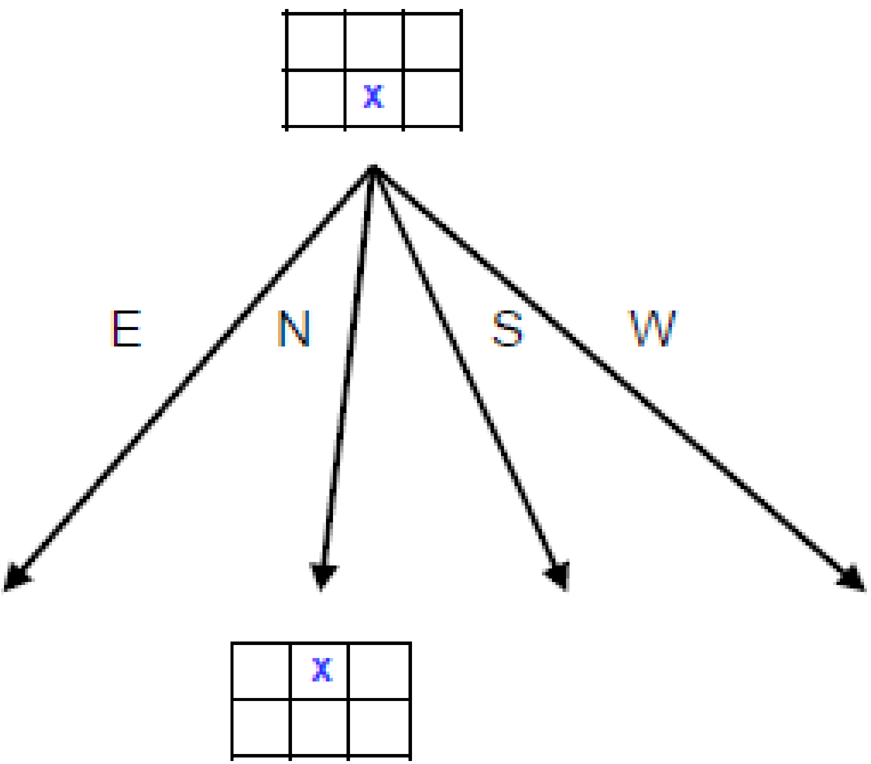
Markov Decision Processes
Where Agents begin to Learn

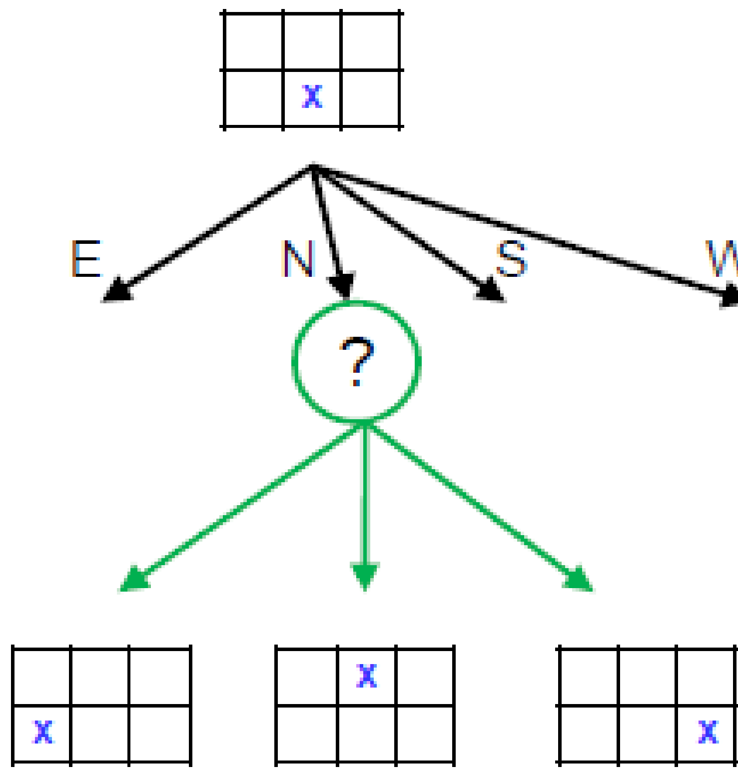
With many slides from Dan Klein and Pieter Abbeel and Stuart Russel

Deterministic vs. Stochastic

Deterministic Grid World



Stochastic Grid World



Big Picture

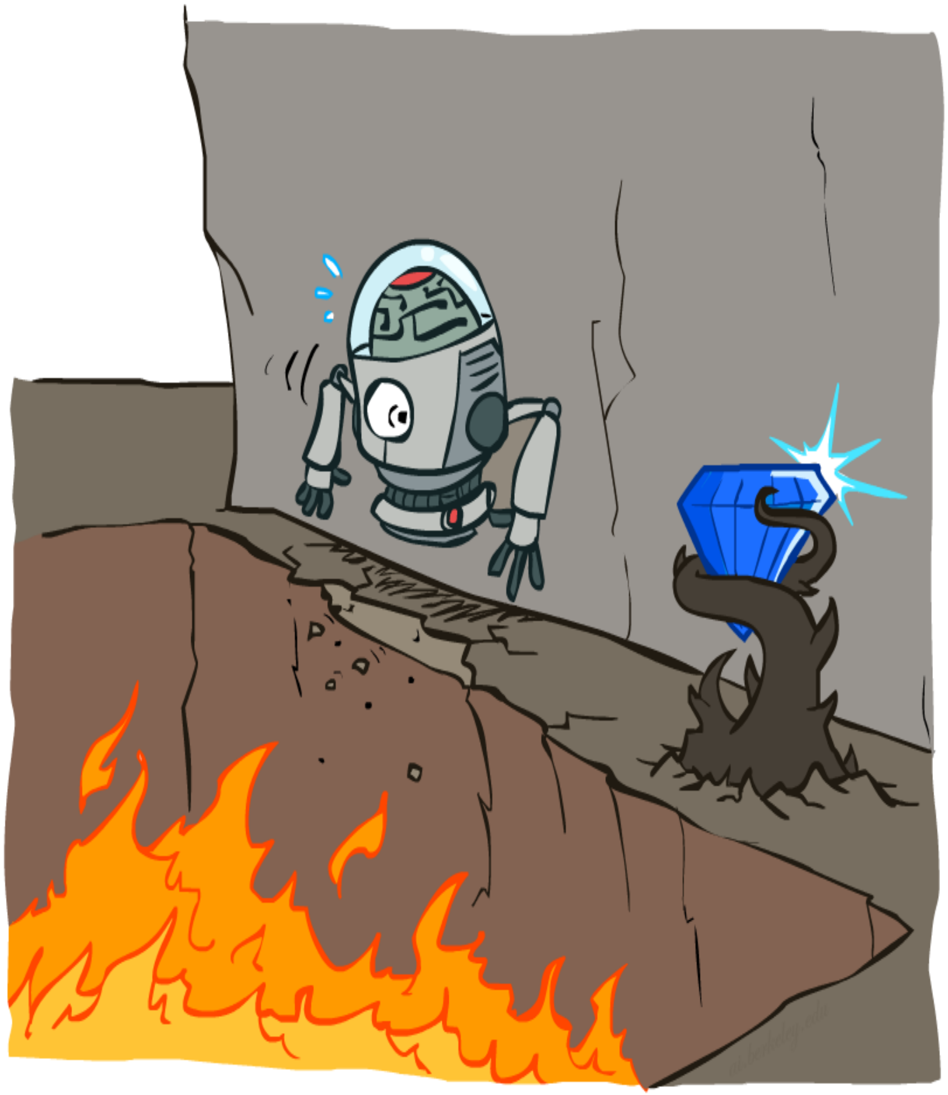
- AI as Planning:
  - Model of the world known (utilities, action outcomes)
  - Deterministic search: UCS, A*, MiniMax
  - Non-deterministic search → ExpectiMax → MDPs

