Solving Networks in Time domain

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In this handout I will try to give you detailed steps regarding solving networks in time domain.

If you have any doubts on this topic, please come to the extra class on October 10th

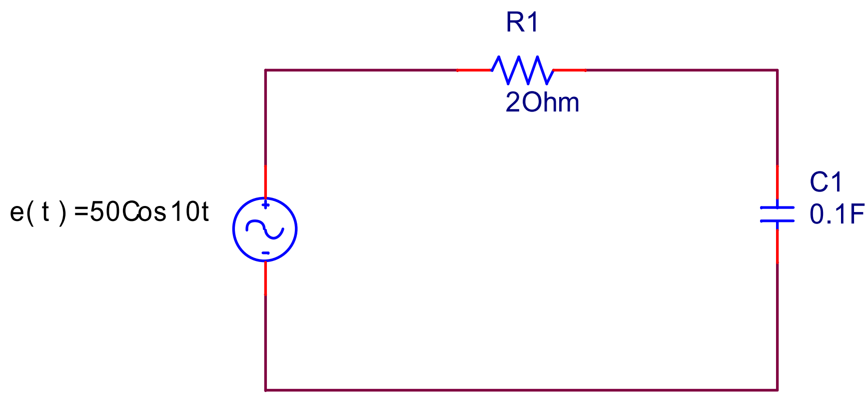

Figure 1: Circuit.

Let's start with the circuit shown in Figure 1. A voltage source is connected in series with a resistor and a capacitor, with the capacitor having an initial voltage of 5V. Look at the value of the voltage source, it is given as $e(t) = cos(10t)$. The voltage (e(t)) is given as a function of time. In other words *e(t)* tells us how the voltage is changing as time passes. It is in the form of a cosine wave with angular frequency 10 rad/sec. From this information, we can determine how other parameters in the circuit, such as the current or the voltage across the resistor or the capacitor, varies as time passes. Since we are trying to find all these parameters with reference to time, we can call the procedure to find these parameters as the time-domain analysis. We will be using techniques in solving differential equations to analyse the circuits in time domain.

Let's now try to find the current in the circuit using time domain analysis. Let's call the current in the circuit as *i(t)*. Writing *i(t)* for current always is slightly lengthy. So we use *small letter i* to represent the time varying functions. Now we can go ahead and write the KVL for the circuit using the time domain v-i relationships for resistor, inductor and capacitor.

$$e = iR + \frac{1}{C}\int_0^t i\,dt + v_c(0) \tag{1}$$

$v_c(0)$ is the initial voltage in the capacitor. Never forget it in the equation. Substituting the values

$$\mathit{50cos(10t)} = 2i + 10\int_0^t i\,dt + 5 \tag{2}$$

There is an integral term in the equation. To remove it, we will have to differentiate the whole equation w.r.t time. Then we will get,

$$\mathit{-500sin(10t)} = 2\frac{di}{dt} + 10 * i \tag{3}$$

or

$$\frac{di}{dt} + 5i = \mathit{-250sin(10t)} \tag{4}$$

So we got the required differential equation. Now if we solve this equation, we will get the expression for i. Since the differential equation contains only a first order differential term ($\frac{d}{dt}$), it is called a first-order differential equation. Hence this circuit can be also called as a first-order system. Since all terms are linear in the equation, this is a first order linear differential equation.

Homogeneous and inhomogeneous differential equations

A differential equation is said to be homogeneous if all the terms in the equation contains either the dependent variables (here i) or its derivatives. Otherwise it is called an inhomogeneous differential equation. Hence $\frac{di}{dt} + 5i = \mathit{-250sin(10t)}$ is inhomogeneous due to the *-250sin(10t)* term. The homogeneous equation for this differential equation is

$$\frac{di}{dt} + 5i = 0 \tag{5}$$

Note that if the right-hand side of the equation is any other constant other than 0, the equation is inhomogeneous. More details about this you will learn soon in MA101.

