Real sequences and series : Ex.1.7 (*Babylonian method*)

Let $u_0 = 1$ and define, for $n \in \mathbb{N}$, $u_{n+1} = \dfrac{u_n + \frac{2}{u_n}}{2}$.

(i) Show that the sequence $u = (u_n)_{n\in\mathbb{N}}$ is well defined.

(ii) Compute the first 6 terms (by using your calculator or smartphone). What do you think the limit will be?

(iii) Show that u is converging[1], and compute its limit.

Solution

Lemma 1. *$\forall n \in \mathbb{N}, u_n > 0$, so u_{n+1} is well defined.*

Proof 1. *Obvious, by a quick induction.*

- *If $n = 0$, $u_0 = 1 > 0$,*
- *If $n \geqslant 0$, and $u_n > 0$, then $\dfrac{u_n}{2} > 0$ and $\dfrac{1}{u_n}$ exists and is positive. So $u_n = \dfrac{u_n}{2} + \dfrac{1}{u_n} > 0$.* □

First values

For question (ii), cf. Figure 1 or this array:

n	u_n
0	$u_0 = 1$
1	$u_1 = 1.5$
2	$u_2 \simeq 1.4166666666666665186$
3	$u_3 \simeq 1.4142156862745096646$
4	$u_4 \simeq 1.4142135623746898698$
5	$u_5 \simeq 1.4142135623730949234$
6	$u_6 \simeq 1.4142135623730949234$

Compute the limit

Lemma 2. $\forall n \in \mathbb{N}, u_{n+1} - \sqrt{2} = \dfrac{(u_n - \sqrt{2})^2}{2u_n} \quad (> 0)$.

Proof 2. *Let $n \in \mathbb{N}$.*

$$
\begin{aligned}
u_{n+1} - \sqrt{2} &= \frac{u_n}{2} + \frac{1}{u_n} - \sqrt{2} \\
&= \frac{u_n^2}{2u_n} + \frac{2}{2u_n} - \frac{2\sqrt{2}u_n}{2u_n} \\
&= \frac{u_n^2 + 2 - 2\sqrt{2}u_n}{2u_n} \\
u_{n+1} - \sqrt{2} &= \frac{(u_n - \sqrt{2})^2}{2u_n} > 0. \square
\end{aligned}
$$

Remark 1. *So we have $\forall n \in \mathbb{N}, \sqrt{2} \leqslant u_n$ (with Lemma 2 it is obvious, because $\frac{1}{2u_n} > 0$).*

Lemma 3. $\forall n \geqslant 1, u_{n+1} \leqslant u_n \leqslant u_1$.

Proof 3. *Let $n \in \mathbb{N}$. By the previous Remark 1, $\sqrt{2} \leqslant u_n$, so $\frac{2}{u_n} \leqslant u_n$. Therefore, $u_{n+1} = \frac{u_n + \frac{2}{u_n}}{2} = \frac{u_n}{2} + \frac{1}{u_n} \leqslant u_n$.*

So the sequence $(u_n)_{n \geqslant 1}$ is non-increasing (in fact, you can prove also that it is decreasing, i.e. *$u_{n+1} < u_n$, because $u_n \in \mathbb{Q}, \forall n \in \mathbb{N}$ so Remark 1 gives $\sqrt{2} < u_n$). So $\forall n \in \mathbb{N}, u_n \leqslant u_1 = \frac{3}{2} = 1.5$.* $\square$

Conclusion: So that sequence is decreasing, lower-bounded (by $\sqrt{2}$), so it converges to a limit $l > 0$. The only possible value for l is $\sqrt{2}$ (because it verifies $l = \frac{l + \frac{2}{l}}{2} \Leftrightarrow l^2 = 2$).

Remark 2. *This method to approximate the numerical value of $\sqrt{2}$ is quite an old one. That Wikipédia page explains a little more about the history of this method, often know as the Hero's method, or Babylonian method.*

Remark 3. *It can be generalized to approximate the square root of any $A > 0$. Just take $u_0 = \lceil A \rceil (= \lfloor A \rfloor + 1)$ and $\forall n \in \mathbb{N}, u_{n+1} = \frac{u_n + \frac{A}{u_n}}{2}$.*

Bonus: *do again the same exercise for the general case, showing that $u_n \underset{n \to +\infty}{\longrightarrow} \sqrt{A}^+$ (*i.e. *by being always greater than $\sqrt{A}$).*

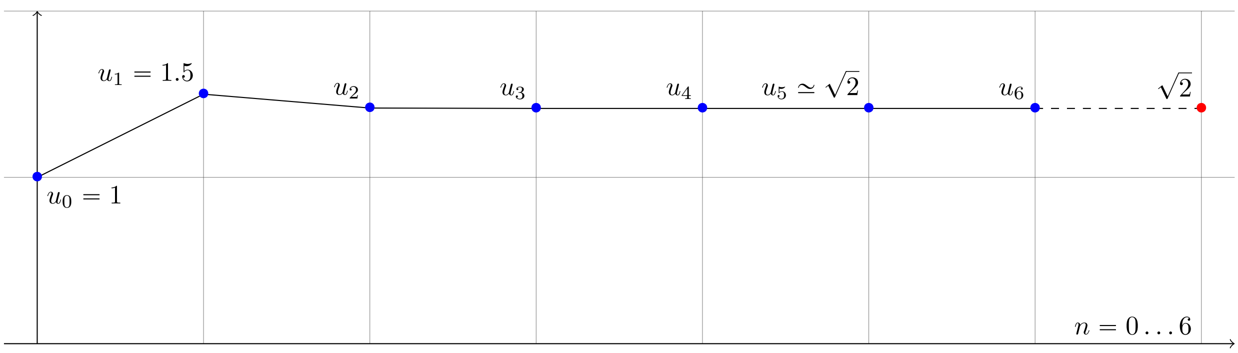

Figure 1: The first values of u_n, from $n = 0$ to $n = 6$.