Notes on Bayesian Update in the Detective Game

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Consider the case in which the agent investigates one suspect with prior probability of being guilty π_0 (before the current round of investigation)

$$\pi_0 = Pr(guilty)$$

and remember that the *conditional* probabilities of receiving a signal N (no final evidence) are

$$p_g = Pr(N|guilty) = 0.75, \qquad p_i = Pr(N|innocent) = 1.$$

It follows that the *unconditional* probability of receiving a signal N is

$$p_N = p_g \cdot \pi_0 + p_i \cdot (1 - \pi_0) \tag{1}$$

that is the weighted average of the two conditional probabilities above, with the weights being equal to the probabilities of the suspect being guilty and innocent, respectively.

We want to compute the posterior of the agent after receiving the signal.
If the signal is Y (yes final evidence), the posterior will be $\pi_Y = 1$ (it also follows from Bayes rule, since $Pr(Y|innocent)$=0).
If the signal is N, the Bayesian agent updates own prior by reducing the likelihood that the suspect is guilty, and the posterior becomes

$$\pi_N = Pr(guilty|N) = \frac{Pr(N|guilty) \cdot Pr(guilty)}{Pr(N)} = \frac{p_g \cdot \pi_0}{p_N} \tag{2}$$

Since $p_N \geq p_g$ (unconditional probability greater than the conditional probability) we can immediately notice that $\pi_N \leq \pi_0$, as expected, with equality only if we are already certain about who is guilty ($\pi = 0$ or $= 1$).

For sake of completeness, consider the case in which you are investigating the *other* suspect. So π_0 can be Pr(Red Guilty) as before, but you are investigating the blue suspect instead. Now $p_g = 1$ and $p_i = 0.75$, the opposite from before (you always find N if red is guilty, and you find Y with 0.25 if red is innocent). If the signal is Y, the posterior will be $\pi_Y = 0$ (you are sure that the red suspect is not guilty). If the signal is N, π_N is computed by following (1) and (2), but with the new guilty/innocent probabilities.
Symmetrical to the previous result, here $p_N \leq p_g$ and therefore $\pi_N \geq \pi_0$.