- AI as Learning:
  - Model of world *partially* known (rewards? outcomes?)
  - rewards, action outcomes unknown → RL

Rough Plan (Next 2-3 weeks)

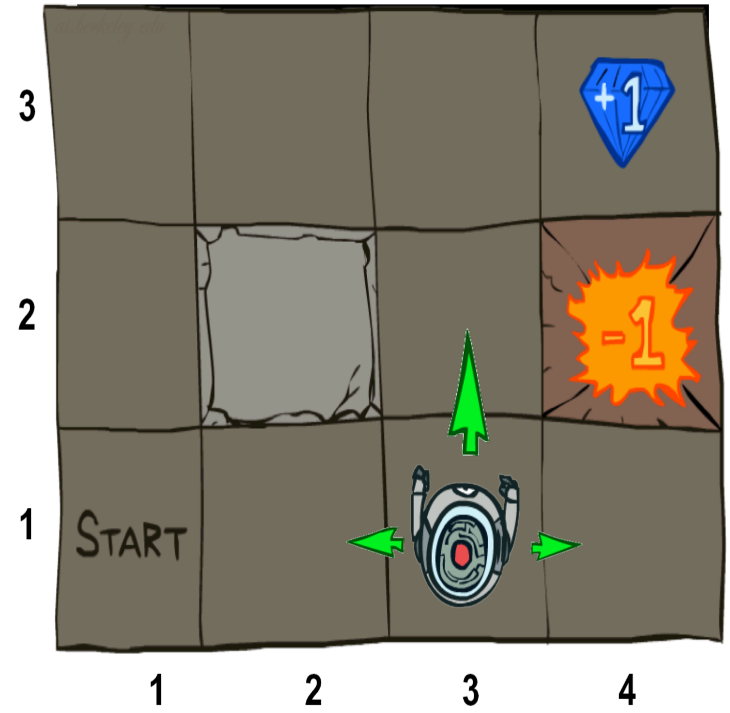
- Markov Decision Processes (MDPs)
  – MDP formalism
  – Value Iteration
  – Policy Iteration
- Reinforcement Learning (RL)
  – Relationship to MDPs
  – Several learning algorithms
  – RL applications to games, "real world"
- **Midterm (February 28?)**

Non-Deterministic Search



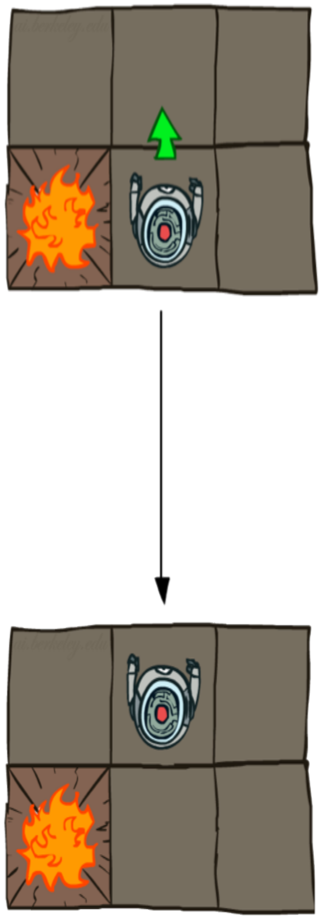
Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North
    (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

