PH101
Lecture17

04.09.14

Recap

Taylor's series expansion of P.E. $U\,(r)$ about a point r_0

$$U(r) = U(r_0) + \underbrace{(r - r_0)\left.\frac{dU}{dr}\right|_{r_0}}_{} + \frac{1}{2}(r - r_0)^2 \left.\frac{d^2U}{dr^2}\right|_{r_0} + \ldots\ldots$$

= 0 because U is minimum at $r = r_0$ for a parabolic P.E.

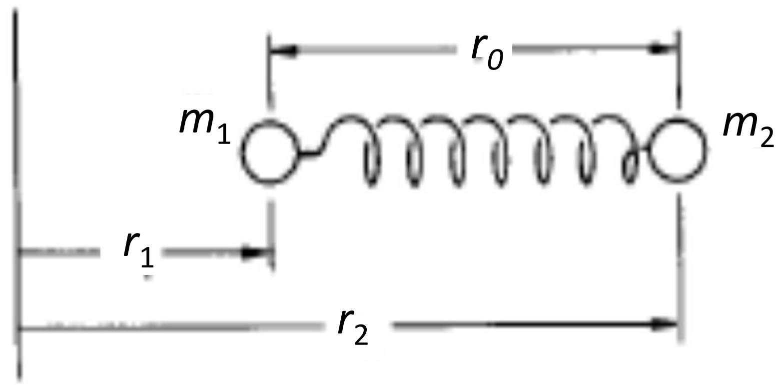
For sufficiently small displacements, terms beyond the 3rd term can be ignored $\Rightarrow$

$$U(r) \cong U(r_0) + \frac{1}{2}(r - r_0)^2 \left.\frac{d^2U}{dr^2}\right|_{r_0} \equiv \text{const} + \frac{1}{2}k(r - r_0)^2$$

Effective spring const.

$$k = \left.\frac{d^2U}{dr^2}\right|_{r_0}$$

Frequency of molecular vibrations



Eqs of motion for two atoms are

$$m_1\ddot{r}_1 = k(r - r_0)$$

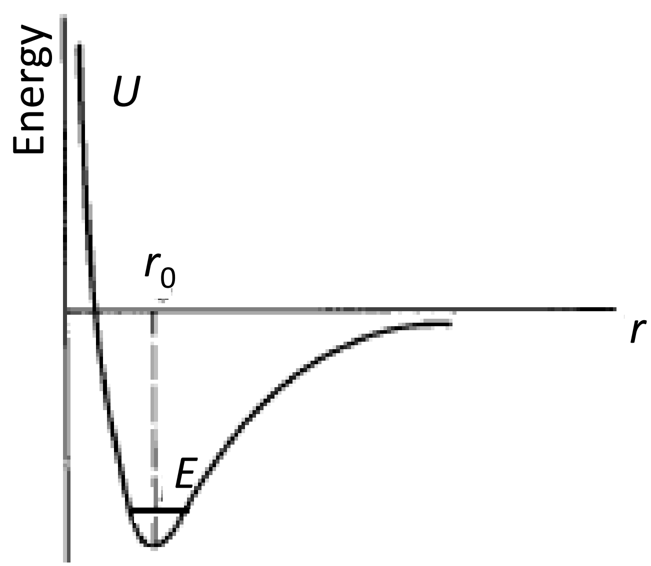
$$m_2\ddot{r}_2 = -k(r - r_0)$$

where $r = r_2 - r_1$

is the instantaneous separation of the atoms

which we assume to be small enough

$\Rightarrow$ separation $\sim$ close to equilibrium length r_0 of a spring of spring constant k



To solve divide the eqs respectively by m_1 and m_2 and subtract 1st from the 2nd

$$\ddot{r}_2 - \ddot{r}_1 \equiv \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0)$$

$$\ddot{r}_2 - \ddot{r}_1 \equiv \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0) = -k\left(\frac{m_1 + m_2}{m_1 m_2}\right)(r - r_0)$$

$$\ddot{r} = -\frac{k}{\mu}(r - r_0)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Dimension of μ? Called reduced mass

For a harmonic oscillator, eq of motion is

$$\ddot{x} = -\frac{k}{m}(x - x_0)$$

What is its freq of oscillation?

