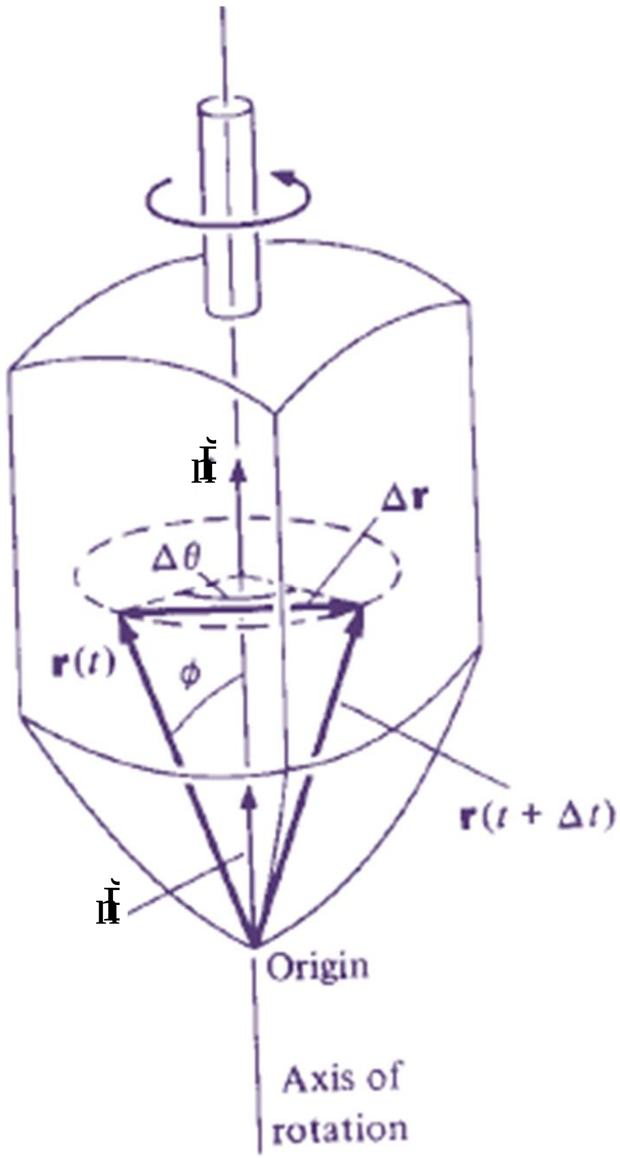
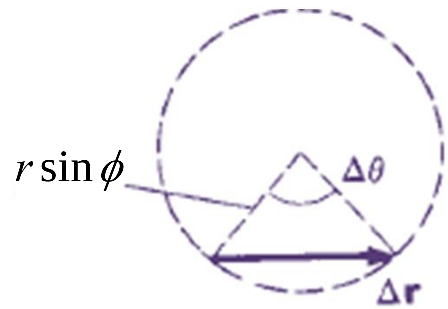
PH101
Lecture 23

