First half of Differential Calculus.

Remark 1. *These partial solutions are only there to help, to give a hint or some element of answers.*

Some problems are not corrected because they should really be easy enough for you to work on them by your own, or because they are not really exciting.

Some remarks might not be as serious *as what you are used to, so deal with it.*

1 A. Homework.

Ex.A.1) Cf. the figures.

(a) Here, $D = [-3, 3]$, and f is differentiable and continuous everywhere except 0, and at 0 it is neither continuous neither differentiable,

(b) Here, $D = [-3, 3]$, and f is continuous everywhere, differentiable except ± 2.

Ex.A.2) Let $x \in \mathbb{R}$. For $h \neq 0$, $\dfrac{y(x+h) - y(x)}{h} = \dfrac{(x+h)^3 - x^3}{h} = \dfrac{x^3 + hx^2 + h^2x + h^3 - x^3}{h} = x^2 + h(x + h) \underset{h \to 0}{\longrightarrow} x^2$. So $y'(x) = x^2$ (as we know). But for any real x, $x^2 \geqslant 0$, so the slope of the function y will never become negative.

Ex.A.3) We do the same, to find that $f'(x) = 4x - 13$ for any $x \in \mathbb{R}$. So, for the tangent at the point x of the curve of f to acquire a slope of -1, we solve $f'(x) = -1$. But $f'(x) = -1$ iff $4x - 13 = -1$ iff $x = (-1 + 13)/4 = 12/4 = 3$. Therefore, at the point $x = 3$, the curve will have the tangent line of equation $y = f(3) + f(3)(x - 3) = -16 + (-1) * (x - 3) = -16 + 9 - x = -7 - x$.

Ex.A.4) As asked, we do it with two different methods:

(a) With the product rule : for $x \in \mathbb{R}, y'(x) = (\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+1))(x + 5 + \dfrac{1}{x}) + (x^2 + 1)(\dfrac{\mathrm{d}}{\mathrm{d}x}(x + 5 + \dfrac{1}{x})) = (2x)(x + 5 + \dfrac{1}{x}) + (x^2 + 1)(1 - \dfrac{1}{x^2}) = 2x^2 + 10x + 2 + x^2 + 1 - 1 - \dfrac{1}{x^2} = 3x^2 + 10x + 2 - \dfrac{1}{x^2}$,

(b) First we simplify rule : for $x \in \mathbb{R}, y(x) = (x^2 + 1)(x + 5 + \dfrac{1}{x}) = x^3 + 5x^2 + x + x + 5 + \dfrac{1}{x} = x^3 + 5x^2 + 2x + 5 + \dfrac{1}{x}$. So with differentiate every term and $y'(x) = 3x^2 + 10x + 2 - \dfrac{1}{x^2}$ (ouf, we find the same!).

Ex.A.5) We simply differentiate : if $y(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$, then for any $x \in \mathbb{R}$, $y'(x) = -\dfrac{6(x^2-2)}{(x-2)^2(x-1)^2}$.

Ex.A.6) We do the same here : if $p(q) = (\dfrac{q^2+3}{12q})(\dfrac{q^4-1}{q^3})$, then $p'(q) = \dfrac{q^6 + q^2 + 6}{6q^5}$. and so $p''(q) = -\dfrac{5}{q^6} - \dfrac{1}{2q^4} + \dfrac{1}{6}$ for any $q \in \mathbb{R}^*$ (of course q has to be different than 0).

Ex.A.7) Same basic problem : if $p(q) = \dfrac{q^2+3}{(q-1)^3 + (q+1)^3}$, well defined if $q \in \mathbb{R}^*$ then $p'(q) = -\dfrac{1}{2q^2}$, and so $p''(q) = \dfrac{1}{4q^3}$ for any $q \in \mathbb{R}^*$.

Ex.A.8) Let f be the function $f(x) = x^3 - 3x - 2$. At one point x_0, the equation of the tangent line to the curve of f is $y = f(x_0) + f'(x_0)(x - x_0)$ because f is continuous and differentiable on $\mathbb{R}$ so at x_0. For the tangent to be horizontal, $f'(x_0)$ has to be 0. We solve $f'(x_0) = 0$, by computing $f'(x) = 3x^2 - 3$, so $f'(x_0) = 0$ iff $3x_0^2 = 3$ iff $x_0 = \pm 1$. Hence, f has exactly two horizontal tangents, at the point $x_1 = -1$ and $x_2 = 1$, with respective equation $y = 1$ and $y = -6$.

