PH101
Lecture14

28.08.14

How to predict motion of a system from known interactions

For a known force, one can use Newton's 2nd law to obtain acceleration, which after two integrations would yield velocity and position

Problem: **F** is normally given as a function of position e.g. spring force or gravitational force and not as a function of time

Thus one needs to find e.g. $\vec{v}(t)$ from

$$m\frac{d\,\vec{v}}{dt} = \vec{F}(\vec{r})$$

Force is known as a function of position

In 1-D it is not difficult through integration of the eq of motion

For example, in 1-D harmonic oscillator problem one deals with

$$m\frac{d^2x}{dt^2} = F(x) \Rightarrow m\frac{dv}{dt} = F(x)$$

By algebraic manipulation through integration w.r.t . *x* and change of variables

$$m\int_{x_a}^{x_b}\left(\frac{dv}{dt}\right)dx = \int_{x_a}^{x_b} F(x)\,dx$$

$$m\int_{x_a}^{x_b}\frac{dv}{dt}\underbrace{\left(\frac{dx}{dt}\right)}_{\downarrow v}dt = \int_{x_a}^{x_b} F(x)\,dx$$

$$\Rightarrow \int_{t_a}^{t_b}\frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt = \frac{1}{2}mv^2\Big|_{t_a}^{t_b} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F(x)\,dx$$

One could also write it in a form where upper limit is indefinite:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x} F(x)\,dx$$

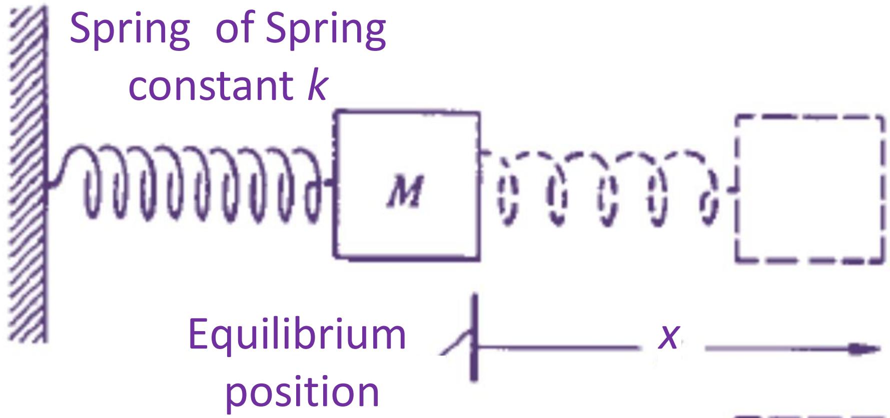
v is the velocity at a position *x*

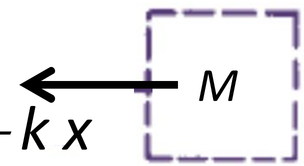
From

$$v = \frac{dx}{dt}$$

x (*t*) can be obtained

Example: Solution for SHM in 1-D





Spring obeys linear restoring force: $F = -k\,x$

Hook's law: $F = -kx$

It is always in a direction to restore to its equilibrium position

$$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -k\int_{x_0}^{x} x\,dx = -\frac{1}{2}k.x^2\Big|_{x_0}^{x}$$

$$= -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$

$$\frac{M}{\cancel{2}}\left(v^2 - v_0^2\right) = -\frac{1}{\cancel{2}}kx^2 + \frac{1}{\cancel{2}}kx_0^2$$

If we assume initial conditions to be that at t = 0, the mass is released from rest $\Rightarrow$ v_0 = 0 at distance x_0 from the origin

Thus

$$v^2 = -\frac{k}{M}x^2 + \frac{k}{M}{x_0}^2$$

& since

$$\frac{dx}{dt} = v$$

$$\Rightarrow \frac{dx}{dt} = v = \left[\frac{k}{M}\left(x_0^2 - x^2\right)\right]^{1/2}$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{\left(x_0^2 - x^2\right)}} = \left[\frac{k}{M}\right]^{1/2} \int_0^t dt = \sqrt{\frac{k}{M}}\,t$$

$$\left.\sin^{-1}\left(\frac{x}{x_0}\right)\right|_{x_0}^{x} = \omega t$$

$$\sin^{-1}\left(\frac{x}{x_0}\right) - \underbrace{\sin^{-1}(1)}_{=\pi/2} = \omega t$$

$$\Rightarrow x = x_0 \cos(\omega t)$$

One can check that it indeed satisfies initial conditions:

At $t = 0$, $x = x_0$, & $\dfrac{dx}{dt} = v_0 = \omega \sin 0 = 0$

In the eq.

$$\frac{1}{2}mv_{\mathrm{b}}^{2}-\frac{1}{2}mv_{\mathrm{a}}^{2}=\int_{x_{\mathrm{a}}}^{\mathrm{x}}F(x)\,dx$$

Physically the quantity

$$\frac{1}{2}mv^{2}$$

Represents K.E. *K*

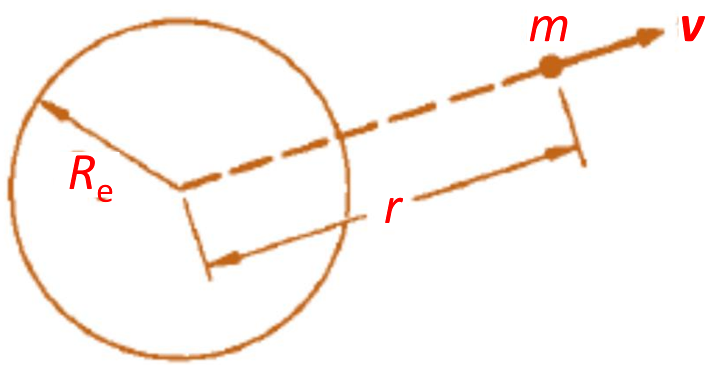
$$\Rightarrow \qquad K_{\mathrm{b}}-K_{\mathrm{a}}=\Delta K=\underbrace{\int_{x_{\mathrm{a}}}^{x_{\mathrm{b}}}F(x)\,dx}\quad =W_{\mathrm{ba}}$$

Work done by the force on the particle as it moves from x_a to x_b

Above relation is known as *work-energy* theorem (in 1-D)

In SI units, work & energy is expressed in Joules; 1 J = 1 kg-m^2/s^2

A mass m is shot vertically upward from the surface of the earth with an initial speed v_0. Assuming that the only force is gravity, find its maximum altitude and escape velocity and also energy needed to eject A 100 kg spacecraft from the surface of earth



Gravity is an inverse square force field

force on m is

$$F = -G\frac{M_e m}{r^2}$$

Problem is equivalently 1-D in the variable r !

Let at $r = R_e$ Initial vel. of the particle is v_0

As per work-energy theorem,

$$K(r) - K(r_e) = \Delta K = \int_{R_e}^{r} F(r)\, dr$$

$$\Delta K = \frac{1}{2} m \left[v(r)^2 - v_0^2 \right] = -GM_e m \int_{R_e}^{r} \frac{dr}{r^2}$$

$$= -GM_e m \left. \frac{1}{r} \right|_{R_e}^{r} = GM_e m \left(\frac{1}{r} - \frac{1}{R_e} \right)$$

Naturally at the highest point,

$$v(r) = 0$$

$$\Rightarrow$$

$$v_0^2 = -\frac{2GM_e m}{m} \left(\frac{1}{r_{max}} - \frac{1}{R_e} \right)$$

$$\Rightarrow v_0^2 = 2GM_{\text{e}}\left(\frac{1}{R_{\text{e}}} - \frac{1}{r_{\max}}\right) = 2\underbrace{\frac{GM_{\text{e}}\,R_{\text{e}}^2}{R_{\text{e}}^2}}_{=g}\left(\frac{1}{R_{\text{e}}} - \frac{1}{r_{\max}}\right)$$

$$\Rightarrow v_0^2 = 2g\,R_{\text{e}}^2\left(\frac{1}{R_{\text{e}}} - \frac{1}{r_{\max}}\right)$$

$$\Rightarrow r_{\max} = \frac{R_e}{\left(1 - \frac{v_0^2}{2gR_{\text{e}}}\right)}$$

Escape velocity implies $r_{\max} \to \infty$

$$\Rightarrow \left(1 - \frac{v_{\text{escape}}^2}{2g\,R_{\text{e}}}\right) = 0$$

$$v_{\mathrm{escape}} = \sqrt{2gR_{\mathrm{e}}}$$

$$= \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \text{ m/s}$$

$$= \sqrt{1.25 \times 10^8}$$

$$= 1.1 \times 10^4 \text{ m/s}$$

Energy reqd to eject 100 kg satellite:

