Remark 1. *These partial solutions are only there to help, to give a hint or some element of answers.*

Some problems are not corrected because they should really be easy enough for you to work on them by your own, or because they are not really exciting.

Some remarks might not be as serious *as what you are used to, so deal with it.*

1 A. Homework.

Ex.A.1) This is based on simple RIEMANN sums, with left-endpoint values for every rectangle of basis 10 sec. What we can say is quickly : $v(t) = d'(t)$ so $d(120) - d(0) = \int_0^{120} v(t)\mathrm{d}t$. But $d(0) = 0m$, so the distance we are looking for is $d(120) = \int_0^{120} v(t)\mathrm{d}t$.

(a) This integral is approximated by left rectangles : $\int_0^{120} v(t)\mathrm{d}t \simeq 10 * v(0s) + 10 * (v(10s)) + \cdots + 10 * (v(110s)) = 1745.0\ m = 1.74\ km$.

(b) This integral is approximated by right rectangles : $\int_0^{120} v(t)\mathrm{d}t \simeq 10 * v(10s) + 10 * (v(20s)) + \cdots + 10 * (v(120s)) = 1920.0\ m = 1.92\ km$.

Ex.A.2) Similarly, we also use RIEMANN sums here.

(a) We can write $v(5s) = v(5s) - v(0s)$ because $v(0s) = 0$. But $v(5s) - v(0s) = \int_{0s}^{5s} a(t)\mathrm{d}t$ because $a(t) = v'(t)$.
We only know that the acceleration is decreasing, so the left-endpoint approximation (with left RIEMANN rectangles) will be an upper-bound, and the right-endpoint approximation will give a lower-bound.

$$\text{(Left)} \int_{0s}^{5s} a(t)\mathrm{d}t \leqslant 1s * a(0s) + 1s * a(1s) + \dots 1s * a(4s) = 23.33\ m/s,$$
$$\text{(Right)} \int_{0s}^{5s} a(t)\mathrm{d}t \geqslant 1s * a(1s) + 1s * a(2s) + \dots 1s * a(5s) = 14.15\ m/s.$$

(b) Similarly, we write that $d(0s) = 0m$, so $d(3s) = d(3s) - d(0s) = \int_{0s}^{3s} v(t)\mathrm{d}t$ because $d'(t) = v(t)$.
So thanks to the monotony of the integral, and because v is increasing (indeed, $a(t) > 0$ for any $t > 0$), we know that $d(3s) \leqslant 1s * v(1s) + 1s * v(2s) + 1s * v(3s)$. We find upper-bounds of $v(1s), v(2s)$ and $v(3s)$ thanks to the same method as previously.
Finally, $d(3s) \leqslant 45.82\ m$.

Ex.A.3) Let $f(x) = \dfrac{1}{1+x^2}$, well defined and continuous on $[0, 1]$.

- On $[0, 1]$, $\frac{1}{2} \leqslant f(x) \leqslant 1$ (by the method you like), so $\frac{1}{2} = \int_0^1 \frac{1}{2}\mathrm{d}x \leqslant \int_0^1 f(x)\mathrm{d}x \leqslant \int_0^1 1\mathrm{d}x = 1$ thanks to the Max-Min inequality (or thanks to the monotony of the integral: if $f \leqslant g$ then $\int_a^b f \leqslant \int_a^b g$).
- We do the same on $\left[0, \frac{1}{2}\right]$, to find the bounds $\frac{4}{5} \leqslant f(x) \leqslant 1$ (by the method you like), so $\frac{2}{5} = \int_0^{\frac{1}{2}} \frac{4}{5}\mathrm{d}x \leqslant \int_0^{\frac{1}{2}} f(x)\mathrm{d}x \leqslant \int_0^{\frac{1}{2}} 1\mathrm{d}x = \frac{1}{2}$ (thanks to the Max-Min inequality).

