PH101
Lecture 20 + 21

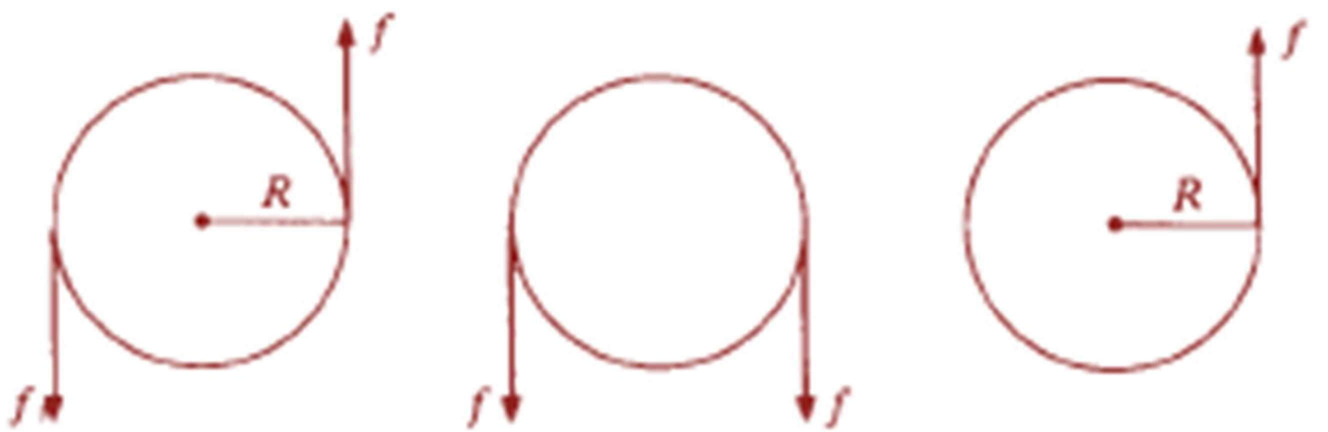
12.09.14 + 15.09.14

Torque and force are two different quantities even though we have been referring to analogy between them

One thing: from the very definition τ and **F** are mutually orthogonal

There could be special cases in which τ could be finite on a system even with zero net force and vice versa

For example



$\tau = 2Rf$ $\qquad$ $\tau = 0$ $\qquad$ $\tau = Rf$

$F = 0$ $\qquad$ $F = 2f$ $\qquad$ $F = f$

Torque is related to rate of change of **L**

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right)$$

$$= \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} \times \vec{p} + \vec{r} \times \underbrace{\frac{d\vec{p}}{dt}}_{= F}$$

$$\Rightarrow \vec{r} \times \vec{F}$$

$$\Rightarrow \vec{v} \times m\,\vec{v} = 0$$

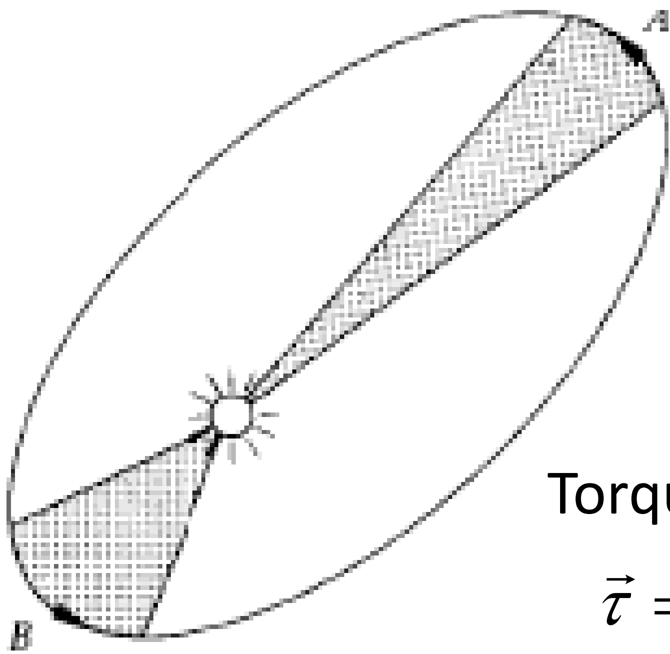
Thus

$$\tau = \frac{d\vec{L}}{dt}$$

$\Rightarrow$ If torque =0, angular momentum **L** will be constant and hence **L** is conserved