Ex.A.9) Let a, b, n, R be some constants, and we define $P(V) = \dfrac{nRT}{V - nb} - \dfrac{an^2}{V^2}$, well defined for $V \in \mathbb{R} \setminus \{0, nb\}$. For such a V, P is continuous and differentiable at the point V, and we compute $P'(V) = -\dfrac{nRT}{(V-nb)^2} + \dfrac{an^2}{2V^3}$.

Ex.A.10) Cf. the figure 3. Prof. Arya did this example in class.
On one hand, B is zero when A has a global extremum, and B is affine while we observe that A looks like a quadratic parabolic function. So B represents the derivative of A. On the other hand, A is zero in two points, where C appears to have a global maximum (first point) and minimum (second point), so we conclude that A represents the derivative of C.

Conclusion: if C represents the position s, A is the velocity $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$ and B is the acceleration $a = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}$.

Ex.A.11) We answer point by point to all the questions (as indicated by `?`).

- First and second derivatives of s with respect to the variable t,
- Period is 2π. Amplitude of displacement, velocity and acceleration are all 5 (units),
- At $t = 0.25$, 0.5 and 1 respectively,
- Frequency. Circular frequency, ie number of full circles traversed when one circle traversal implies covering an angle of 2π,
- At $s = 0$, movement in either direction,
- At $s = -5$ and $+5$,
- At the extreme points of velocity, the acceleration, ie rate of change of velocity is 0.

Ex.A.12) Order 4. Just compute $y'(x)$, $y''(x)$, $y'''(x) = y^{(3)}(x)$, $y''''(x) = y^{(4)}(x)$ to find this out.

Ex.A.13)

- We recall that $\sec(x) = \dfrac{1}{\cos x}$ and $\tan(x) = \dfrac{\cos x}{\sin x}$ when there are defined. We have $\tan'(x) = 1 + \tan^2(x) = \dfrac{1}{\sin^2(x)}$ and $\sec'(x) = \dfrac{\sin(x)}{\cos^2(x)} = \dfrac{\tan(x)}{\cos(x)}$.
  Therefore if $y(x) = (\sec(x)+\tan(x))(\sec(x)-\tan(x))$ for $x \in \mathbb{R}\backslash(\dfrac{\pi}{2} + \pi\mathbb{Z} \cup \pi\mathbb{Z}) = D_f$, a first method is just to differentiate as a product, and another method is to expand and simplify y and *then* differentiate.
  - *First method:* For $x \in D_f$, $y'(x) = (\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec(x)+\tan(x)))(\sec(x)-\tan(x)) + (\sec(x)+\tan(x))(\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec(x)-\tan(x))) = (\dfrac{\tan(x)}{\cos(x)} + 1 + \tan^2(x))(\sec(x)-\tan(x)) + (\sec(x)+\tan(x))(\dfrac{\tan(x)}{\cos(x)} - 1 - \tan^2(x)) = 0$ after simplification.
  - *Second method:* For $x \in D_f$, $y(x) = (\sec(x)+\tan(x))(\sec(x)-\tan(x)) = \sec^2 - \tan^2(x)$, so $y'(x) = 2(\sec(x)(\dfrac{\tan(x)}{\cos(x)}) - \tan(x)(1+\tan^2(x))) = \cdots = 0$.
- And then if $y(x) = (\sin(x)+\cos(x))\sec(x)$ for $x \in \mathbb{R}\backslash(\pi\mathbb{Z}) = D_f$, $y(x) = 1 + \dfrac{\sin x}{\cos x}$. So $y'(x) = \dfrac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \sec^2(x)$.

Ex.A.14) For $\theta \in \mathbb{R}\backslash\left\{\dfrac{\pi}{2} + \pi\mathbb{Z} \cup \pi\mathbb{Z}\right\} = D_r$, if $r(\theta) = \sec(\theta)\csc(\theta)$, then we have $r(\theta) = \dfrac{1}{\cos(\theta)\sin(\theta)}$, so r is indeed differentiable on its domain. We compute directly $r'(\theta) = \dfrac{\sin^2(\theta) - \cos^2(\theta)}{\cos^2(\theta)\sin^2(\theta)} = \sec^2(\theta) - \csc^2(\theta)$.

Ex.A.15) We give the result : if $y(x) = (4x+3)^4(x-1)^{-3}$, y is defined on $\mathbb{R}\backslash\{-1\}$, and on its domain, $y'(x) = \dfrac{(4x+3)^3(4x+7)}{(x+1)^4}$.

Ex.A.16) We give the result : if $f(\theta) = \left(\dfrac{\sin(\theta)}{1+\cos(\theta)}\right)$, f is defined on $\mathbb{R}\backslash\{\pi\mathbb{Z}\}$, and on its domain, $f'(\theta) = \dfrac{2\sin(x)}{(1+\cos(x))^2}$.