18.09.14

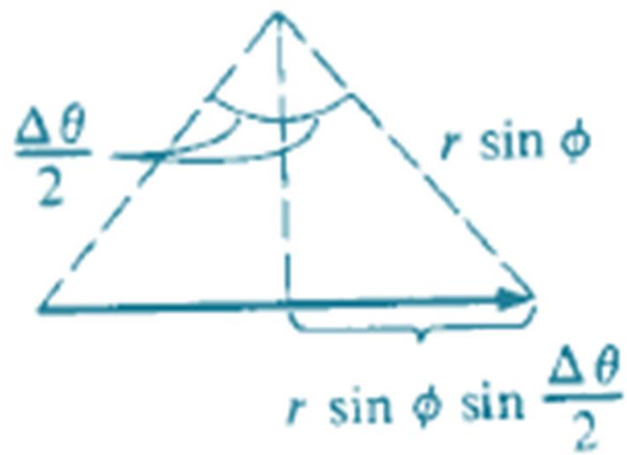
Recap

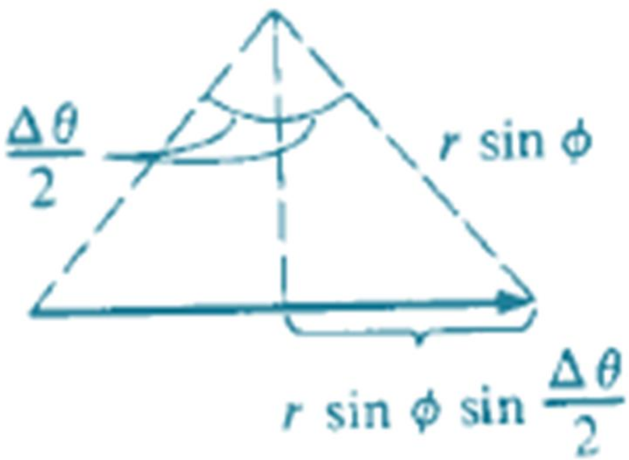


A vector **r** from the origin to any Particle tends to sweep out a cone; semi-angle ϕ is constant



Tip of **r** moves on a circle of radius r sin ϕ





$$\left|\Delta\vec{r}\right| = 2r\sin\phi\sin\frac{\Delta\theta}{2}$$

$$\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$$

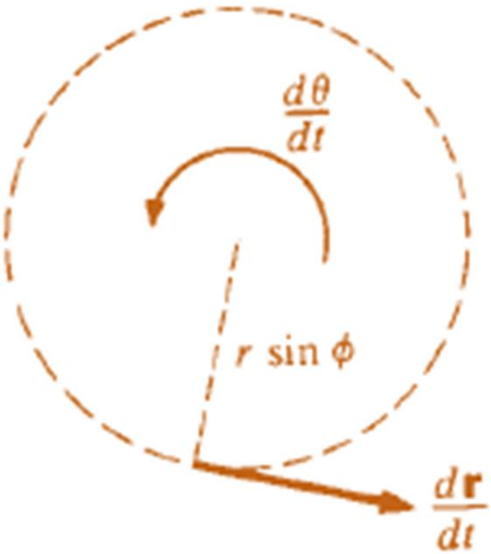
$$\Rightarrow \left|\Delta\vec{r}\right| \approx r\sin\phi\,\Delta\theta$$

If $\Delta\theta$ occurs over a time Δt

$$\Rightarrow \frac{\left|\Delta\vec{r}\right|}{\Delta t} \approx r\sin\phi\left(\frac{\Delta\theta}{\Delta t}\right)$$

which in the limit $\Delta t \to 0$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right| \approx r\sin\phi\left(\frac{d\theta}{dt}\right)$$



In the limit,

$\frac{d\vec{r}}{dt}$ is tangential to the circle

whose magnitude is $\left|\frac{d\vec{r}}{dt}\right| \approx r\sin\phi\left(\frac{d\theta}{dt}\right)$

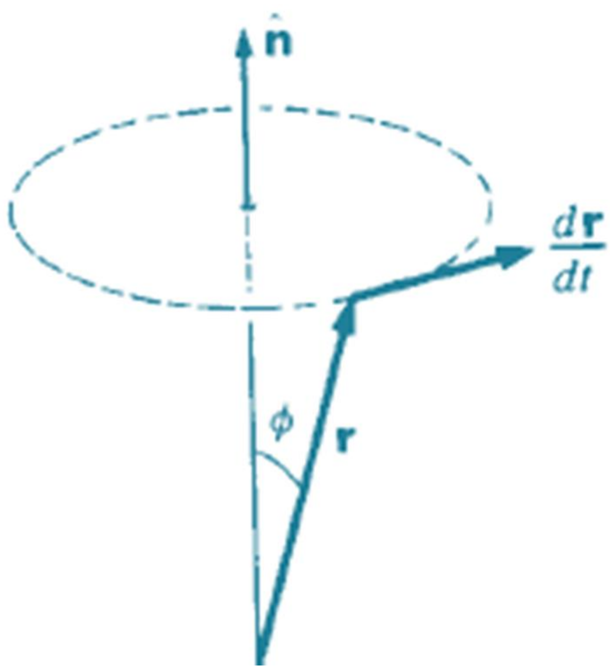
and direction?

