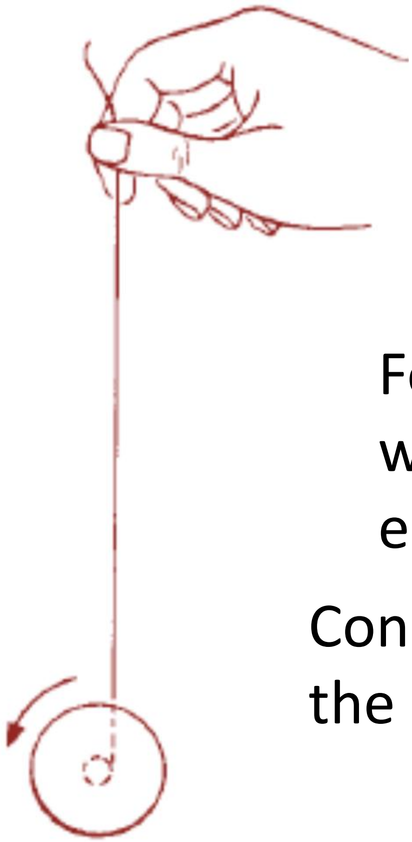
PH101
Lecture19

11.09.14

Angular Momentum and Fixed Axis Rotation



If we consider rotation, an ordinary object does not simply rotate, it wobbles, shakes, and bends

Consider running up and down of a Yo-Yo down a string as the winding spool winds and unwinds

For simplicity we consider motion of a rigid body, which is example of a non-existent ideal object, which essentially retains its shape as it moves about

Consider a line, which we take it as an *axis* about which the body in question is rotating

How will one define rotation?

Let us mark a point somewhere on the object (except ?) on the axis

And track where the point has moved after a certain time. How?

If we measure its change of angle i.e. variation in angle with time!

$\Rightarrow$ Rotation amounts to studying angular change of the position of the body from one time to another

From kinematics point of view in rotation angle changes with time

$\Rightarrow$ analogous to position & velocity in one 1-D

We may associate angular position and angular velocity in rotational motion

In fact almost every quantity in rotation has its analog in 1-D displacement!

Angle θ which defines how far the body has gone *around* replaces the *distance* i.e. how far it has gone *along*

Likewise we may ascribe the notion of *velocity of turning* through

$$\frac{d\theta}{dt} = \omega$$

$\Rightarrow$ How much the angle changes in 1 sec

Just like velocity $v \Rightarrow$ How far it moves in 1 sec

We can go on

For example

$$\frac{d\omega}{dt} = \alpha(\text{say}) \qquad \Rightarrow \frac{d^2\theta}{dt^2} = \text{Angular acceleration}$$