Ex.A.17) Same "mechanical" problem : if $y(x) = (1-\sqrt{x})^{-1}$, then y is defined for $x \geqslant 0$ and $x \neq 1$, ie $x \in (0,1) \cup (1,+\infty)$. On this domain, $y'(x) = \dfrac{1}{2(\sqrt{x}-1)^2\sqrt{x}}$. Yup, so exciting!

Ex.A.18) Thanks to the "chain rule", we know that if $g = f \circ u$, then $g'(-1) = u'(-1).f'(u(-1))$. Or $u(-1) = 0$, $u'(-1) = 1/2$ and $f'(0) = -4$, so $g'(-1) = -2$.

Ex.A.19) Again the same kind of reasoning. Let $h(x) = \sqrt{f^2(x) + g^2(x)}$, then thanks to these values at $x = 2$, we can claim that h is also differentiable at this point. And we have $h'(x) = (2f'(x)f(x)+2g'(x)g(x))\dfrac{1}{2\sqrt{f^2(x) + g^2(x)}}$, so at $x = 2$, we have $h'(2) = (2f'(2)f(2)+2g'(2)g(2))\dfrac{1}{2\sqrt{f^2(2) + g^2(2)}} = (2.(\dfrac{1}{3}).8+2.(-3).2)\dfrac{1}{2\sqrt{8^2 + 2^2}} = \dfrac{\frac{16}{3} - 12}{2\sqrt{68}} = -\dfrac{5}{3\sqrt{17}} \simeq -0.40426$.

Ex.A.20) Nothing to say : if $y(x) = x^{3/2}$, y is defined on $[0, +\infty)$, and differentiable on $(0, +\infty)$, and we find $y'(x) = \dfrac{3}{2}\sqrt{x}$ with the two methods.

Ex.A.21) Let $h(x) = f(g(x)) = (f \circ g)(x)$, and assume that at the point $x = 1$, h has a horizontal tangent. But h is differentiable at this point, so we conclude that $h'(1) = 0$. However, $h'(1) = g'(1)f'(g(1))$, so either $g'(1)$ or $f'(g(1))$ is zero! That means either the tangent to the graph of g at $x = 1$ is horizontal, either the tangent to the graph of f at $u = g(1)$ is.

Ex.A.22) You should work by your own to *properly justify* on this one:
a) cannot be true, *b*) could, *c*) is always true, and *d*) can also be true.

2 B. Tutorial for class workout.

Ex.B.1) Easy. Just do it yourself carefully.

Ex.B.2) First of all, a, b, c have to be such that the two curves touch each other at the point $x = 1$.
Let $f_1(x) = x^2 + ax + b$ and $f_2(x) = cx - x^2$. At the point $x = 1$, $f_1(1) = 1 + a + b$ and $f_2(1) = c - 1$. So we have one equation, $a + b + 1 = c - 1$.
These two curves have to take the value $y = 0$ at $x = 1$. So $c - 1 = 0$, ie $c = 1$, and $a + b = -1$.
Now we need to have the same slope, that is $f_1'(1) = f_2'(1)$. But $f_1'(1) = 2 + a$ and $f_2'(1) = c - 2$. So this is a second equation : $2 + a = c - 2$. We solve these two, giving $a = c - 4 = -3$, so $b = -a - 1 = 2$.
Conclusion, there is only one solution : $\underline{a = -3, b = 2, c = 1}$.

Ex.B.3) Let $x \in \mathbb{R}$. For $h \neq 0$, we expand $\dfrac{\cos(x + h) - \cos(x)}{h}$ with the usual identity of $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$:

$$\frac{\cos(x + h) - \cos(x)}{h} = \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} = \cos(x)\frac{\cos(h) - 1}{h} - \sin(x)\frac{\sin(h)}{h}.$$

Now we use the two known identities : $\dfrac{\cos(h) - 1}{h} \underset{h\to 0}{\to} 0$ and $\dfrac{\sin(h)}{h} \underset{h\to 0}{\to} 1$. Therefore, $\dfrac{\cos(x + h) - \cos(x)}{h} \underset{h\to 0}{\to} -\sin(x)$, which is the derivative of cos at the point x as you knew already.

Ex.B.4) We solve this one quickly:

(a) For $y(x) = \sin(2x)$, the equation of the tangent at the origin is $y = 2x$, for the other $y(x) = -\sin(x/2)$, the equation is $y = -x/2$. Feel free to draw and discuss more in details!

(b) To conclude.

Ex.B.5) To conclude.

3 C. Additional Exercises.

Ex.C.1) To conclude.