Perpendicular to the plane of

$\vec{r}$ and $\hat{n}$

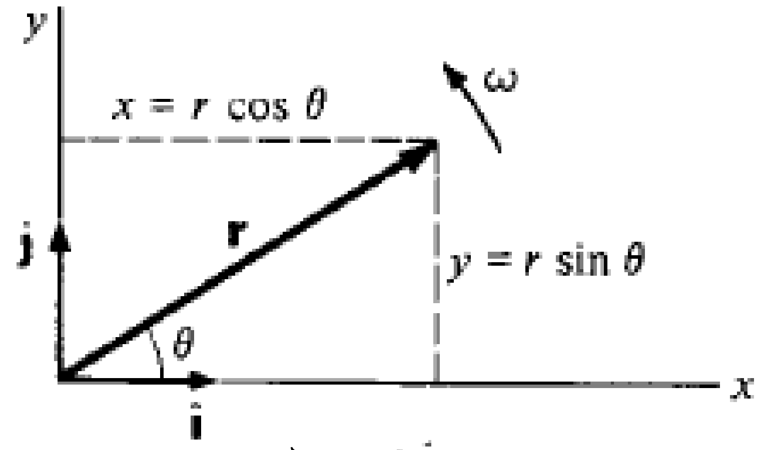
$$\Rightarrow \underbrace{\frac{d\vec{r}}{dt}}_{=\mathbf{v}} = \hat{n}\times\vec{r}\,\frac{d\theta}{dt}$$

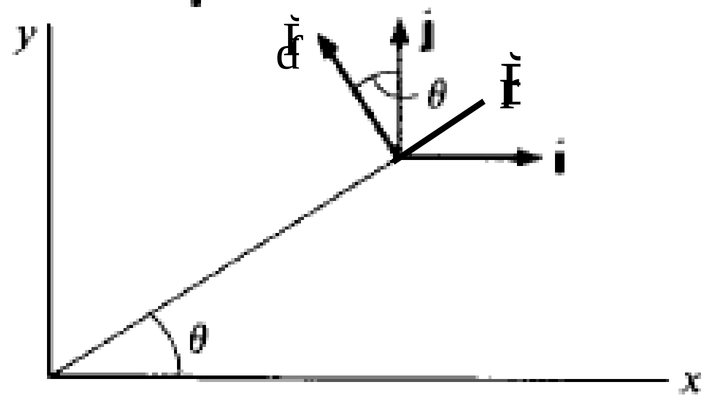
$$\hat{n}\frac{d\theta}{dt} = \vec{\omega}$$

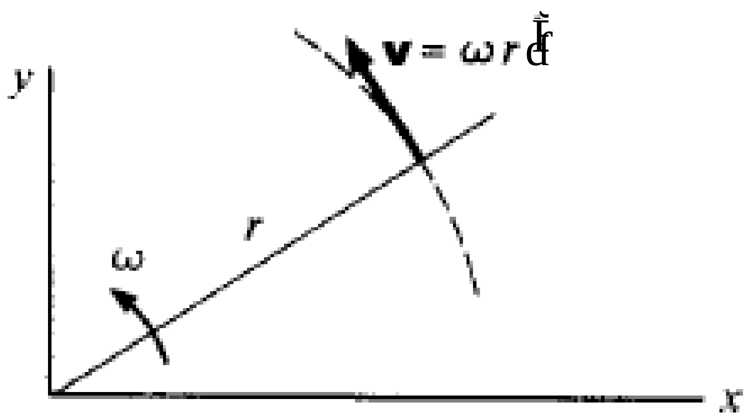


$$\Rightarrow \vec{v} = \vec{\omega}\times\vec{r}$$

Let us relate $\vec{v} = \vec{\omega} \times \vec{r}$ to rotation in the *xy* plane