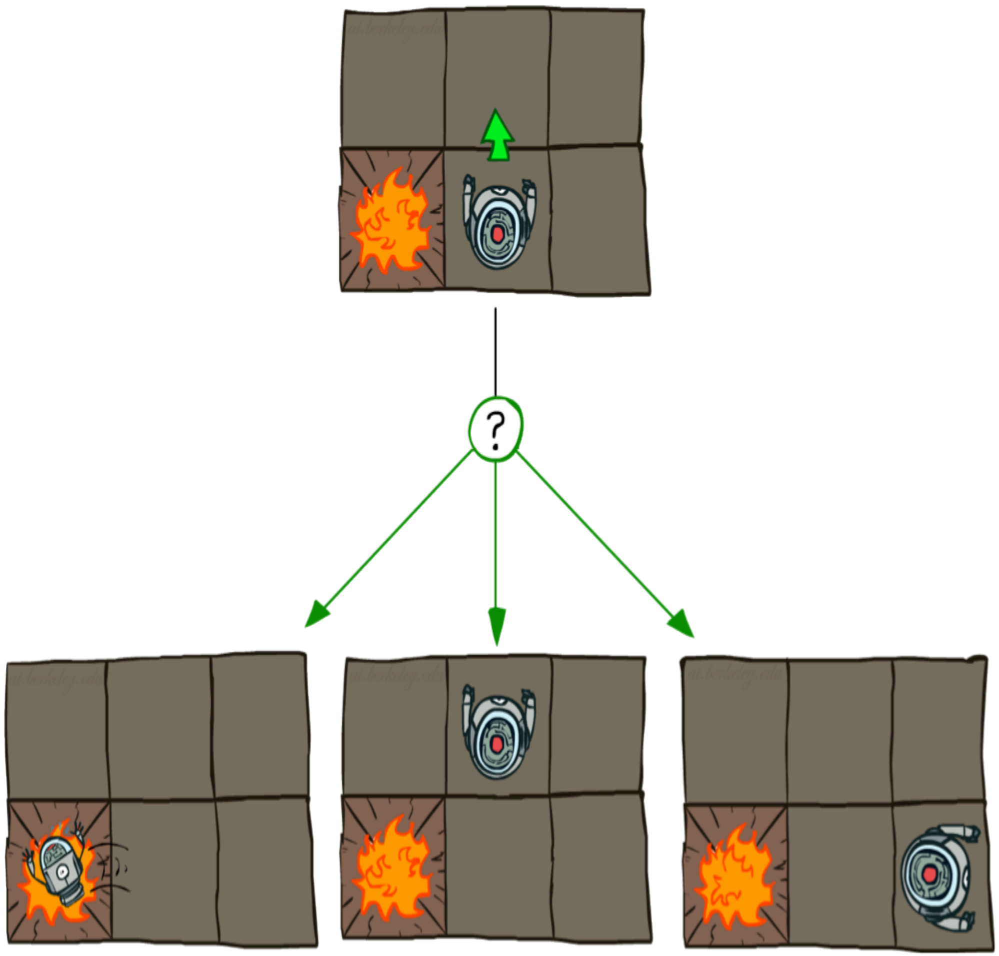


Grid World Actions

Deterministic Grid World

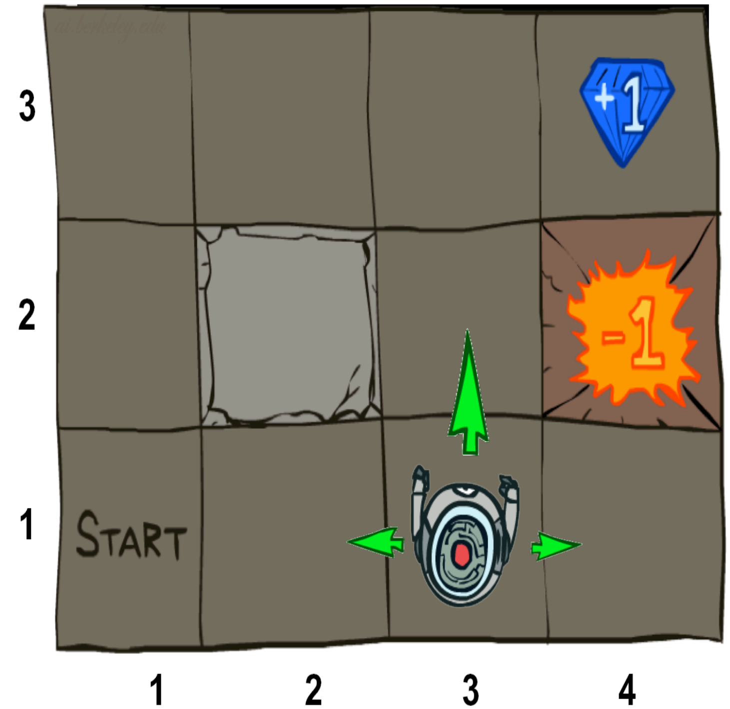


Stochastic Grid World



Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function $T(s, a, s')$
    - Probability that a from s leads to s', i.e., $P(s' | s, a)$
    - Also called the model or the dynamics
  - A reward function $R(s, a, s')$
    - Sometimes just $R(s)$ or $R(s')$
  - A start state
  - Maybe a terminal state



- MDPs are non-deterministic search problems
  - One way to solve them is with expectimax search
  - We'll have a new tool soon

What is Markov about MDPs?

- "Markov" generally means that given the present state, the future and the past are independent
- For Markov decision processes, "Markov" means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

- This is just like search, where the successor function could only depend on the current state (not the history)

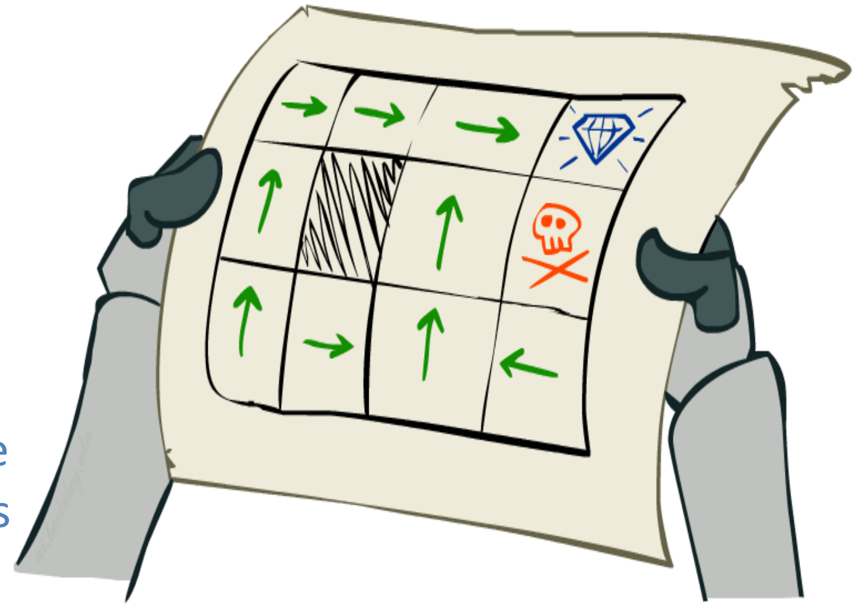
Andrey
Markov
(1856-1922)

Policies

- In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal

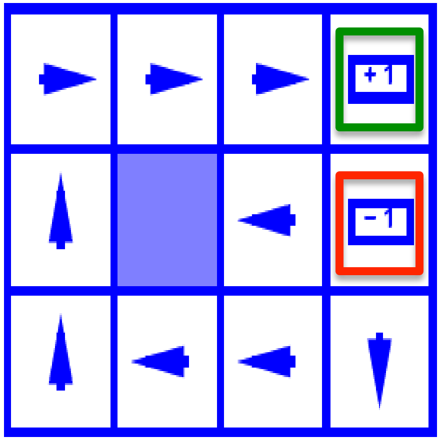
- For MDPs, we want an optimal policy $\pi^*: S \rightarrow A$
  - A policy π gives an action for each state
  - An optimal policy is one that maximizes expected utility if followed
  - An explicit policy defines a reflex agent

- Expectimax didn't compute entire policies
  - **It computed the action for a single state only**