Here we need to develop concepts of

Å Torque

Å Angular momentum, and

Å Moment of inertia

Equivalent to concepts of

Å Force

Å Linear momentum, and

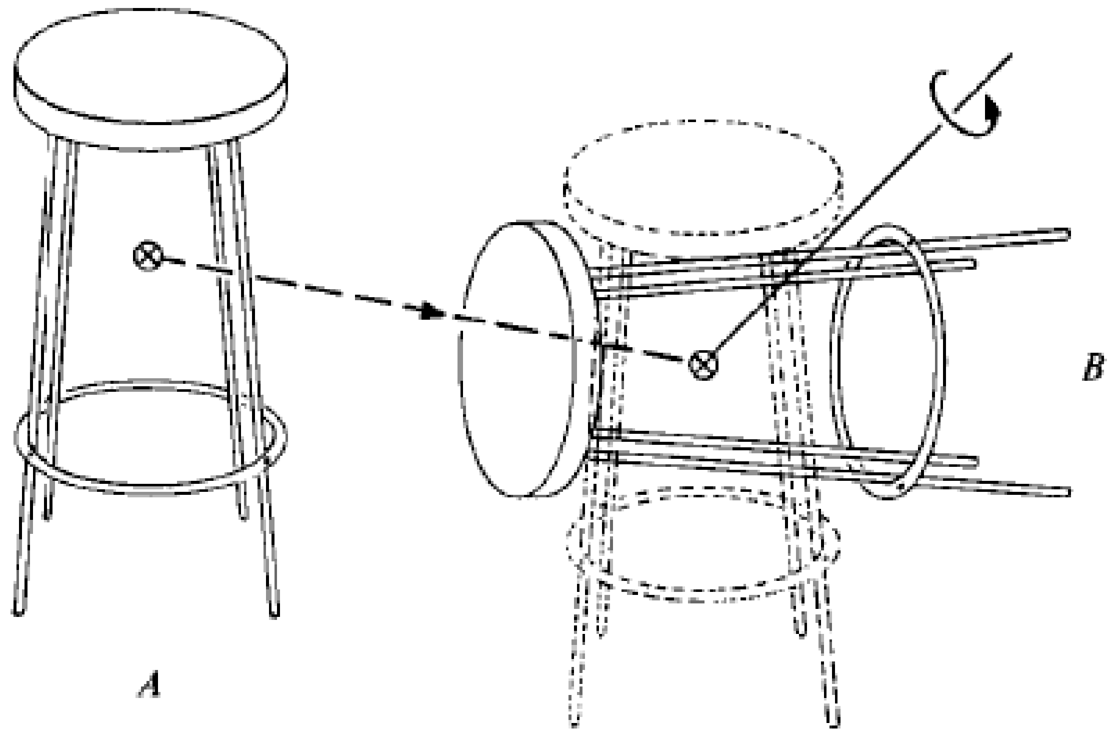
Å Center of mass

In order to analyze the general motion of a rigid body under combination of applied forces, Chasles' theorem is very useful

According to this theorem:

An arbitrary displacement of a rigid body can always be represented by a combination of a translation of its center of mass and a rotation about its center of mass

To move a body from *A* to a desired position *B*



Step 1: translate it so as to make the center of mass coincide

Step 2: introduce rotation about appropriate axis through the center of mass

Thus the problem reduces to finding

1. Center of mass motion
2. Rotational motion about the center of mass

For the time being let us leave aside extended bodies and find out the useful quantity angular momentum of a particle defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

r is the position vector of the particle w.r.t. a given coord. system having momentum **p**

Unit of **L** is kg.m^2 /s in the SI system

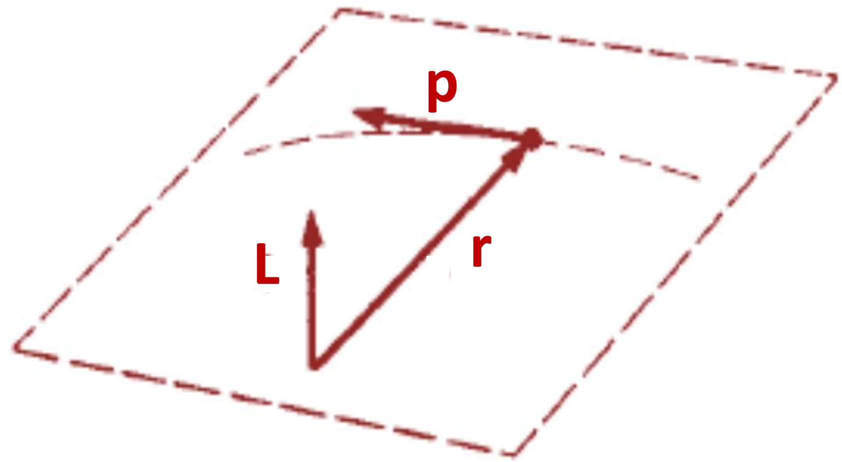
What will be the orientation i.e. direction of **L** ?