Solving differential equations

Solving linear differential equations is composed of two parts namely finding the the complementary functions and then the particular integral.

Finding the complementary function

To find the complementary solution, we use the homogeneous equation. In our case, it is

$$\frac{di}{dt} + 5i = 0 \tag{6}$$

Now replace the operator $\frac{d}{dt}$ by D and find the solution for D. The solution for the first order equation will be always in the form Ke^{st}, where K is a constant and s is the solution for D. How an operator $\frac{d}{dt}$ can be represented as a variable, you will better understand in MA101. Just understand that it is possible since the solutions are in the form of Ke^{st}, whose derivatives are nothing but Ke^{st} multiplied by s the times you differentiate. For a second order differential equation, the solution will be in the form $K_1e^{s_1t}$+$K_2e^{s_2t}$, where s_1 and s_1 are the roots of the equation expressed using D. We will see an example for the second order system later. So our homogeneous equation can be written as

$$Di + 5i = 0 \tag{7}$$

Solving this we get D = -5 and hence the complementary function is

$$i_c = Ke^{-5t} \tag{8}$$

Finding the particular integral

Now we need to find the particular integral. The particular integral depends on the type of input is given to the system. Whenever the input is in the form of sin(ωt) or cos(ωt), the particular integral will be in the form Acos(ωt)+Bsin(ωt). Again the proof you will see in the MA101. In our example, the input is *-250sin(10t)*, hence ω=10. Hence

$$i_p = Acos(10t) + Bsin(10t) \tag{9}$$

$$\frac{d\,i_p}{dt} = -10Asin(10t) + 10Bcos(10t) \tag{10}$$

Substituting equations 9 and 10 in 4,

$$-10Asin(10t) + 10Bcos(10t) + 5Acos(10t) + 5Bsin(10t) = -250sin(10t) \tag{11}$$

Comparing the coefficients of the sine and cosine terms,

$$-10A + 5B = -250 \tag{12}$$

$$10B + 5A = 0 \tag{13}$$

From 12 and 13, A = 20, B=-10. Substituting in 9, the particular integral is

$$i_p = 20cos(10t) - 10sin(10t) \tag{14}$$

Final solution

The final solution to the equation is the sum of complementary function and the particular integral, thus

$$\begin{aligned} i &= i_c + i_p \\ i = Ke^{-5t} &+ 20cos(10t) - 10sin(10t) \end{aligned} \tag{15}$$

Now the only unknown is K. To solve for K, we need to use the initial conditions. Hence substitute 15 in 2. If you substitute it in any other equation, you will not be able to solve it as your K terms will get cancelled. Try your self. After substitution, the equation becomes

$$\begin{aligned} 50cos(10t) &= 2(Ke^{-5t} + 20cos(10t) - 10sin(10t)) + 10\int_0^t Ke^{-5t} + 20cos(10t) - 10sin(10t)\,dt + 5 \\ &= 2Ke^{-5t} + 40cos(10t) - 20sin(10t) + 10\Big[K\frac{e^{-5t}}{-5} + 20\frac{sin10t}{10} + 10\frac{cos10t}{10}\Big]_0^t + 5 \\ &= 2Ke^{-5t} + 40cos(10t) - 20sin(10t) - 2Ke^{-5t} + 20sin10t + 10cos10t + 2K - 10 + 5 \end{aligned} \tag{16}$$

Now substituting at t=0

$$\begin{aligned} 50 = 2K + 40 - 2K + 10 + 2K - 5 \\ K = \frac{5}{2} \end{aligned} \tag{17}$$

Substituting this in 15, the total solution becomes

$$\boxed{i(t) = \frac{5}{2}e^{-5t} + 20cos(10t) - 10sin(10t)}$$

The part $\frac{5}{2}e^{-5t}$ is called the transient response. A function *f(t)* is called transient, if

$$\lim_{t\to\infty} f(t) = 0 \tag{18}$$

The transients are generated not only because of the initial conditions in the circuit but also due to the discontinuities in the input. For example in our circuit, the transients are generated by the input voltage in the capacitor and the input cosine wave. The part *20cos(10t)-10sin(10t)* is called the steady-state response, since this is the only response remaining in the system after the transients die down.