An interesting application of conservation of angular momentum lies in derivation of Kepler's 2nd law announced by Kepler in 1609

Area swept out by the radius vector joining sun to a planet in a given time is independent of the location of the planet in its orbit



Shaded areas in the figure is assumed to nominally represent area swept out In a given time at two different seasons in a year

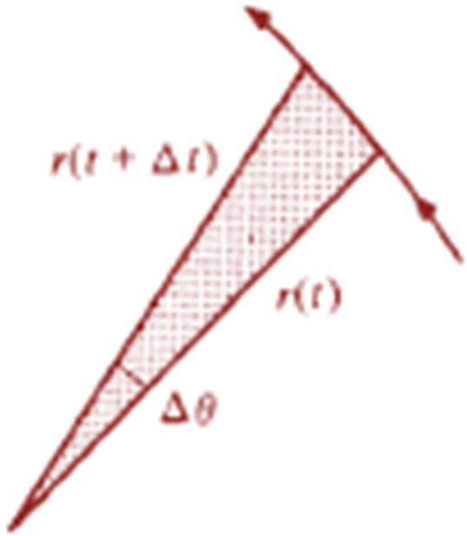
For a central force, $\vec{F} = f(r)\hat{r}$

Torque on the particle about the origin:

$$\vec{\tau} = \vec{r} \times f(r)\hat{r} = 0 \Rightarrow \tau = \frac{d\vec{L}}{dt} = 0$$

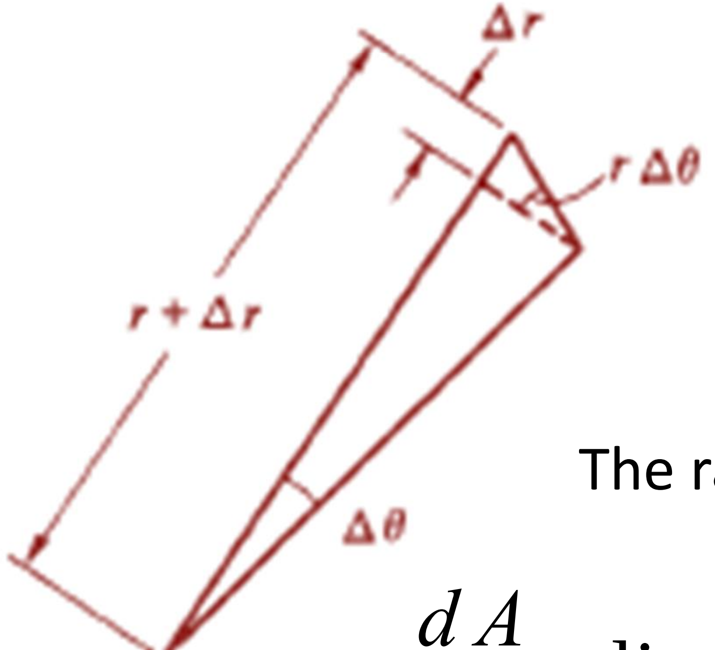
$\Rightarrow$ Angular momentum of the particle is constant in both magnitude and direction

$\Rightarrow$ motion is confined in a plane or else direction of **L** with time will change



In polar coords., if positions at times t and $t + dt$ are denoted as (r, θ) and $(r + \Delta r, \theta + \Delta\theta)$ and during which time area swept out is shown as the shaded area

For small $\Delta\theta$ the area is approxly that of a triangle as shown in the adjacent figure:



$$\Delta A \approx \frac{1}{2}(r + \Delta r)(r\,\Delta\theta)$$

$$= \frac{1}{2} r^2 \Delta\theta + \frac{1}{2} r\,\Delta r\,\Delta\theta$$

The rate at which area is swept out:

