# First Midterm Examination Solutions

**Q1.**

(i) The zero input response should be always decaying in nature and since it contains only R, L and C elements, it will be exponential in nature

$\Rightarrow e^{-t}, e^{-t}\cos\omega_0 t, te^{-t}$.

(ii) Given $\frac{d^2i}{dt^2} + \frac{di}{dt} + i = e_s(t)$

To find the zero input response, assign the input voltage as zero

$$\Rightarrow \frac{d^2i}{dt^2} + \frac{di}{dt} + i = 0 \cancel{e_s(t)}$$

$$\Rightarrow D^2 + D + 1 = 0 \quad \text{—} (1)$$

The general solution for this second order diff. equation is in the form $K_1 e^{S_1 t} + K_2 e^{S_2 t}$

$$\Rightarrow i(t) = K_1 e^{S_1 t} + K_2 e^{S_2 t}.$$

$S_1$ and $S_2$ are the roots of the quadratic equation given in (1)

From (1), $D = \frac{-1 \pm \sqrt{1 - 4\times1\times1}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

$$\Rightarrow \boxed{i(t) = K_1 e^{\frac{-1+\sqrt{3}i}{2}t} + K_2 e^{\frac{-1-\sqrt{3}i}{2}t}.}$$

Now to solve for $K_1$ and $K_2$, we need the initial current in the Inductor. Since it is not given, the above equation is the general solution

Since the given equation contains second first and zeroth order components, the network may contain resistor, inductor and capacitor

iii. The necessary and sufficient condition for linearity is that the element should satisfy superposition

Given $r = i + i^2$

For $i = i_1$

$\Rightarrow r_1 = i_1 + i_1^2$.

For $i = i_2$

$\Rightarrow r_2 = i_2 + i_2^2$

Now $i = \alpha i_1 + \beta i_2$

$r' = \alpha i_1 + \beta i_2 + (\alpha i_1 + \beta i_2)^2$

$= \alpha i_1 + \beta i_2 + \alpha^2 i_1^2 + \beta^2 i_2^2 + 2\alpha\beta i_1 \cdot i_2 \neq \alpha r_1 + \beta r_2$

Hence the network is <u>non-linear</u>

(iv) $V(s) = \frac{s^2+s+1}{s+1} I(s)$

For finding $i(t) = I\cos(\omega t + \theta)$, represent it as $i(t) = Ie^{j\omega t} \cdot e^{j\theta}$

Since $V(s) - I(s)$ relation is when the input signal is $e^{st}$ form, comparing $s = j\omega$

$V(s) = \frac{(j\omega)^2 + (j\omega) + 1}{j\omega + 1} \cdot I(s)$

$= \frac{-\omega^2 + j\omega + 1}{j\omega + 1} \cdot I(s)$

$$\boxed{V(t) = \frac{(1-\omega^2) + j\omega}{1 + j\omega} \cdot I\cos(\omega t + \theta)}$$

We can further simplify this by multiplying the denominator by the complex conjugate and by later finding the amplitude and phase angle.

Q2

(i) Total number of nodes, n: 4

$\Rightarrow$ Number of independent node equations $= (n-1) = \underline{\underline{3}}$

(ii) Total number of nodes, n : 4

Total number of branches, b : 6

Number of mesh equations, $b-n+1 = \underline{\underline{3}}$

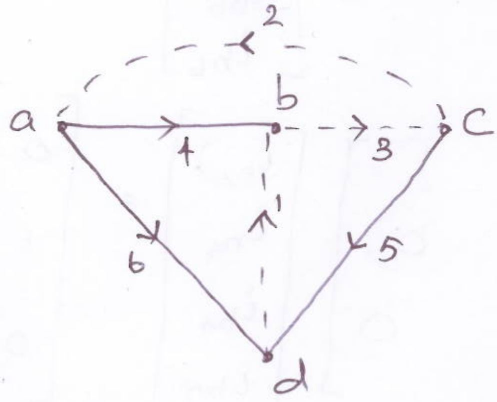
(iii)

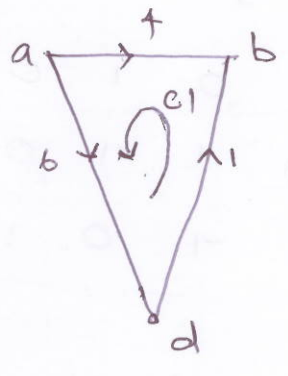
$$A_B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