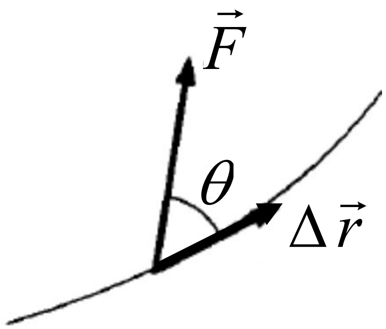
$$W = \frac{1}{2} M\, v_{\mathrm{escape}}^2$$

$$= \frac{1}{2}(100)\left(1.1 \times 10^4\right)^2 = 6 \times 10^9 \text{ J}$$

<u>Generalization of the problem to find</u> $m\dfrac{d\vec{v}}{dt} = \vec{F}(\vec{r})$

Let us find out what happens when the particle moves through a short distance $\Delta\vec{r}$

If we assume that Δr is so small that effectively $\vec{F}$ remains const.



$$\underbrace{\vec{F}.\Delta\vec{r}} = m\frac{d\vec{v}}{dt}.\Delta\vec{r}$$

$$\vec{F}.\Delta\vec{r} = F\Delta r\cos\theta$$

For sufficiently small Δr , **v** is approximately const.

$$\Rightarrow \Delta\vec{r} \approx \vec{v}\Delta t$$

Time reqd. to cover Δr

$$m\frac{d\vec{v}}{dt}.\Delta\vec{r} = m\frac{d\vec{v}}{dt}.\vec{v}\Delta t$$

$$m\frac{d\vec{v}}{dt}.\Delta\vec{r} = m\frac{d\vec{v}}{dt}.\vec{v}\Delta t$$

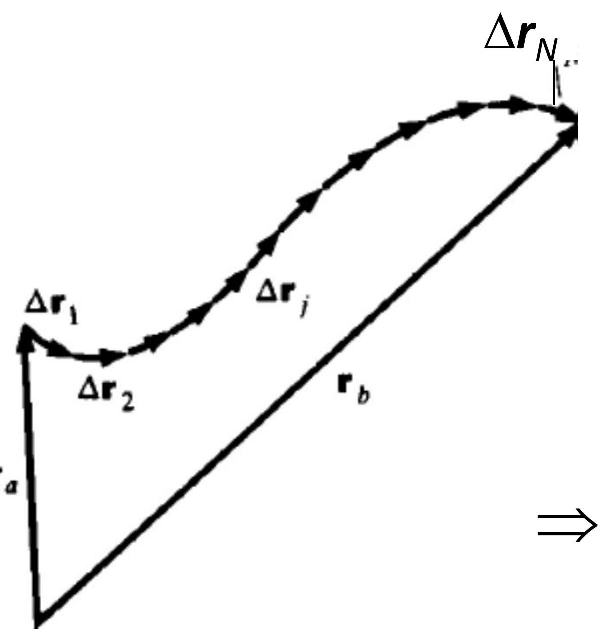
Use of the vector identity

$$\vec{v}.\frac{d\vec{v}}{dt} = \frac{1}{2}\frac{d}{dt}\left(v^2\right)$$

$$\frac{1}{2}\frac{d}{dt}\left(v^2\right) = \frac{1}{2}\frac{d}{dt}\left(\vec{v}.\vec{v}\right) = \frac{1}{2}\left(\vec{v}.\frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt}.\vec{v}\right) = \frac{1}{2}2\vec{v}.\frac{d\vec{v}}{dt}$$

will yield

$$\vec{F}.\Delta\vec{r} = m\frac{dv}{dt}.\vec{v}\,\Delta t = \frac{m}{2}\frac{d}{dt}\left(v^2\right)\Delta t$$



trajectory

$\mathbf{r}_a$ to $\mathbf{r}_a$ divide into *N* short Segments‘; for each segment we can write

$$\vec{F}\left(\vec{r}_j\right).\,\Delta\vec{r}_j = m\frac{1}{2}\frac{d}{dt}\left({v_j}^2\right)\Delta t_j$$

$$\Rightarrow \sum_{j=1}^{N}\vec{F}\left(\vec{r}_j\right).\,\Delta\vec{r}_j{}_{\,j} = \sum_{j=1}^{N} m\frac{1}{2}\frac{d}{dt}\left({v_j}^2\right)\Delta t_j$$

$$\Rightarrow \sum_{j=1}^{N} \vec{F}(\vec{r}_j).\,\Delta\vec{r}_j{}_{\,j} = \sum_{j=1}^{N} m\frac{1}{2}\frac{d}{dt}\left(v_j{}^2\right)\Delta t_j$$

