**Second half of Differential Calculus.**

**Remark 1.** *These partial solutions are only there to help, to give a hint or some element of answers.*

*Some problems are not corrected because they should really be easy enough for you to work on them by your own, or because they are not really exciting.*

*Some remarks might not be as* serious *as what you are used to, so deal with it.*

# 1 A. Homework.

Ex.A.1) Let $l(t)$ be the length of the rectangle, given by $l(t) = 12 - 2t$ for $t \in [0, 6s]$. And similarly $w(t)$ is the width, by $w(t) = 5 + 2t$ for $t \in [0, 6s]$.

- The area is $A(t) = l(t)w(t)$, so $A'(t) = -2(5 + 2t) + 2(12 - 2t) = -34$.
- The perimeter is $p(t) = 2(l(t) + w(t))$, so $p'(t) = 0$ : the perimeter is constant.
- The length of the diagonal of the rectangle is $d(t) = \sqrt{l^2(t) + w^2(t)}$ thanks to Pythagorean theorem. So for $t \in [0, 6s)$, $d$ is differentiable, and $d'(t) = \dfrac{-2(12-2t) + 2(5-2t)}{\sqrt{l^2(t) + w^2(t)}} = -\dfrac{14}{\sqrt{(12-2t)^2 + (5+2t)^2}} < 0$ (nothing interesting to say about this one).

The area and the length of the diagonal are decreasing, the perimeter is constant.

Ex.A.2) Cf. figure 1. Let call $\theta$ this angle, we have $\cos(\theta) = \dfrac{12}{13}$, so $\theta = \arccos(12/13)$.

(a) The top of the ladder is sliding down the wall at the velocity $\tan(\theta) * 5$ ft/sec, so at approximately 2.08 ft/sec.

(b) The area can be expressed as $13 * \text{base}/2$, so the rate of change of the area is $33.5\text{ft}^2/\text{sec}$.

(c) $\theta'(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}(\arccos((12 - 5t)/13)) = \dfrac{5}{1 - ((12 - 5t)/13)^2}$.

(d) Nothing else to add on this real world based problem. The main interest is your ability to mathematically model a physical problem.

Ex.A.3) Here, $k > 0$, and $f$ is defined for any $x > -1$.
What is asked is a simple application of the Taylor expansion formula for $a = 0$, $n = 1$ : $f(x) \simeq L(x)$ if $x \simeq 0$, with $L(x) = f(0) + f'(0)x$, but $f(0) = 1$ and $f'(0) = k$.

(a) $(1.0002)^{1/3} \simeq 1 + 50 * 0.0002 = 1.01$ because $0.0002$ is small regarding to 1,

(b) $(1.009)^{1/3} \simeq 1 + \dfrac{1}{3} * 0.009 = 1.003$ because $0.009$ is small also regarding to 1.

(c) One can verify numerically that the absolute errors made here are small regarding 1 (less than 1%).

Ex.A.4) Way out of the syllabus, do not even try to solve it.

Ex.A.5) Please understand the $\mathrm{d}x$ as a $\Delta x$, this one has *nothing* to do with the **purely formal** $\mathrm{d}x$ used in the notations $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x)$ or $\displaystyle\int_a^b f(x)\mathrm{d}x$.
So, $x_0 = -1, \Delta x = 0.1, f(x) = 2x^2 + 4x - 3$.

(a) The change $\Delta f$ is simply $f(-0.9) - f(-1) = 0.02$,

(b) As $f'(x) = 4x + 4$, we have $f'(x_0) = 0$, the estimate $\delta f = f'(x_0)\Delta x$ is simply 0.

(c) **Warning:** here again, the question paper use a notation that you might find confusing: $\mathrm{d}f$ is a special notation, which have a **purely formal** use in the notation $f'(x) = \dfrac{\mathrm{d}f}{\mathrm{d}x}(x)$, so naming this estimate $\mathrm{d}f$ can be confusing. Just name it $\delta f$ if you prefer.

(d) The approximation error is 2%, which is *not that bad*.

Ex.A.6) Le $f$ be defined and continuous on $[a, b] \subset \mathbb{R}$, differentiable on $(a, b)$. Assume that $f$ acquires a local minimum at the point $c \in (a, b)$. At this point, we know that we can write $f'(c)$ as two different limit : the left ($h < 0$) and right ($h > 0$) limits.