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2}\left(r^2 \frac{\Delta\theta}{\Delta t} + r\,\frac{\Delta\theta\,\Delta r}{\Delta t} \right)$$

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2}\left(r^2 \frac{\Delta\theta}{\Delta t} + r \frac{\Delta\theta\,\Delta r}{\underbrace{\Delta t}_{\text{Very small}}} \right)$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

Velocity in polar coords.: $\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$

$\Rightarrow$ Angular momentum:

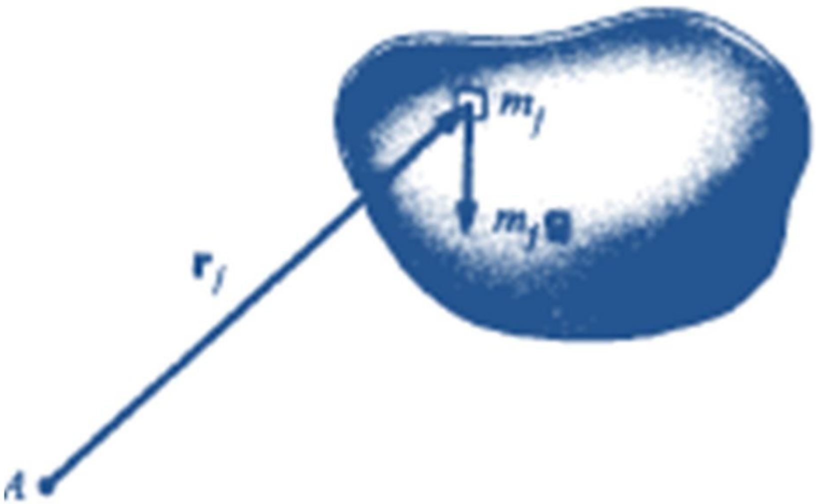
$$\vec{L} = \vec{r} \times m\,\vec{v} = r\,\hat{r} \times m\left(\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}\right)$$

$$\Rightarrow \vec{L} = m\,r^2\dot{\theta}\,\hat{k} \qquad \Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{L_z}{2m}$$

Thus it will be a const because for any central force L_z is a constant

Torque on a Conical pendulum will be discussed in the review class

Torque on a body like pendulum due to gravity

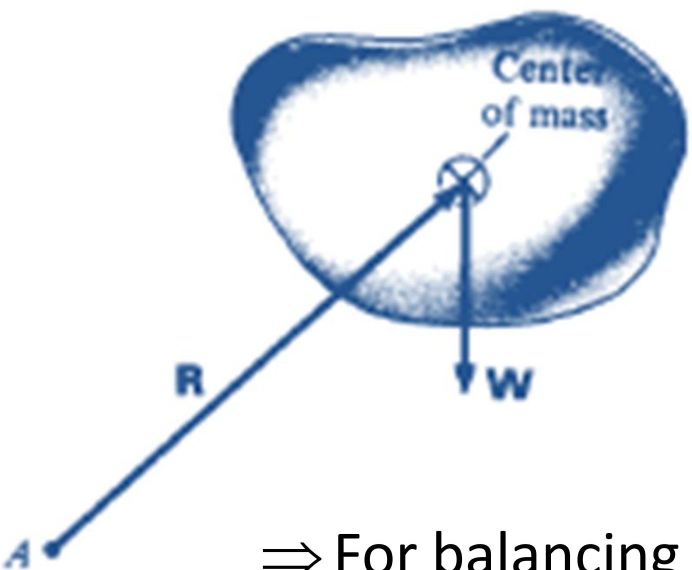


If we regard the body of mass M as a collection of particles each of mass m_j

Then torque on j^{th} particle:

$$\vec{\tau}_j = \vec{r}_j \times m_j \vec{g}$$

$\Rightarrow$ Total torque on the body:

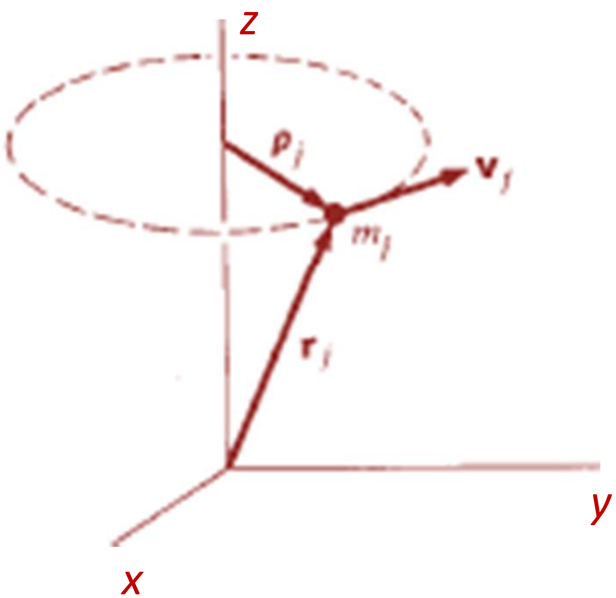


$$= \left(\underbrace{\sum_j m_j \vec{r}_j} \right) \times \vec{g}$$

$$\Rightarrow \vec{\tau} = M\vec{R} \times \vec{g}$$

$$\Rightarrow \vec{\tau} = \vec{R} \times M\vec{g} = \vec{R} \times \vec{W}$$

$\Rightarrow$ For balancing an object the pivot point shld be at the center of mass



v is perpendicular to **r**

$\Rightarrow$ For the *j*th particle of the rotating body body about the *z* axis

$$\left|\vec{v}_j\right| = \left|\frac{d\vec{r}}{dt}\right| = \omega\rho_j$$

Perpendicular distance from the axis of rotation

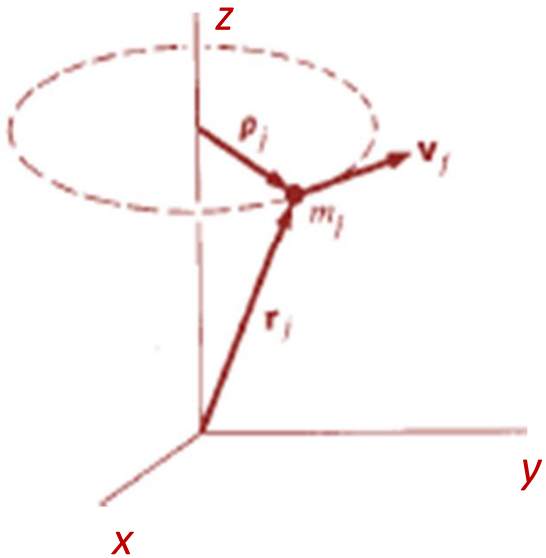
$$\rho_j = \left(x_j^2 + y_j^2\right)^{\frac{1}{2}}$$

while distance from the origin:

$$r_j = \left(x_j^2 + y_j^2 + z_j^2\right)^{\frac{1}{2}}$$

Angular momentum of the *j*th particle:

$$\vec{L}(j) = \vec{r}_j \times \vec{p}_j = \vec{r}_j \times m_j\vec{v}_j$$



$$L_z(j) = m_j v_j \times (\text{distance to } z \text{ axis})$$

$$= m_j v_j \rho_j$$

$$L_z(j) = m_j(\omega\rho_j)\rho_j$$

Total momentum of the body:

$$L_z = \sum_j L_z(j) = \sum_j \underbrace{m_j \rho_j^2}_{I} \omega = I\omega$$

$$I \equiv \sum_j m_j \rho_j^2$$

I is a purely geometrical quantity

Called *moment of inertia*

I depends on both distribution of mass and location of the axis of rotation

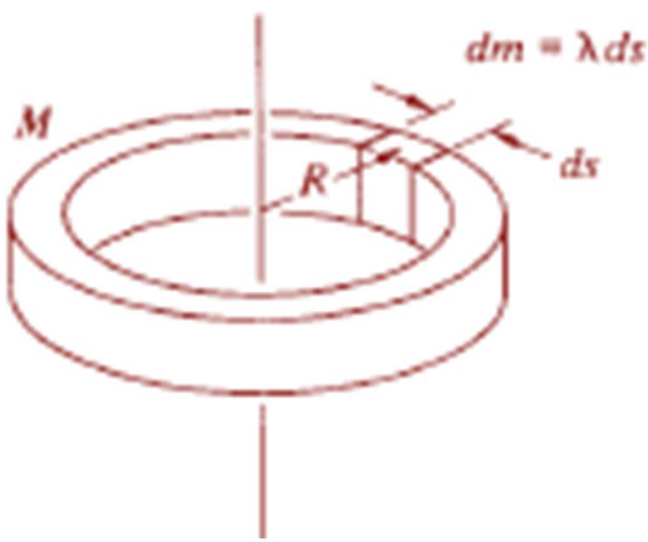
For a continuous distribution of mass

$$I \equiv \sum_j m_j \rho_j^2 \rightarrow \int \rho^2 dm$$

$$I = \int \rho^2 dm = \int \left(x^2 + y^2\right) \underbrace{dm}_{= wdV}$$

$$I = \int \left(x^2 + y^2\right) wdV$$

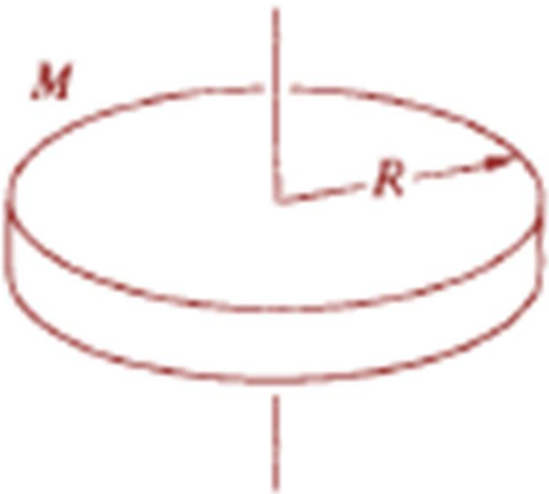
M.I. for a thin hoop of mass *M* and radius *R*
& axis through center perpendicular to its plane?

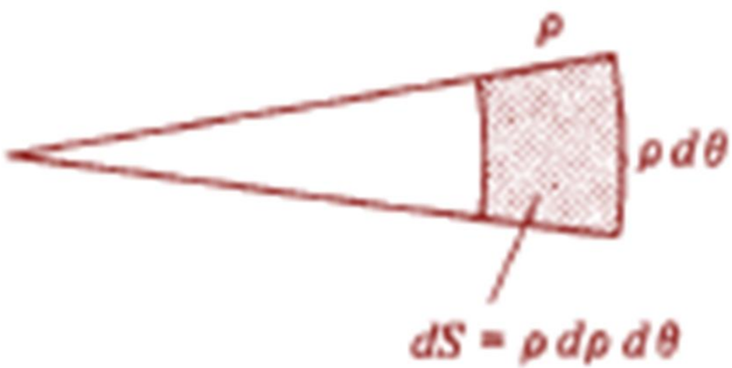


$$I = \int_0^{2\pi R} R^2 \frac{M}{2\pi R} ds$$

$$I = R^2 \frac{M}{2\pi R} s\Big|_0^{2\pi R} = MR^2$$

Corresponding problem for a disc of mass M and radius R about the axis perpendicular to its plane & through the center?





