Exercise

Recall that $\mathbb{R}_+^* \overset{\text{def}}{=} (0, +\infty)$ is the set of positive real numbers (*i.e.* $\mathbb{R}_+^* = \{x : x \in \mathbb{R}, x > 0\}$).
Let $a, b \in \mathbb{R}_+^*$ ($a > 0, b > 0$).
Consider the set $E = \left\{(-1)^n a + \dfrac{b}{n} : n \in \mathbb{N}^*\right\}$. (Of course $E \subset \mathbb{R}$.)

1. (for A group) Determine (if they exist) the **maximum**, the **supremum**, and *one* **upper bound** of this set E.
2. (for B group) Determine (if they exist) the **minimum**, the **infimum**, and *one* **lower bound** of this set E.

Solution for A group (max)

Just to check: First of all, in the definition of E, $n > 0$ because in $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, so $\dfrac{b}{n}$ is well defined.

Studying E: Then, for any $n \in \mathbb{N}^*$, let $u_n = (-1)^n a + \dfrac{b}{n}$.

We have $u_{2n} = a + \dfrac{b}{2n}$ and $u_{2n-1} = -a + \dfrac{b}{2n-1}$.

The two sequences $(u_{2n})_{n\in\mathbb{N}^*}$ and $(u_{2m-1})_{m\in\mathbb{N}^*}$ are decreasing, because $b > 0$.

So if we define $E_{\text{even}} \overset{\text{def}}{=} \{u_{2n} : n \in \mathbb{N}^*\}$ and $E_{\text{odd}} \overset{\text{def}}{=} \{u_{2m-1} : m \in \mathbb{N}^*\}$, then[1] $E = E_{\text{even}} \cup E_{\text{odd}}$.

But because the sequences decrease, both E_{even} and E_{odd} have a greatest element:

- $\max E_{\text{even}} = u_2$ is well defined (for $n = 1$), *i.e.* $\max E_{\text{even}} = a + \dfrac{b}{2}$,
- and $\max E_{\text{odd}} = u_1$ is well defined (for $m = 1$), *i.e.* $\max E_{\text{odd}} = -a + \dfrac{b}{1} = b - a$.

Therefore E also have a greatest element, given by $\max E = \max\left(\max(E_{\text{even}}), \max(E_{\text{odd}})\right)$.

We conclude with $\max E = \max(a + \dfrac{b}{2}, b - a)$. So E upper-bounded and $\sup E = \max E$.

One upper bound is $a + b$, and another one (less precise) is $a + b + 2014$ (indeed, $\forall x \in E, x \leqslant a + b$ so $\forall x \in E, x \leqslant a + b + 2014$). Any number bigger than $a + b$ is **an** upper bound!

Bonus: We precise the notation with $E_{a,b} \overset{\text{def}}{=} \left\{(-1)^n a + \dfrac{b}{n} : n \in \mathbb{N}^*\right\}$.

Find one pair a, b such that $\max E_{a,b} = a + \dfrac{b}{2}$ and another pair a', b' such that $\max E_{a',b'} = b' - a'$.

[1] In fact, $E = E_{\text{even}} \uplus E_{\text{odd}}$: these two sets form a partition of E.

Answer: With $a = 100, b = 1$, we have $\max E_{100,1} = a + \frac{b}{2} = 100 + \frac{1}{2}$. With $a' = 2, b' = 50$, we have $\max E_{2,50} = b' - a' = 50 - 2 = 48$.

Solution for B group (min)

Just to check: First of all, in the definition of E, $n > 0$ because in $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, so $\frac{b}{n}$ is well defined.

Studying E: Then, for any $n \in \mathbb{N}^*$, let $u_n = (-1)^n a + \frac{b}{n}$.

We have $u_{2n} = a + \frac{b}{2n}$ and $u_{2n-1} = -a + \frac{b}{2n-1}$.

The two sequences $(u_{2n})_{n\in\mathbb{N}^*}$ and $(u_{2m-1})_{m\in\mathbb{N}^*}$ are decreasing, because $b > 0$.

And we know that $u_{2n} \underset{n\to+\infty}{\to} a$ and $u_{2m-1} \underset{m\to+\infty}{\to} -a$, because $\frac{b}{k} \underset{k\to+\infty}{\to} 0$.

So if we define $E_{\text{even}} \overset{\text{def}}{=} \{u_{2n} : n \in \mathbb{N}^*\}$ and $E_{\text{odd}} \overset{\text{def}}{=} \{u_{2m-1} : m \in \mathbb{N}^*\}$, then[2] $E = E_{\text{even}} \cup E_{\text{odd}}$.

But because the sequences decrease (strictly) so:

- $\inf E_{\text{even}} = \lim_{n\to+\infty} u_{2n} = a$, and E_{even} does not have a smallest element,
- $\inf E_{\text{odd}} = \lim_{m\to+\infty} u_{2m-1} = -a$, and E_{odd} does not have a smallest element.

And because of the partition[2], $\inf E = \min(\inf(E_{\text{even}}), \inf(E_{\text{odd}}))$. We conclude by saying that E is lower-bounded and $\inf E = \min(a, -a) = -a$ (because $a > 0$), and E does not have a smallest element.

Therefore one lower bound is $-a$, and another one (less precise) is $-a - 7$ (indeed, $\forall x \in E, x \geqslant -a$ so $\forall x \in E, x \geqslant -a - 7$). Any number smaller than $-a$ is **a** lower bound!

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[2] In fact, $= E_{\text{even}} \uplus E_{\text{odd}}$: these two sets form a partition of E.