(left) So $f'(c) = \lim\limits_{h \to 0^-} \dfrac{f(c+h) - f(h)}{h}$. But at the point $c$, $f$ acquires a (local) minimum by hypothesis, so we know that $f(c+h) \leqslant f(h)$, ie $f(c+h) - f(h) \leqslant 0$. But when $h \to 0^-$, $h < 0$, so $f'(c) \leqslant 0$.

(left) So $f'(c) = \lim\limits_{h \to 0^+} \dfrac{f(c+h) - f(h)}{h}$. But at the point $c$, $f$ acquires a (local) minimum by hypothesis, so we know that $f(c+h) \geqslant f(h)$, ie $f(c+h) - f(h) \geqslant 0$. But when $h \to 0^+$, $h > 0$, so $f'(c) \geqslant 0$.

And now we see that we have both $f'(c) \leqslant 0$ and $f'(c) \geqslant 0$, so $f'(c) = 0$.
Obviously, the same can be done with a (local) maximum.

Ex.A.7) The function $f(x) = \dfrac{x}{x^2+1}$ is continuous and differentiable on $\mathbb{R}$.
For $x \in \mathbb{R}$, $f'(x) = \dfrac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \dfrac{1 - x^2}{(x^2+1)^2}$.
So $f'(x) = 0$ iff $x = \pm 1$. Thanks to the first derivative test, that function $f$ *can have* a local extremum only at the points $x = -1$ or $x = 1$. Graphically, or by differentiating twice, we check that $x = -1$ is a global minimum, where $f(-1) = -\dfrac{1}{2}$ and $x = 1$ is a global maximum, where $f(1) = \dfrac{1}{2}$.

<u>Remark:</u> one other way to prove this was to remember that $\forall a, b \in \mathbb{R}, |ab| \leqslant \dfrac{a^2+b^2}{2}$, so here with $a = x, b = 1$, we add directly that $-\dfrac{1}{2} \leqslant f(x) \leqslant \dfrac{1}{2}$. And we just check that at $x = \pm 1$, these extrema values are taken.

Ex.A.8) Let $f(x) = -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}$ if $x \leqslant 1$ and $f(x) = x^3 - 6x^2 + 8x$ if $x > 1$. $f$ is continuous and differentiable on $(-\infty, 1)$ and on $(1, +\infty)$, the only point we should look more carefully is $x = 1$.
$f(1) = -\dfrac{1}{4}1^2 - \dfrac{1}{2}1 + \dfrac{15}{4} = 3$ and $\lim\limits_{x \to 1^+} x^3 - 6x^2 + 8x = 1^3 - 6 * 1^2 + 8 * 1 = 3$. So $f$ is continuous at $x = 1$.
On the left, $f'(x) = -\dfrac{1}{2}x - \dfrac{1}{2}$ so $f'(1) = -1$; and on the right, $f'(x) = 3x^2 - 12x + 8$ so $\lim\limits_{x \to 1^+} f'(x) = -1$.
So $f$ is also differentiable at $x = 1$.
Therefore $f$ is continuous and differentiable everywhere on $\mathbb{R}$.
Now we can use the first derivative test, and look for the critical point. Let $x \in \mathbb{R}$.

If $x \leqslant 1$ $f'(x) = 0$ iff $-\dfrac{1}{2}x - \dfrac{1}{2} = 0$ iff $x = -1$.
At this point, $f''(-1) = -\dfrac{1}{2} < 0$, so it is a maximum.
Without any more knowledge, we cannot tell between local or global maximum, but the left part of $f$ is a quadratic polynomial, of main coefficient being negative.
So we can conclude that at $x = -1$, the left part of $f$ have a global maximum.
But as $\lim\limits_{x \to +\infty} f(x) = +\infty$, we can also conclude that it is not a global maximum for $f$, only local.

If $x > 1$ $f'(x) = 0$ is a quadratic equation, of discriminant $\Delta = 12^2 - 4*3*8 = 144 - 12*8 = 144 - 96 = 48 > 0$, so there is exactly two real solutions, $x_{1,2} = \dfrac{12 \pm \sqrt{48}}{6}$, with $x_1 < x_2$.
At $x_1$, $f''(x) = 6x - 12$ will take the a negative value, and at $x_2$, $f''(x_2)$ will be positive. So $x_1$ is a local maximum, and $x_2$ a local minimum.

