Complex numbers : Ex.2.4

Resolve in $\mathbb{C}$ the following equation (z being the unknown): $z^4 + z^3 + z^2 + z + 1 = 0$. One could introduce the new variable $Z := z + \frac{1}{z}$. Then show that the solutions are all 5^{th} roots of unity (*i.e.* in the set $\mathbb{U}_5$), and use this to compute the value of $\cos\left(\frac{2\pi}{5}\right)$.

Solution Let call (E) this equation $z^4 + z^3 + z^2 + z + 1 = 0$, and S the set of its solutions. We know that S has exactly four elements (D'Alembert-Gauss' theorem).

Lemma 1. *$z \in S \Rightarrow z \neq 0$ and $z \neq 1$.*

Proof 1. *Obvious, because $0^4 + 0^3 + 0^2 + 0 + 1 = 1 \neq 0$ and $1^4 + 1^3 + 1^2 + 1 + 1 = 5 \neq 0$.* $\square$

Lemma 2. *$z \in S \Rightarrow z^5 = 1$ (*i.e. $z \in \mathbb{U}_5$*).*

Proof 2. *If $z^4 + z^3 + z^2 + z + 1 = 0$, then by multiplying by $z - 1$ on both sides, we get $(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$, and by using the well-known identity $(z-1)(z^4 + z^3 + z^2 + z + 1) = z^5 - 1$, so $z^5 - 1 = 0$,* i.e. *$z^5 = 1$ (which can be written as $z \in \mathbb{U}_5$ by definition of $\mathbb{U}_k, k \in \mathbb{N}^*$)).* $\square$

Remark 1. *Here, we can already state that $S = \mathbb{U}_5 \backslash \{1\}$, directly from these two lemmas (1 and 2).*

Trying to find S Let $z \in \mathbb{C}$.

$$z \in S \Leftrightarrow z \in S \text{ and } z \neq 0$$

We can assume $z \neq 0$, so $Z(z) \stackrel{\text{def}}{=} z + \frac{1}{z} \in \mathbb{C}$ is well defined.

$$\begin{aligned} z \in S \Leftrightarrow & z \in S \text{ and } Z(z)^2 + Z(z) - 1 = 0 \\ \Leftrightarrow & z \in S \text{ and } Z(z) = \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

Let call $\phi_{1,2} \stackrel{\text{def}}{=} \frac{-1 \pm \sqrt{5}}{2} \in \mathbb{R}$. We have $\phi_2 < \phi_1$.

$$\begin{aligned} z \in S \Leftrightarrow & z \in S \text{ and } z + \frac{1}{z} = \phi_{1,2} \\ \Leftrightarrow & z \in S \text{ and } z^2 - \phi_{1,2} z + 1 = 0 \end{aligned}$$

For ϕ_1 If $z^2 - \phi_1 z + 1 = 0$, then $z = z_{1,2} = \frac{\phi_1 \pm i\sqrt{\Delta_1}}{2}$ if $\Delta_1 \stackrel{\text{def}}{=} -\frac{5 + \sqrt{5}}{2} < 0$.
Conversely, we verify that these two values $z_{1,2}$ are solutions of (E) (and in $\mathbb{U}_5$, *i.e.* $z_{1,2}^5 = 1$, even if we already know that thanks to Lemma 2).

For ϕ_2 If $z^2 - \phi_2 z + 1 = 0$, then $z = z_{3,4} = \dfrac{\phi_2 \pm i\sqrt{\Delta_2}}{2}$ if $\Delta_2 \overset{\text{def}}{=} -\dfrac{5-\sqrt{5}}{2} < 0$.
Conversely, we verify that these two values $z_{3,4}$ are solutions of (E) (and in $\mathbb{U}_5$, *i.e.* $z_{3,4}^5 = 1$, even if we already know that thanks to Lemma 2).

> *Conclusion*: Therefore, we conclude that $S = \{z_1, z_2, z_3, z_4\} = \left\{\dfrac{\phi_1 \pm i\sqrt{\Delta_1}}{2}, \dfrac{\phi_2 \pm i\sqrt{\Delta_2}}{2}\right\}$.

Computing $\cos(\frac{2\pi}{5})$ We know that $\mathbb{U}_5 = \{e^{ik\pi/5}, k \in [0,4]\}$. Therefore, $\cos(\frac{2\pi}{5})$ is the real part of one of the 5^{th} root of unity. Thanks to Remark 1, $S = \mathbb{U}_5 \backslash \{1\}$, so $\cos(\frac{2\pi}{5}) = \max(\mathcal{R}e(z), z \in S) = \mathcal{R}e(z_1) = \dfrac{\phi_1}{2} = \dfrac{\sqrt{5}-1}{4}$. *(A small drawing could help).*

> *Conclusion*: Hence $\cos(\frac{2\pi}{5}) = \dfrac{\sqrt{5}-1}{4} \simeq 0.309016$.

Remark 2. *You can numerically check this with your calculator.*

Remark 3. *This method is similar to the one historically used by* GAUSS *to compute* $\cos(\frac{2\pi}{17})$*, to prove that the polygon with* 17 *sides is constructible with a straightedge and a compass, cf. on wikipédia.*