L involves vector cross product

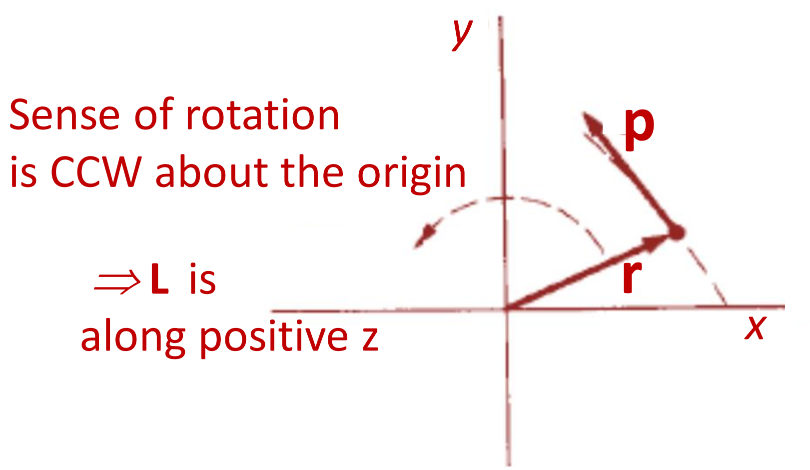
$\Rightarrow$ Direction of **L** will be dictated by RHL for vector multiplication

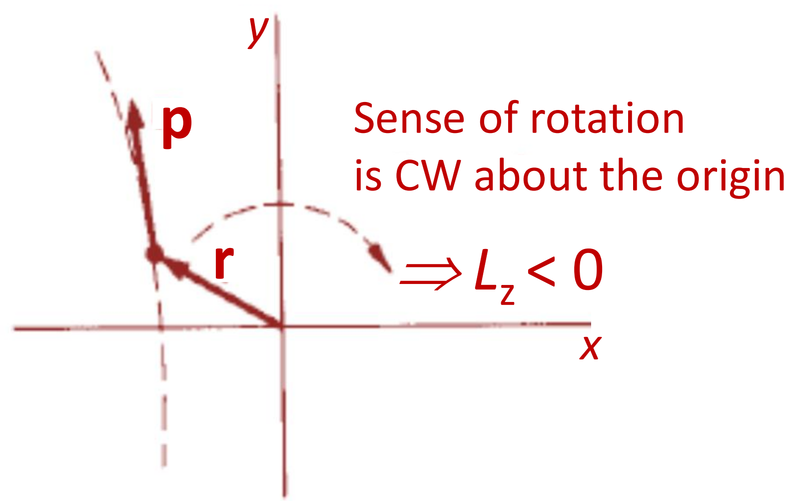
By definition of cross product $\Rightarrow$ **L** is perpendicular to the plane containing **r** and **p**

It shows the trajectory & instantaneous **r** & **p** of a particle



Note: only direction and magnitude of **L** matters and not its location!



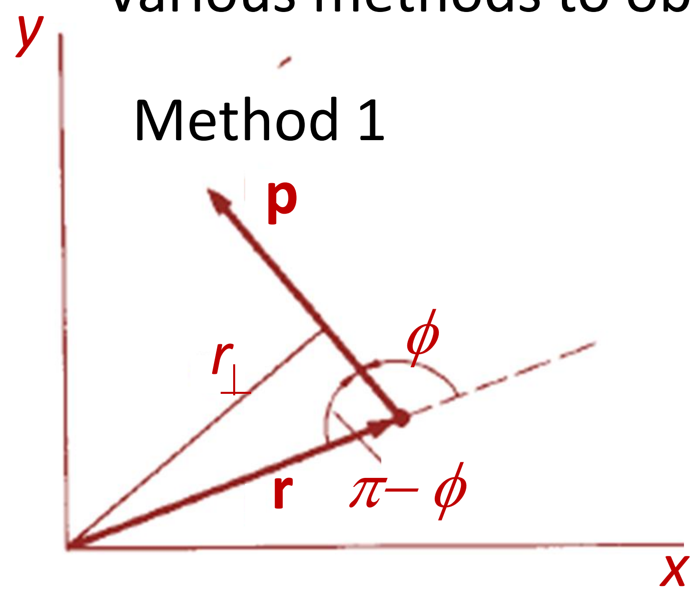


Note: Sense of rotation is well defined even if trajectory is a straight line

Except when it aims at the origin $\Rightarrow$ **p** and **r** are along same direction!

