Roll No: Section:

Mahindra École Centrale

MA 101: Calculus and Introduction to Analysis
Mid Term Examination I, September 8, 2014
Duration: 120 min; Each question carries 20 marks

Problem 1. Let E be a subset of $\mathbb{R}$. Determine whether each of the following statements is true or false. Justify the true statements by providing proofs and the false statements by counter examples.

(1.a) If E is empty ($E = \emptyset$) then it has a maximum.

(1.b) If E has a maximum then it is lower-bounded.

(1.c) If E has a supremum then it has a maximum.

(1.d) If E has a minimum then is is lower-bounded.

Problem 2. (2.a) Determine the modulus and the (principal) argument for $z = \frac{1+i}{1-i}$.

(2.b) Solve (for z) the equation $z^2 = 2i$.

(2.c) Let u be a non-zero complex number. Then solve (for z) the equation $e^z = u$. Is the solution unique?

Problem 3. (3.a) Determine if the following limit exists $\lim\limits_{n\to\infty} \frac{\frac{n}{2}+\sin(n)}{3n+4\sqrt{n}+1}$. Compute the limit if it exists.

(3.b) Show that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+3)}$ is converging. *Bonus question*: Compute the sum.

(3.c) Determine if the following series converge or diverge: (1) $\sum\limits_{n=1}^{\infty} \frac{n!}{2^{2n}}$, (2) $\sum\limits_{n=1}^{\infty} e^{-n}$, and (3) $\sum\limits_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$.

Problem 4. (4.a) Let $c \in \mathbb{R}$. Show that $\lim\limits_{x\to c} x^2 = c^2$ using ε–δ definition.

(4.b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$, and $g(x) = \begin{cases} 2 - x^2, & |x| \le 2, \\ 2, & |x| > 2, \end{cases}$.
Determine the formula for $h(x) = f(g(x))$. Are f, g, h continuous?

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be such that $f(100) = g(0) = 1$ and $g(x - y) = g(x)g(y) + f(x)f(y)$ for all x and y in $\mathbb{R}$.

(5.a) Show that $(f(x))^2 + (g(x))^2 = 1$ for all $x \in \mathbb{R}$.

(5.b) Show that $f(0) = g(100) = 0$ (you may use the previous result).

(5.c) Show that $g(-x) = g(x)$ for all $x \in \mathbb{R}$.

(5.d) Show that $g(100 - x) = f(x)$ for all $x \in \mathbb{R}$.

Do the functions f and g remind of you some standard functions?