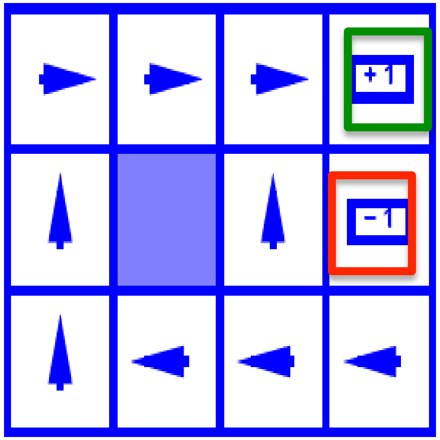


Optimal policy when R(s, a, s') = -0.03 for all non-terminals s

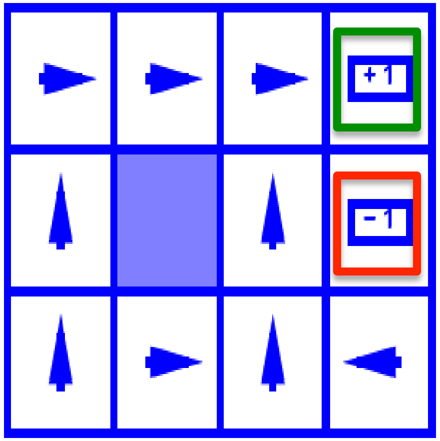
Optimal Policies



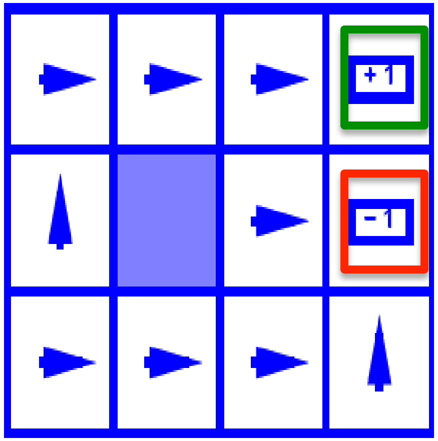
R(s) = -0.01



R(s) = -0.03

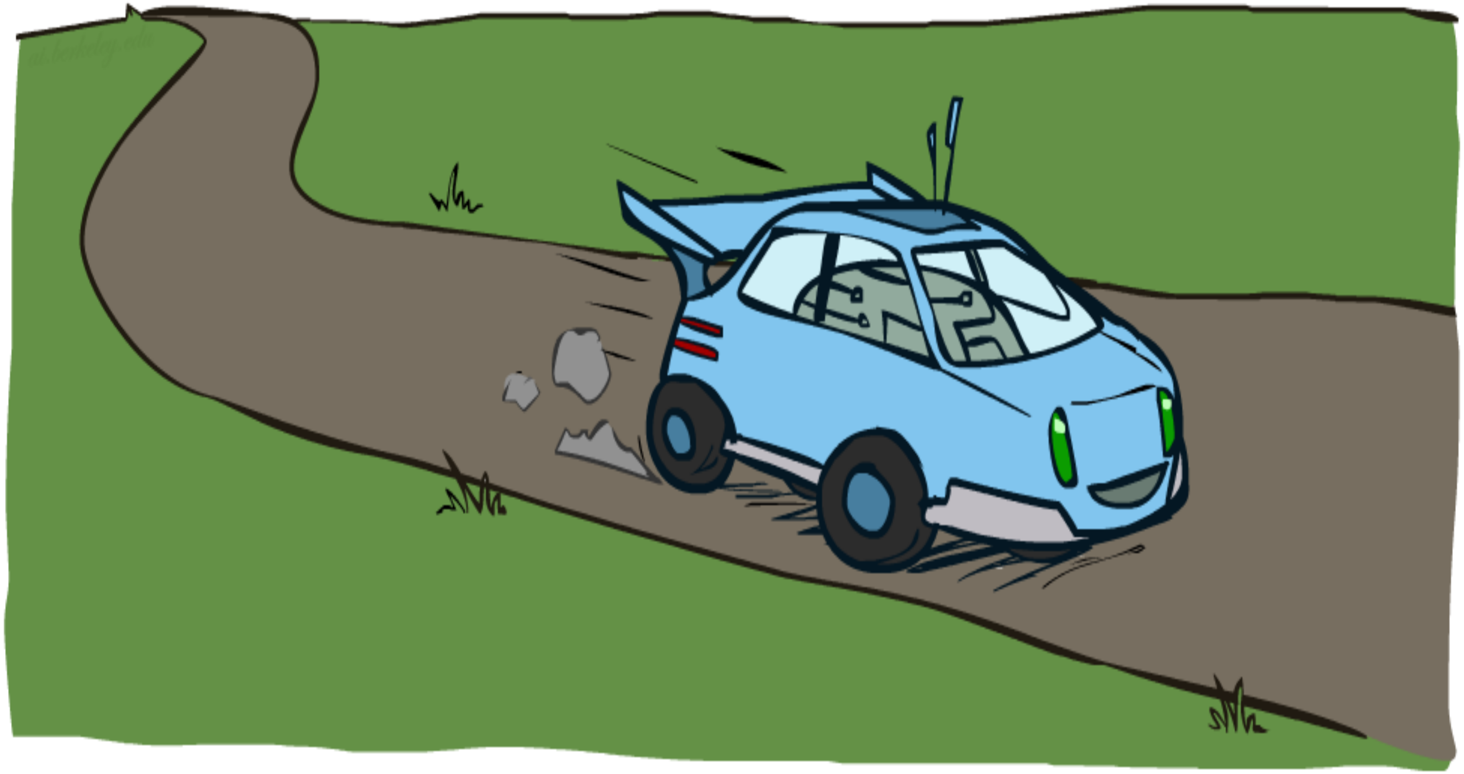


R(s) = -0.4



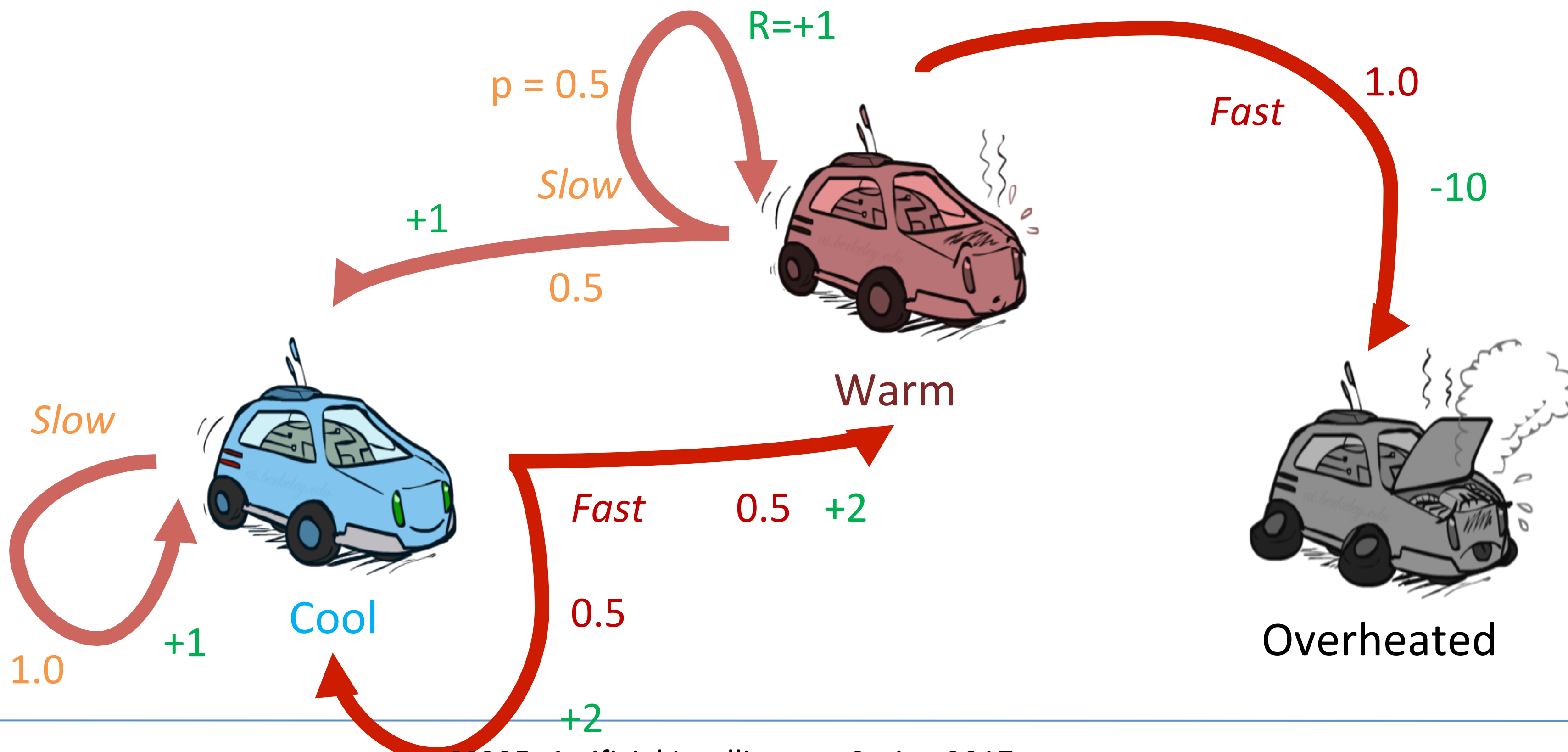
R(s) = -2.0

Example: Racing

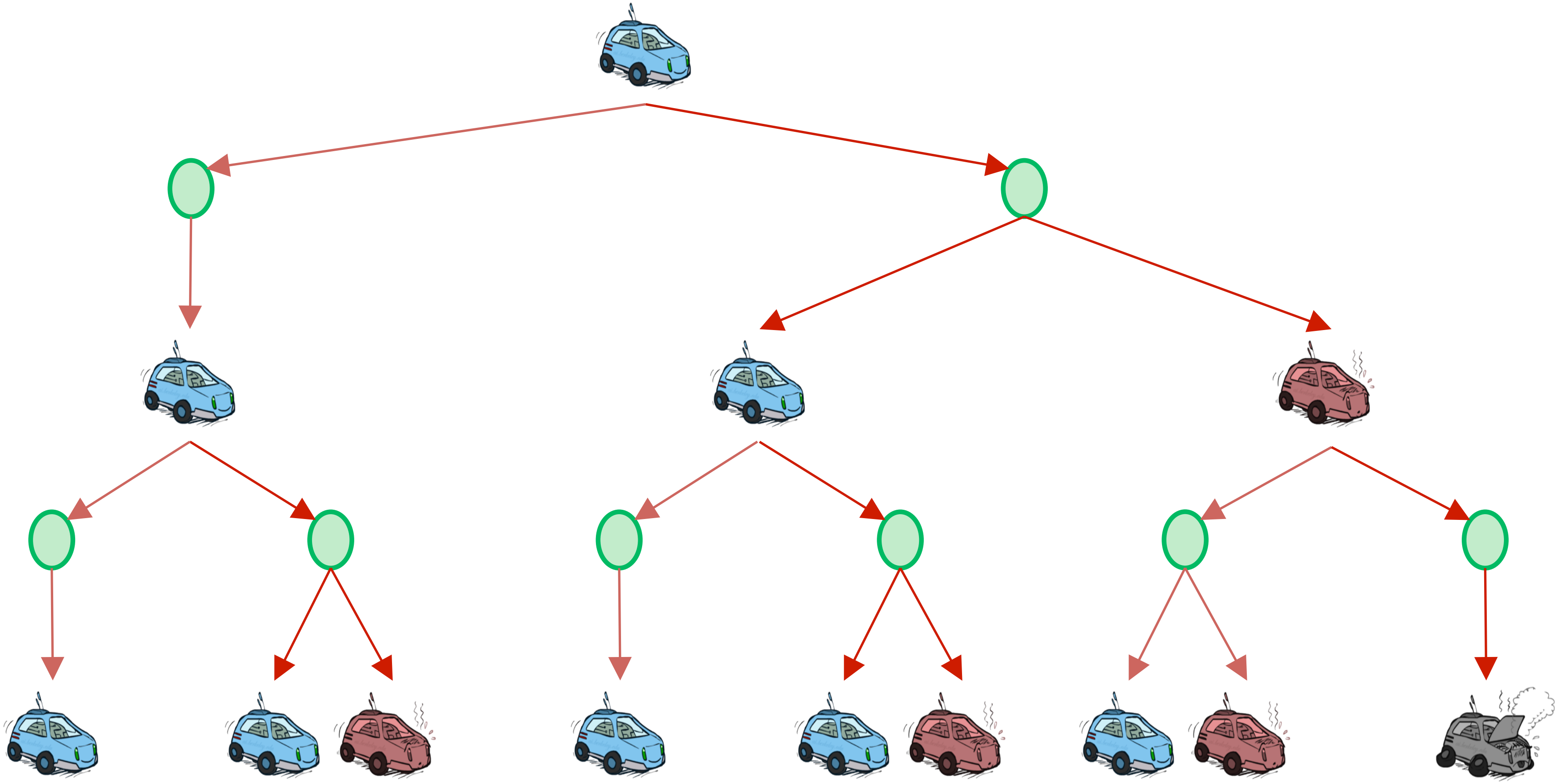