For a particle rotating about *z* axis

We have

$$\vec{\omega} = \omega \hat{k}$$
$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\Rightarrow \quad \vec{v} = \vec{\omega} \times \vec{r} = \omega \hat{k} \times \left(x\hat{i} + y\hat{j} \right)$$

$$\Rightarrow \quad = \omega \left(x\hat{j} - y\hat{i} \right)$$

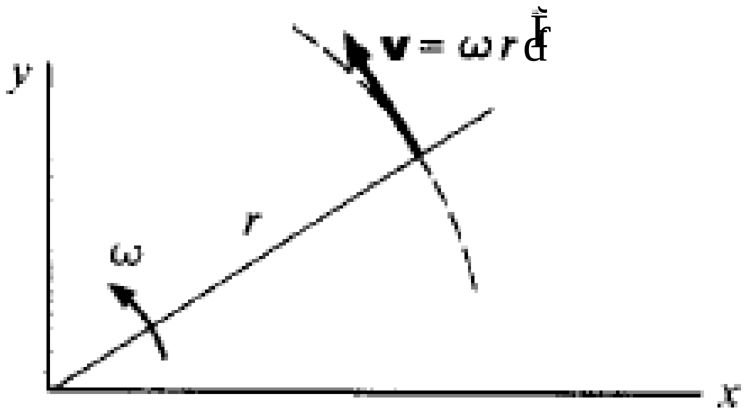
In polar coordinates

$$\vec{v} = \omega r \left(\underbrace{\hat{j}\cos\theta - \hat{i}\sin\theta}_{} \right)$$

A unit vector in the tangential direction $\hat{\theta}$

Thus

$$\vec{v} = \omega r \hat{\theta}$$



$$\vec{v} = \omega r \hat{e}_\theta$$

This is vel of a particle moving in a circle of radius *r* at angular vel ω

Since we are mostly used to rotation about a fixed axis, it is difficult to appreciate that ω is indeed a vector

