RLC Circuit Analysis in Time Domain

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In this handout we try to analyse a more complicated circuit involving R,L and C. If you haven't read the previous handout on "time domain analysis", please read it before reading this handout

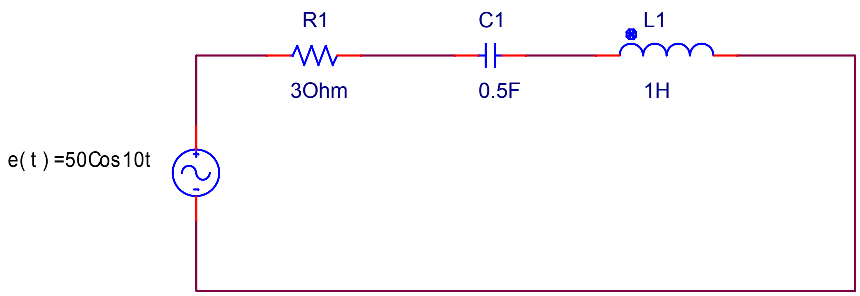

Figure 1: Circuit.

In the previous handout we saw how differential equations can be used to solve networks in time domain. We will continue with time domain analysis for solving a second order system. The circuit under analysis is shown in in Figure 1. From the previous handout you may remember we use the initial conditions in the circuit to determine the constant values for the complementary function. For a first order system, we need to determine the value of a single constant hence we need a single initial condition. In case of a second order, there will be two constants hence we need two initial conditions. These initial conditions will be the voltage across the capacitor and the current in the inductor. Let's assume the capacitor has an initial voltage of 5V and the inductor has an initial current of 1A. Now we can write the KVL for the circuit using the time domain v-i relationships.

$$e = iR + \frac{1}{C}\int_0^t i\,dt + L\frac{d\,i}{dt} + v_c(0) \tag{1}$$

$v_c(0)$ is the initial voltage in the capacitor. Substituting the values

$$50cos(10t) = 3i + 2\int_0^t i\,dt + \frac{d\,i}{dt} + 5 \tag{2}$$

To remove the integral term, we need to differentiate it again w.r.t time. Then we will get,

$$\textit{-500sin(10t)} = 3\frac{di}{dt} + 2i + \frac{d^2\, i}{dt^2} \tag{3}$$

or

$$\frac{d^2\, i}{dt^2} + 3\frac{di}{dt} + 2i = \textit{-500sin(10t)} \tag{4}$$

Since the differential equation contains a second order differential term ($\frac{d^2}{dt^2}$), the system is called a second order system. Again since all terms are linear in the equation, this is a second order linear differential equation.

Complementary function

To find the complementary solution, we use the homogeneous equation.

$$\frac{d^2\, i}{dt^2} + 3\frac{di}{dt} + 2i = 0 \tag{5}$$

Now replace the operator $\frac{d}{dt}$ by D and $\frac{d^2}{dt^2}$ by D^2 and solve for D. The solution for the second order homogeneous equation will be always in the form $K_1e^{s_1t} + K_2e^{s_2t}$, where K_1 and K_2 are constants and s_1 and s_2 are the solutions for D. So our homogeneous equation can be written as

$$(D^2 + 3D + 2)i = 0 \tag{6}$$

Solving this we get D = -1 or D = -2 and hence the complementary function is

$$i_c = K_1e^{-t} + K_2e^{-2t} \tag{7}$$

Some times you will get the solution to D as complex numbers. If they are complex in nature, both roots will be always complex conjugates. Again the real part of the solutions should be always negative. (This is easy to see since the coefficient of D will be the resistance value, which will be always positive. So during solving the quadratic equation, the *-b* term will be always negative.) Then only the response will be decaying in nature as we know the transient response due to initial condition should be always decaying in nature. Since you get complex conjugates as the power of e, finally you will be able to write the transient response as sine or cosine wave multiplied by some e^{-kt} term. It will look like a decaying sine or cosine wave or like a damped oscillation. I will try to put another example showing this condition.