Example: Racing

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward



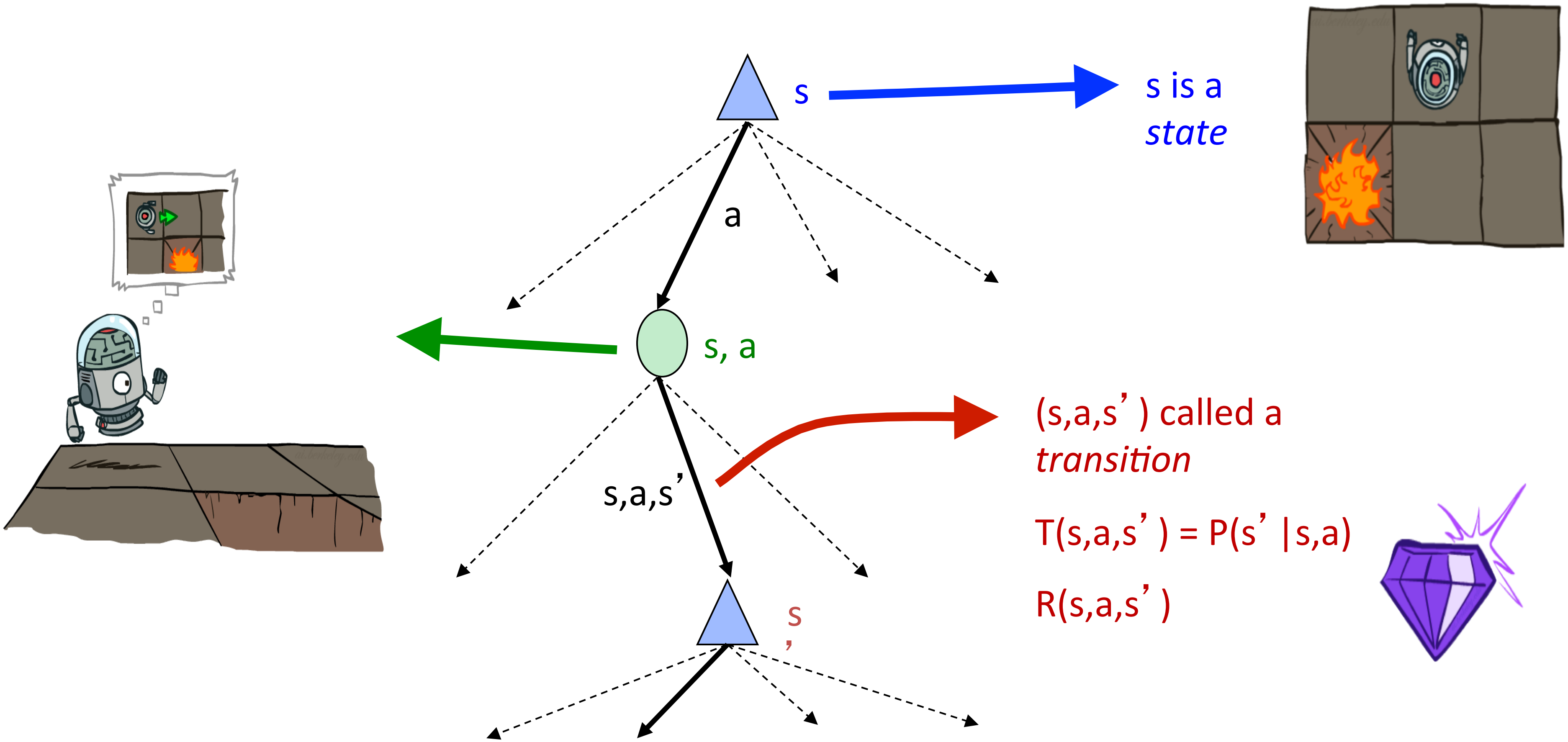
Racing Search Tree



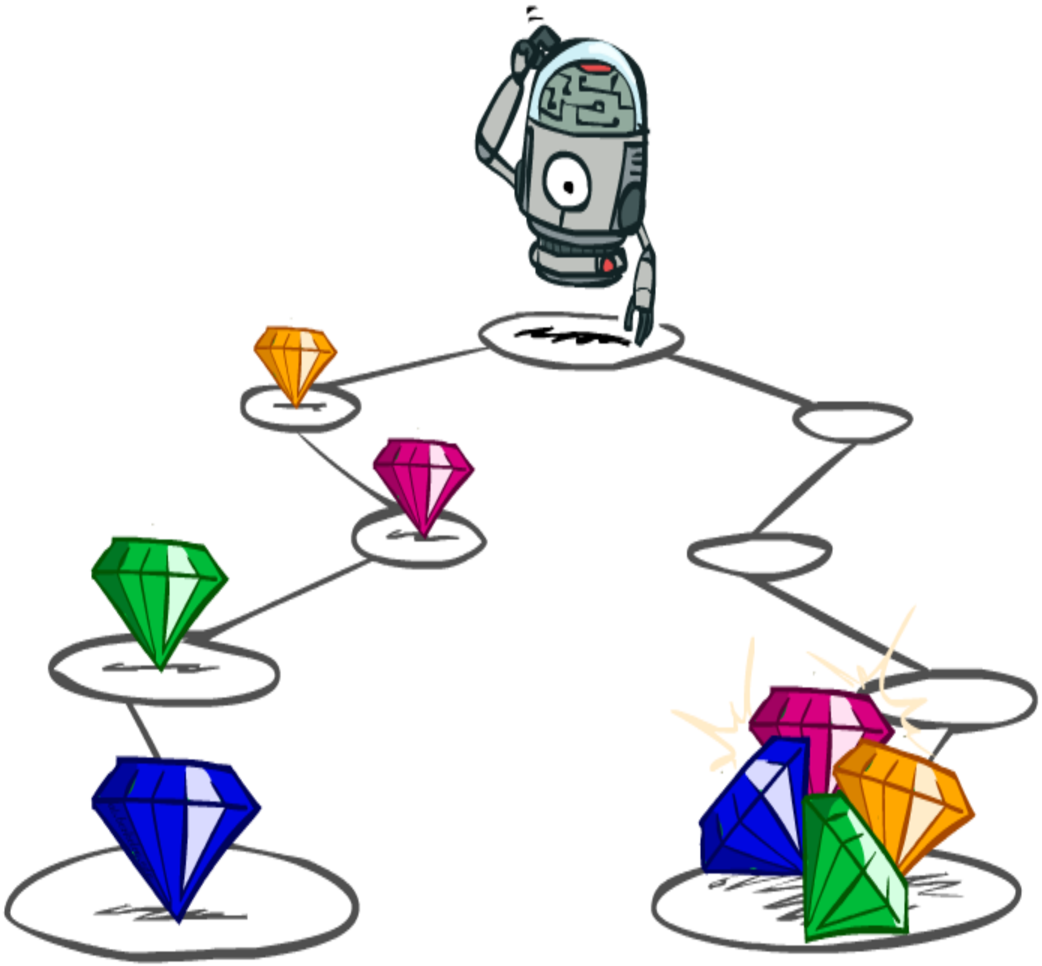
... potentially infinite depth …

MDP Search Trees

- Each MDP state projects an expectimax-like search tree



Utilities of Sequences



Utilities of Sequences

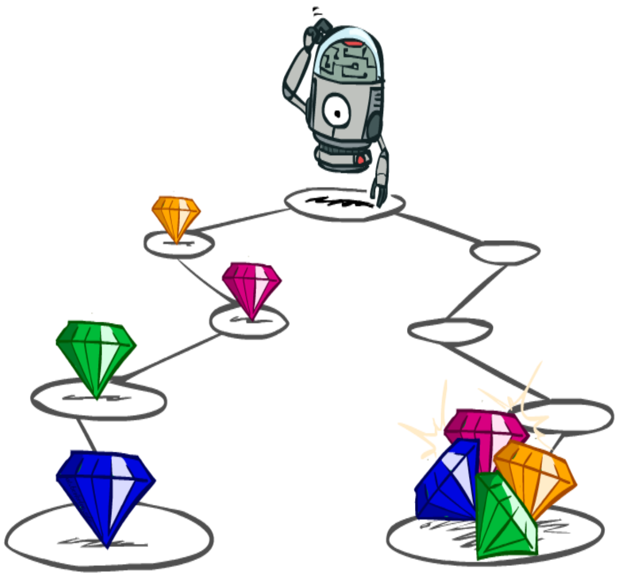
