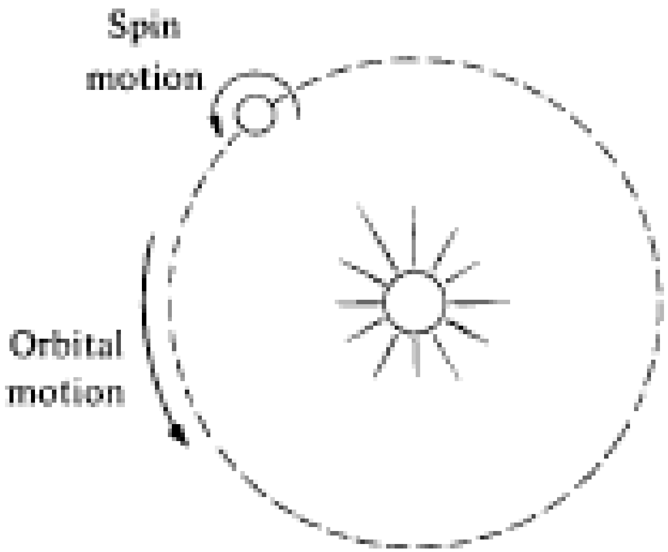
PH101
Lecture 22

16.09.14

$$L_z = \underbrace{I_0\omega}_{\text{Spin}} + \Big(\underbrace{\vec{R}\times M\vec{V}}_{\text{Orbital}}\Big)_z$$

Example earth:



Spin angular momentum is independent of the coordinate system
It is intrinsic to the body

Orbital angular momentum disappears if the origin is along the line of motion

Torque: $\vec{\tau} = \sum_j \vec{r}_j \times \vec{f}_j$

$$= \sum_j \left(\vec{r}_j^{\,\prime} + \vec{R}\right) \times \vec{f}_j$$

$$= \sum_j \underbrace{\left(\vec{r}_j^{\,\prime} \times \vec{f}_j\right)} + \vec{R} \times \vec{F}$$

Total appl. force: $\vec{F} = \sum_j \vec{f}_j$

Torque about the c.m
due to various ext. forces

2nd term is the torque due to the external forces acting at the c.m.

For fixed axis rotation $\vec{\omega} = \omega \hat{k}$

$$\tau_z = \tau_0 + \left(\vec{R} \times \vec{F}\right)_z$$

z comp of the torque about the c.m.

We had

$$L_z = I_0\omega + \left(\vec{R}\times M\vec{V}\right)_z$$

$$\Rightarrow \quad \frac{dL_z}{dt} = I_0\frac{d\omega}{dt} + \frac{d}{dt}\left(\vec{R}\times M\vec{V}\right)_z$$

$$\Rightarrow \quad \frac{dL_z}{dt} = I_0\alpha + \left(\vec{R}\times M\vec{a}\right)_z$$

$$\Rightarrow \quad \frac{dL_z}{dt} = \tau_z = I_0\alpha + \left(\vec{R}\times M\vec{a}\right)_z$$

$$= \tau_0 + \left(\vec{R}\times\vec{F}\right)_z$$

$$\downarrow$$

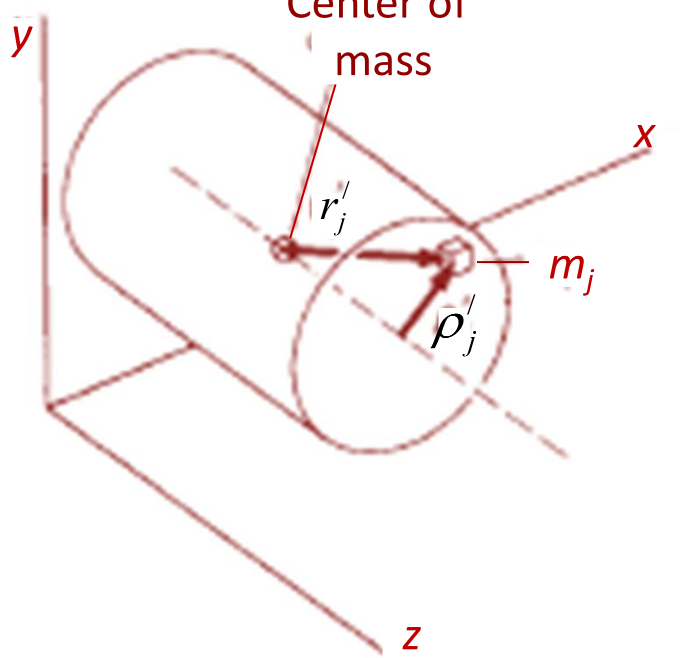
Thus rotational motion about c.m. depends only on the torque about the c.m. independent of translational motion