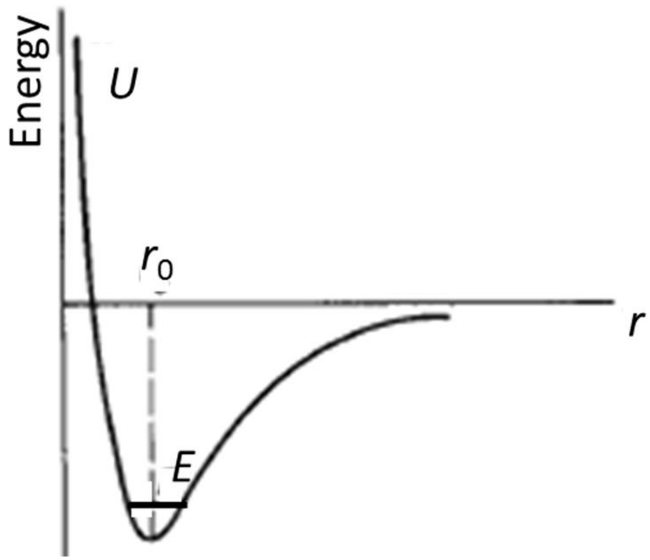
$$\omega = \sqrt{\frac{k}{m}}$$

From this analogy

Freq of vibration of the molecule

$$\omega_{\text{vib}} = \sqrt{\frac{k}{\mu}}$$

$$\mu \equiv m$$



$$\text{and } k = \left.\frac{d^2U}{dr^2}\right|_{r_0}$$

υ_{vib} typically lies in the near-infra red (3 x 10^{13} Hz)

Vibrational motion is a characteristic of a molecule and identifies the light emitted by the molecule

υ_{vib} yields value of $\frac{d^2U}{dr^2}$ at the P.E. minimum

For HCl molecule, *k* is 5 x 10^5 dynes/cm

$\Rightarrow$ All such bound systems behave like a harmonic oscillator for small displacements

For larger amplitudes , higher order terms in Taylor's series introduce certain anharmonicity , which modifies the P.E. curve

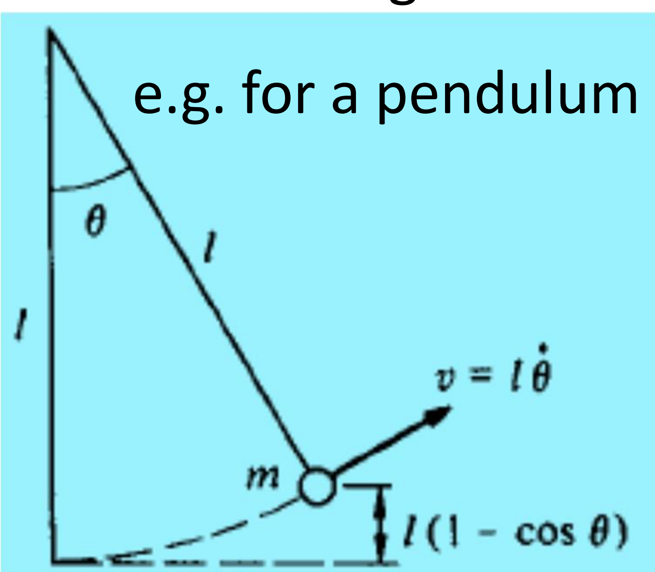
Harmonic oscillator approxn is thus quite ubiquitous even down to internal motions in nuclei!