If σ = mass per unit area

$$dS = \rho d\rho d\theta$$

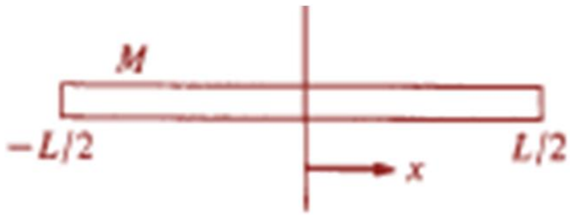
$$I = \int \rho^2 \sigma dS$$

$$= \left(\frac{M}{\pi R^2}\right) \iint \rho^3 d\rho d\theta$$

$$\int_0^{2\pi} d\theta \text{ and } \int_0^R d\rho$$

$$\Rightarrow I = \frac{MR^2}{2}$$

For a uniform thin stick of mass *M* and length *L* and axis perpendicular to the stick and through the mid-point



$$I = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx =$$

$$\Rightarrow I = \frac{M}{L}\left(\frac{1}{3}\right)\left(\frac{2L^3}{8}\right) = \frac{ML^2}{12}$$

Repeat about an axis at one end

$$I = \int_0^L x^2 \frac{M}{L} dx = \frac{M}{L}\frac{L^3}{3} = \frac{ML^2}{3}$$

Parallel axis theorem

$$I = I_0 + Ml^2$$

$$I = \frac{Ml^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$\Rightarrow \; I = \frac{ML^2}{3}$$

Dynamics of pure rotation without any translation about an axis

Example: motion of a door on its hinges or spinning of fan blades with an angular velocity ω about the *z* axis

Recollect
$$L_z = \sum_j L_z(j) = \sum_j m_j \rho_j^2 \omega = I\omega$$

For an external torque τ

$$\tau_z = \frac{dL}{dt} = \frac{d}{dt}\left(I\omega\right)$$

$$= I\frac{d\omega}{dt} = I\alpha$$

Angular acceleration

In case of pure rotation about the *z* axis

$$\tau = I\alpha$$

K.E. of a body undergoing pure rotation

$$K.E. = \sum \frac{1}{2} m_j v_j^2 = \frac{1}{2}\sum_j m_j \rho_j^2 \omega^2 \quad = \frac{1}{2} I\omega^2$$

Motion in the presence of both translation and rotation

Example: A rolling drum

Recall Chasle's theorem

Arbitrary displacement of a rigid body can be found out by a translation of its center of mass plus a rotation about its center of mass

If we restrict to axis of rotation always as parallel to *z* axis

$$L_z = \underbrace{I_0\omega}_{\text{Due to its rotation about its c.m.}} + \underbrace{\left(\vec{R}\times M\vec{V}\right)_z}_{\text{Contribution from motion of the c.m. w.r.t. origin of the inertial coord. system}}$$

R is the position vector of c.m.

V velocity of the c.m. = $\dot{\vec{R}}$

Consider the body to be an aggregation of N particles, each of mass m_j and position vectors $\mathbf{r}_j$

Thus angular momentum $\mathbf{L}$:

$$\vec{L} = \sum_{j=1}^{N} \left(\vec{r}_j \times m_j \dot{\vec{r}}_j \right)$$

$$\vec{R} = \frac{\sum m_j \vec{r}_j}{M}$$

Position vector of the center of mass

Introducing c.m. coordinates through $\vec{r}_j = R + \vec{r}_j^{\,/}$

$$\vec{L} = \sum_{j=1}^{N} \left(\vec{R} + \vec{r}_j^{\,/} \right) \times m_j \left(\dot{\vec{R}} + \dot{\vec{r}}_j^{\,/} \right)$$

$$\vec{L} = \vec{R} \times \sum m_j \dot{\vec{R}} + \sum m_j r_j^{/} \times \dot{\vec{R}} + \vec{R} \times \sum m_j \dot{\vec{r}}_j^{\,/} + \sum m_j r_j^{/} \times \dot{\vec{r}}_j^{\,/}$$

$$\vec{L} = \vec{R} \times \sum m_j \dot{\vec{R}} + \sum m_j r_j^{/} \times \dot{\vec{R}} + \vec{R} \times \sum m_j \dot{\vec{r}}_j^{/} + \sum m_j r_j^{/} \times \dot{\vec{r}}_j^{/}$$

$$\sum m_j r_j^{/} = \sum m_j \left(\vec{r}_j - R\right) = \sum m_j \vec{r}_j - M\vec{R} \ = \ 0$$