So what is the current in the circuit at t=0?

$$i(0) = \frac{5}{2} + 20 = \frac{45}{2}A$$

This result can be used to determine whether your answer is right or wrong to some extend. Look back at the circuit in Fig.1. At t=0, there is some voltage in the capacitor (5V) and the input voltage is the peak of the cosine wave (50V). Now if we assume the clock-wise direction of current flow is positive, we can find the current in the circuit due to the initial capacitor voltage by assuming there is no external input (zero input condition). Then the current will be $\frac{V_c}{R}$ = -$\frac{5}{2}$A. The minus sign is there since the current due to this initial voltage will be flowing the in the anti-clock-wise direction. Now the current due to the input voltage can be found by assuming there is no input voltage across the capacitor. Since at t=0, the peak of the cosine wave applied across the capacitor (50V). Hence the current in the circuit will be $\frac{V_{in}}{R}$ = $\frac{50}{2}$A. So the total current in the circuit is the sum of this zero input and zero state currents (super position principle). i(0) = -$\frac{5}{2}$ + $\frac{50}{2}$ = $\frac{45}{2}$A, which is same as what we got by substituting t=0 in our differential equation solution. So it is very much possible that we did not make any mistakes during solving the equations. Again one more important observation is the angular frequency (ω) terms in the solution is same as in the input ω (10). It should be like that since all the RLC circuits are linear in nature and will not change the frequency of the signals.

A different perspective : Solving networks using zero input (natural) and zero state (forced) responses

We can view the total response of the network as the sum of response due to input conditions (zero state) and that of the initial conditions (zero state).

Zero input response

When we want to find the zero input response, we assume there is no input to the circuit. So the equation for our example circuit becomes

$$2i + 10\int_0^t i\,dt + 5 = 0 \tag{19}$$

Differentiating

$$2\frac{di}{dt} + 10 * i = 0 \tag{20}$$

This differential equation has only complementary function since there is no input.

$$Di + 5i = 0 \tag{21}$$

Solving this we get D = -5 and hence the complementary function is

$$i_i = Ke^{-5t} \tag{22}$$

i_i is to represent current due to initial conditions. Now apply the initial conditions now itself. So substitute it in equation 19. From this we will get K = -$\frac{5}{2}$

$$i_i = -\frac{5}{2}e^{-5t} \tag{23}$$

Zero state response

Here we assume that the initial conditions are zero. Hence by applying KVL to the circuit,

$$50cos(10t) = 2i + 10\int_0^t i\,dt \tag{24}$$

Now differentiating to remove the integral term

$$\text{-}500sin(10t) = 2\frac{di}{dt} + 10 * i \tag{25}$$

This equation is same as 3. So finding the solution is exactly what we did in the previous section with complementary function and particular integral. Once you do it, you will get the equation

$$i_s = Ke^{-5t} + 20cos(10t) - 10sin(10t) \tag{26}$$

i_s represents current due to the inputs. To find K, now you substitute this equation in 25 and solve. You will get K as 5. Hence

$$i_s = 5e^{-5t} + 20cos(10t) - 10sin(10t) \tag{27}$$

Total response

The total response is the sum of i_i and i_s.

$$\boxed{i(t) = -\frac{5}{2}e^{-5t} + 5e^{-5t} + 20cos(10t) - 10sin(10t)}$$

The result is same as what we got by solving the differential equations directly. But in this form we are able to separately see what is the response due to the initial condition and what is the response due to the input. Also the transients due to initial conditions and the inputs are separately visible.