For the simple case $U = \frac{1}{2}kx^2; K = \frac{1}{2}m\dot{x}^2; \omega = \sqrt{\frac{k}{m}}$

In terms of a general variable *q* appropriate to a problem

$$U = \frac{1}{2}Aq^2 + \text{const}; K = \frac{1}{2}B\dot{q}^2$$

e.g. for a pendulum



$$U = mgl(1 - \cos\theta) = mgl.2\sin^2\frac{\theta}{2} \approx mgl\frac{\theta^2}{2}$$

$$K = \frac{1}{2}ml^2\dot{\theta}^2$$

From

$$U = \frac{1}{2}Aq^2 + \text{const}; K = \frac{1}{2}B\dot{q}^2$$

By comparison with the case of a harmonic oscillator of a mass *m* on a spring of spring const *k*, frequency in the general case

$$\omega = \sqrt{\frac{A}{B}}$$

Total energy of the system

$$E = U + K = \frac{1}{2}Aq^2 + \text{const} + \frac{1}{2}B\dot{q}^2$$

the system being conservative $\frac{dE}{dt} = 0$

$$\Rightarrow \frac{1}{\cancel{2}}A.\cancel{2}q\dot{q} + \frac{1}{\cancel{2}}B.\cancel{2}\dot{q}\ddot{q} = 0 \quad \Rightarrow Aq + B\ddot{q} = 0$$

$$\Rightarrow \ddot{q} + \frac{A}{B}q = 0 \qquad \Rightarrow \omega = \sqrt{\frac{A}{B}}$$

Work-energy theorem for nonconservative force

For a body falling through air under gravity, force

$$\vec{F} = \vec{F}^{c} + \vec{F}^{nc}$$

gravitational force (conservative) and air friction (nonconservative)

Total work done

$$W_{ba}^{T} = \oint_{a}^{b} \vec{F} . d\vec{r}$$

$$= \oint_{a}^{b} \vec{F}^{c} . d\vec{r} + \oint_{a}^{b} \vec{F}^{nc} . d\vec{r}$$

$$= -U_b + U_a + W_{ba}^{nc}$$

Thus work-energy theorem takes the form

$$K_b - K_a = -U_b + U_a + W_{ba}^{nc} \qquad \Rightarrow E_b - E_a = W_{ba}{}^{nc}$$

$$\Rightarrow W_{ba}^{nc} = K_b + U_b - \left(K_a + U_a\right)$$

Thus $E = K + U$ is no longer const but depends on the state of the system

Power

$$P = \frac{dW}{dt}$$ i.e. rate of doing work

$$\vec{F}.d\vec{r} = dW \Rightarrow$$ Power delivered by the force :

$$P = \frac{dW}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \vec{F}.\vec{v}$$

