

PROBLEM 1A

Question. True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m .

Solution. **False.** Let us consider a group containing 3 men (m_1, m_2, m_3) and 3 women (w_1, w_2, w_3).

Let the preferences of m_1, m_2 and m_3 be as follows:

	1 st	2 nd	3 rd
m_1	w_2	w_3	w_1
m_2	w_2	w_3	w_1
m_3	w_1	w_3	w_2

Let the preferences of w_1, w_2 and w_3 be as follows:

	1 st	2 nd	3 rd
w_1	m_2	m_1	m_3
w_2	m_3	m_2	m_1
w_3	m_2	m_1	m_3

If we run the stable matching on the above set of men and women, we get the following matches (stable pairs):

$$\begin{matrix} (m_1 & w_3) \\ (m_2 & w_2) \\ (m_3 & w_1) \end{matrix}$$

As we see, in this particular outcome, there is no pair (m, w) such that m and w are ranked first on each other's preference lists.

In fact, no matter which man we start the algorithm from, we do not find the claim to be true.

- For m_1 to be paired with his top preference, he would have to be paired with w_2 . However, the top preference of w_2 is m_3 .
- Similarly, in the case of m_2 , even though his top preference is w_2 , the top preference of w_2 is m_3 .
- Lastly, for m_3 , even though his top preference is w_1 , the top preference of w_1 is m_2 .

Therefore, we cannot say that in every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m . \square

PROBLEM 1B

Question. True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m . Then in every stable matching S for this instance, the pair (m, w) belongs to S .

Solution. **True.** Proof by contradiction:

Let us consider a group containing 3 men (m_1, m_2, m_3) and 3 women (w_1, w_2, w_3).

Let the preferences of m_1, m_2 and m_3 be as follows:

	1 st	2 nd	3 rd
m_1	w_2	w_3	w_1
m_2	w_3	w_1	w_2
m_3	w_1	w_2	w_3

Let the preferences of w_1, w_2 and w_3 be as follows:

	1 st	2 nd	3 rd
w_1	m_1	m_2	m_3
w_2	m_3	m_1	m_2
w_3	m_2	m_1	m_3

Here, we see that both m_2 and w_3 are ranked first on each other's preference lists.

Now, let us assume that there exists a stable matching S' such that m_2 and w_3 are **not** paired with each other.

We know two facts about the Stable Matching algorithm:

- Men will only propose to women **in the order of their preference list**. Therefore, in the case of man m_2 , proposals will only happen in the order $w_3 \rightarrow w_1 \rightarrow m_2$.

- Every man will get a chance to propose. Therefore, m_2 will surely get a chance to propose to w_3 - and as m_2 is the top preference of w_3 , she will always match with him - as **women are never left unmatched, they always trade up.**

As a result, we cannot find a scenario where m_2 and w_3 do not end up together. **This fact remains true regardless of which group does the proposals.**

Even if, for the sake of the argument, we assume that there does exist a matching where m_2 and w_3 do not end up together, we can see that they would form an **unstable pair** - that is, they would both end up in matches where they prefer each other over their current partners.

This contradicts the definition of Stable Matching.

Therefore, in every stable matching S for a man m and a woman w , such that m is ranked first on the preference list of w and w is ranked first on the preference list of m , the pair (m, w) belongs to S . \square

PROBLEM 2 - OPTION B

Question. Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

Solution. Let us consider a group of 3 men A, B, C and 3 women X, Y, Z .

Let their respective preferences be as follows:

	1 st	2 nd	3 rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1 st	2 nd	3 rd
X	B	A	C
Y	A	B	C
Z	A	B	C

After we run the Gale-Shapely algorithm on the above set of participants, we get the following stable matching:

$$\begin{matrix} (A & X) \\ (B & Y) \\ (C & Z) \end{matrix}$$

If we take a look at woman X , we see that her top choice is man B but she ends up with man A .

Now, suppose woman X lies about her preferences and says that her preference order is $B \rightarrow C \rightarrow A$ instead. Now we have the new, updated list of preferences as follows:

	1 st	2 nd	3 rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1 st	2 nd	3 rd
X	B	C	A
Y	A	B	C
Z	A	B	C

When we run the Gale-Shapely algorithm on this new set of preferences, we get the following stable matching:

$$\begin{matrix} (A & Y) \\ (B & X) \\ (C & Z) \end{matrix}$$

In this scenario, we see that **woman X actually ends up with man B , her first preference**. By lying, woman X has actually hoodwinked the algorithm!

Therefore, this is an example where a switch in preferences improves the partner of a woman.

Consequently, we can also conclude that **yes, a man or a woman can end up better off by lying about his or her preferences.** \square