3.a ∴ To show the existence of $\lim_{n\to\infty} \frac{\frac{n}{2}+\sin(n)}{3n+4\sqrt{n}+1}$

Let $a_n = \frac{\frac{n}{2}+\sin(n)}{3n+4\sqrt{n}+1}$

Divide numerator and denominator by n.
we get, and $n \neq 0$

$$a_n = \frac{\frac{1}{2}+\frac{\sin(n)}{n}}{3+\frac{4}{\sqrt{n}}+\frac{1}{n}}.$$

using Thm. 2.3(a), we know that $\lim_{n\to\infty} \frac{1}{n} \to 0$
and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \to 0$, which means the sequences $\{\frac{1}{n}\}$ and $\{\frac{1}{\sqrt{n}}\}$ converge to 0.

we know that,

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}.$$

Therefore using Thm. 2.2(e) ∴ we know that

$$\lim_{n\to\infty}\left(-\frac{1}{n}-\frac{1}{n}\right) = \lim_{n\to\infty}\left(-\frac{1}{n}\right) - \lim_{n\to\infty}\left(\frac{1}{n}\right)$$

$$= 0 - 0 \quad \left\{ \text{using Thm. } 2.2(c) \right.$$

$$= 0$$

Therefore from 2.2(e)

$$\lim_{n\to\infty}\left(-\frac{1}{n}-\frac{\sin(n)}{n}\right) = 0. \Rightarrow \lim_{n\to\infty} \frac{\sin(n)}{n} = 0.$$

Therefore

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty}\left(\frac{1}{2}+\frac{\sin(n)}{n}\right)}{\lim_{n\to\infty}\left(3+\frac{4}{\sqrt{n}}+\frac{1}{n}\right)}$$

$$= \frac{\lim_{n\to\infty}\frac{1}{2} + \lim_{n\to\infty}\frac{\sin(n)}{n}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty}\frac{4}{\sqrt{n}} + \lim_{n\to\infty}\frac{1}{n}} \quad \left\{ \text{using Thm. } 2.2(c) \right.$$

$$= \frac{\frac{1}{2}+0}{3+0+0} = \frac{1}{6} \quad \#$$

3.b To test the convergence of $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

and. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Here, $a_n, b_n > 0$. $\forall n$.

We know, that,

$$\frac{1}{n(n+3)} \leq \frac{1}{n^2}, \forall n.$$

The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (Any series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p>1$ is convergent)

Hence, by comparison test (Thm 2.5(a)), the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ also converges.

To compute the sum.

The ~~series~~ given series can be written as.

$$\lim_{n\to\infty} \frac{1}{3}\left[\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+3}\right]$$

Next we compute,

$$S_n = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+3}$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} \cdots + \frac{1}{n} - \left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6} \cdots + \frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n}\right) - \frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}$$

$$S_n = \frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

Therefore the sum is $\lim_{n\to\infty} \frac{1}{3} S_n$.

$$= \frac{1}{3} \lim_{n\to\infty} S_n$$

$$= \frac{1}{3} \lim_{n\to\infty} \left(\frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{1}{3}\left(\lim_{n\to\infty} \frac{11}{6} - \lim_{n\to\infty} \frac{1}{n+1} - \lim_{n\to\infty} \frac{1}{n+2}\right.$$

$$= \frac{1}{3}\left(\frac{11}{6} + 0 - 0\right)$$

$$= \frac{11}{18} \quad \#$$

3.C (1) Given Series, $\sum_{n=1}^{\infty} \frac{n!}{2^n}$.

Let $a_n = \frac{n!}{2^n}$. Here $a_n > 0$, $\forall n$.
& $a_n \neq 0$, $\forall n \geq 1$

$$a_{n+1} = \frac{(n+1)!}{2^{n+1}}$$

using Ratio test. (Thm. 2.5(d))

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} = \frac{(n+1).\cancel{n!}}{2.\cancel{2^n}} \times \frac{\cancel{2^n}}{\cancel{n!}}$$

$$= \frac{(n+1)}{2}$$

Hence $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n+1}{2}\right) = +\infty > 1$

Hence by Ratio test the series $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ diverges. #

2). Given series $\sum_{n=1}^{\infty} e^{-n}$.

Let $a_n = e^{-n}$, $a_{n+1} = e^{-(n+1)}$ | Here. $a_n > 0$. $\forall n$ & $a_n \neq 0$, $\forall n \geq 1$

using Ratio Test (Thm. 2.5(d))

$$\frac{a_{n+1}}{a_n} = \frac{e^{-(n+1)}}{e^{-n}} = \frac{\cancel{e^{-n}}.e^{-1}}{\cancel{e^{-n}}} = e^{-1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} e^{-1} = e^{-1} < 1$$

Hence by Ratio test, the given series $\sum_{n=1}^{\infty} e^{-n}$ Converges. #

Method 2 using root test (Thm. 2.5(c))

$$a_n^{1/n} = \left(e^{-n}\right)^{1/n} = e^{-1}$$

$$\& \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} e^{-1} = e^{-1} < 1$$

Hence by Root test, the given series $\sum_{n=1}^{\infty} e^{-n}$ Converges.

3) Given series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$.

Let $a_n = \frac{2+(-1)^n}{2^n}$, $a_n > 0$, $\forall n$.

Note that,

$$\frac{2+(-1)^n}{2^n} \leq \frac{3}{2^n}.$$

Let $b_n = \frac{1}{2^n}$, $b_n > 0$, $\forall n$.

& $a_n \leq 3 b_n$.

By Comparison test, if $\sum_{n=1}^{\infty} b_n$ converges, the series $\sum_{n=1}^{\infty} a_n$ Converges.

Next we test convergence of ~~S~~ series $\sum_{n=1}^{\infty} b_n$.

$$b_n = \frac{1}{2^n}, \quad b_n > 0, \ \forall n$$

$$\& \ b_n \neq 0, \ \forall n \geq 1$$

$$b_{n+1} = \frac{1}{2^{n+1}}$$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2.\cancel{2^n}} \times \frac{\cancel{2^n}}{1}$$

$$= \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1$$

Hence by Ratio test (Thm 2.5 d) the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ Converges.

since series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ Converges, By Comparison test (Thm 2.5(a)), the series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$ also Converges. #