In the limit of each segment $\rightarrow 0$ & no. of segment $\rightarrow \infty$

$$\int_{r_a}^{r_b} \vec{F}.d\vec{r} = \int_{t_a}^{t_b} \frac{m}{2}\frac{d}{dt}\left(v^2\right)dt$$

$$= \frac{m}{2} v^2 \Big|_{v_a}^{v_b} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}.d\vec{r} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$: is called line integral

Recollect $$v^2 = v_x^2 + v_y^2 + v_z^2$$

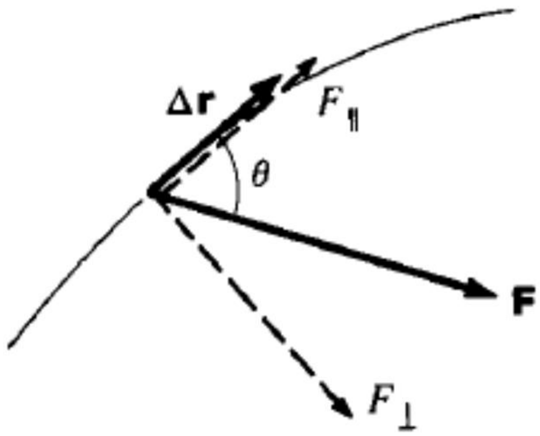
In 1-D, it was $$v^2 = v_x^2$$

$$\vec{F} = M\,\ddot{\vec{R}} = M\,\frac{dV}{dt}$$

$V = \dot{\vec{R}}$: Vel of the center of mass

$$\int_{\vec{R}_a}^{\vec{R}_b} \vec{F}\,.\,d\vec{R} = \frac{1}{2} M V_b^2 - \frac{1}{2} M V_a^2$$

Work-energy theorem



$$\Delta W = \vec{F} \, . \, \Delta \vec{r} = F \cos\theta \, \Delta r = F_{\mathrm{II}} \, \Delta r$$

Work done by the force in a small displacement

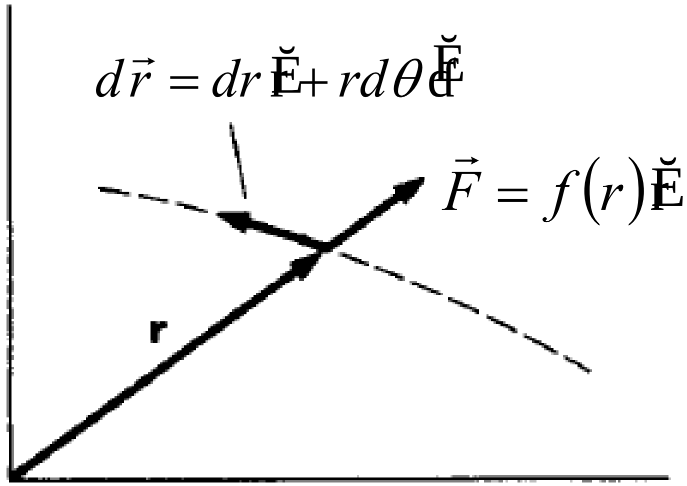
Perpendicular component does no work!

Work done for a finite displacement = sum of contributions from each ΔW from each segment of the path

If **F** is the sum of several forces: $\vec{F} = \sum_i \vec{F}_i$

$$\Rightarrow \qquad (W_i)_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}_i \, . \, d\vec{r}$$

Work done by a central force $\Rightarrow$ It is a function of *r* alone



$$W_{ba} = \oint \vec{F}.d\vec{r} = \int_a^b f(r)\,\hat{r}.\left(dr\,\hat{r} + r\,d\theta\,\hat{\theta}\right) = \int_a^b f(r)\,dr$$

Work done depends only initial and final positions besides the functional dependence of *f*(*r*) and not on the particular path

This is in contrast to force of sliding friction, where work done would be different for different paths