- What preferences should an agent have over reward sequences?

  [1, 2, 2] or [2, 3, 4]

- More or less?

  [0, 0, 1] or [1, 0, 0]

- Now or later?



Discounting

- It’s reasonable to maximize the sum of rewards
- It’s also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially

1

Worth Now

γ

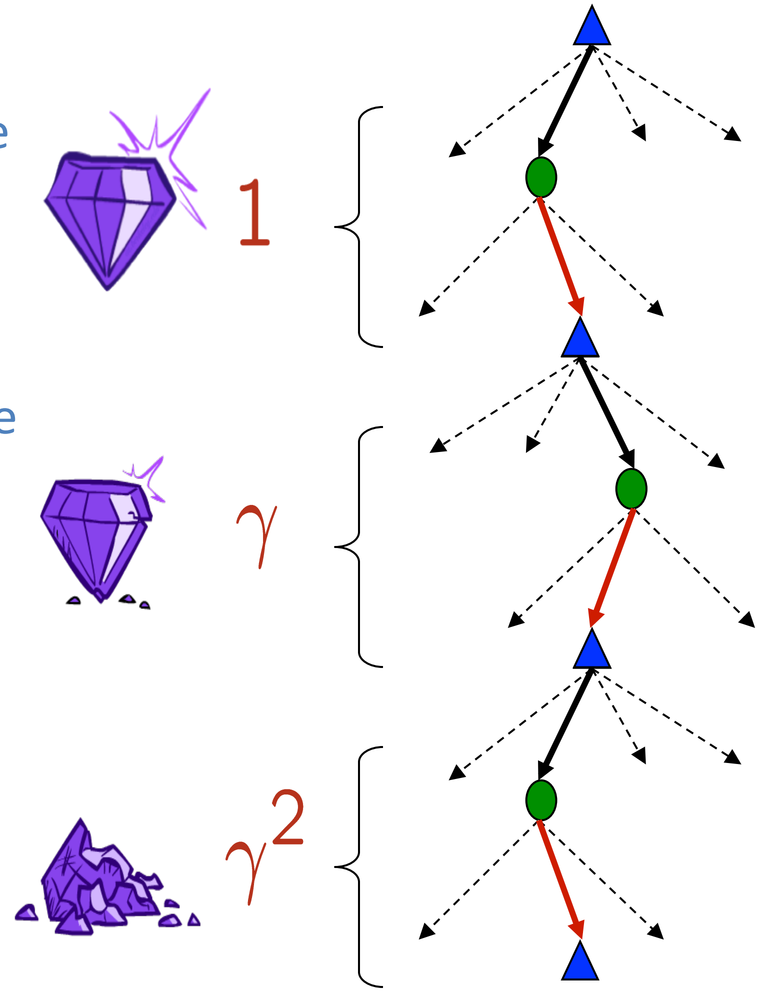
Worth Next Step

γ^2

Worth In Two Steps

Discounting

