PH101

Work-Energy/Momentum

Dr. Murtaza

05-09-2014

Overview

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

$$\sum \vec{F}_i = \vec{0} = \frac{\overrightarrow{dP}}{dt} \Rightarrow m\,\vec{v} = \text{constant}$$

$$\Downarrow$$

$$\int_a^b m\,\frac{\overrightarrow{\mathrm{dv}}}{\mathrm{dt}}\,dx = \int_a^b F(x)dx$$

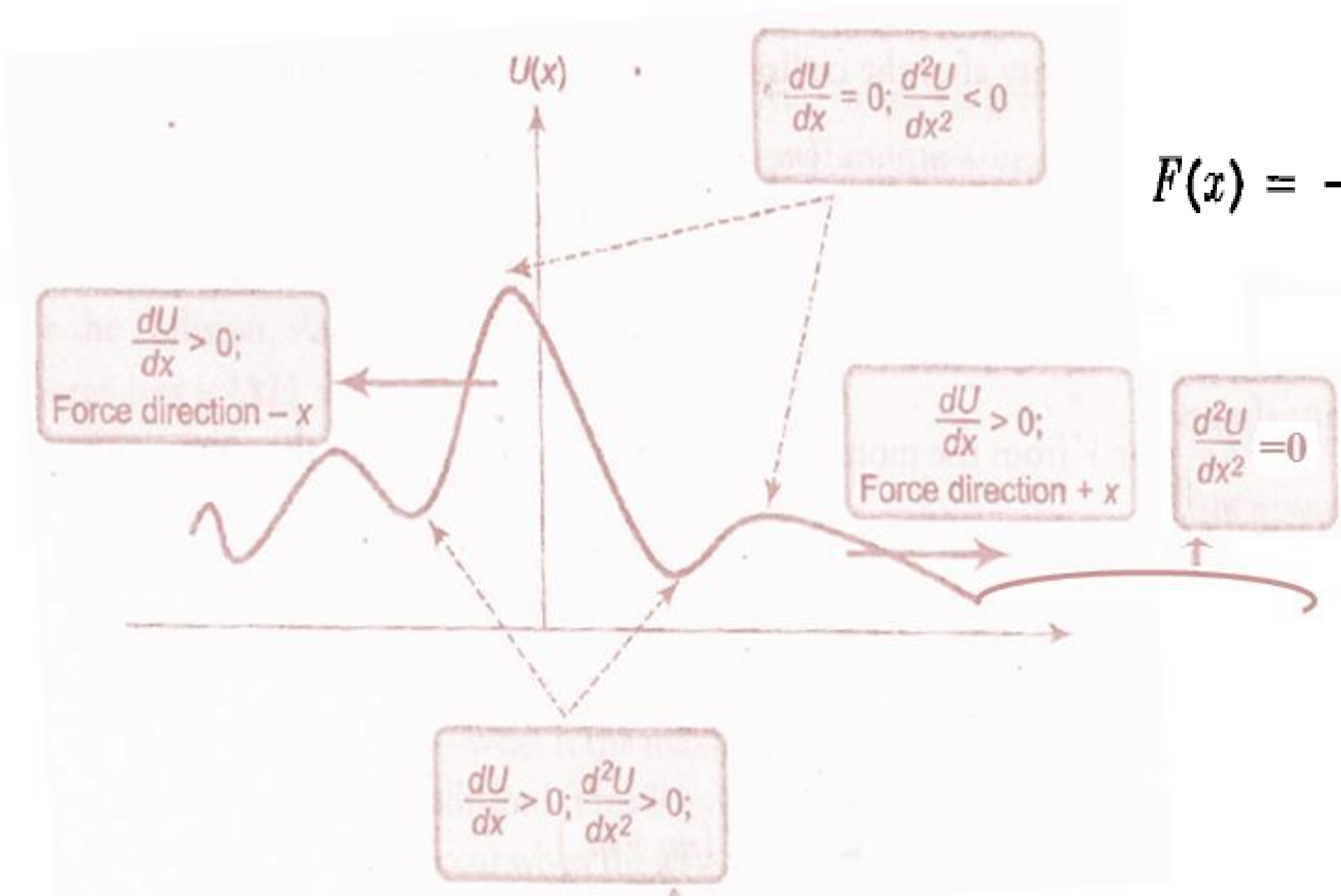
$$\Downarrow$$

$$\frac{1}{2}m\,{v_b}^2 - \frac{1}{2}m\,{v_a}^2 = \int_a^b F(x)dx = work\ energy\ theorem$$

