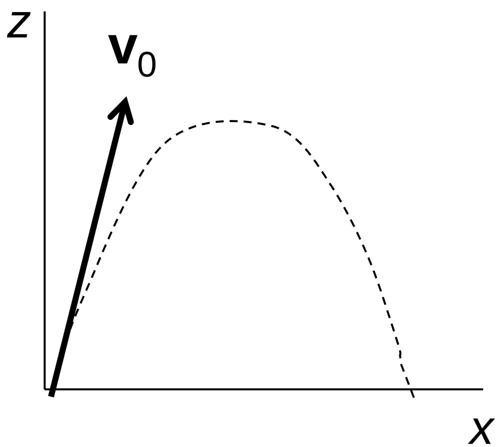
PH101
Lecture 4

08.08.14

A correction on the last but one slide in Lecture 3

Example on motion in a uniform gravitational field:

A freely moving ball under the influence of earth's gravity experiences a constant downward acceleration of *g*. Assuming that it is confined to *xz* plane, obtain its trajectory by assuming *z*-axis to be vertically upwards.



$$\vec{a} = -g\,\hat{\mathrm{k}}$$

Assuming it to be released at $t = 0$ with an initial velocity $\mathbf{v}_0$ and assuming position vector $\mathbf{r}_0 = 0$

$$x = v_{0x} t$$

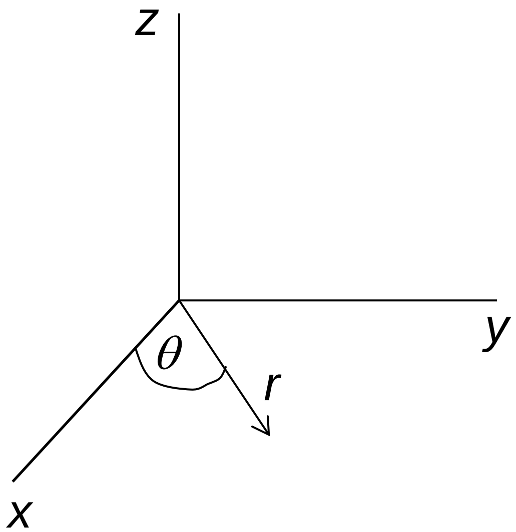
$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\, t + \frac{1}{2}\vec{a}\, t^2$$

$$z = v_{0z} t - \frac{1}{2} g t^2$$

$$\Rightarrow \text{trajectory}: z = v_{0z}\frac{x}{v_{0x}} - \frac{1}{2} g \frac{x^2}{v_{0x}^2} = \frac{v_{0z}}{v_{ox}} x - \frac{g}{2v_{0x}^2} x^2$$

Motion in plane polar coordinates

Cartesian coordinates are **not** suitable to model/discuss circular motion

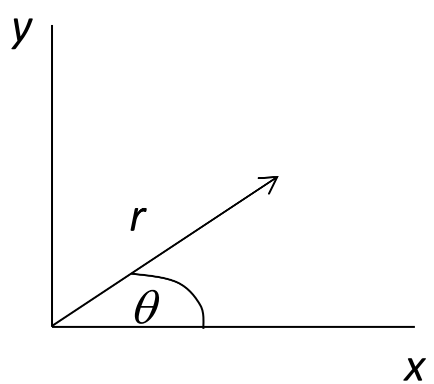


Polar coordinates

The cylindrical coordinate system, in which *z* axis is identical to cartesian system

Position in the xy-plane is given by the distance *r* from the *z* axis and its angle θ with the *x* axis