If we express $\omega = \omega_1 + \omega_2$

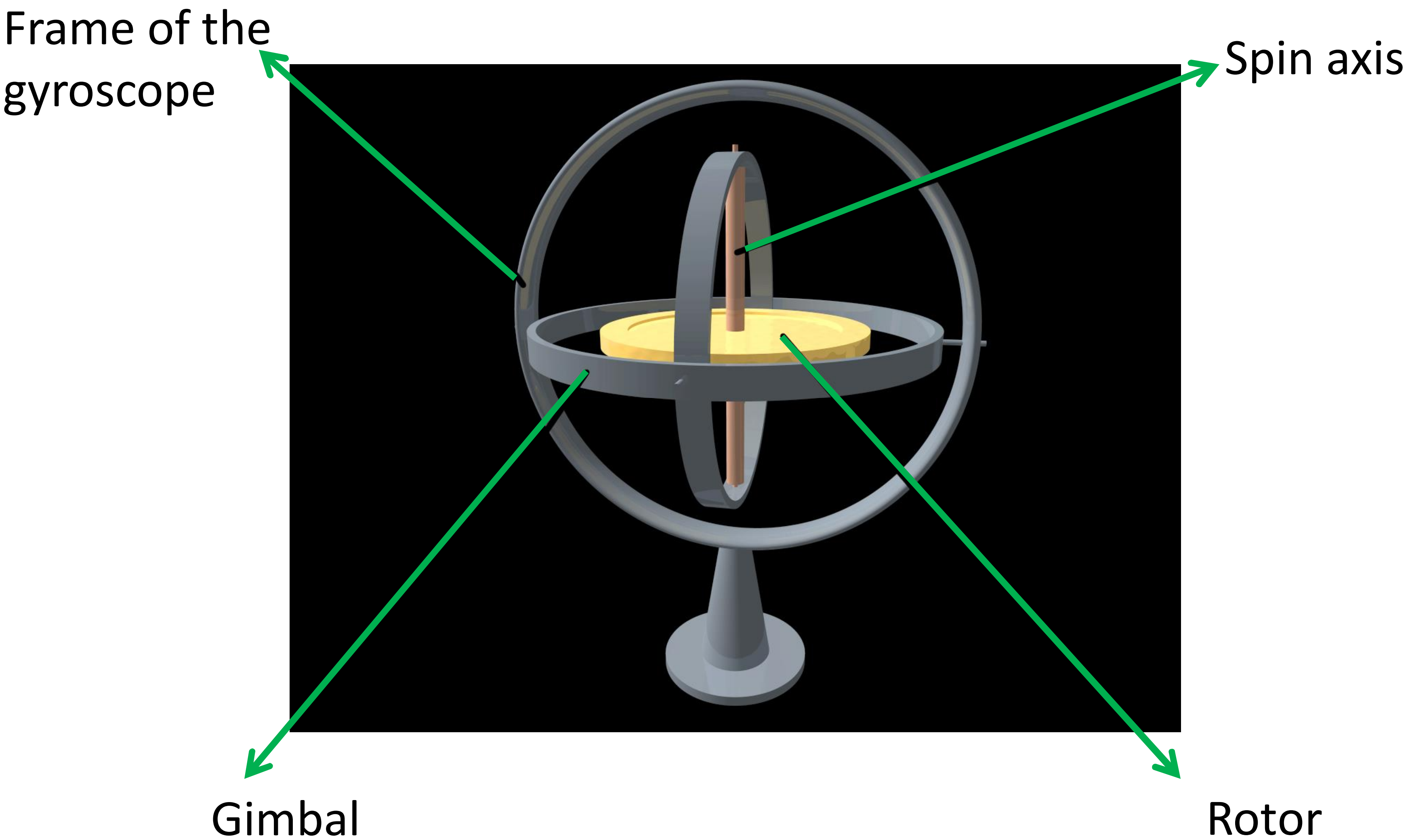
Like any other vector

$$\vec{v} = \vec{\omega} \times \vec{r} = \left(\vec{\omega}_1 \times \vec{r}\right) + \left(\vec{\omega}_2 \times \vec{r}\right)$$

will be also valid

Read the example 7.3 given in Kleppner's book to appreciate vector nature of angular velocity

<u>The Gyroscope</u>

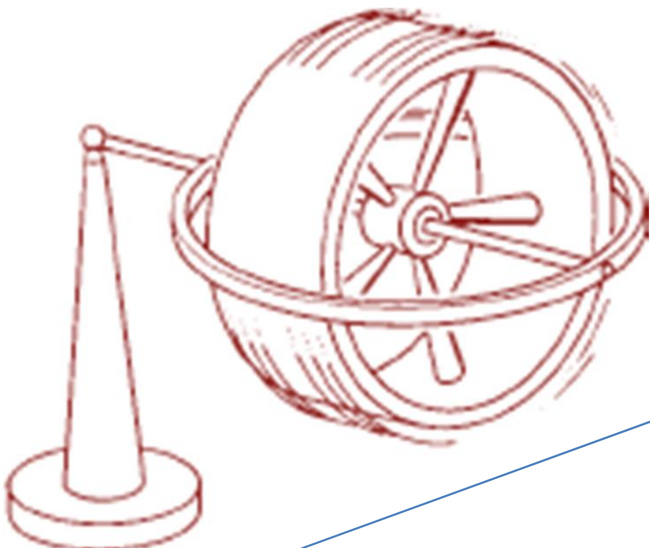


Invented in 1852 by French Experimental Physicist Leone Foucault

To investigate rotation of the earth

It is a device/instrument that is used to measure or maintain orientation based on applications of basic concepts of **L**, τ, and time derivative of a vector

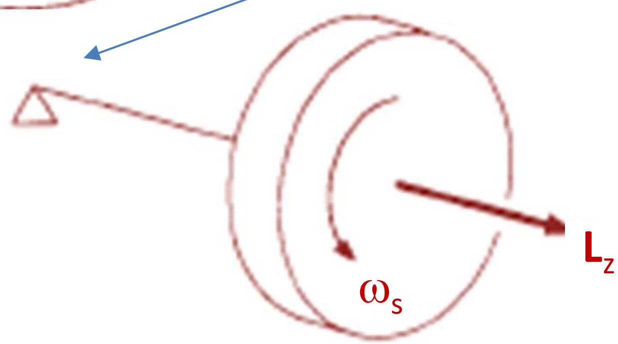
It is essentially a spinning flywheel and a suspension that allow the axle to take any orientation