$\Rightarrow$ Expression for τ_0 even if the body is accelerating

$$\vec{r}_j = \vec{R} + \vec{r}_j^{\,/}$$

$$\Rightarrow \frac{d}{dt}\left(\vec{r}_j\right) = \dot{\vec{R}} + \dot{\vec{\rho}}_j^{\,/} = \dot{\vec{\rho}}_j^{\,/} + \vec{V} \quad \Rightarrow K.E. = \frac{1}{2}\sum m_j\left(\dot{\vec{\rho}}_j^{\,/} + \vec{V}\right)^2$$

$$= \frac{1}{2}\sum m_j \dot{\rho}_j^{/\,2} + \frac{1}{2}\sum m_j 2\dot{\vec{\rho}}_j^{\,/}.\vec{V} + \frac{1}{2}\sum m_j V^2$$



But $\dot{\rho}_j^{/} = \rho_j^{/}\omega$

$$\Rightarrow K.E. = \frac{1}{2}\sum \underbrace{m_j \rho_j^{/\,2}}_{\equiv I_0}\omega^2 + \frac{1}{2}MV^2$$

$$\Rightarrow K.E. = \frac{1}{2}I_0\omega^2 + \frac{1}{2}MV^2$$

1st term: K.E. of spin

2nd term: due to orbital c.m. motion

Summary:

For pure rotational motion about an axis (without any translation):

$$L = I\omega$$

$$\tau = I\alpha$$

$$K = \frac{1}{2} I\omega^2$$

For rotational + translational motion :

$$L_z = I_0\omega + \left(\vec{R} \times M\vec{V}\right)_z$$

$$\tau_z = \tau_0 + \left(\vec{R} \times \vec{F}\right)_z; \quad \tau_0 = I_0\alpha$$

$$K = \frac{1}{2} I_0\omega^2 + \frac{1}{2} MV^2$$

Work-energy theorem for a rigid body:

For a particle undergoing translational motion, work-energy theorem was

$$K_b - K_a = W_{ba}; \quad W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} \underbrace{\vec{F}.d\vec{r}}$$

Line integral $\oint$

It can be generalized for a rigid body and can be shown that this theorem divides into two parts: translational + rotational

For the translational part

From Newton's law , eq of motion for the c.m.

$$\vec{F} = M\frac{d^2\vec{R}}{dt^2} = M\frac{d\vec{V}}{dt}$$

Work done in displacing the c.m. through $d\vec{R} = \vec{V}dt$

$\Rightarrow$ For translational motion

$$\vec{F}.d\vec{R} = M\frac{d\vec{V}}{dt}.\vec{V}dt = d\left(\frac{1}{2}MV^2\right)$$

Thus line integral

$$\Rightarrow \int_{\vec{R}_a}^{\vec{R}_b} \vec{F}.d\vec{R} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2$$

Work associated with rotational K.E. :

Eq of motion for fixed axis rotation about the c.m. is: $\tau_0 = I_0\alpha = I_0 \dfrac{d\omega}{dt}$

Rotational K.E. has the form $\dfrac{1}{2} I_0 \omega^2$

This suggests as in case of translational motion of c.m. that we may make use of

$$\omega = \frac{d\theta}{dt} \Rightarrow d\theta = \omega\, dt$$

And multiply eq of motion by $d\theta$

$$\Rightarrow \tau_0\, d\theta = I_0 \frac{d\omega}{dt} \omega\, dt$$

$$= d\left(\frac{1}{2} I_0 \omega^2\right)$$

$$\Rightarrow \int_{\theta_a}^{\theta_b} \tau_0\, d\theta = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_a^2$$

