1 Exercises to do before tutorials.

Ex.1.1) Show that the sequence $(\sqrt{n})_{n\in\mathbb{N}}$ diverges to $+\infty$.

Ex.1.2) Consider sequences given below by their terms u_n ($\forall n \in \mathbb{N}$). Check if each of these sequences are convergent or divergent. If they are convergent then compute the limits.

(i) $u_n \overset{\text{def}}{=} \dfrac{n^2+3n-4}{4n^2+5}$,

(ii) $u_n \overset{\text{def}}{=} \dfrac{\sin(n^2)}{n}, n \neq 0$,

(iii) $u_n \overset{\text{def}}{=} \dfrac{n^4+7n}{5n^4+\cos(n^2)-\frac{1}{n}}, n \neq 0$,

(iv) $u_n \overset{\text{def}}{=} \dfrac{a^n-b^n}{a^n+b^n}$, with $a, b > 0$.

Ex.1.3) Is the product sequence of two lower-bounded sequences still lower-bounded?

Ex.1.4) Show that a non-decreasing sequence is lower-bounded.

Ex.1.5) Let u be an integer sequence (*i.e.* $u \in \mathbb{N}^{\mathbb{N}}$). Prove that u converges *if and only if*[1] u is constant after a certain number of terms (*i.e.* $\exists n_0 \in \mathbb{N} / (\forall n \geqslant n_0, u_n = u_{n_0})$).

Ex.1.6) Use one of the convergence tests saw in lectures to explore the convergence/divergence of the following series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}$ (for $x \in \mathbb{R}$),

(iii) $\displaystyle\sum_{n=2}^{\infty} \frac{n!}{(n+2)!}$,

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1000n+1}$,

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{1000^n}{n!}$.

Ex.1.7) (*Babylonian method*) Let $u_0 = 1$ and define, for $n \in \mathbb{N}$, $u_{n+1} = \dfrac{u_n + \frac{2}{u_n}}{2}$.

(i) Show that the sequence $u = (u_n)_{n\in\mathbb{N}}$ is well defined.

(ii) Compute the first 6 terms (by using your calculator or smartphone). What do you think the limit will be?

(iii) Show that u is converging[2], and compute its limit.

[1] *If and only if* will be abbreviated "**iff**".

[2] One could show that $(u_n)_{n\geqslant 1}$ is decreasing and lower-bounded by 1.

2 Exercises to do during or after tutorials.

Ex.2.1) Let u be a bounded sequence, satisfying: $\forall n \in \mathbb{N}^*, 2u_n \leqslant u_{n-1} + u_{n+1}$. Let $v = (v_n)_{n\in\mathbb{N}}$ be given by $\forall n \in \mathbb{N}, v_n \overset{\text{def}}{=} u_{n+1} - u_n$. Show[3] that v converges, and compute its limit.

Ex.2.2) Let u be a convergent sequence. Let v be a sub-sequence of u. Show that then v converges (to the same limit).

Ex.2.3) Show the following equalities:

(i) $$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1,$$

(ii) $$\sum_{n=1}^{\infty} \frac{2^n+3^n}{6^n} = \frac{3}{2},$$

(iii) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2},$$

(iv) $$\sum_{n=2}^{\infty} \frac{\log((1+\frac{1}{n})^n(1+n))}{(\log n^n)(\log(n+1)^{n+1})} = \log_2(\sqrt{e}).$$

Ex.2.4) Consider $u \in \mathbb{R}^{\mathbb{N}}$ a real sequence (*i.e.*a sequence with values being reals), and assume that the three sequences $(u_{2n+1})_{n\in\mathbb{N}}$, $(u_{2n})_{n\in\mathbb{N}}$, $(u_{3n})_{n\in\mathbb{N}}$ are all converging. Then show that u converges (to the same limit, obviously).

3 Bonus exercises.

Ex.3.1) Let u be a bounded sequence and v be a sequence diverging to $+\infty$. Show that $u + v$ diverges to $+\infty$.

Ex.3.2) Let u be a convergent sequence and v be a divergent sequence. Show that $u + v$ diverges. What can you say about the product sequence $(u.v)_n \overset{\text{def}}{=} u_n.v_n, \forall n \in \mathbb{N}$?

Ex.3.3) Let u and v be two sequences converging to l et l', respectively. We consider the two sequences w and w' defined by: $\forall n \in \mathbb{N}$, $w_n \overset{\text{def}}{=} \min(u_n, v_n)$, $w'_n \overset{\text{def}}{=} \max(u_n, v_n)$. Show that both the sequences w and w' converge and compute their limits.

Ex.3.4) Let u be a non-decreasing real sequence. If we assume that u has a sub-sequence which converges, then show that u also converges. (A *sub-sequence* is of the form $u_\phi = (u_{\phi(n)})_{n\in\mathbb{N}}$ if $\phi : \mathbb{N} \mapsto \mathbb{N}$ is a non-decreasing function.)

[3] First prove that v is monotonic and bounded, then prove by contradiction that its limit is zero.