- We again do the same on $\left[\frac{1}{2}, 1\right]$, to find the bounds $\frac{1}{2} \leqslant f(x) \leqslant \frac{4}{5}$ (by the method you like), so $\frac{1}{4} = \int_{\frac{1}{2}}^{1} \frac{1}{2}\mathrm{d}x \leqslant \int_{\frac{1}{2}}^{1} f(x)\mathrm{d}x \leqslant \int_{\frac{1}{2}}^{1} \frac{4}{5}\mathrm{d}x = \frac{2}{5}$ (thanks to the Max-Min inequality).
- We combine these two bounds to have an improved estimate of the initial integral : $\frac{2}{5} + \frac{1}{4} = \frac{13}{20} \leqslant \int_0^1 f(x)\mathrm{d}x \leqslant \frac{1}{2} + \frac{2}{5} = \frac{18}{20}$.
- (**Bonus:**) In fact, we know that $\int_0^1 \frac{1}{1+x^2}\mathrm{d}x = \Big[\arctan(x)\Big]_0^1 = \frac{\pi}{4}$ (here, $\arctan(x) = \tan^{-1}(x)$ in the arctangent, or tangent inverse function). With the approximation $\frac{\pi}{4} \simeq 0.7853$, we can check that the inequality $\frac{1}{2} \leqslant \frac{\pi}{4} \leqslant 1$ is indeed true, and the better inequality is also true : $\frac{13}{20} \leqslant \frac{\pi}{4} \leqslant \frac{9}{10}$.

Ex.A.4) $\int_0^{\pi} 1 + cos(x)\mathrm{d}x = \pi + \Big[\sin(x)\Big]_0^{\pi} = \pi$.

Ex.A.5) $\int_{\pi/2}^{0} \frac{1+cos(2t)}{2}\mathrm{d}t = -\frac{\pi}{2} + \left[\frac{\sin(2t)}{4}\right]_{-\pi/2}^{0} = -\frac{\pi}{2}$.

Ex.A.6) $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\mathrm{d}t = 10\sqrt{3} \simeq 17.321$ by direct integration of the expanded polynomial.

Ex.A.7) Similarly $\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}}\mathrm{d}u = \Big[2\sqrt{u} - u\Big]_9^4 = 3$.

Ex.A.8) The general case is $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(t)\mathrm{d}t\right) = a'(x)f(a(x)) - b'(x)f(b(x))$ which is a direct consequence of the **Fundamental Theorem of Calculus** (cf. lectures, or Wikipédia).

a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\sqrt{x}} \cos(t)\mathrm{d}t\right) = \frac{1}{2\sqrt{x}}\cos(\sqrt{x})$,

b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\sin x} 3t^2\mathrm{d}t\right) = \cos(x)3\sin^2(x)$.

Ex.A.9) For $k > 0$, we have $\int_0^{\pi/k} \sin(kx)\mathrm{d}x = \left[\frac{\cos(kx)}{k}\right]_0^{\pi/k} = \frac{1}{k} - (-\frac{1}{k}) = \frac{2}{k}$ as wanted.
Remark: one arch of the curve $y = \sin(kx)$ is indeed between 0 and π/k, and not between 0 and π.

Ex.A.10) We use one property[1] of the integral : $\int_{-2}^{2} f(x)\mathrm{d}x = \left(\int_{-2}^{-1} f(x)\mathrm{d}x\right) + \left(\int_{-1}^{1} f(x)\mathrm{d}x\right) + \left(\int_{1}^{2} f(x)\mathrm{d}x\right)$.
And now, on every of these three intervals, we just use the given expression of f :

- $\int_{-2}^{-1} f(x)\mathrm{d}x = \int_{-2}^{-1} 1\mathrm{d}x = 1$,
- $\int_{-1}^{1} f(x)\mathrm{d}x = \int_{-1}^{1} 1 - x^2\mathrm{d}x = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 0$,
- $\int_{1}^{2} f(x)\mathrm{d}x = \int_{1}^{2} 2\mathrm{d}x = 2$.

So $\int_{-2}^{2} f(x)\mathrm{d}x = 1 + 0 + 2 = 3$.

[1] This property is called *additivity on intervals*, see `https://en.wikipedia.org/wiki/Integral#Conventions` if required.

