1 Exercises to do before tutorials.

Ex.1.1) Show that sum $(f+g)$ of two functions (f, g) which are individually continuous at a point (x_0) is continuous at that point.

Ex.1.2) Compute the following left-hand side limits, right-hand side limits, and limits (if they exist):

(a) $\lim_{x\to 0} \frac{|x|}{x}$,

(b) $\lim_{x\to 0} \frac{\sqrt{x^2}}{x}$,

(c) $\lim_{x\to 1} \frac{2x^2-3x+1}{x-1}$,

(d) $\lim_{h\to 0} \frac{(t+h)^2-t^2}{h}$ ($t \in \mathbb{R}$ is fixed),

(e) $\lim_{x\to a} \frac{\sin x - \sin a}{x-a}$ ($a \in \mathbb{R}$ is fixed),

(f) $\lim_{x\to 0} \frac{\sin 5x}{\sin x}$.

Ex.1.3) Use the fact[1] that $|\sin(x)| < |x|$ for all $0 < |x| < \frac{\pi}{2}$ to show that sin is continuous at 0. Next, use the identity $\cos 2x = 1 - 2\sin^2 x$ ($\forall x \in \mathbb{R}$) to show that cos is also continuous at 0. Finally, use the standard formula for $\sin(x+h)$ and $\cos(x+h)$ to show that sin and cos are continuous everywhere in $\mathbb{R}$.

Ex.1.4) Let f be a polynomial function of degree $n > 0$ such that the last and first coefficients have opposite signs. Prove[2] that there exists a positive x (*i.e.* $x \in \mathbb{R}_+^*$) such that $f(x) = 0$.

2 Exercises to do during or after tutorials.

Ex.2.1) Let f and g be given below and let $h = f \circ g$. In each case, determine the domains of f, g, and h and given direct formula for h. Finally, investigate the continuity of h:

(i) $f(x) = x^2 - 2x$, $g(x) = x+1$,

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2$,

(iii) $f(x) = \sqrt{x}$, $g(x) = x + \sqrt{x}$,

(iv) $f(x) = \sqrt{x}$, $g(x) = \sin(x)$.

Ex.2.2) If n is an odd positive integer ($n \in 2\mathbb{N}+1$) and $a < 0$ prove that there is exactly one negative $b < 0$ such that $b^n = a$. And what happens if n is even?

[1] You can also *try to* prove that fact, by the mean of your choice.

[2] You should start by giving name to all these coefficients, and reason by contradiction (if f has no zero, then ...)

Ex.2.3) Let $f = \tan$ (on a certain domain you should precise). Although $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, there is no x in $[\pi/4, 3\pi/4]$ such that $f(x) = 0$.
Why does this not contradict the *intermediate value theorem*?

Ex.2.4) Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto x \sin\left(\dfrac{1}{x}\right)$ if $x \neq 0$.
What should be the value of $f(0)$ so that f is continuous at 0.

Ex.2.5) For $a, b \in \mathbb{R}, a < b$, let $f : [a, b] \to \mathbb{R}$ be such that[3] $\forall x, y \in [a, b], |f(x) - f(y)| \leqslant L \times |x - y|$, for some $L \geqslant 0$. Show that f is continuous everywhere.

Ex.2.6) Let $f : \mathbb{Z} \to \mathbb{R}$ be well-defined. Show that f is continuous[4] everywhere on $\mathbb{Z}$.

3 Bonus exercises.

Ex.3.1) Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto \sin\left(\dfrac{1}{x}\right)$ if $x \neq 0$. Can we choose $f(0)$ such that f is continuous at 0? What about limit?

Ex.3.2) For $x \in \mathbb{R}$, show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent (this shows that e^x is well-defined for all $x \in \mathbb{R}$). Next, show that there exists $\lambda > 0$ such that $\mathrm{e}^x \leqslant 1 + \lambda x$ for all $x \in [0, 1]$. Finally, show that $\exp : x \mapsto \mathrm{e}^x$ is continuous at 0, and subsequently everywhere in $\mathbb{R}$.

Ex.3.3) Use the fact that $\exp : x \mapsto \mathrm{e}^x$ is continuous to show that $\ln : x \mapsto \ln x$ is continuous in its domain.

Ex.3.4) Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$. Assume that $\forall x \in [0, 1], 0 \leqslant f(x) \leqslant 1$. Show that there exists $c \in [0, 1]$ such that $f(c) = c$.

Ex.3.5) Is it possible to have a function $f : \mathbb{R} \to \mathbb{R}$, well-defined everywhere, but continuous **nowhere**?
Remember that $\mathbb{Q}$ is the set of rationals numbers, dense in $\mathbb{R}$.
Show that $\mathbb{1}_{\mathbb{Q}}$ ($\mathbb{R} \to \mathbb{R}, x \mapsto 1$ if $x \in \mathbb{Q}, 0$ otherwise) is continuous nowhere, by using a sequence $(r_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}, r_n \underset{n\to+\infty}{\longrightarrow} \alpha$ for a $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, to show that $\mathbb{1}_{\mathbb{Q}}$ is not continuous at α, (and conversely $(\beta_n)_{n\in\mathbb{N}} \in (\mathbb{R}\backslash\mathbb{Q})^{\mathbb{N}}, b_n \underset{n\to+\infty}{\longrightarrow} q$ for a $q \in \mathbb{Q}$, for non-continuity at q).

[3] f is said to be L-Lipschitz, or Lipschitz continuous of ratio L on $[a, b]$.
[4] You should use the "$\varepsilon, \delta(\varepsilon)$" definition, because if $\varepsilon < 1, x, y \in \mathbb{Z}$, then $(|x-y| < \varepsilon \implies x = y)$.