Ex.A.9) Let $f : [a, b] \to \mathbb{R}$, with $a, b \in \mathbb{R}$ such that $a < b$, and assume that $f$ is continuous on $[a, b]$, differentiable on $(a, b)$, with $f(a) = f(b)$. Lets show that there is a $c \in (a, b)$ such that $f'(c) = 0$.
$f$ is continuous on $[a, b]$, so thanks to the **Extreme Value Theorem**, $f$ is bounded and attains its bounds on $[a, b]$. Let $M$ be the maximum of $f$.

- First case, if $M > f(a)$. Let $c$ be a point such that $f(c) = M$, thanks to the second conclusion of the EVT. $c$ is a point where $f$ has a local maximum, and $c \neq a, b$ because $f(c) \neq f(a), f(b)$ (so $c \in (a, b)$). Therefore, the **First Derivative Theorem** gives that $f'(c) = 0$. Done!
- Second case, if $M = f(a)$. Consider now $m$ the minimum of $f$.
  - If $m < f(a)$, we do the same as previously, and the EVT gives one point $c \in (a, b)$ such that $f(c) = m$, and so $f'(c) = 0$. Done!
  - In this last case, $f$ is constant on $[a, b]$, and as $a < b$, one could define $c$ as $\frac{a+b}{2}$ and we have $f'(c) = 0$. Done!

In every of these cases, we conclude with having a $c \in (a, b)$ satisfying $f'(c) = 0$.

Ex.A.10) We have two cases here:

a) That $f$ is well defined, continuous and differentiable on $[\frac{1}{2}, 2]$, and on this interval, $f'(x) = 1 - \frac{1}{x^2}$, and $f(\frac{1}{2}) = f(2) = \frac{5}{2}$, so the average rate $\frac{f(b) - f(a)}{b - a}$ is 0 here. Hence, the only possible $c$ is 1 (because $1 - \frac{1}{x^2} = 1$ iff $x^2 = 1$ iff $x = 1$ because $x > 0$ anyway).

b) That second $f$ is well defined and continuous on $[1, 3]$, and differentiable on $(1, 3]$.

Ex.A.11) $f$ is not continuous at $x = 1$.

Ex.A.12) $a, b, m$ have to be such that $f$ is continuous on $[0, 2]$ <u>and</u> differentiable on $(0, 2)$.

- Continuity at $x = 0$. We have $\lim_{x \to 0^+} f(x) = a$, so $a$ has to be 3.
- Continuity at $x = 1$. We have $f(1) = m + b$, and $\lim_{x \to 1^-} f(x) = 2 + a$, so $m + b = 2 + a$ is another condition.
- Differentiability at $x = 0$ is not required, but at $x = 1$ it is. From the left, $f'(x) = -2x + 3$ if $0 < x < 1$, so $\lim_{x \to 1^-} f'(x) = 1$.
  From the right, $f'(x) = m$ if $1 \leqslant x \leqslant 2$, so $f'(1) = m$, and so $m = 1$ is another condition.

Finally, we conclude by saying that $a = 3, m = 1, b = 2 + a - m = 4$ are the only possible values for $a, b, m$ for $f$ to verify the hypotheses of the mean value theorem on $[0, 2]$.

Ex.A.13) The zero of the given polynomial are (clearly) $z_0 = 0, z_1 = 9$ and $z_2 = 24$. The zero of its derivative are $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$, with $a = 3, b = -66, c = 216$, and $\Delta = b^2 - 4ac = 66^2 - 4 * 3 * 216 = 1764 = 42^2 > 0$. So $x_{1,2} = \frac{66 \pm 42}{6}$, hence $x_1 = 4$, $x_2 = 18$. We can see that $z_0 < x_1 < z_1 < x_2 < z_2$ : the zero of $y'$ are interlaced between the zeros of $y$.

Ex.A.14) This problem is the general case of the previous one. Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = x^n + \sum_{k=n-1}^{0} a_k x^k$ and $p'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$ its derivative.
Let $\alpha, \beta$ be two consecutive zeros of $p$ (ie, no other zero of $p$ between them). Then we have $p(\alpha) = 0 = p(\beta)$, and $p$ is continuous on $[\alpha, \beta]$, and differentiable on $(\alpha, \beta)$ because it is of class $\mathcal{C}^1$ on $\mathbb{R}$.
So we can apply Rolle's Theorem to conclude that there is a $c \in (\alpha, \beta)$ such that $p'(c) = 0$.
<u>Conclusion:</u> so between every two zeros of $p$ there lies a zero of $p'$.