Since $\sum m_j \vec{r}_j^{/} = 0$

Thus $\sum m_j \dot{\vec{r}}_j^{/} = 0$

The first term $\vec{R} \times \sum m_j \dot{\vec{R}} = \vec{R} \times M \vec{V}$

Thus $\vec{L} = \vec{R} \times M\vec{V} + \sum r_j^{/} \times m_j \dot{\vec{r}}_j^{/}$

1st term: ang. momentum due to motion of c.m.

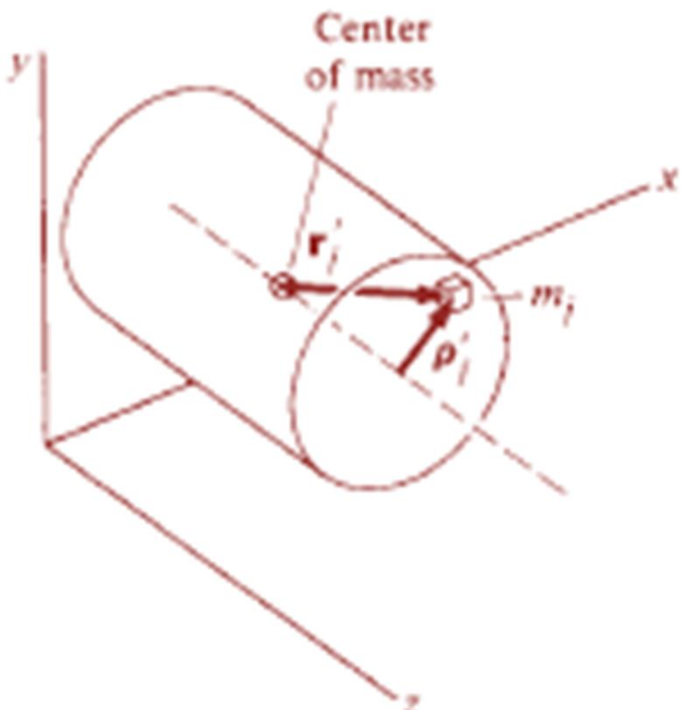
2nd term: ang. momentum due to motion around c.m.

Only way the particles to move w.r.t c.m. is to rotate the body as a whole

$$\text{Thus } \vec{L} = \vec{R} \times M\vec{V} + \sum r_j' \times m_j \dot{\vec{r}}_j'$$

If we restrict to fixed axis rotation about *z* axis

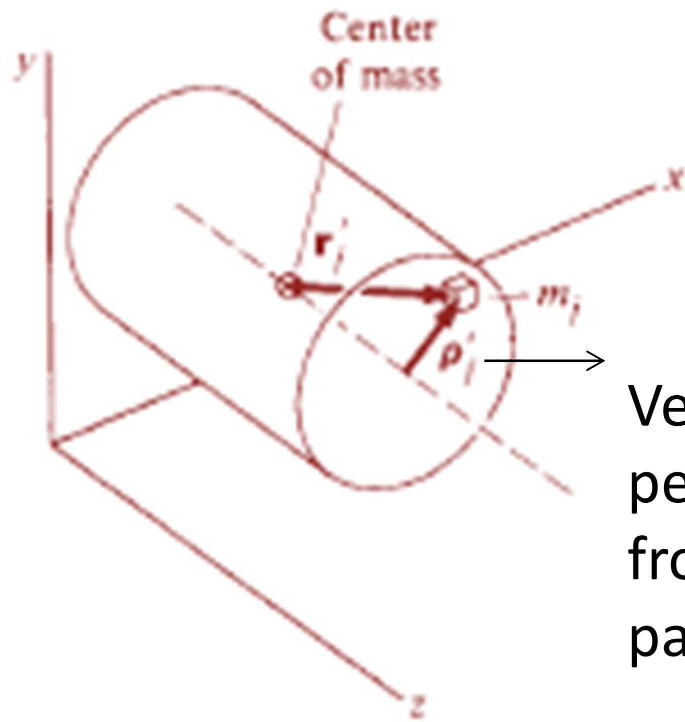
$$L_z = \left(\vec{R} \times M\vec{V}\right)_z + \left(\sum r_j' \times m_j \dot{\vec{r}}_j'\right)_z$$