Various methods to obtain **L** of a particle:

Method 1



$$\vec{L} = \vec{r}\times\vec{p} = rp\sin\phi\,\hat{k}$$

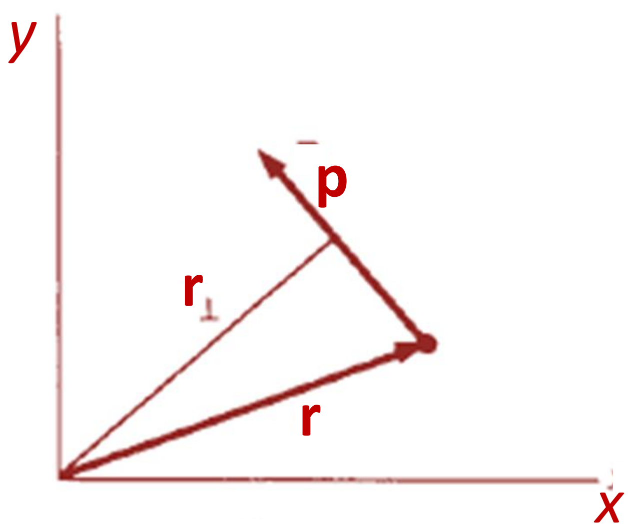
$$\Rightarrow L_z = rp\sin\phi$$

$\Rightarrow$ For motion in *xy*-plane, **L** is along *z*

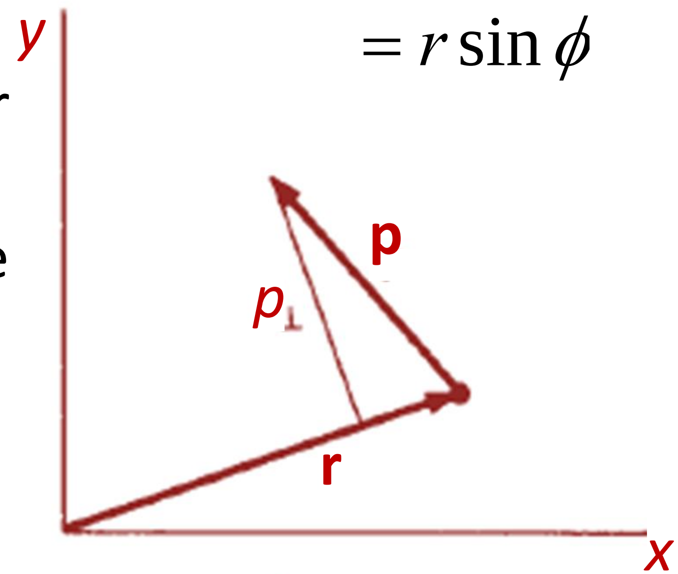
Its magnitude is $r_\perp = r\sin(\pi-\phi)$