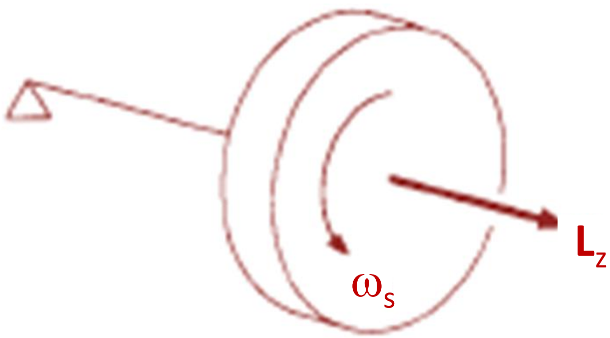
Typically it comprises a spinning wheel or disc in which the axle is free to assume any orientation.

End of the axis rests on a pylon

The triangle represents the free pivot

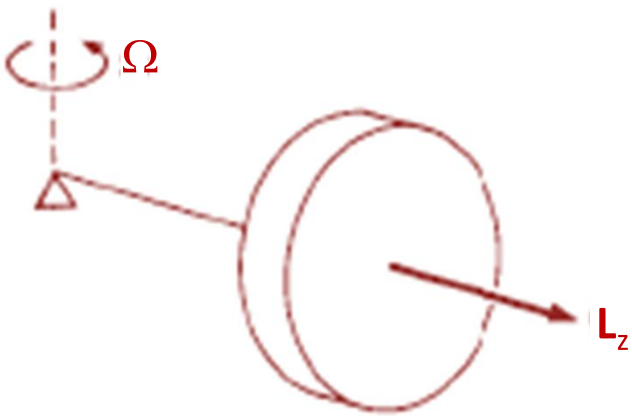


If the flywheel spins in the direction shown, direction of thumb will point towards the direction of angular momentum

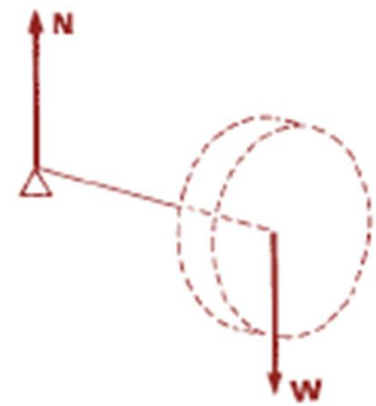


If the gyroscope is released horizontally with its one end supported by the pivot It slowly settles down to a *uniform precession*

$\Rightarrow$ The axle slowly rotates about the vertical with a constant angular velocity Ω



Why it does not fall?



$$\underbrace{\text{Total vertical force}}_{= N - W}$$

For $N = W$, c.m. can not fall!

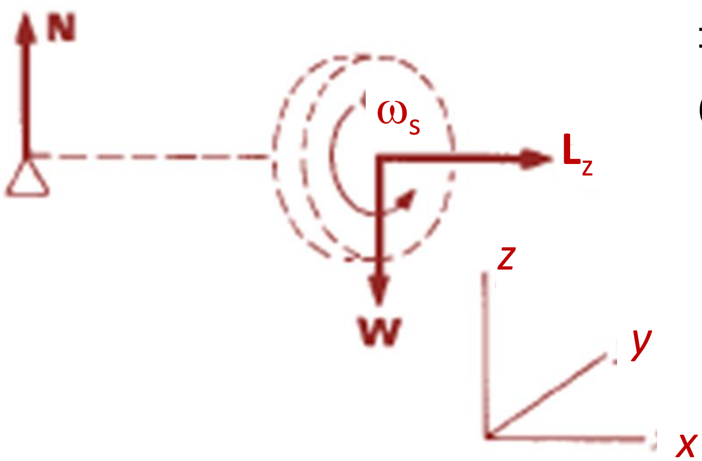
Why it does not swing about the pivot like a pendulum?