Ex.A.11) For a continuous function f, we recognized a RIEMANN sum[2].
The nth partition is $\{x_1^{(n)}, \dots, x_k^{(n)}, \dots, x_n^{(n)}\} = \left\{ \frac{k}{n} : 1 \leqslant k \leqslant n \right\}$ for any $n > 0$.
So the norm of the partition is $\frac{1}{n}$, which is indeed going to 0 when $n \to +\infty$.
So we can conclude directly, by using what have been seen in the lectures, that

$$\frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right) = \sum_{k=1}^{n} f(x_k^{(n)})\Delta x_k^{(n)} \underset{n\to+\infty}{\longrightarrow} \int_{\lim_{n\to+\infty} x_1^{(n)}}^{\lim_{n\to+\infty} x_n^{(n)}} f(t)\mathrm{d}t = \int_0^1 f(t)\mathrm{d}t.$$

Ex.A.12) By the previous result, applied to $f : x \mapsto x^{15}$, this limit will be $\int_0^1 f(t)\mathrm{d}t = \int_0^1 t^{15}\mathrm{d}t = \left[\frac{t^{16}}{16}\right]_0^1 = \frac{1}{16}$.

Ex.A.13) Thanks to the **Fundamental Theorem of Calculus**, such f is continuous, twice differentiable.

$f(0) = 3 + \int_0^0 \frac{1+\sin(t)}{2+t^2}\mathrm{d}t = 3$. So p has to satisfies $p(0) = 3$, but $p(0) = a$. So $a = 3$.

$f'(0) = \frac{1+\sin(0)}{2+0^2} = \frac{1}{2}$. So p has to satisfies $p'(0) = \frac{1}{2}$, but $p'(0) = b$. So $b = \frac{1}{2}$.

$f''(0) = \frac{1}{2}$. So p has to satisfies $p''(0) = \frac{1}{2}$, but $p''(0) = 2c$. So $c = \frac{1}{4}$.

Ex.A.14) For more details on this, please take a look at Thomas Calculus, section 5.3 (Volumes of Solids of Revolution-Disks and Washers). The main formula we will use is this one (numbered as (2), page 380 of Thomas Calculus 9th edition) :

> **Theorem 1** (Volume of a Solid of Revolution (Rotation About the y-axis))**.** *The volume of the solid generated by revolving about the y-axis the region between the y-axis and the graph of the continuous function $x = R(y)$, $c < x < d$, is :*
>
> $$V = \int_c^d \pi(\mathit{radius}(y))^2\mathrm{d}y = \int_c^d \pi(R(y))^2\mathrm{d}y.$$

You should try to illustrate these three examples with a drawing (not easy on 3D, try to do your best).

a. In this first case, x will move from -1 to 1, and the radius is $R(x) = 1 - x^2$ (distance from the line $y = 1$ to the curve $y = x^2$, at that point x).
So the total volume is $V_1 = \int_{-1}^1 \pi(R(x))^2\mathrm{d}x = \int_{-1}^1 \pi(1-x^2)^2\mathrm{d}x = \pi\int_{-1}^1 (1 - 2x^2 + x^4)\mathrm{d}x = \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^1 = \cdots = \frac{16\pi}{15}$.

b. Here we use the more general result (the one which consider a volume with “a whole in the middle”, called the *Washer method*, cf. Thomas Calculus for example).
In this case, x will move from -1 to 1, and the inner radius is $r(x) = 1$ (distance from the line $y = 2$ to the line $y = 1$) and the outer radius is $R(x) = 2 - x^2$ (distance from the line $y = 2$ to the curve $y = x^2$, at that point x).
So the total volume is $V_2 = \int_{-1}^1 \pi(R(x))^2 - (r(x))^2\mathrm{d}x = \int_{-1}^1 \pi((2-x^2)^2 - 1)\mathrm{d}x = \pi\int_{-1}^1 (4 - 4x^2 + x^4 - 1)\mathrm{d}x = \pi\left[3x - \frac{4x^3}{3} + \frac{x^5}{5}\right]_{-1}^1 = \cdots = \frac{56\pi}{15}$.
As expected, we find $V_2 > V_1$ (this inequality is clear on a drawing).