**Remark 2.** *(Out of the syllabus!) For a more general version of this result, for a complex polynomial, one could be interested to take a look at* Lucas*'s Theorem (also called the* Gauss-Lucas*'s Theorem), for example on this Wikipédia page* `https://en.wikipedia.org/wiki/Gauss-Lucas_theorem`.

Ex.A.15) Let $a, b \in \mathbb{R}_+^*$, with $a < b$. The function $f : [a, b] \to \mathbb{R}, x \mapsto \frac{1}{x}$ is well defined and continuous on $[a, b]$, and differentiable on $(a, b)$. We know $f'(x) = -\frac{1}{x^2}$, let solve $f'(c) = \frac{f(b) - f(a)}{b - a}$ iff $-\frac{1}{c^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab}$ iff $c^2 = ab$ iff $c = \sqrt{ab}$ because $a, b > 0$.

Ex.A.16) (quickly) $f'' = g''$ iff there exists $a, b \in \mathbb{R}$ such that $\forall x \in D, f(x) = g(x) + ax + b$.

Ex.A.17) (quickly) Cf. the figure 2.

(a) When the slope $s'$ is positive, resp. when it is negative.
(b) The velocity $s'(t) = 0$ for approximately $t = 2.2s$, $t = 6s$, $t = 9.5s$.
(c) The acceleration $s''(t) = 0$ for approximately $t = 4s, 8s, 12s$ (inflection points).
(d) Acceleration is negative for $t \in [0, 4s] \bigcup [8s, 12s]$ and positive for $t \in [4, 8s] \bigcup [12s, 15s]$.

Ex.A.18) The critical points of $f$ (where $f'(x) = 0$) are $x = 1, 2, 4$. At these points, we just have to compute the sign of $f''(x)$ to conclude. For any $x \in \mathbb{R}, y''(x) = 2(x-1)(x-2)(x-4) + (x-1)^2((x-4)+(x-2))$

$x = 1$ $y''(1) = 0$, so 1 is a point of inflection.
$x = 2$ $y''(2) = -2 < 0$ so 2 is a point of (local) maximum.
$x = 4$ $y''(4) = 18 > 0$ so 4 is a point of (local) minimum.

Ex.A.19) I am from France, we do not use feet. We use, as every one else is supposed to, **meters**.
This problem is not that hard, so you can do it yourself.

Ex.A.20) Cf. figure 4.

a) $y'(0) = d$, and graphically we clearly see that $y'(0) = 0$ as the curve has a horizontal tangent at $x = 0$.
b) $y'(-L) = 3aL^2 - 2bL + c$, and graphically again, $y'(-L) = 0$.
c) If $y(-L) = H$, and $y(0) = 0$, we check that the proposed cubic polynomial satisfies exactly the same 4 conditions (values of $y$ and $y'$ at $-L$ and 0) as $y$, so by uniqueness, we can indeed conclude that :

$$\forall x \in \mathbb{R}, y'(x) = H\left(2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right).$$

Ex.A.21) To conclude.

Ex.A.22) The limit is 0, with both method.

Ex.A.23) That limit is $an$, for any $n > 0$.

Ex.A.24) That limit is 1.

Ex.A.25) Boring, and purely mechanical. Just compute $f(-2), f'(-2), f''(-2), f'''(-2), f^{(4)}(-2)$.
Finally, we have, for every[1] $x \in \mathbb{R}$, we have $f(x) = x^4+x^2+1 = (x+2)^4 - 8(x+2)^3 + 25(x+2)^2 - 36(x+2) + 21$.

Ex.A.26) The MacLaurin expansion of $e^x$ is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{k \geqslant 0} \frac{x^k}{k!}.$$

At $x = 1$, going up to the third order, this expansion gives the approximation $e \simeq 1 + \frac{1^1}{1!} + \frac{1^2}{2!} = 2.5$. So with the approximation $e = 2.732$, the absolute error is $\left|\frac{2.732 - 2.5}{2.732}\right| \simeq 0.0803$, so an error of approximately 8%, which is not really good.

With the remainder formula, we have an upper bound of the form $M\frac{x^3}{3!}$ where $M$ is an upper bound of $e^x$ for $x \in (0.9, 1.1)$ (for example). So we can take $M = 2.732^1.1 \simeq 3.02$, $x = 1$, and so the bound is approximately 0.5. <u>Not exactly sure of what the question paper is asking here.</u>

[1] This is a special case when a finite order Taylor approximation is an equality everywhere, only for polynomials!