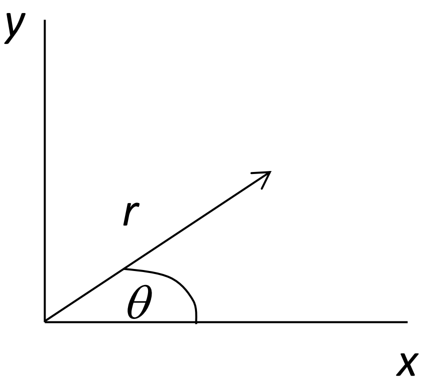
r and θ: plane polar coordinates

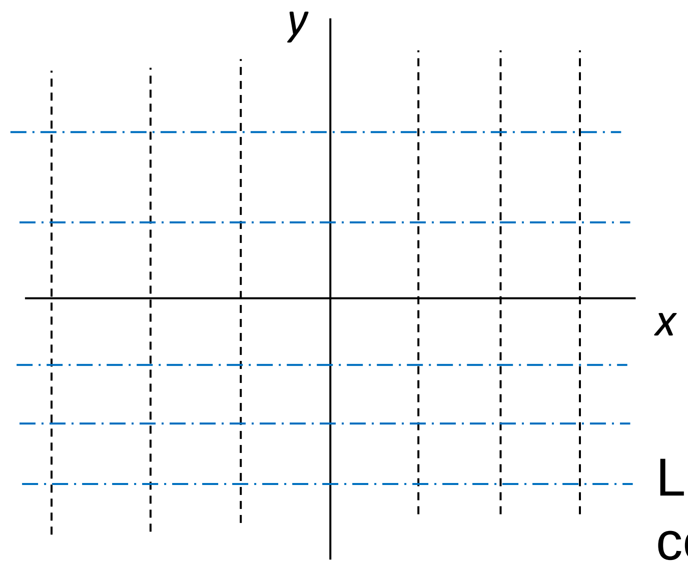


$$r = \sqrt{x^2 + y^2}$$

$$\frac{r\cos\theta}{r\sin\theta} = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\frac{y}{x}$$