$$= r\sin\phi$$

$r_\perp$ is the perpendicular distance betn origin and the line of **p**



$$\Rightarrow L_z = r_\perp p$$

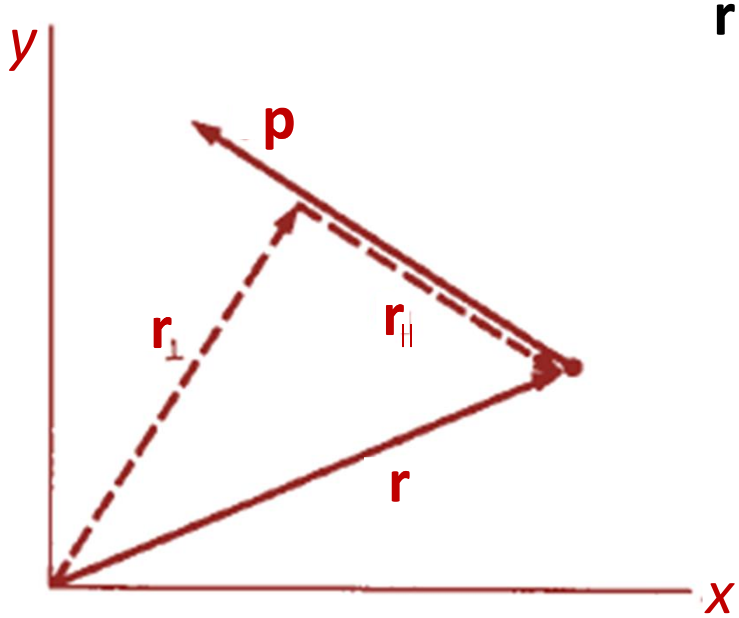


$$\Rightarrow L_z = r\,p_\perp$$

$\downarrow$

Component of **p** Perpendicular to **r**

Method 2



$\mathbf{r} = \mathbf{r}_{||} + \mathbf{r}_\perp$

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$

$= (\mathbf{r}_{||} + \mathbf{r}_\perp) \times \mathbf{p}$

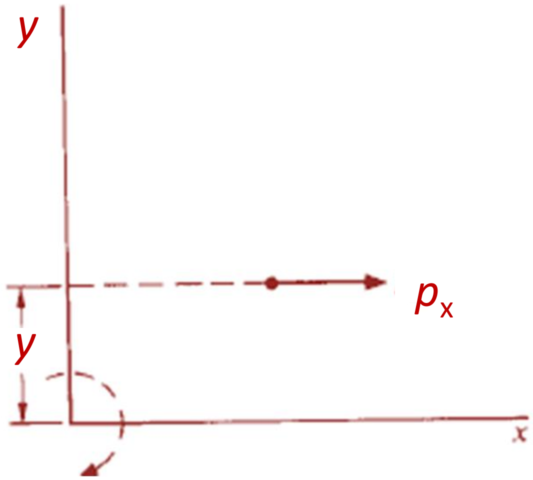
$= \mathbf{r}_\perp \times \mathbf{p}$

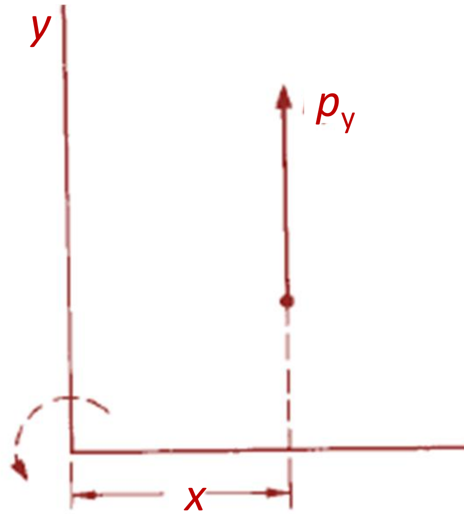
Due to angle of 90° between them

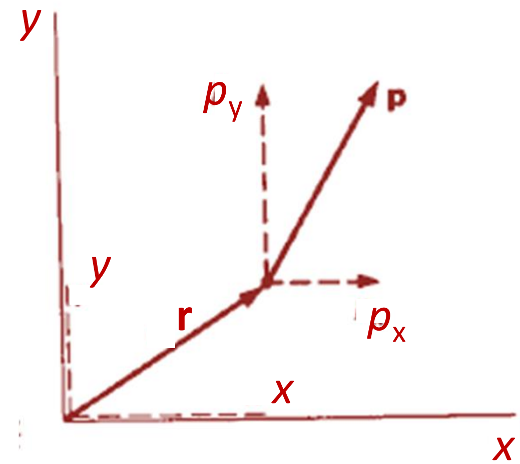
$$\Rightarrow \; L_z = |\vec{r}_\perp||\vec{p}|$$