[2]Take a look to that Wikipédia page for more details : `https://en.wikipedia.org/wiki/Riemann_sum#Right_Riemann_Sum`.

c. Here we use again the *Washer method.*
In this last case, x will move from -1 to 1, and the inner radius is now $r(x) = 1 + x^2$ (distance from the line $y = -1$ to the curve $y = x^2$, at that point x) and the outer radius is $R(x) = 2$ (distance from the line $y = -1$ to the line $y = 1$).
So the total volume is $V_3 = \int_{-1}^{1} \pi(R(x))^2 - (r(x))^2 \mathrm{d}x = \int_{-1}^{1} \pi(4 - (1 + x^2)^2)\mathrm{d}x = \pi \int_{-1}^{1} (3 - 2x^2 - x^4)\mathrm{d}x = \pi \left[3x - \frac{2x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \cdots = \frac{64\pi}{15}$.
Finally, we find $V_3 > V_2$, but it is somehow *harder* to graphically understand this inequality on a drawing.

2 B. Tutorial for class workout.

Ex.B.1) On $[a, b]$, this operator av can be computed by $av(f) = \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t$.

a. So $av(f + g) = av(f) + av(g)$ is obvious by the *linearity* of the integral,
b. So $av(kf) = k.av(f)$ is also almost obvious by the *linearity* of the integral (as k is a constant),
c. If $f \leqslant g$ on $[a, b]$ (ie $\forall x \in [a, b], f(x) \leqslant g(x)$), then $g - f$ is a positive function. But we know that the integral from a to b (with $a < b$) of a positive function is itself positive. So $av(g - f) \leqslant 0$, but $av(g - f) = av(g) - av(f)$, so $av(f) \leqslant av(g)$ is direct (we just proved the *monotony* of the integral).

Ex.B.2) Let $r > 0$ and $n \in \mathbb{N}^*$. We consider a n-sided regular polygon in a circle of radius r, as illustrated in this figure[3] :

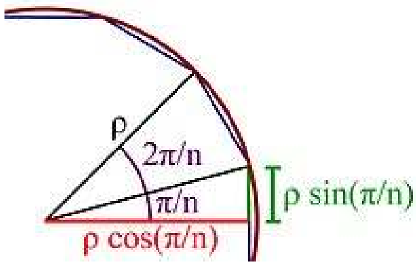

Figure 1: n-sided regular polygon

This figure shows that the area of one of the n triangles that constitute the polygon is $a_n = \frac{1}{2} \times r \times r \sin(\theta)$, with $\theta = \frac{2\pi}{n}$. So the entire area is $A_n = n \times a_n = \frac{nr^2}{2} \sin(\frac{2\pi}{n})$.
For $n \to +\infty$, we have $A_n = \frac{nr^2}{2} \sin\left(\frac{2\pi}{n}\right) = \pi r^2 \left(\frac{\sin(\frac{2\pi}{n})}{\frac{2\pi}{n}}\right) \underset{n\to+\infty}{\to} \pi r^2$, because $\frac{\sin(x)}{x} \underset{x\to 0}{\to} 1$.
Remark: so of course we find that the limit of the area is πr^2, the area of the circle of radius r, because the n-sided regular polygon will fill the circle entirely as $n \to +\infty$.

Ex.B.3) Let f be[4] such that $x \sin(\pi x) = \int_0^{x^2} f(t)\mathrm{d}t$ for $x \in \mathbb{R}$.

For the left side, let $g : \mathbb{R} \to \mathbb{R}, x \mapsto x \sin(\pi x)$. g is continuous and differentiable on $\mathbb{R}$. We have $g'(x) = \sin(\pi x) + x\pi \cos(\pi x)$ for every $x \in \mathbb{R}$.

[3] Such a figure, and an other one of the entire circle being divided as n pieces can really help in this problem. Usually, doing figures and small plots of function **is an excellent reflex!**
[4] We should also justify properly why such a f exists. . .