Constant coordinate lines in cartesian & plane polar coordinates

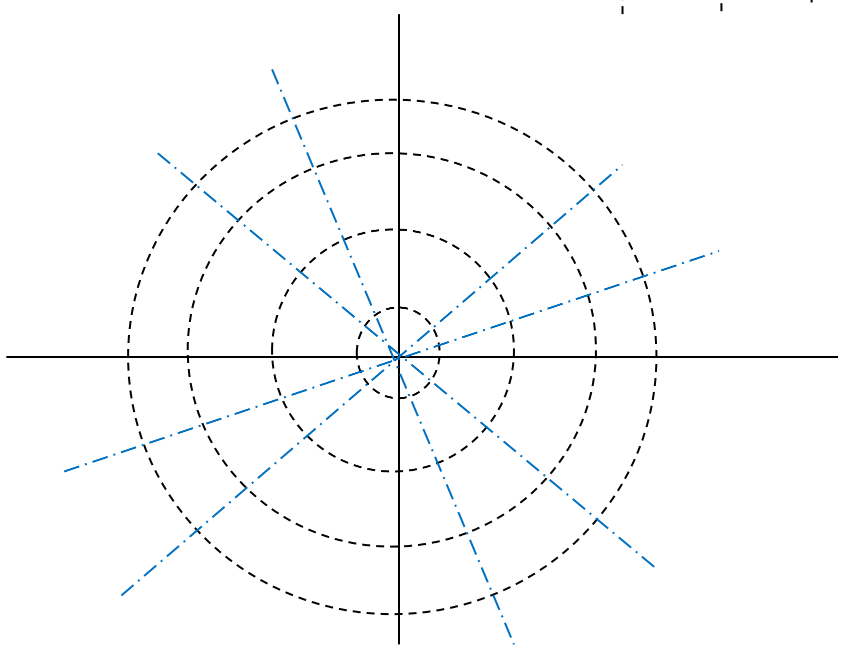




---- : x = constant

---- : y = constant

Lines of constant x & constant y are st lines $\perp$ to each other



---- : r = constant

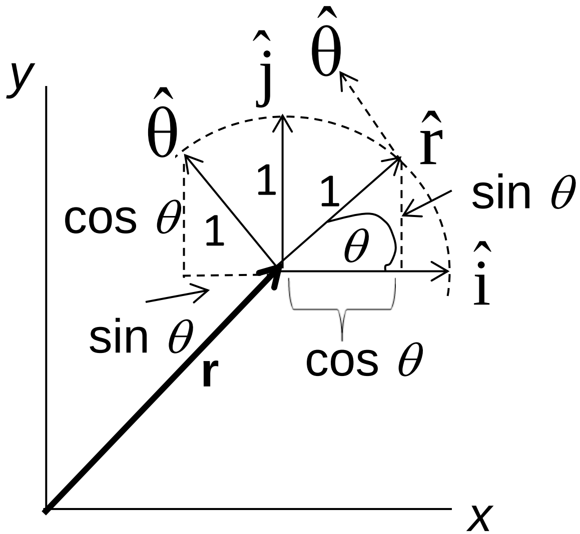
---- : θ = constant

Lines of constant θ are also st lines

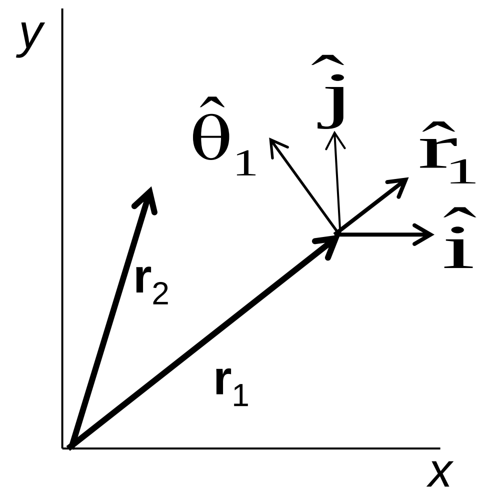
Lines of constant r are concentric circles

Base vectors in plane polar coordinates are

$$\hat{\mathrm{r}}\ \&\ \hat{\theta}\ \text{and are perpendicular to each other}$$



They point in the direction of increasing *r* and increasing θ and they both vary with position.



$$\hat{\mathrm{r}} = \hat{\mathrm{i}}\cos\theta + \hat{\mathrm{j}}\sin\theta$$

$$\hat{\theta} = -\hat{\mathrm{i}}\sin\theta + \hat{\mathrm{j}}\cos\theta$$

Naturally $\hat{\mathrm{r}}.\hat{\theta} = 0$

Check for $\theta = 0$ and $\pi/2$

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}}$$

In plane polar coordinates: $\vec{r} = r\,\hat{\mathrm{r}}$

$$\Rightarrow \quad x\,\hat{\mathrm{i}} + y\,\hat{\mathrm{j}} = r\left(\hat{\mathrm{i}}\cos\theta + \hat{\mathrm{j}}\sin\theta\right)$$

$$\Rightarrow \quad \left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$ As one would expect

$$\vec{r} = r\,\hat{\mathrm{r}}$$

*x a*nd *y* are needed in cartesian coordinates to specify position vector **r** !

$\hat{\mathbf{r}}$ is not a fixed vector

Orientation θ of $\hat{\mathbf{r}}$ is needed as well as *r* to find out how far one is from the origin

$$\Rightarrow \vec{r} = r\,\hat{\mathrm{r}}\left(\theta\right)$$

$\hat{\mathrm{r}}$ and $\hat{\theta}$ $\perp$ to each other and also are unit vectors

$$\Rightarrow |\hat{\mathrm{r}}| = 1 \text{ and } |\hat{\theta}| = 1$$

Thus if

$$\vec{A} = A_r \hat{\mathrm{r}} + A_\theta \hat{\theta}$$

$$\vec{B} = B_r \hat{\mathrm{r}} + B_\theta \hat{\theta}$$

$$\vec{A}.\vec{B} = A_r B_r + A_\theta B_\theta$$

Is this true everywhere in space ?

$\hat{\mathrm{r}}$ & $\hat{\theta}$ must refer to the same point in space!

Velocity in polar coordinates

Position **r** of a particle in plane polar coordinates is

$$\vec{r} = r\,\hat{\mathrm{r}}$$

r changes as the particle moves

$$\vec{v} = \frac{d\,\vec{r}}{dt} = \frac{dr}{dt}\,\hat{\mathrm{r}}$$

$\hat{\mathrm{r}}$ depends on θ

As the particle moves θ changes with time and hence $\hat{\mathrm{r}}$ also changes

Correct **v** is

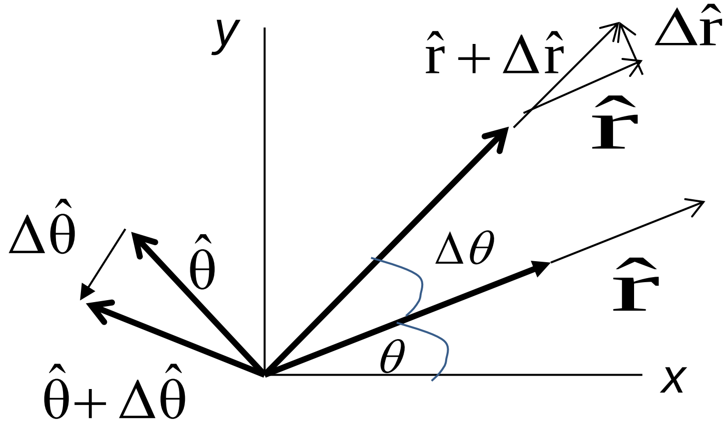
$$\vec{v} = \left(\frac{dr}{dt}\right)\hat{\mathrm{r}} + r\left(\frac{d\,\hat{\mathrm{r}}}{dt}\right)$$