LHS represents work done by the applied torque

$\Rightarrow$ General work-energy theorem for a rigid body:

$$K_b - K_a = W_{ba} \qquad \text{where } K = \frac{1}{2}MV^2 + \frac{1}{2}I_0\omega^2$$
$$\downarrow$$

Work done on the rigid body as it moves from position *a* to position *b*

Since

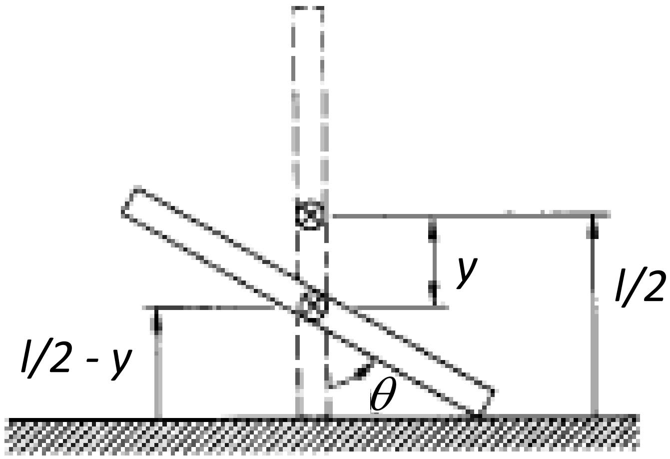
$$\int_{\vec{R}_a}^{\vec{R}_b} \vec{F}.\,d\vec{R} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2$$

and

$$\int_{\theta_a}^{\theta_b} \tau_0\, d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2$$

We may conclude that work-energy theorem consists of two independent theorems: one for translation & the other for rotation

Consider a falling stick & find velocity of its c.m. as a function of position



As it falls its c.m. drops through *y*

During which the stick has rotated through θ

Initially

$$E = K_0 + U_0 = \frac{Mgl}{2}$$

At a later time, K.E. is

$$K = \frac{1}{2} I_0 \omega^2 + \frac{1}{2} M v^2 = \frac{1}{2} I_0 \dot{\theta}^2 + \frac{1}{2} M \dot{y}^2$$

and P.E. is

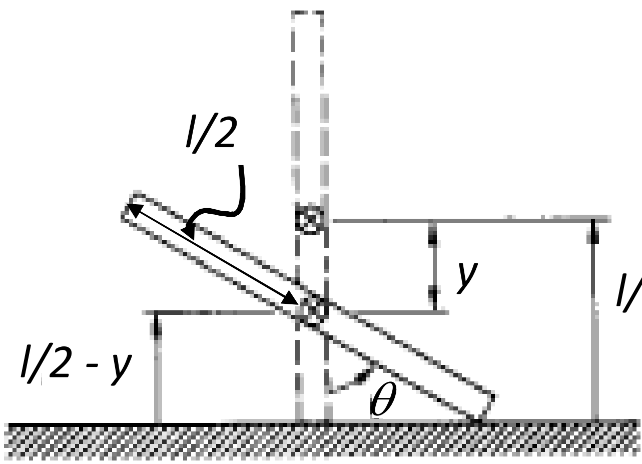
$$U = M g \left(\frac{l}{2} - y \right)$$

What is the dissipative force?

$\Rightarrow$ Mechanical energy: *K* + *U* must be conserved

$$K + U = K_0 + U_0 = \frac{Mgl}{2}$$

$$\Rightarrow \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I_0 \dot{\theta}^2 + Mg\left(\frac{l}{2} - y\right) = \frac{Mgl}{2}$$



$$\frac{{}^{l}\!/_{2} - y}{{}^{l}\!/_{2}} = \cos\theta \quad \Rightarrow y = \frac{l}{2}(1 - \cos\theta)$$

$$\Rightarrow \dot{y} = \frac{l}{2} \sin\theta\, \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{2}{l \sin\theta} \dot{y}$$