Indeed if it is released when its flywheel is stationary and not spinning

It would swing vertically like a pendulum instead of precessing horizontally

Precession occurs only when it is spinning rapidly

Major part of the system's angular momentum is contributed by the rapid spin about the horizontal axis and very little comes from the slow turning about the vertical axis



$\Rightarrow$ Almost entire angular momentum of the Gyroscope lies in $\mathbf{L}_s$

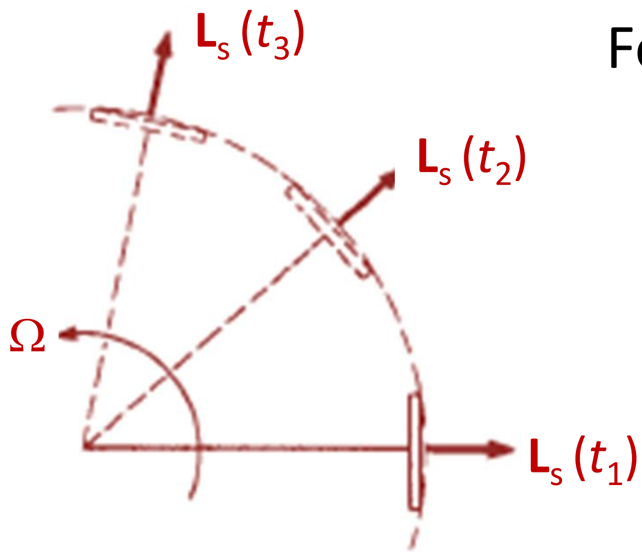
$$\left|\vec{L}_s\right| = \underbrace{I_0}_{\text{M.I. of the flywheel about its axle}} \omega_s$$

Note: During precession about z axis, the gyroscope does have an angular momentum along z albeit very small in magnitude

For uniform precession orbital $\mathbf{L}_p$ remains constant both in magnitude & direction

$\Rightarrow$ It plays no role in the dynamics of the gyroscope and hence may be ignored

As the gyroscope precesses, $\mathbf{L}_s$ rotates with it



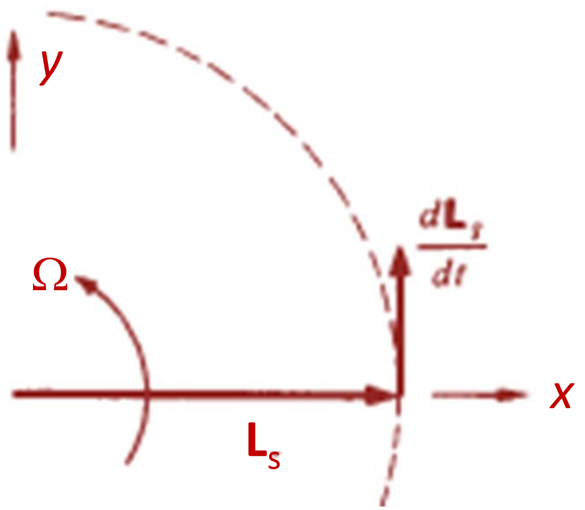
For an angular vel. Ω, rate of change of $\mathbf{L}_s$ is

$$\left|\frac{d\vec{L}_s}{dt}\right| = \Omega L_s$$

Only way a vector of constant magnitude **A** can change with time is only through rotation!

If $d\theta/dt$ represents instantaneous rate of rotation, then

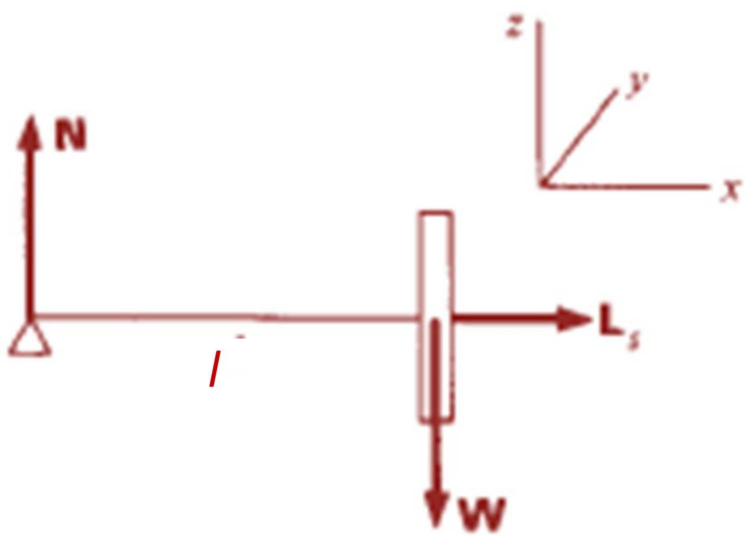
$$\left|\frac{d\vec{L}_s}{dt}\right| = L_s\frac{d\theta}{dt} = L_s\Omega$$