Particular integral

Now we will find the particular integral. Since the input is in the form of sin(ωt), the particular integral will be in the form Acos(ωt)+Bsin(ωt) with ω=10. Hence

$$i_p = Acos(10t) + Bsin(10t) \tag{8}$$

$$\frac{d\,i_p}{dt} = -10Asin(10t) + 10Bcos(10t) \tag{9}$$

$$\frac{d^2\,i_p}{dt^2} = -100Acos(10t) - 100Bsin(10t) \tag{10}$$

Substituting equations 8, 9 and 10 in 4,

$$-100Acos(10t) - 100Bsin(10t) - 30Asin(10t) + 30Bcos(10t) + 2Acos(10t) + 2Bsin(10t) = -500sin(10t) \tag{11}$$

Comparing the coefficients of the sine and cosine terms,

$$-30A - 98B = -500 \tag{12}$$

$$-98A + 30A = 0 \tag{13}$$

From 12 and 13, A = 1.428, B=4.664. Substituting in 8, the particular integral is

$$i_p = 1.428cos(10t) + 4.664sin(10t) = 4.87cos(10t - 73) \tag{14}$$

Final solution

The final solution to the equation is the sum of complementary function and the particular integral, thus

$$\begin{gathered} i = i_c + i_p \\ i = K_1e^{-t} + K_2e^{-2t} + 4.87cos(10t - 73) \end{gathered} \tag{15}$$

Now we need to find the values of K_1 and K_2 for which we can use the initial conditions. It is given that the inductor has 1A current when t=0. Since inductor will oppose any change in current, the whole current in the circuit at t=0 will be the current in the inductor. Thus substituting this condition in 15,

$$1 = K_1 + K_2 + 4.87cos(-73) \tag{16}$$

or

$$K_2 = -0.42 - K_1 \tag{17}$$

Now we have a relation between K_1 and K_2.

$$i = K_1e^{-t} - (0.42 + K_1)e^{-2t} + 4.87cos(10t - 73) \tag{18}$$

Now substitute this in equation 2

$$\mathit{50cos(10t)} = 3(K_1e^{-t} - (0.42 + K_1)e^{-2t} + 4.87cos(10t - 73)) + 2\int_0^t K_1e^{-t} \\ -(0.42 + K_1)e^{-2t} + 4.87cos(10t - 73)\,dt + \frac{d\,K_1e^{-t} - (0.42 + K_1)e^{-2t} + 4.87cos(10t - 73)}{dt} + 5 \tag{19}$$

After substitution, the equation becomes

$$
\begin{aligned}
50cos(10t) = 3(K_1e^{-t} - (0.42 - K_1)e^{-2t} + 4.87cos(10t - 73)) \\
+2\Big[- K_1e^{-t} + \frac{(0.42 + K_1)e^{-2t}}{2} + .487sin(10t - 73)\Big]_0^t \\
+K_1e^{-t} + (0.42 + K_1)e^{-2t}2 - 4.87sin(10t - 73)10 + 5
\end{aligned}
\tag{20}
$$

Now substituting at t=0

$$
\begin{aligned}
50 = -0.42 + 3 \times 4.87cos(73) + K_1 + 48.7sin(73) + 5 \\
K_1 = 50 + 0.42 - 3 \times 4.87cos(73) - 48.7sin(73) - 5 \\
K_1 = -5.42
\end{aligned}
\tag{21}
$$

Then from equation 17, K_2=5 Let me give you a small observation when you are applying the initial conditions. If there are integral terms in the equation to used to apply the initial condition (for example equation 19), you can ignore that integration since finally we will be considering at t=0. So if you observe, every term after the integration will cancel away when we apply this condition. This is obvious since in this case we are integrating t=0 to 0. Substituting this in 15, the total solution becomes

$$\boxed{i(t) = -5.42e^{-t} + 5e^{-2t} + 4.87cos(10t - 73)}$$