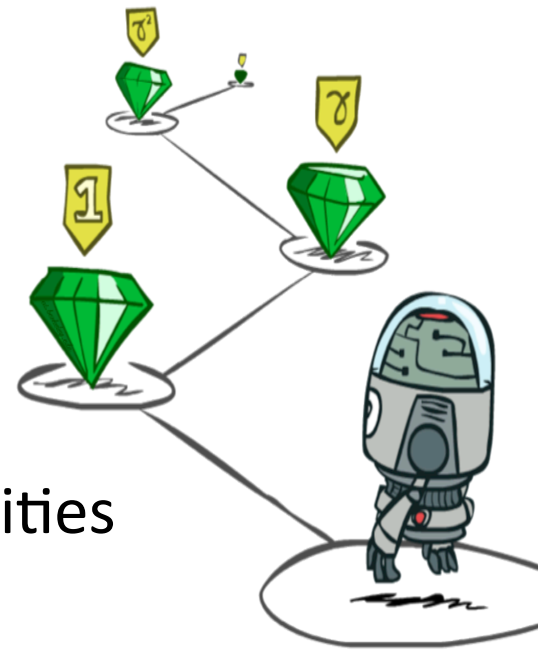
- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Sooner rewards probably do have higher utility than later rewards
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])



Stationary Preferences

- Theorem: if we assume stationary preferences:

$$[a_1, a_2, \ldots] \succ [b_1, b_2, \ldots]$$
$$\Updownarrow$$
$$[r, a_1, a_2, \ldots] \succ [r, b_1, b_2, \ldots]$$