M.I. of a stick

$$I_0 = \frac{Ml^2}{12}$$

$$\Rightarrow \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I_0\dot{\theta}^2 + Mg\left(\frac{l}{2} - y\right) = \frac{Mgl}{2}$$

and $$\dot{\theta} = \frac{2}{l\sin\theta}\dot{y}$$

Thus

$$\frac{1}{2}\cancel{M}\dot{y}^2 + \frac{1}{2}\frac{\cancel{M}L^2}{12}\left(\frac{2}{l\sin\theta}\right)^2\dot{y}^2 + \cancel{M}g\left(\frac{l}{2} - y\right) = \cancel{M}g\frac{l}{2}$$

$$\dot{y}^2 = \frac{2gy}{\left[1 + 1/\left(3\sin^2\theta\right)\right]}$$

Motion of rigid bodies which can be rotated about any axis

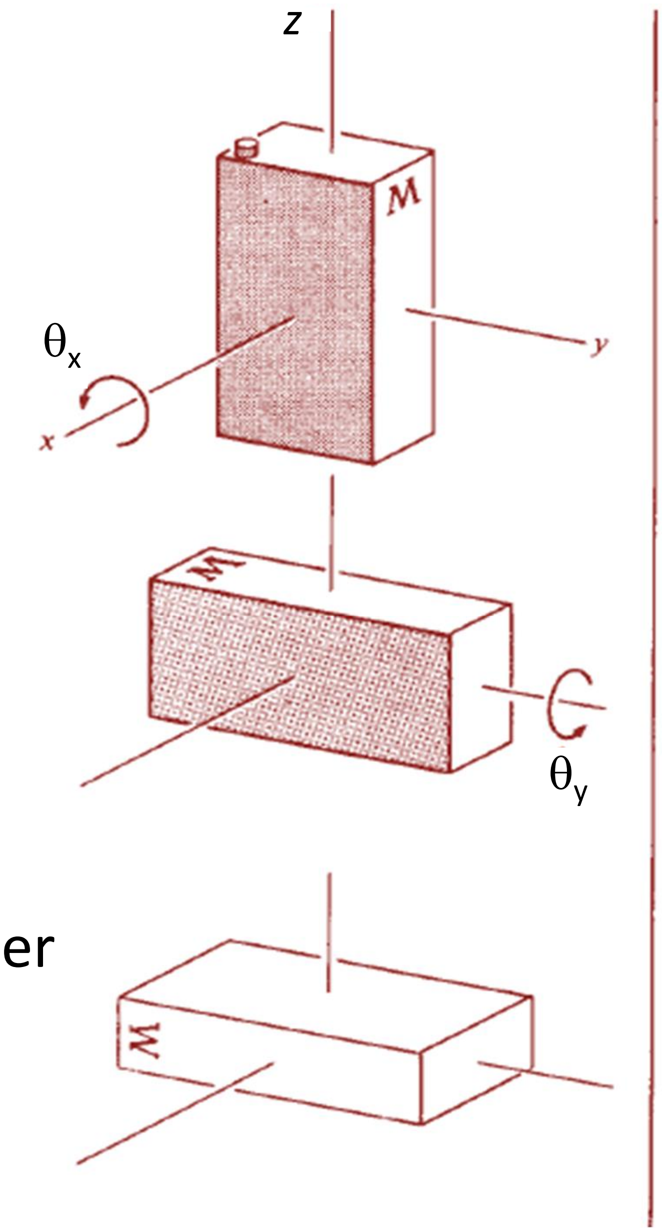
Vector nature of ω and **L**

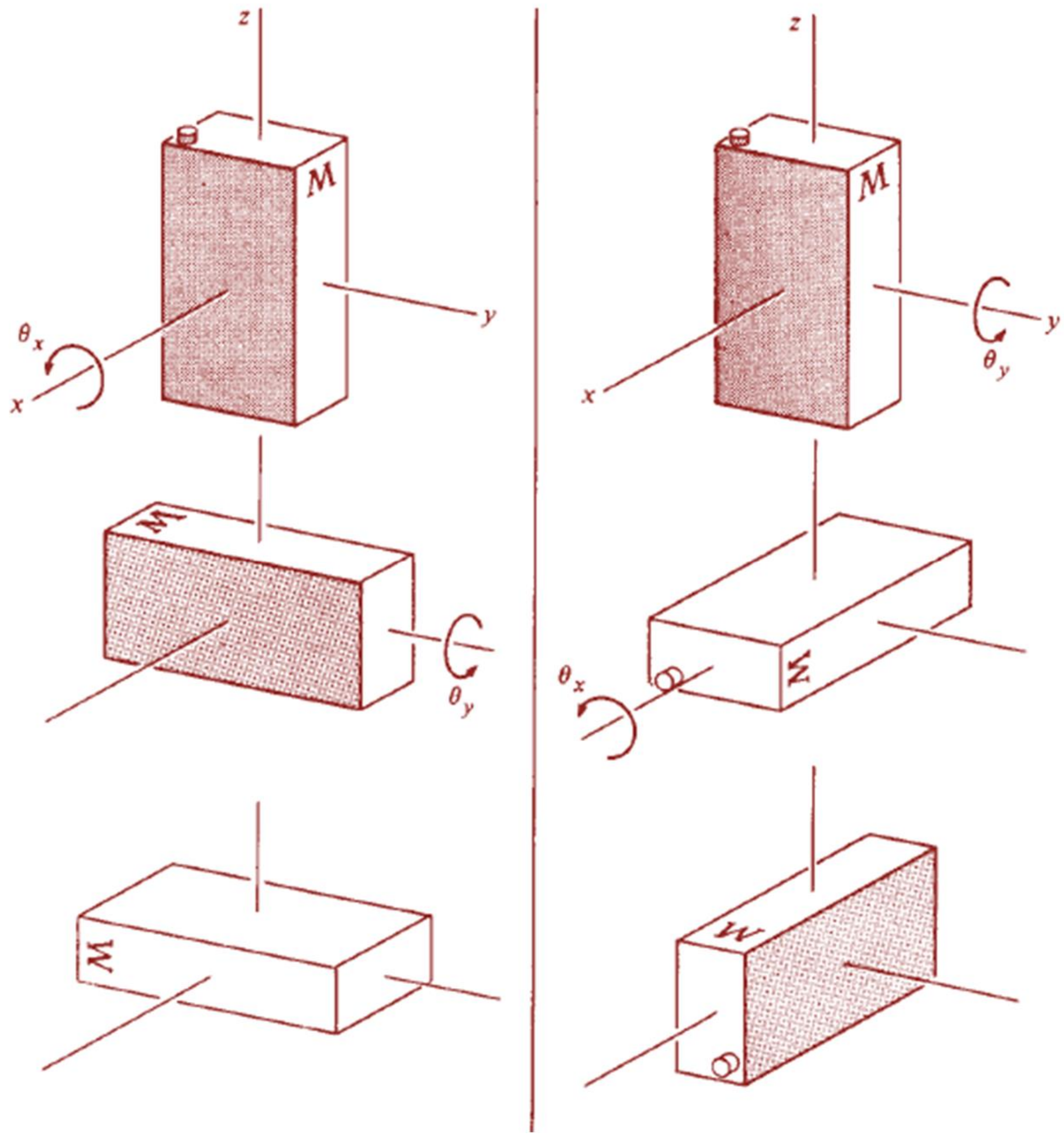
Consider a can of syrup in which alphabet *M* appears