1 W = 1 J/s

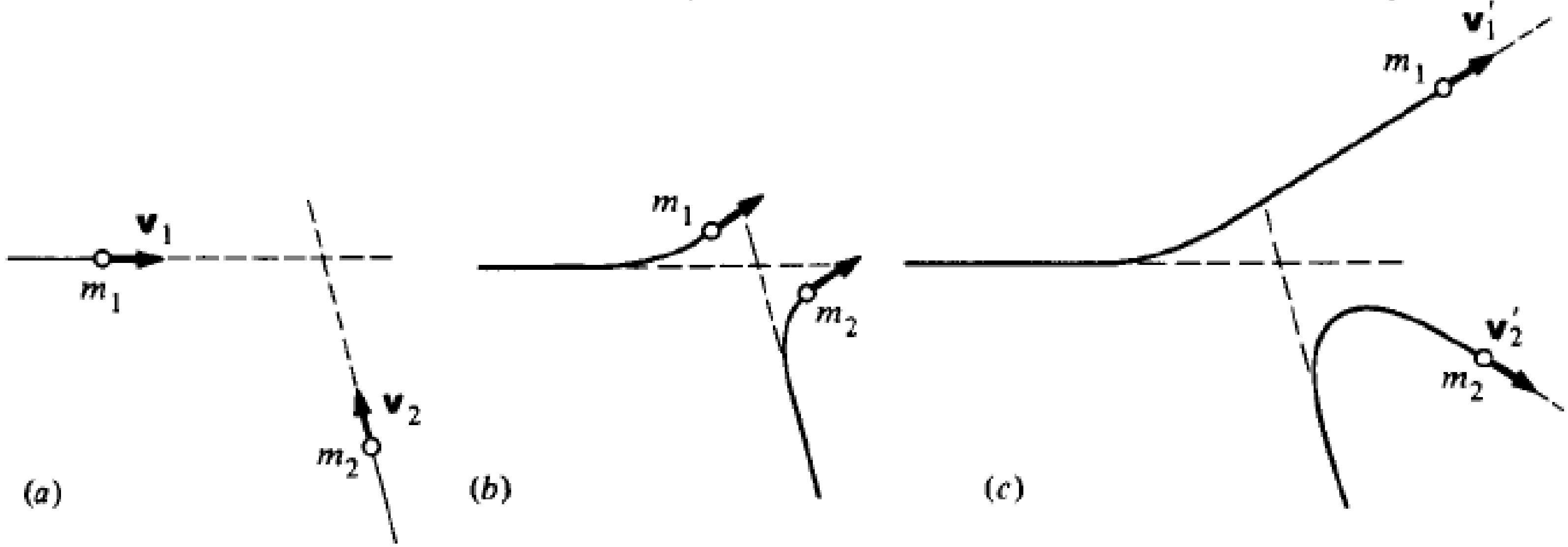
When the sun is overhead, it supplies ~ 1000 W/m^2 of energy to Earth's surface

Conservation laws and particle collisions

Scattering experiments e.g. of Rutherford in 1911

Scattering of α particles from a thin gold foil

Variation in no. of scattered α particles with deflection angle



a) Each particle is effectively free

b) Momentum & energy of each particle changes. Why?

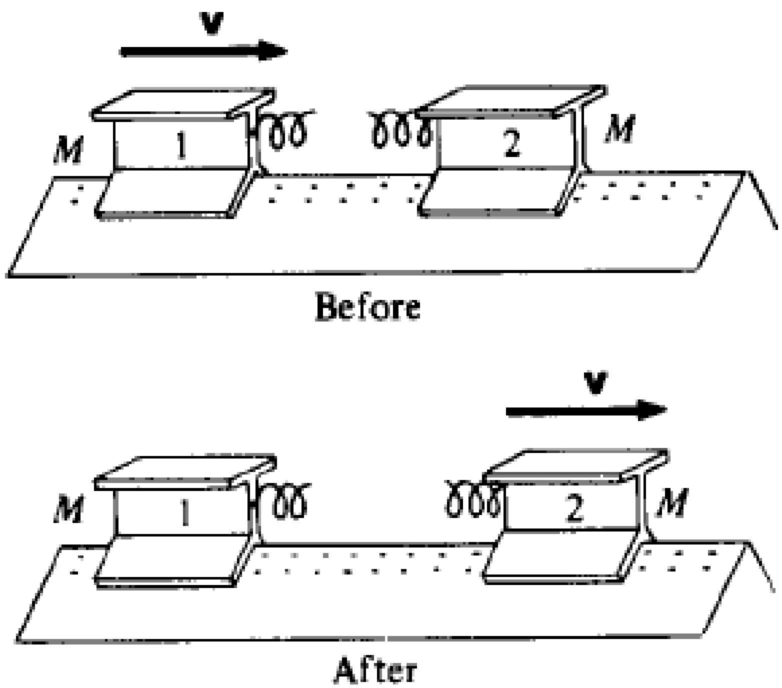
c) Particles are again free & move along new directions and velocities

Momentum is conserved since external forces could be neglected

$$\vec{P}_i = \vec{P}_f$$

For a two body collision

$$\Rightarrow \quad m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1^{\,/} + m_2\vec{v}_2^{\,/}$$