$$\Downarrow$$

$$U_b - U_a = \int_a^b F(x)dx$$

Learning About Force And Motion From The Potential Energy Curve

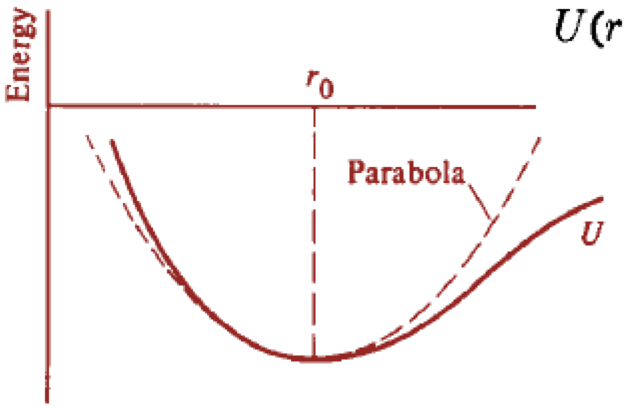


$$F(x) = -\frac{dU}{dx}$$

Oscillations about stable point

Suppose that we expand $U(r)$ about r_0, the position of the potential minimum. Then

$$U(r) = U(r_0) + (r - r_0)\left.\frac{dU}{dr}\right|_{r_0} + \frac{1}{2}(r - r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r_0} + \cdots .$$



U is a minimum at r_0, $(dU/dr)\,|_{r_0} = 0$.

$$U(r) = U(r_0) + \frac{1}{2}(r - r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r_0}$$

This is the potential energy of a harmonic oscillator,

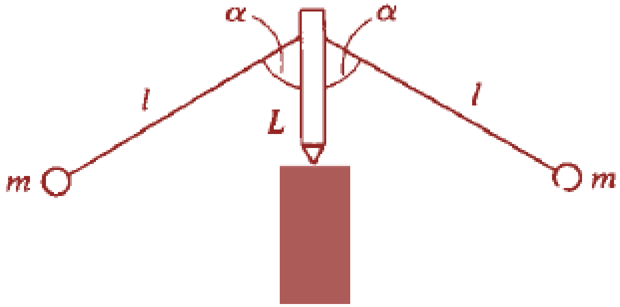
$$U(x) = \text{constant} + \frac{kx^2}{2}.$$

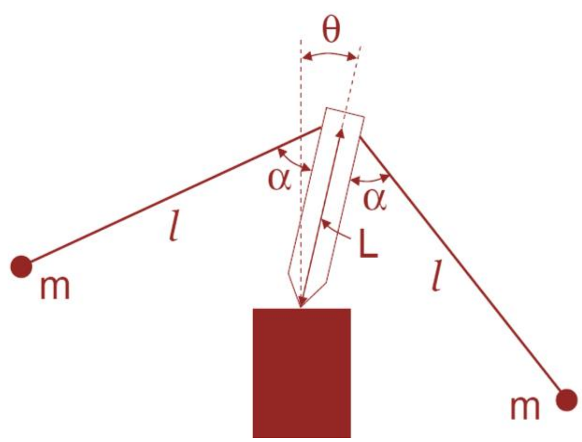
We can even identify the effective spring constant:

$$k = \left.\frac{d^2U}{dr^2}\right|_{r_0}$$

Teeter Toy

A teeter toy consists of two weights of mass m, which hang from a peg attached to two arms at a fixed angle α. We wish to determine the stability of the teeter toy when the peg is rotated from the vertical by an angle θ. You may consider the arms and peg as massless.



