_0)^2 + C$$

Conventionally, PE is chosen to be 0 at equilibrium position r_0

$\Rightarrow U(r_0) = 0 \Rightarrow C = 0$

Thus

$$U(r) = \frac{1}{2}k(r - r_0)^2$$

What does P.E. reveals about force?

P.E. can be found out from

$$U_a - U_b = -\int_a^b \vec{F}.d\vec{r}$$

Where the integral is over any path from $\mathbf{r}_a$ to $\mathbf{r}_b$

In many cases it is easier to characterize a force in terms of P.E. functn. Rather than specifying each of its components, this is simpler

Consider a 1-D system like mass on a spring

$$U_a - U_b = -\int_{s_a}^{s_b} F(x)\,dx$$

Consider the change ΔU as the particle moves from x to $x + \Delta x$

$$U(x+\Delta x) \; - U(x) \; = \Delta U = -\int_{x}^{x+\Delta x} F(x)\, dx$$

Assuming Δx to be so small that effectively F (x) effectively remains constant across Δx

$$\Rightarrow \Delta U \approx -F(x)(x+\Delta x - x) = -F(x)\Delta x$$

$$\Rightarrow F(x) = -\frac{\Delta U}{\Delta x}$$

$$\mathrm{Lt}\; \Delta x \to 0,\;\; F(x) = -\frac{dU}{dx}$$