Let us find out what happens when it is rotated by $\pi/2$ around *x*-axis

Repeat the same by rotating by $\pi/2$ around *y*-axis

Repeat the same two steps in reverse order



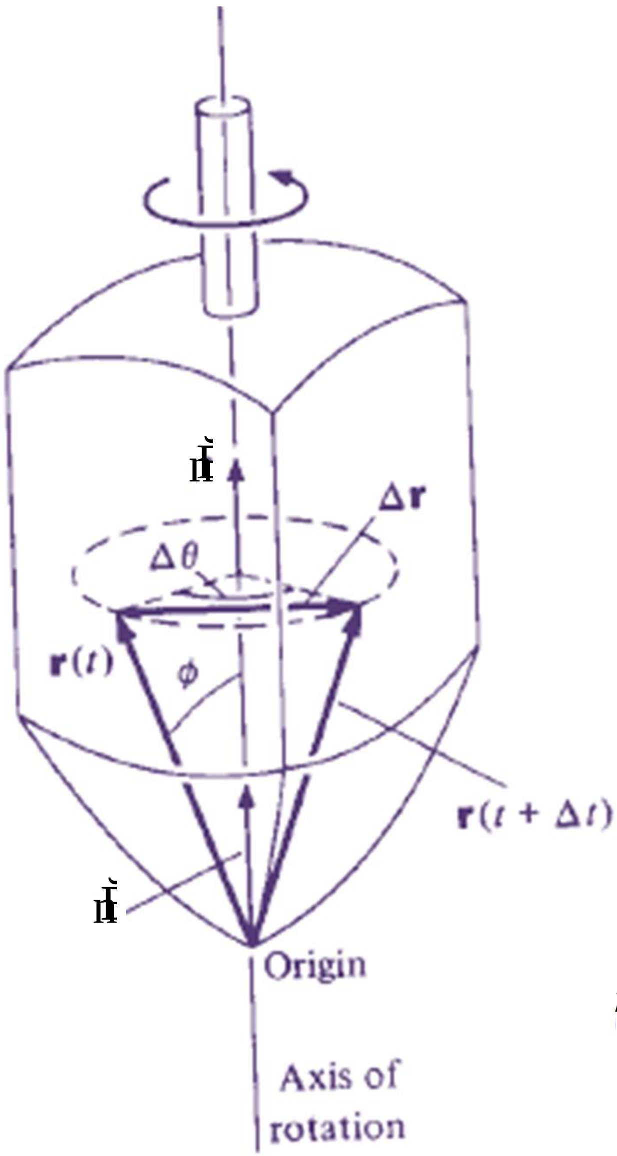


It can be seen that

$$\theta_x \vec{i} + \theta_y \vec{j} \neq \theta_y \vec{j} + \theta_x \vec{i}$$

$\Rightarrow$ Angular position can not be represented by a vector!

But angular velocity is perfectly a vector! $\omega = \frac{d\theta_x}{dt}\hat{i} + \frac{d\theta_y}{dt}\hat{j} + \frac{d\theta_z}{dt}\hat{k}$

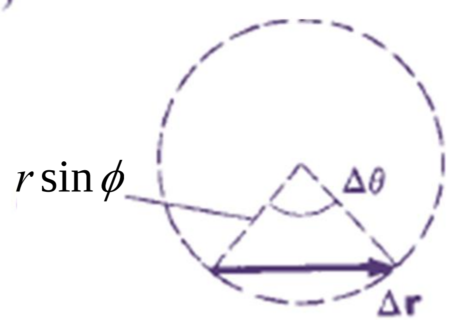


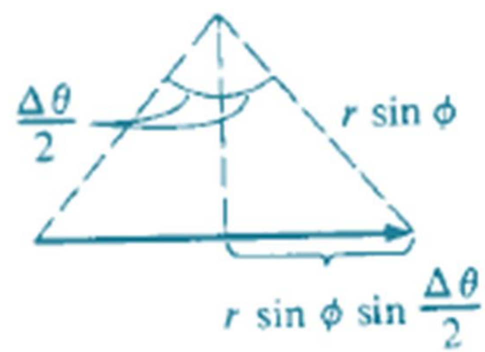
Consider rotation of a rigid body about some axis, represented by $\hat{n}$

Choose a coordinate system fixed in apace with its origin on the axis

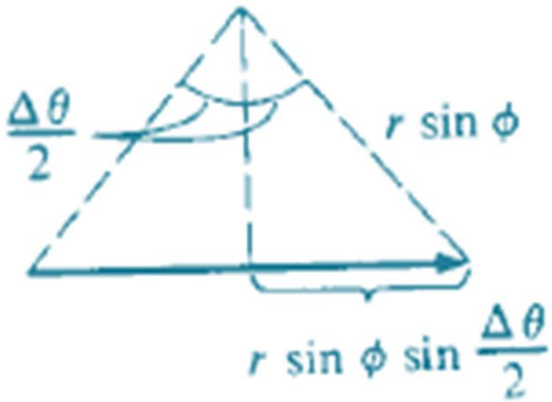
As the body rotates, each particle of it Describes a circle about its axis of rotation