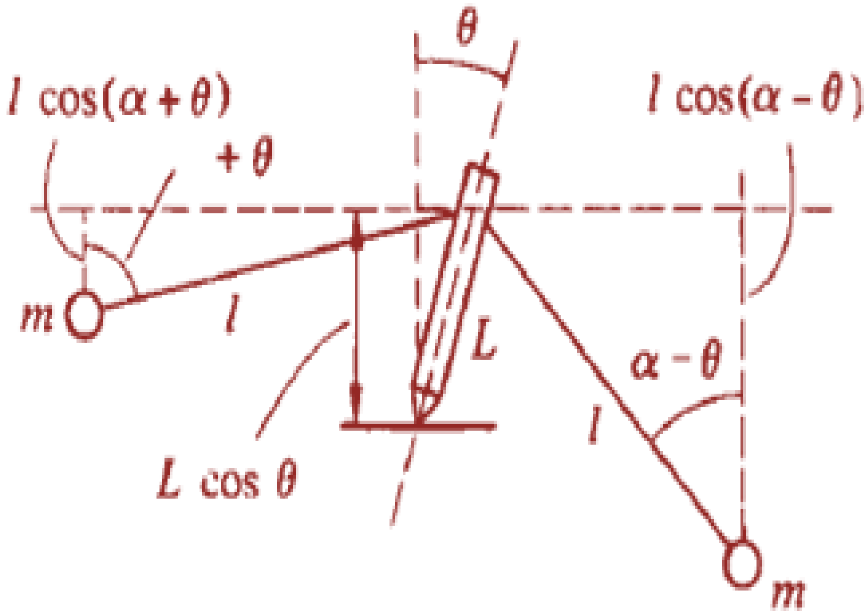


Energy and stability

(A) The potential energy $U(\theta)$ is given by

$$U(\theta) = mg(y_l + y_2)$$

$$U(\theta) = mg[L\cos\theta - l\cos(\alpha+\theta)] + mg[L\cos\theta - l\cos(\alpha-\theta)]$$



Using the identity $\cos(\alpha \pm \theta) = \cos\alpha\cos\theta \mp \sin\alpha\sin\theta$

$$U(\theta) = 2mg\cos\theta(L - l\cos\alpha)$$

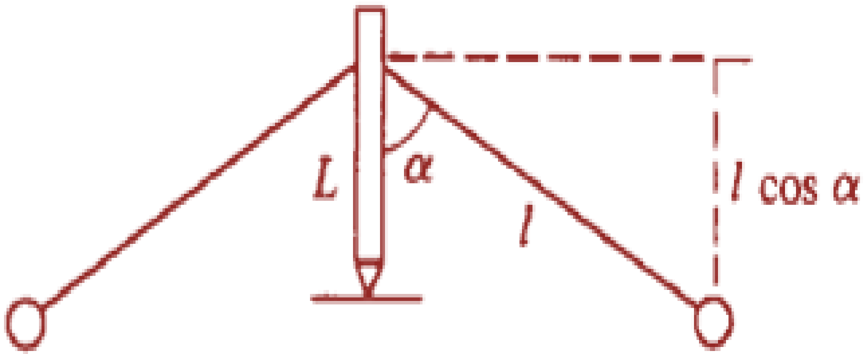
Equilibrium occurs when

$$\frac{dU}{d\theta} = -2mg\sin\theta(L - l\cos\alpha)$$

$$= 0.$$

The solution is $\theta = 0$, as we expect from symmetry. (We reject the solution $\theta = \pi$ on the grounds that θ must be limited to values less than $\pi/2$.) To investigate the stability of the equilibrium position, we must examine the second derivative of the potential energy. We have

$$\frac{d^2U}{d\theta^2} = -2mg\cos\theta(L - l\cos\alpha)$$



At equilibrium,

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = -2mg(L - l\cos\alpha)$$

For the second derivative to be positive, we require $L - l\cos\alpha < 0$, or

$L < l\cos\alpha$.