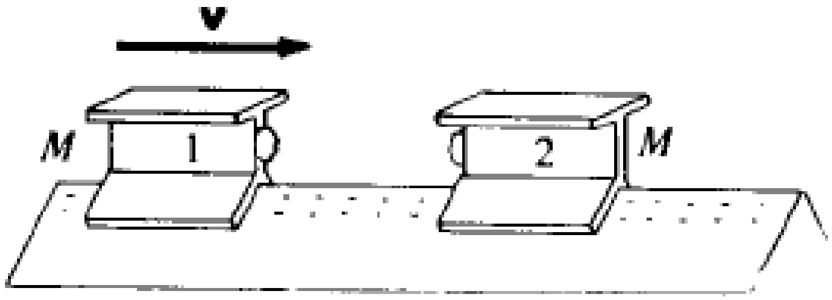
Total K.E. is conserved

↓

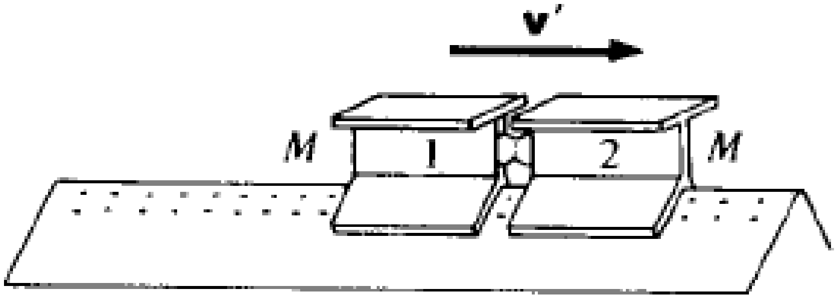
Elastic collisions

If the interaction forces are conservative (here the spring force)

Collision betn. two riders of mass *M* on a linear air track which interact via good coil springs



Before



After

Replace the springs by lumps of sticky putty

By conservation of momentum

$$M\,\vec{v} = 2M\,\vec{v}^{\,/}$$

$$v^{/} = \frac{v}{2}$$

Initial K.E.= $\dfrac{1}{2}M\,v^{2}$

Final K.E.= $(2M)\dfrac{v^{/2}}{2} = M\,\dfrac{v^{2}}{4}$

$\Rightarrow$ Half of initial K.E.! Why?

Interaction forces are nonconservative *Inelastic* collision

$$K_i = K_f + Q$$

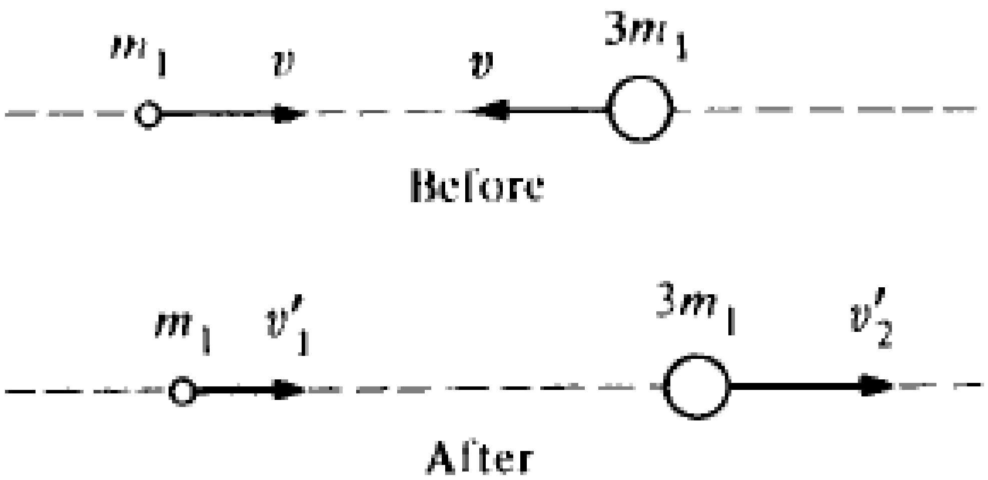
$$Q = K_i - K_f$$

Amount of K.E. converted to some other form

$\Rightarrow$ energy eq becomes

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 + Q$$

Elastic collision of two balls



Assume equal and opposite velocities before collision

$$\Rightarrow \quad m_1 v - 3m_1 v = m_1 v_1' + 3m_1 v_2'$$

$$\frac{1}{2}m_1 v^2 + \frac{1}{2}(3m_1)v^2 = \frac{1}{2}m_1 v'^2_1 + \frac{1}{2}(3m_1)v'^2_2$$