Likewise $$L_z = |\vec{r}||\vec{p}_\perp|$$

Method 3







1[st] along *x*

$L_z = -y\, p_x$

2[nd] along *y*

$L_z = x\, p_y$

Simultaneously in *xy* plane

$L_z = x\, p_y - y\, p_x$

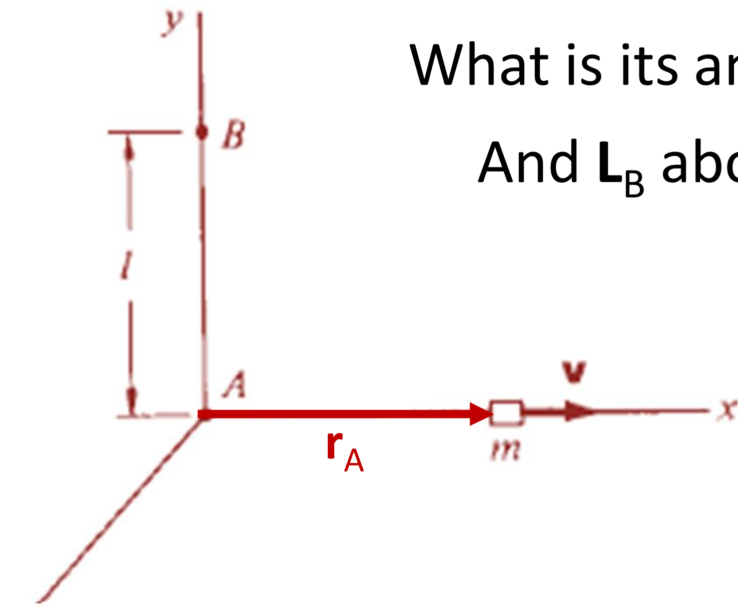
$$\vec{L} = \vec{r} \times \vec{p}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & 0 \\ p_x & p_y & 0 \end{vmatrix} = \left(x\, p_y - y\, p_x\right)\hat{k}$$

r = (*x*, *y*, 0)

p = (p_x, p_y, 0)

Consider a sliding block along *x* direction with a velocity **v**

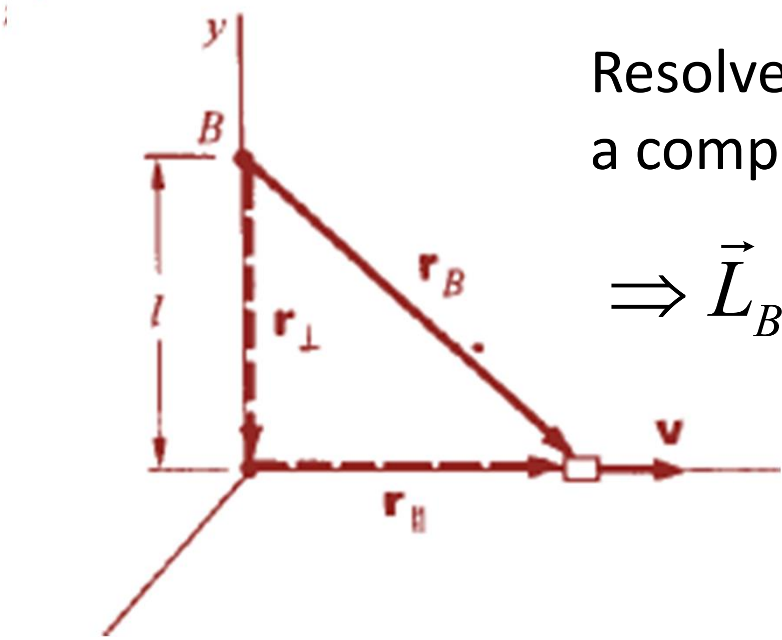


What is its angular momentum $\mathbf{L}_A$ about origin A?

And $\mathbf{L}_B$ about origin B?

$$r_A = x\hat{i}$$

$$\Rightarrow \vec{L}_A = 0$$



Resolve $\mathbf{r}_B$ into a component parallel to **v** and a component perpendicular to **v**

$$\Rightarrow \vec{L}_B = m\,\vec{r}_B \times \vec{v} = mlv\hat{k}$$

Along positive *z* direction

$$r_B = x\hat{i} - l\hat{j}$$

$$r_B = x\hat{i} - l\hat{j}$$

$$\Rightarrow \quad \vec{L}_B = m\vec{r}_B \times \vec{v}$$

$$= m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & -l & 0 \\ v & 0 & 0 \end{vmatrix} = mlv\,\hat{k}$$

Torque

In order to consider *dynamics* of rotation, a new concept of force needs to be introduced

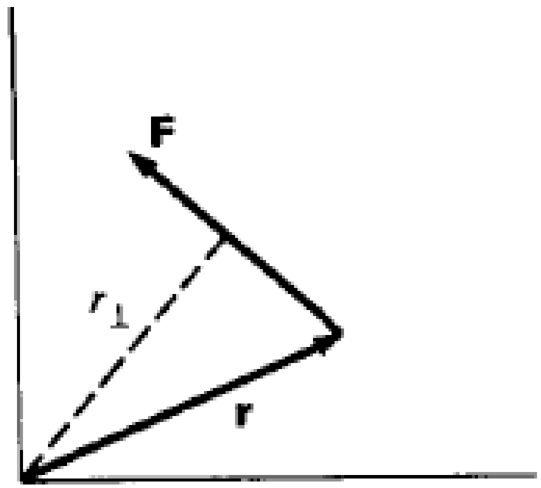
Let us call it *torque*

which has same relationship to rotation as force does to linear motion

$\Rightarrow$ Torque is a rotary force/twisting force that makes something rotate