At $x = -1$, going up to the order 4, the expansion gives the approximation $\frac{1}{\mathrm{e}} \simeq 1 + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} = 0.5 - \frac{1}{6} = 0.1667$. So with the approximation $\frac{1}{\mathrm{e}} = 0.3678$, the absolute error is $\left|\frac{0.3678 - 0.1667}{0.3678}\right| \simeq 0.5469$, so an error of approximately 55%, which is really too much!

Ex.A.27) Should be done more in details...

a) For $k > 0, x > -1$, $f : x \mapsto (1+x)^k$ is continuous and differentiable. The second order TAYLOR's formula gives $f(x) \simeq 1 + kx + k(k-1)\frac{x^2}{2}$ when $x \simeq 0$, which is the quadratic approximation of $f(x)$.

b) Let $k = 3$. *Numerically*, we find that the error $|f(x) - (1 + kx + k(k-1)\frac{x^2}{2})|$ is less than 1% if $x < 0.01$ itself.

c) Let $k = 3$. *Numerically*, we find that the error $|f(x) - (1 + kx + k(k-1)\frac{x^2}{2})|$ is less than 10% if $x < 0.4642$ itself.

d) Remark: the last two question should be done more precisely with the Remainder Formula. The solution was done by numerically finding an approximately good threshold to have the error less than 1% or 10%.

## 2 B. Tutorial for class workout.

Ex.B.1) Gross!

Ex.B.2) Out of the syllabus.

Ex.B.3) See the figure, it is pretty easy anyway. Part *a*) was done in class.

Ex.B.4) Obvious by successive application of ROLLE's theorem.
General result is also easy.
We can use to prove that a cubic polynomial cannot have more than three real zeros, because its third derivative is a constant so do not have any zero.

Ex.B.5) If you tried carefully that problem, you might have found that the statement was not precise enough. What is $a, x, f, c_2$? Which variable is depending of the others? This can be your job, to try to improve this.

Ideally, the exercise should say[2] : let $f : \mathbb{R} \to \mathbb{R}$, of class $\mathcal{C}^2$, and one $a \in \mathbb{R}$. For any $x \in \mathbb{R}$, there is one $c_2$ between $a$ and $x$ (or between $x$ and $a$, depending on their respective position) such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2. \tag{2.1}$$

So this can be used as the question is asking to have a condition on the sign of $f''$ on an entire interval.

Assume $f$ to be continuous, with continuous first and second derivatives, and $f'(a) = 0$.

a. If $f'' \leqslant 0$ throughout a (small) interval $I = [\alpha, \beta]$ whose interior $(\alpha, \beta)$ contains $a$, then we can use the equation 2.1, with $a$ being fixed, and **any** $x \in I$, and $c_2$ between $a$ and $x$, so in the interior of $I$. Then, we have $f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2 = f(a) + \frac{f''(c_2)}{2}(x-a)^2$. But $f''(c_2) \leqslant 0$, and $(x-a)^2 \geqslant 0$, so $f(x) \leqslant f(a)$ for any $x \in I$.
This is exactly saying that $f$ has a local **maximum** at the point $a$.

b. Exactly the same can be done with a local minimum, as if $f''(c_2) \geqslant 0$, then $f(x) \geqslant f(a)$ for any $x$ in this small interval $I$.

[2] That statement is called the TAYLOR's theorem with an explicit formula for the remainder. If needed, take a look at that Wikipédia page: `https://en.wikipedia.org/wiki/Taylor'_theorem#Explicit_formulae_for_the_remainder` for examples and a proof.

Ex.B.6) A little bit out of the syllabus, as the Taylor expansion as seen in lectures are only for a real variable. Anyway, what could have been possible is simply use the generic form of the Taylor expansion of $\exp(x)$ valid for any $x \in \mathbb{C}$, and the ones for cos and sin valid on $\mathbb{R}$, and use the basic property of $i^2 = 1, i^4 = -1$ to conclude.

## 3 C. Additional Exercises.

Ex.C.1) To be concluded.

Ex.C.2) For a fixed integer $m \in \mathbb{N}$, at the point $x = 0$, the TAYLOR expansion is the binomial expansion given by the binomial theorem. For $|x| < 1$, it obviously converges as there is a finite number of terms $(m + 1)$: $(1 + x)^m = \sum_{k=0}^{m} \binom{k}{m} x^k$