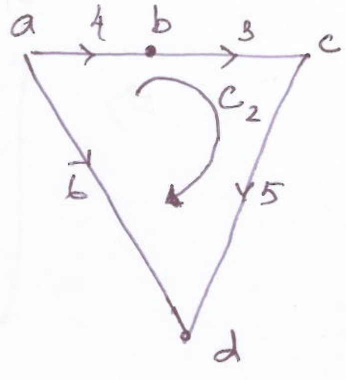
Now considering 'd' as the reference node,

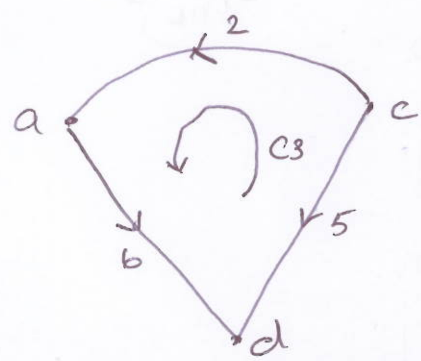
$$A = \begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{bmatrix}$$

(iv)



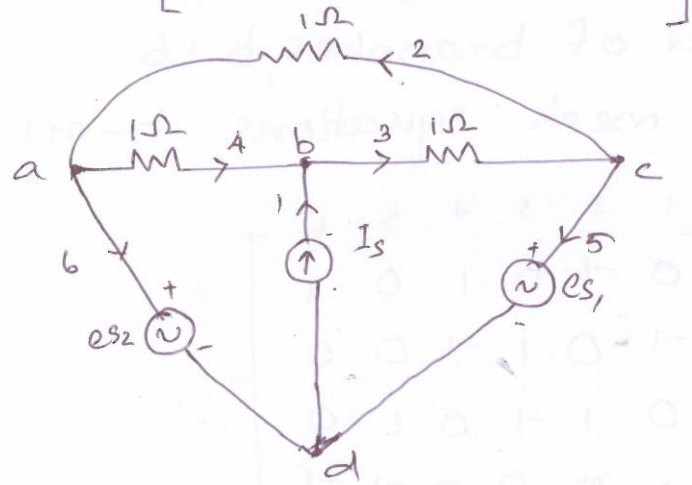



$$B_f = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & \\ 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Q3.




Applying KCL,

$$A i_b = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{b_1} \\ i_{b_2} \\ i_{b_3} \\ i_{b_4} \\ i_{b_5} \\ i_{b_6} \end{bmatrix} = 0,$$

$$i_{b_1} = i_s$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{b_2} \\ i_{b_3} \\ i_{b_4} \\ i_{b_5} \\ i_{b_6} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} i_s .$$

$$-i_{b_2} + i_{b_4} + i_{b_6} = 0$$

$$\boxed{i_{b_6} = i_{b_2} - i_{b_4}}$$

$$i_{b_2} - i_{b_3} + i_{b_5} = 0 .$$

$$\boxed{i_{b_5} = i_{b_3} - i_{b_2}}$$

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_{b_2} \\ i_{b_3} \\ i_{b_4} \end{bmatrix} = i_s \quad - ①$$

Writing branch currents interms of branch voltages,

$$\begin{bmatrix} i_{b_2} \\ i_{b_3} \\ i_{b_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{b_2} \\ V_{b_3} \\ V_{b_4} \end{bmatrix} \quad - ②$$

Applying indirect KVL,

$$V_b = A^T V_n$$

$$\begin{bmatrix} V_{b_1} \\ V_{b_2} \\ V_{b_3} \\ V_{b_4} \\ V_{b_5} \\ V_{b_6} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

$$\boxed{\begin{array}{l} V_{b_6} = e_{s_2} = V_a \\ V_{b_5} = e_{s_1} = V_c \end{array}}$$

$$V_{b_1} = -V_b.$$

$$\Rightarrow \begin{bmatrix} V_{b_2} \\ V_{b_3} \\ V_{b_4} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} \quad - ③$$

Substituting ② and ③ in ①

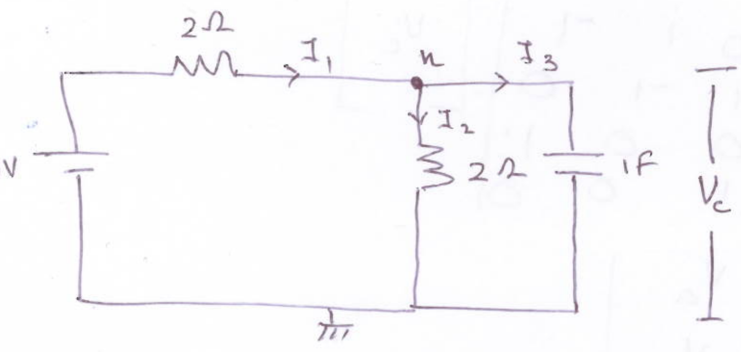
$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_{s_2} \\ V_b \\ e_{s_1} \end{bmatrix} = i_s$$