Eliminate v_1' from 1st eq and substitute in the 2nd eq

$$\Rightarrow \quad 12v\,v_2' + 12v_2'^2 = 0 \qquad \Rightarrow \quad v_2'(v_2' + v) = 0$$

$\text{Solutions are}: v_2' = -v \;\&\; v_2' = 0$

$$v_1' = v$$

$$v_1' = -2v$$

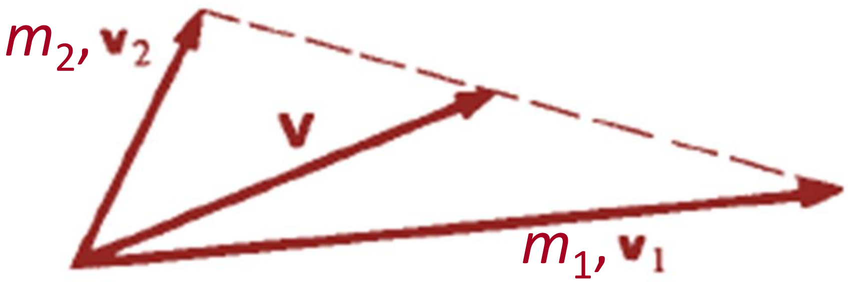
$\Rightarrow$

Initial condition

$\Rightarrow$

After collision, m_1 moves with twice its initial velocity to the left while the heavier one remains at rest

Collision in center of mass coordinates

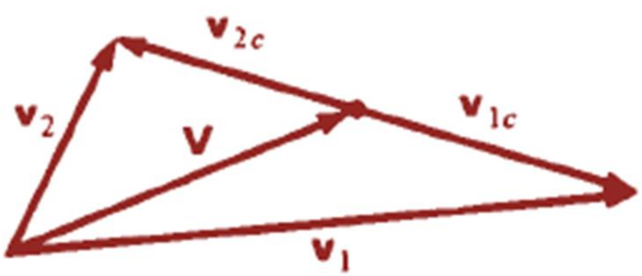


If $\mathbf{v}_1$ and $\mathbf{v}_2$ correspond to velocities of two masses m_1 and m_2

Center of mass velocities is given by

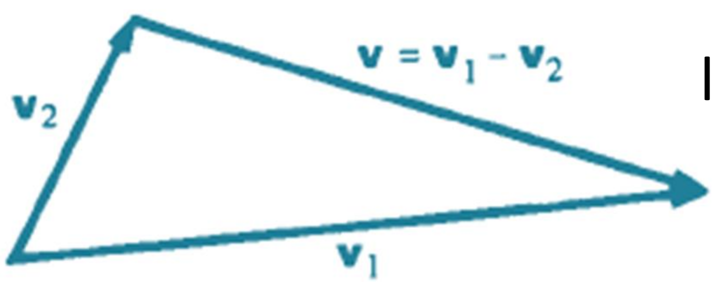
$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

velocities in center of mass system will be given by



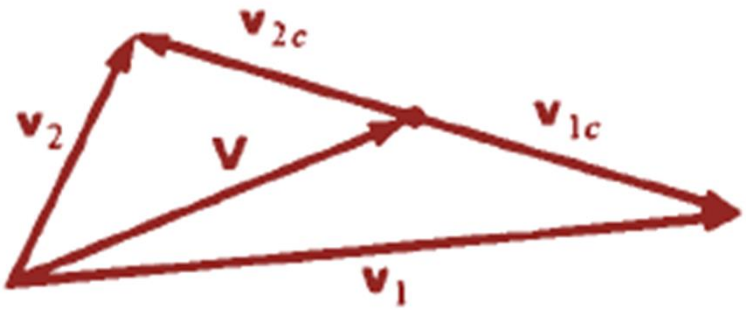
$$\vec{v}_{1c} = \vec{v}_1 - \vec{V} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$= \frac{m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right)$$