Direction of $\dfrac{d\vec{L}_s}{dt}$?

It is tangential to the horizontal circle that is swept out by $\mathbf{L}_s$

Due to a finite $\dfrac{d\vec{L}_s}{dt}$



Naturally, there will be a torque on the gyroscope due to weight of the flywheel acting at the end of the axle, whose magnitude is

$$\tau = \left|\vec{r} \times \vec{F}\right| = lW$$

And direction parallel to $d\mathbf{L}_s/dt$

$\Rightarrow$ along y

By definition

$$\tau = \left| \frac{d\vec{L}_s}{dt} \right|$$

But

$$\left| \frac{d\vec{L}_s}{dt} \right| = \Omega L_s \qquad \Rightarrow \quad \tau = lW$$

$$\Rightarrow \quad \Omega L_s = lW$$

$$\Rightarrow \quad \Omega = \frac{lW}{L_s} = \frac{lW}{I_0 \omega_s}$$

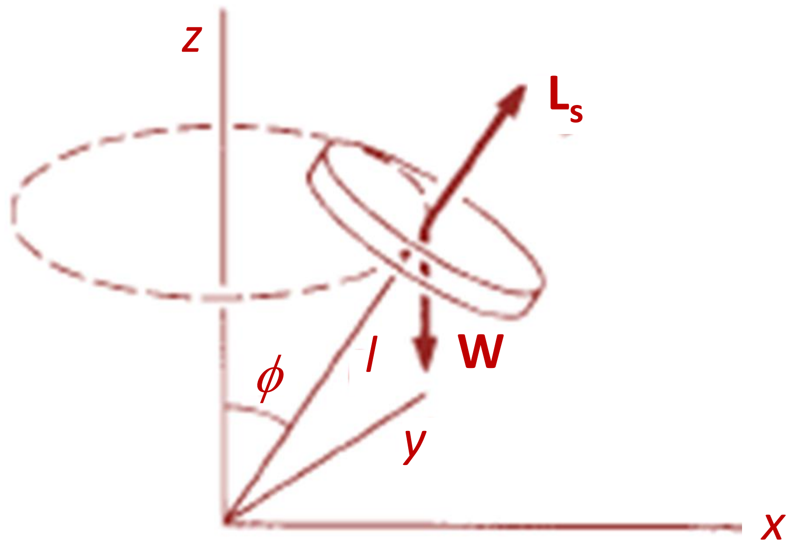
$\Rightarrow$ With slowing down of the flywheel, Ω increases

Can Ω increase indefinitely?

No. Eventually uniform precession will be overtaken by erratic & violent motion

Ω Becomes so large that that angular momentum about vertical axis can no longer be ignored due to frictional torque

If the gyroscope axis is elevated at some angle ϕ with the vertical, will the rate of precession Ω change?



Component of $\mathbf{L}_s$ parallel to *xy* plane changes

but one parallel to *z* axis remains unchanged

Horizontal component of $\mathbf{L}_s$:

$$L_s \sin\phi$$

$$\Rightarrow \left|\frac{d\vec{L}_s}{dt}\right| = \Omega L_s \sin\phi$$

τ due to gravity is horizontal

Its magnitude is: $\tau = l \sin\phi W$

Thus $\Omega L_s \sin\phi = l \sin\phi W \quad \Rightarrow \quad \Omega = \dfrac{lW}{L_s} = \dfrac{lW}{I_0 \omega_s}$ Independent of ϕ