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_{s_2} \\ V_b \\ e_{s_1} \end{bmatrix} = i_s$$

$$\begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} e_{s_2} \\ V_b \\ e_{s_1} \end{bmatrix} = i_s$$

$$\Rightarrow \boxed{V_b = \frac{i_s + e_{s_2} + e_{s_1}}{2}}$$

Q4.




When switch is closed, the capacitor will charge.
Applying KCL at node n,

$$I_1 = I_2 + I_3$$

$$= \frac{10 - V_c}{2} = \frac{V_c}{2} + 1 \cdot \frac{dV_c}{dt} \quad (1)$$

$$= 10 - V_c = V_c + 2\frac{dV_c}{dt}$$

$$2V_c + 2\frac{dV_c}{dt} - 10 = 0$$

$$V_c + \frac{dV}{dt} - 5 = 0 \quad — (2).$$

$$V_c - 5 = -\frac{dV_c}{dt}$$

$$\frac{dV_c}{V_c - 5} = -dt \quad — (3)$$

Integrating both sides

$$\int_1^{V_c} \frac{dV_c}{V_c - 5} = -\int_0^t dt$$

For $V_c$, lower limit is 1, since there is initial voltage of 1V.

$$= \ln\left[V_c - 5\right]_1^{V_c} = -t\Big]_0^t$$

$$= \ln\left[V_c - 5\right] - \ln\left[-4\right] = -t.$$

$$= \frac{\ln\left[V_c - 5\right]}{-4} = -t$$

$$\Rightarrow e^{-t} = \frac{V_c - 5}{4}$$

$$\boxed{V_c = 5 - 4e^{-t}}$$

So at $t = 1$, voltage across capacitor $= \boxed{5 - 4e^{-1}\,V}$

When switch is opened, capacitor will discharge through the $2\Omega$ resistor. Capacitor discharge equation,

$$V_c = V_{init}\, e^{-\frac{t}{RC}}$$

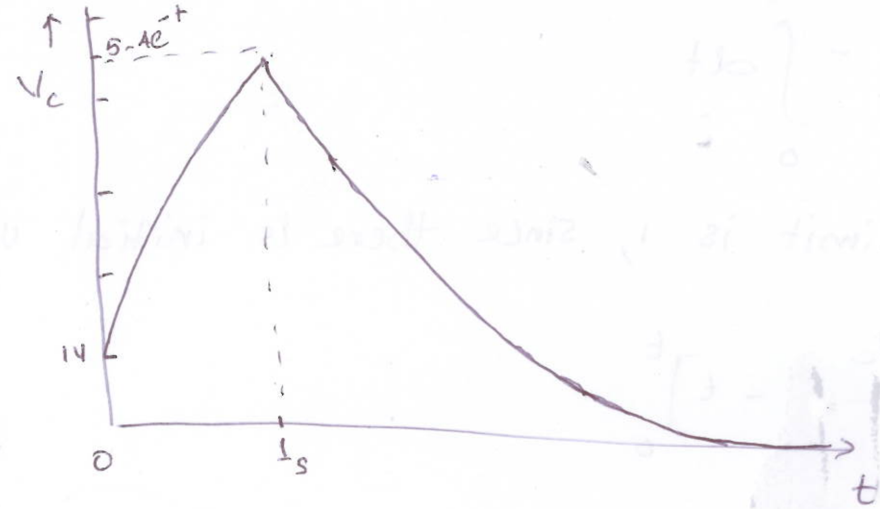
Here $V_{init} = 5 - 4e^{-t}$

$R = 2\Omega$

$C = 1F.$

$\Rightarrow V_c = \left(5 - 4e^{-t}\right) e^{-t/2}$

at $t = \infty$, $V_c = 0$.




Q5. Since the inductor is in series with the current source, same current will be flowing through it

$\Rightarrow I_L = I e^{-\alpha t}$

$V_L = L \frac{dI_L}{dt}$

$= L_1 I e^{-\alpha t} \cdot -\alpha$

$$\boxed{V_L = -\alpha I L_1 e^{-\alpha t}}$$