likewise

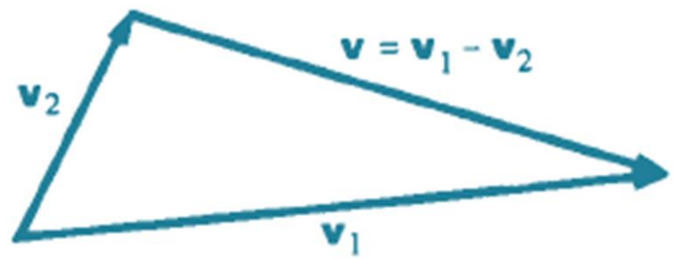
$$\vec{v}_{2c} = \vec{v}_2 - \vec{V} = \frac{-m_1}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right)$$



$\mathbf{v}_{1c}$ and $\mathbf{v}_{2c}$ lie back to back along the relative vel vector

$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$$

$\Rightarrow$ Momenta in the center of mass system



$$\vec{p}_{1c} = m_1\vec{v}_{1c} = \frac{m_1 m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right) = \mu\vec{v}$$

$$\vec{p}_{2c} = m_2\vec{v}_{2c} = \frac{-m_1 m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right) = -\mu\vec{v}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Total momentum in the C system?

0

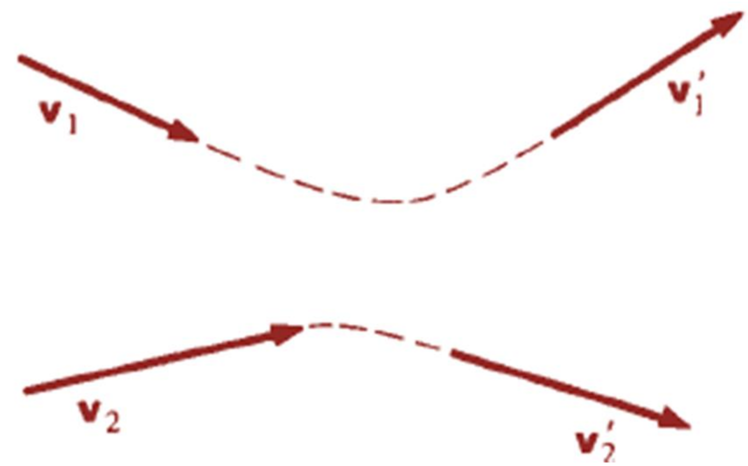
Total momentum in the lab system?

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \left(m_1 + m_2\right)\vec{V}$$

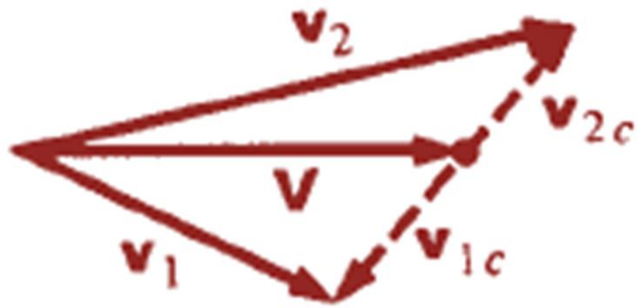
$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{V}$$