In linear motion, one way of defining force is to state how much work it does when it acts through a given displacement

One could extrapolate the analogy between linear and angular Quantities by equating the work that we do in order to turn something a little bit when there are forces acting on it to torque X the angle it turns through

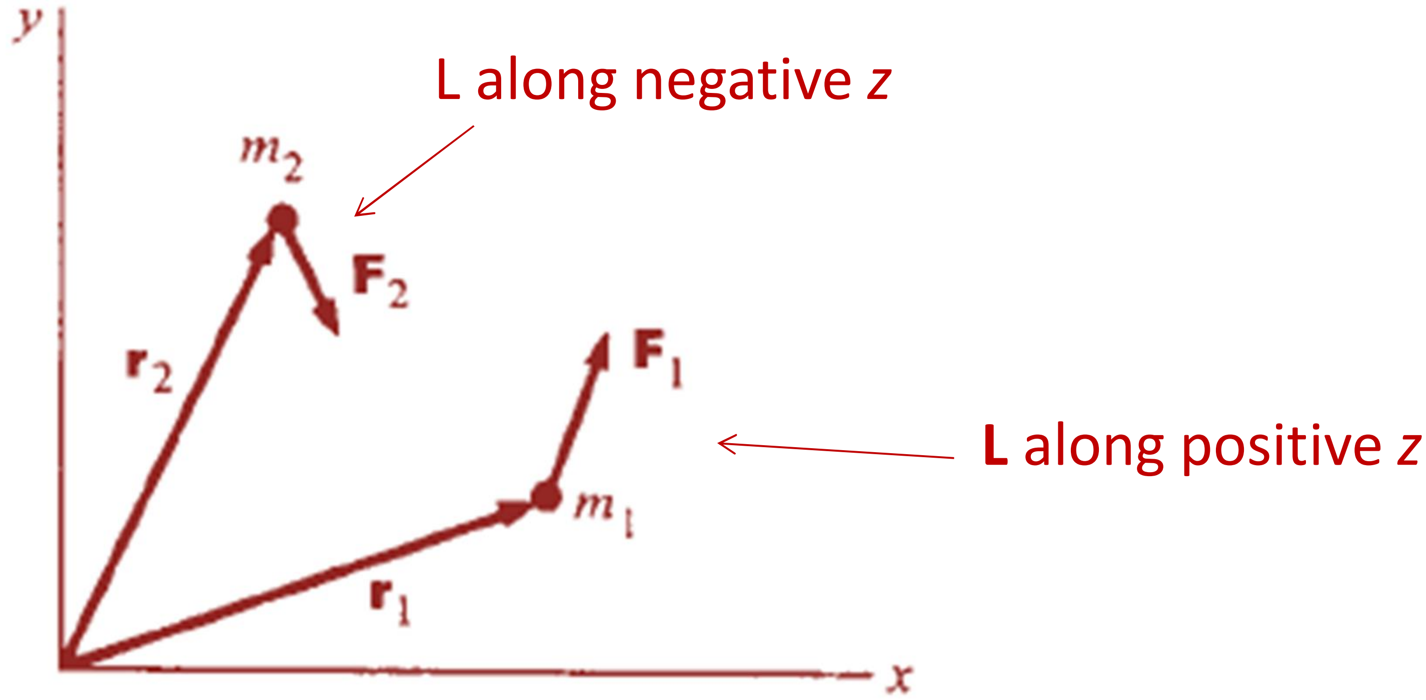


$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}_\perp||\vec{F}|$$

$$|\vec{\tau}| = |\vec{r}\,||\vec{F}_\perp|$$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$



$\Rightarrow$ One could associate a sense of rotation to torque