In order for the teeter toy to be stable, the weights must hang below the pivot.

we used the fact that m and *g* are positive. Looking back at Figure, we see that this condition requires the centre of mass of the teeter toy to be below its pivot point, which is what we would expect (a pendulum has its centre of mass below its pivot, and it is stable; try to balance a pencil on its tip -it is not stable; its centre of mass is above the pivot.)

Small oscillation

the period of oscillation of the toy when it is rocking from side to side.

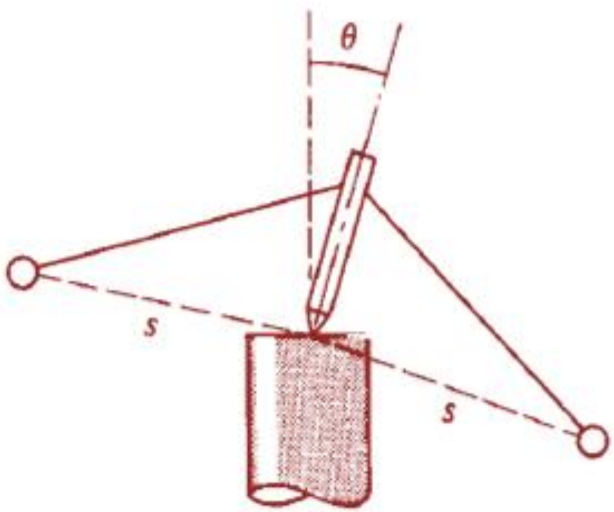
potential energy $$U(\theta) = -A\cos\theta$$

where $A = 2mg(l\cos\alpha - L)$. For stability, $A > 0$.

If we expand $U(\theta)$ about $\theta = 0$, we have

$$U(\theta) = -A\left(1 - \frac{\theta^2}{2} + \cdots\right)$$

since $\cos\theta = 1 - \theta^2/2 + \cdots$.



$$U(\theta) = -A + \tfrac{1}{2}A\theta^2.$$

$$\frac{d^2U}{d\theta^2} = A = \boldsymbol{k}$$

$$\omega^2 = \sqrt{\frac{k}{I}}$$

$$\omega^2 = \sqrt{\frac{2mg(l\,cos\alpha - L)}{2ms^2}} \Rightarrow \boldsymbol{\omega} = \sqrt{\frac{g(l\,cos\alpha - L)}{s^2}}$$

Conclusions:

1- for stability $l\cos\alpha - L > 0$.

2- as $l\cos\alpha - L$ approaches zero, ω approaches zero, and the period of oscillation becomes infinite. In the limit $l\cos\alpha - L = 0$, the system is in neutral equilibrium,

3- if $l\cos\alpha - L < 0$, the system becomes unstable. Thus, a low frequency of oscillation is associated with the system operating near the threshold of stability. This is a general property of stable systems, because a low frequency of oscillation corresponds to a weak restoring force.