Compare with **v** in cartesian coordinates

$$\vec{v} = \frac{d}{dt}\left(x\,\hat{\mathrm{i}} + y\,\hat{\mathrm{j}}\right) = \dot{x}\,\hat{\mathrm{i}} + \dot{y}\,\hat{\mathrm{j}}$$

$\left(\frac{d\,\hat{\mathbf{r}}}{dt}\right)$: $\hat{\mathbf{r}}$ does not change as one moves radially in or out

$\hat{\mathrm{r}}$ changes only if θ changes



$$\left|\Delta\hat{\mathrm{r}}\right| \approx \left|\hat{\mathrm{r}}\right|\Delta\theta = \Delta\theta$$

$$\frac{\left|\Delta\hat{\mathrm{r}}\right|}{\Delta t} \approx \frac{\Delta\theta}{\Delta t}$$ In the limit $\Delta t \to 0$ $$\left|\frac{d\hat{\mathrm{r}}}{dt}\right| = \frac{d\theta}{dt}$$

As θ increases, $\hat{\mathrm{r}}$ swings in the direction $\hat{\Theta}$ $$\frac{d\,\hat{\mathrm{r}}}{dt} = \dot{\theta}\,\hat{\theta}$$

Alternatively

$$\frac{d\hat{\mathrm{r}}}{dt}=\frac{d}{dt}\left(\cos\theta\,\hat{\mathrm{i}}+\sin\theta\,\hat{\mathrm{j}}\right):\left(\frac{d}{d\theta}\cos\theta\right)\frac{d\theta}{dt}=-\sin\theta\,\dot{\theta}$$

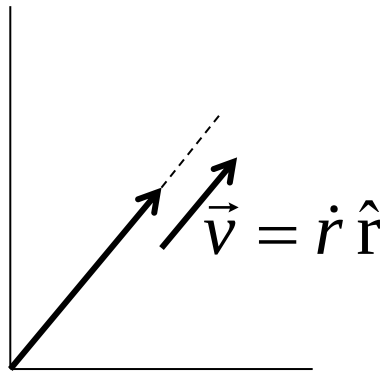
$$\left(\frac{d}{d\theta}\sin\theta\right)\frac{d\theta}{dt}=\cos\theta\,\dot{\theta}$$

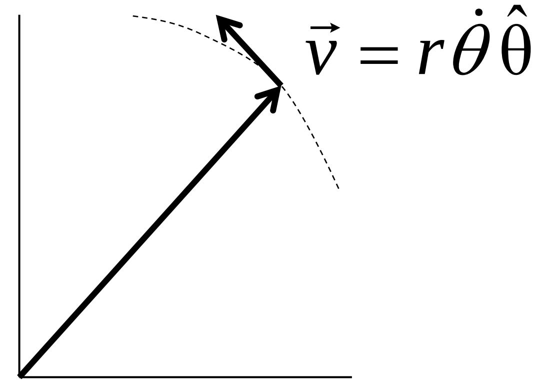
$$\Rightarrow\qquad \frac{d\hat{\mathrm{r}}}{dt}=-\hat{\mathrm{i}}\sin\theta\,\dot{\theta}+\hat{\mathrm{j}}\cos\theta\,\dot{\theta}=(-\hat{\mathrm{i}}\sin\theta+\hat{\mathrm{j}}\cos\theta)\,\dot{\theta}$$

$$\Rightarrow$$

$$\frac{d\hat{\mathrm{r}}}{dt}=\dot{\theta}\,\hat{\theta}$$

Thus, velocity of the particle will be $\vec{v}=\dot{r}\,\hat{\mathrm{r}}+r\dot{\theta}\,\hat{\theta}$


$\vec{v} = \dot{r}\,\hat{\mathrm{r}}$


$\vec{v} = r\dot{\theta}\,\hat{\theta}$

Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}\right) = \ddot{r}\,\hat{r} + \dot{r}\frac{d\,\hat{r}}{dt} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\frac{d\,\hat{\theta}}{dt}$$

$$\frac{d\,\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$$

$$\frac{d\,\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}$$

Centripetal acceleration

Coriolis acceleration

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\hat{\theta}$$

Linear acceleration in the radial direction

Linear acceleration in the tangential direction