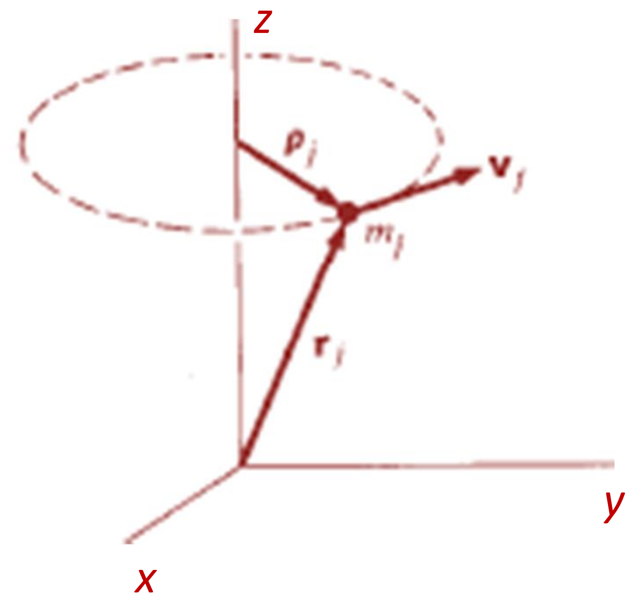
$$\left(\sum r_j' \times m_j \dot{\vec{r}}_j'\right)_z$$

Identical in form to pure rotation

$$\vec{L}(j) = \vec{r}_j \times \vec{p}_j = \vec{r}_j \times m_j \vec{v}_j$$



Vector to m_j perpendicular from an axis parallel to *z* axis



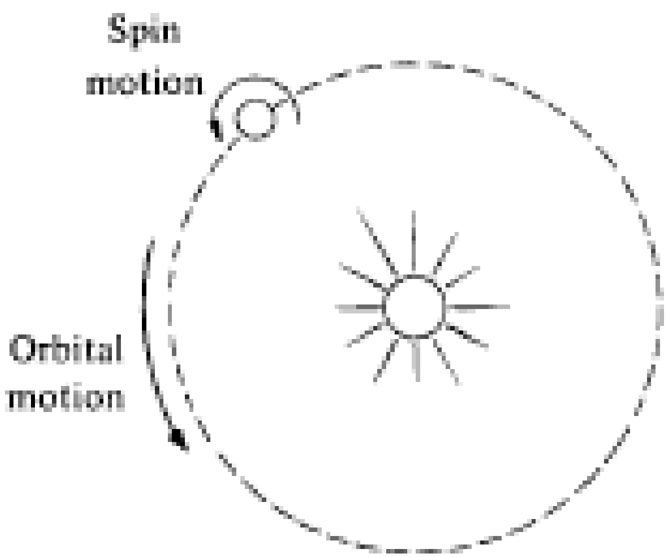
$$\left(\sum r_j' \times m_j \dot{\vec{r}}_j'\right)_z = \left(\sum \rho_j' \times m_j \dot{\vec{\rho}}_j'\right)_z$$

$$= \sum m_j {\rho_j'}^2 \omega = \underbrace{I_0}_{\text{M. I. of the body about this axis}} \omega$$

Thus $$L_z = I_0 \omega + \left(\vec{R} \times M\vec{V}\right)_z$$

$$L_z = \underbrace{I_0\omega}_{\text{Spin}} + \underbrace{\left(\vec{R}\times M\vec{V}\right)_z}_{\text{Orbital}}$$

Example earth:



Spin angular momentum is independent of the coordinate system
It is intrinsic to the body

Orbital angular momentum disappears if the origin is along the line of motion