4- So the angular frequency of small oscillations of the teeter toy is

$$\boxed{\omega = \sqrt{\frac{g(l\cos\alpha - L)}{l^2 + L^2 - 2lL\cos\alpha}}}$$

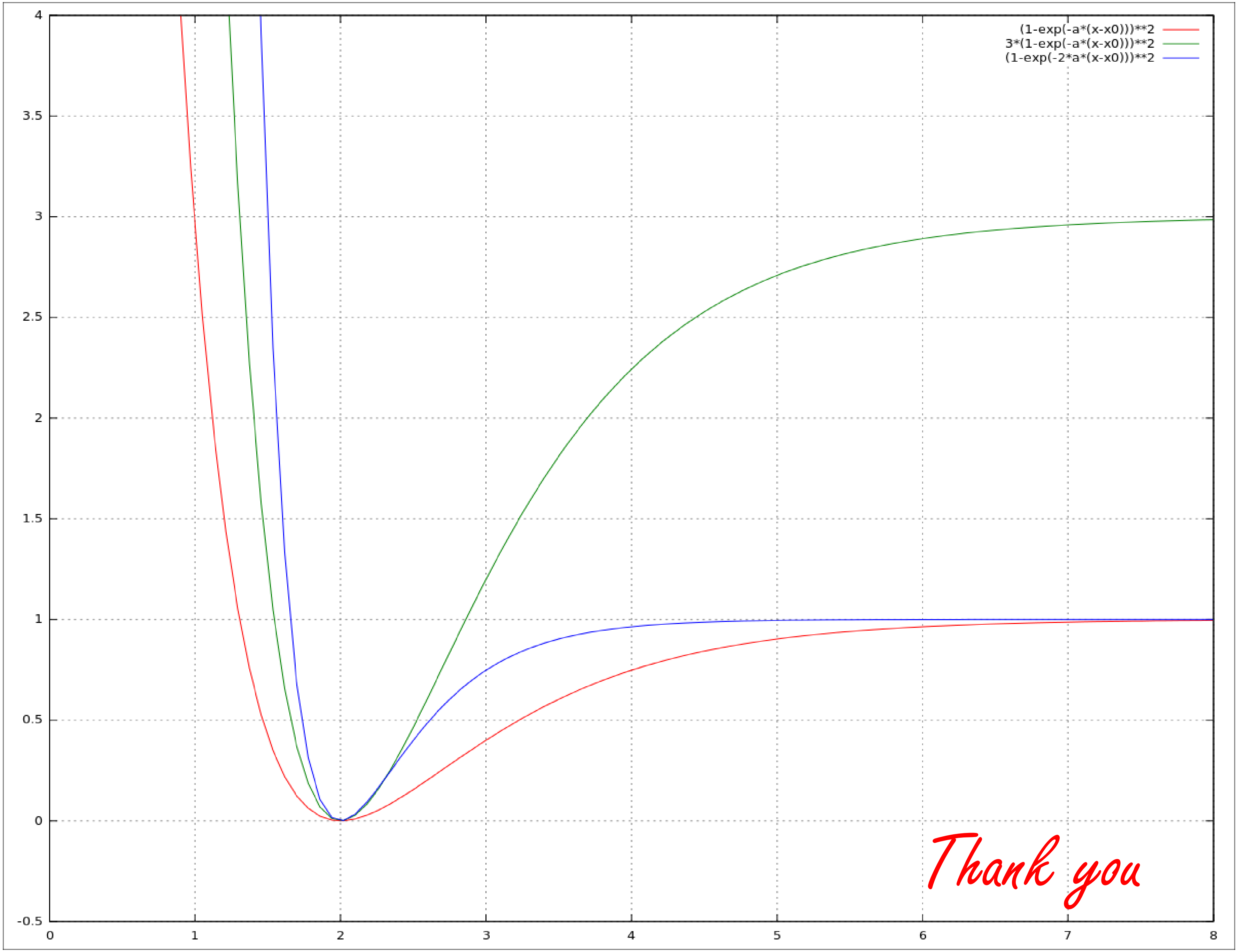
Note that if $\alpha = 0$, this reduces to $\sqrt{g/(l-L)}$, which is what you would expect, since then the teeter toy looks just like a pendulum of length $l - L$.

Exercise

Q.1 **Consider a particle of mass m moving in a potential of the form**

$$U(x)=C\,(1 - e^{-a(x-x_0)})^2$$

(a) If the potential were to be approximated by periodic potential , find corresponding 'spring constant'
(b) Plot the potential U(x) and corresponding approximation to it.
(c) For small oscillation about the equilibrium point, find the frequency of oscillations.



(1-exp(-a*(x-x0)))**2
3*(1-exp(-a*(x-x0)))**2
(1-exp(-2*a*(x-x0)))**2
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
1
2
3
4
5
6
7
8
Thank you