For the right side, let $h : \mathbb{R} \to \mathbb{R}, x \mapsto \int_0^{x^2} f(t)\mathrm{d}t$. We assume that f is RIEMANN integrable, so h is continuous, and that f is continuous, so h is differentiable. Furthermore, we have $h'(x) = 2xf(2x)$ for any $x \in \mathbb{R}$. But $f(4) = f(2x)$ if $x = 2$, so $f(4)$ is $\frac{h'(2)}{2}$.
But $g(x) = h(x)$ so $h'(2) = g'(2) = \sin(2\pi) + 2\pi\cos(2\pi) = 2\pi$.
Hence, $f(4) = \pi$.

3 C. Additional Exercises.

Ex.C.1) **Not sure of how to explain all this.**

First, let change the notation: for $x > 0$, let $h(x) = \int_0^x \frac{1}{\sqrt{1+4t^2}}\mathrm{d}t$.
$t \mapsto \frac{1}{\sqrt{1+4t^2}}$ is continuous on $(0, +\infty)$ so h is well defined, continuous and differentiable on $(0, +\infty)$, and $h'(x) = \frac{1}{\sqrt{1+4x^2}}$.
But $h'(x) = \frac{\mathrm{d}h}{\mathrm{d}x}(x)$ is the $\frac{\mathrm{d}x}{\mathrm{d}y}$ of the question paper. Similarly, $h''(x)$ is the $\frac{\mathrm{d}^2x}{\mathrm{d}y^2}$ from the question paper.
We compute $h''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{1+4x^2}}\right) = -\frac{4x}{(4x^2+1)^{3/2}}$.
So $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ from the question paper is $-\frac{(4x^2+1)^{3/2}}{4x}$.
To be concluded.

Ex.C.2) **This problem is harder than the previous ones.** Assume that f is RIEMANN integrable and continuous on $\mathbb{R}$.
For $x \in \mathbb{R}$, let $g(x) = \int_0^x \left(\int_0^u f(t)\mathrm{d}t\right)\mathrm{d}u$ the left side, and $h(x) = \int_0^x f(u)(x-u)\mathrm{d}u$ the right side.

For the left side, by RIEMANN integrability of f, the application $j : u \mapsto \int_0^u f(t)\mathrm{d}t$ is continuous and so RIEMANN integrable on every bounded interval $[0, x]$. So g is well defined, continuous, and differentiable on $\mathbb{R}$. For one $x \in \mathbb{R}$, as $g(x) = \int_0^x j(u)\mathrm{d}u$, we know that $g'(x) = j(x)$, ie $g'(x) = \int_0^x f(t)\mathrm{d}t$.
For the right side, by RIEMANN integrability of $u \mapsto f(u)(x-u)$, for any $x \in \mathbb{R}$, the function h is well defined for any x, and continuous on $\mathbb{R}$. Now, for $x \in \mathbb{R}$, we can split $h(x)$ in two parts :

$$h(x) = \int_0^x f(u)(x-u)\mathrm{d}u = \left(\int_0^x f(u)x\mathrm{d}u\right) - \left(\int_0^x f(u)u\mathrm{d}u\right) = x\left(\int_0^x f(u)\mathrm{d}u\right) - \left(\int_0^x f(u)u\mathrm{d}u\right).$$

So h is also differentiable, because $u \mapsto f(u)$ and $u \mapsto f(u)u$ are continuous. And $h'(x) = \left(\int_0^x f(u)\mathrm{d}u\right) + x\left([f(u)]_0^x\right) - [f(u)u]_0^x = \left(\int_0^x f(u)\mathrm{d}u\right) + xf(x) - xf(x) = \left(\int_0^x f(u)\mathrm{d}u\right)$. So we recognized that $g'(x) = h'(x)$.
And finally, at $x = 0$, we have directly that $g(0) = 0$ and $h(0) = 0$, because we integrate from 0 to 0.
So $g' = h'$ and they have one value in common (at $x = 0$), hence they are equal : $\forall x > 0, g(x) = h(x)$.
Hence, as wanted, we can conclude that for any f RIEMANN integrable and continuous on $\mathbb{R}$:

$$\boxed{\forall x > 0, \int_0^x \left(\int_0^u f(t)\mathrm{d}t\right)\mathrm{d}u = \int_0^x f(u)(x-u)\mathrm{d}u.}$$