Nihar Shailesh Joshi
njoshi27@uic.edu

Problem 1a

Question. True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair *(m, w)* such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

Solution. **False.** Let us consider a group containing 3 men (m_1, m_2, m_3) and 3 women (w_1, w_2, w_3).

Let the preferences of m_1, m_2 and m_3 be as follows:

	1^{st}	2^{nd}	3^{rd}
m_1	w_2	w_3	w_1
m_2	w_2	w_3	w_1
m_3	w_1	w_3	w_2

Let the preferences of w_1, w_2 and w_3 be as follows:

	1^{st}	2^{nd}	3^{rd}
w_1	m_2	m_1	m_3
w_2	m_3	m_2	m_1
w_3	m_2	m_1	m_3

If we run the stable matching on the above set of men and women, we get the following matches (stable pairs):

$$(m_1 \quad w_3)$$
$$(m_2 \quad w_2)$$
$$(m_3 \quad w_1)$$

As we see, in this particular outcome, there is no pair *(m,w)* such that m and w are ranked first on each other's preference lists.

In fact, no matter which man we start the algorithm from, we do not find the claim to be true.

- For m_1 to be paired with his top preference, he would have to be paired with w_2. However, the top preference of w_2 is m_3.
- Similarly, in the case of m_2, even though his top preference is w_2, the top preference of w_2 is m_3.
- Lastly, for m_3, even though his top preference is w_1, the top preference of w_1 is m_2.

Therefore, we cannot say that in every instance of the Stable Matching Problem, there is a stable matching containing a pair *(m,w)* such that *m* is ranked first on the preference list of *w* and *w* is ranked first on the preference list of *m*. □

Problem 1b

Question. True or false? Consider an instance of the Stable Matching Problem in which there exists a man *m* and a woman *w* such that *m* is ranked first on the preference list of *w* and *w* is ranked first on the preference list of *m*. Then in every stable matching *S* for this instance, the pair *(m, w)* belongs to *S*.

Solution. **True.** Proof by contradiction:

Let us consider a group containing 3 men (m_1, m_2, m_3) and 3 women (w_1, w_2, w_3).

Let the preferences of m_1, m_2 and m_3 be as follows:

	1^{st}	2^{nd}	3^{rd}
m_1	w_2	w_3	w_1
m_2	w_3	w_1	w_2
m_3	w_1	w_2	w_3

Let the preferences of w_1, w_2 and w_3 be as follows:

	1^{st}	2^{nd}	3^{rd}
w_1	m_1	m_2	m_3
w_2	m_3	m_1	m_2
w_3	m_2	m_1	m_3

Here, we see that both m_2 and w_3 are ranked first on each other's preference lists.

Now, let us assume that there exists a stable matching S' such that m_2 and w_3 are **not** paired with each other.

We know two facts about the Stable Matching algorithm:

- Men will only propose to women **in the order of their preference list.** Therefore, in the case of man m_2, proposals will only happen in the order $w_3 \rightarrow w_1 \rightarrow w_2$.

- Every man will get a chance to propose. Therefore, m_2 will surely get a chance to propose to w_3 - and as m_2 is the top preference of w_3, she will always match with him - as **women are never left unmatched, they always trade up.**

As a result, we cannot find a scenario where m_2 and w_3 do not end up together. **This fact remains true regardless of which group does the proposals.**

Even if, for the sake of the argument, we assume that there does exist a matching where m_2 and w_3 do not end up together, we can see that they would form an **unstable pair** - that is, they would both end up in matches where they prefer each other over their current partners.

This contradicts the definition of Stable Matching.

Therefore, in every stable matching *S* for a man *m* and a woman *w*, such that *m* is ranked first on the preference list of *w* and *w* is ranked first on the preference list of *m*, the pair *(m, w)* belongs to belongs *S*. □

Problem 2 - option b

Question. Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

Solution. Let us consider a group of 3 men *A, B, C* and 3 women *X, Y, Z*.

Let their respective preferences be as follows:

	1^{st}	2^{nd}	3^{rd}
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1^{st}	2^{nd}	3^{rd}
X	B	A	C
Y	A	B	C
Z	A	B	C

After we run the Gale-Shapely algorithm on the above set of participants, we get the following stable matching:

(A X)
(B Y)
(C Z)

If we take a look at woman X, we see that her top choice is man B but she ends up with man A.

Now, suppose woman X lies about her preferences and says that her preference order is $\boldsymbol{B} \to \boldsymbol{C} \to \boldsymbol{A}$ instead. Now we have the new, updated list of preferences as follows:

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	**B**	**C**	**A**
Y	A	B	C
Z	A	B	C

When we run the Gale-Shapely algorithm on this new set of preferences, we get the following stable matching:

(A Y)
(**B** **X**)
(C Z)

In this scenario, we see that **woman *X* actually ends up with man *B*, her first preference.** By lying, woman X has actually hoodwinked the algorithm!

Therefore, this is an example where a switch in preferences improves the partner of a woman.

Consequently, we can also conclude that **yes, a man or a woman can end up better off by lying about his or her preferences.** This is an example of **"Machiavelli meets Gale-Shapley".** □