Total momentum in any collision is conserved

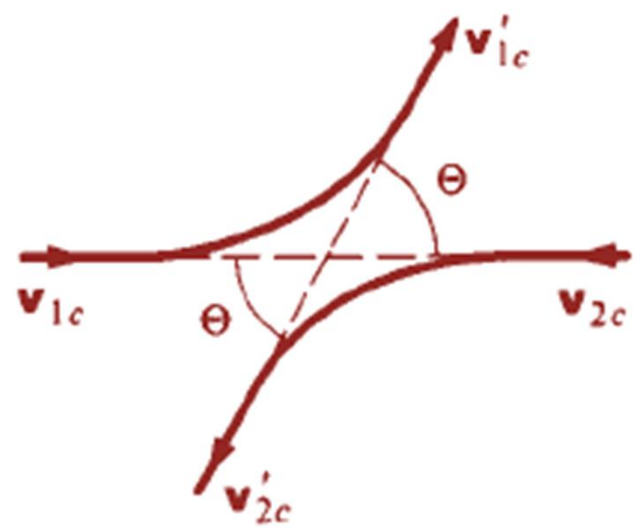
$\Rightarrow$ **V** must be a constant

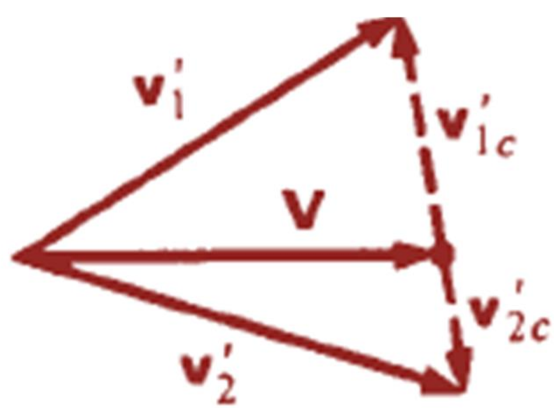


Trajectories of 2 colliding particles with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$



Initial velocities in the Lab and center of mass (C) systems





Velocities post-collision in C system

For elastic collisions, conservation of energy in the C system:

$$\frac{1}{2}m_1 v_{1c}^2 + \frac{1}{2}m_2 v_{2c}^2 = \frac{1}{2}m_1 v'^2_{1c} + \frac{1}{2}m_1 v'^2_{2c}$$

For elastic collisions, since momentum is 0 in the C system:

$$m_1 v_{1c} - m_2 v_{2c} = 0$$

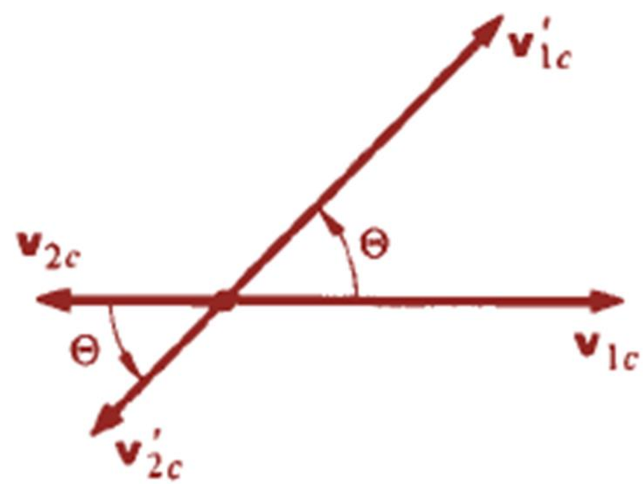
$$m_1 v'_{1c} - m_2 v'_{2c} = 0$$

$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1c}^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v'^2_{1c}$$

$$\Rightarrow v_{1c} = v'_{1c}$$

Similarly

$$v_{2c} = v'_{2c}$$



Vel. vectors simply rotate
In the scattering plane

If m_2 is at rest initially

$$\vec{V} = \frac{m_1}{m_1 + m_2}\vec{v}$$

$$\vec{v}_{1c} = \vec{v}_1 - \vec{V} = \frac{m_2}{m_1 + m_2}\vec{v}_1$$

$$\vec{v}_{2c} = \vec{v}_2 - \vec{V} = \frac{-m_1}{m_1 + m_2}\vec{v}_1$$