A vector **r** from the origin to any particle tends to sweep out a cone; semi-angle ϕ is constant





Tip of **r** moves on a circle of radius r sin ϕ



$$\left|\Delta\vec{r}\right| = 2r\sin\phi\sin\frac{\Delta\theta}{2}$$

$$\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$$

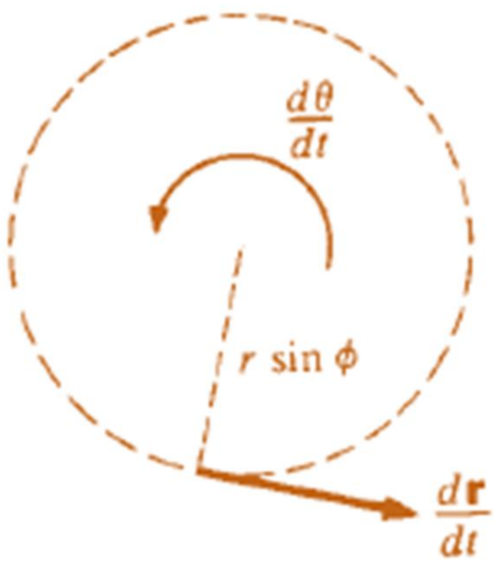
$$\Rightarrow \left|\Delta\vec{r}\right| \approx r\sin\phi\,\Delta\theta$$

If $\Delta\theta$ occurs over a time Δt

$$\Rightarrow \frac{\left|\Delta\vec{r}\right|}{\Delta t} \approx r\sin\phi\left(\frac{\Delta\theta}{\Delta t}\right)$$

which in the limit $\Delta t \to 0$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right| \approx r\sin\phi\left(\frac{d\theta}{dt}\right)$$



In the limit,

$$\frac{d\vec{r}}{dt}$$ is tangential to the circle

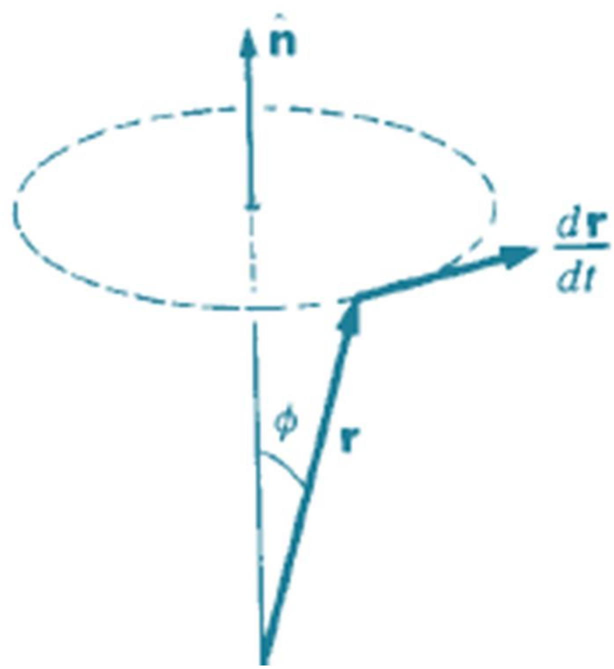
whose magnitude is $$\left|\frac{d\vec{r}}{dt}\right| \approx r\sin\phi\left(\frac{d\theta}{dt}\right)$$

and direction?

Perpendicular to the plane of

$$\vec{r} \text{ and } \hat{n}$$

$$\Rightarrow \underbrace{\frac{d\vec{r}}{dt}}_{=\,\mathbf{v}} = \hat{n}\times\vec{r}\,\frac{d\theta}{dt}$$



$$\hat{n}\frac{d\theta}{dt} = \vec{\omega}$$

$$\Rightarrow \vec{v} = \vec{\omega}\times\vec{r}$$