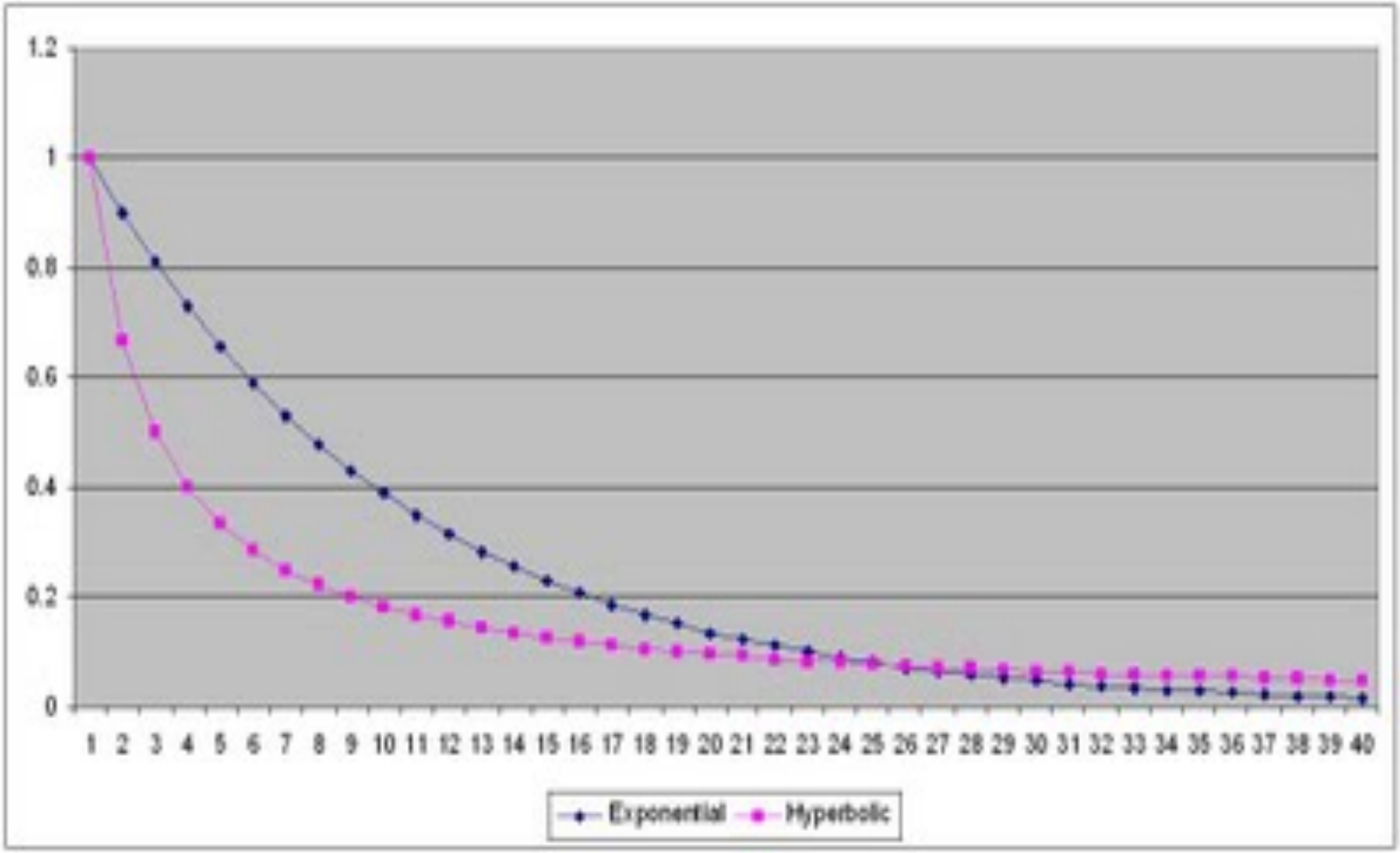
- Then: there are only two ways to define utilities
  - Additive utility:
    $$U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$$
  - Discounted utility:
    $$U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$$

Detour: Temporal/Delay Discounting

- What would you rather have?
  - A. $100 today
  - B. $150 a year from now
- What about:
  - A. $100 in 12 months
  - B. $110 in 13 months
- Humans temporally discount values of rewards
  - https://en.wikipedia.org/wiki/Temporal_discounting
- Delayed gratification: https://www.youtube.com/watch?v=QX_oy9614HQ

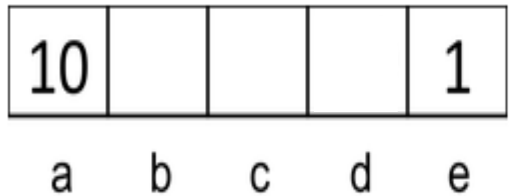
Normative Theory of Discounting

- Money should be discounted at a constant rate over time. (γ)
- This implies that preferences will be consistent over time. (stationary)



Quiz: Discounting

- Given:



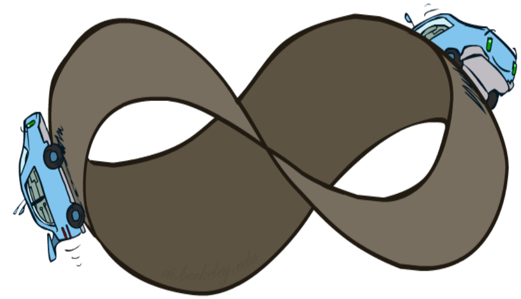
  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic (no noise, for now)

- Quiz 1: For $\gamma = 1$, what is the optimal policy?

- Quiz 2: For $\gamma = 0.1$, what is the optimal policy?

Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?



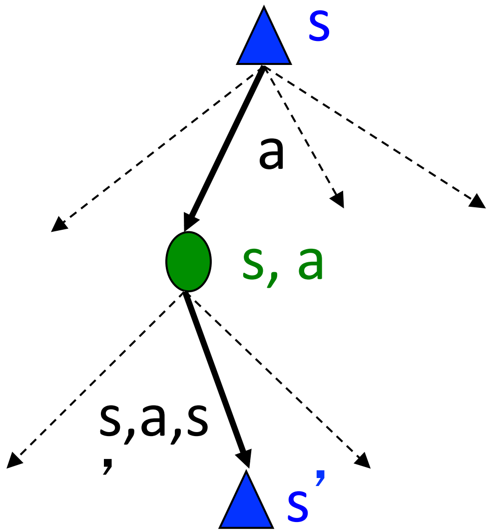
- Solutions:
  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)
  - Discounting: use $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)

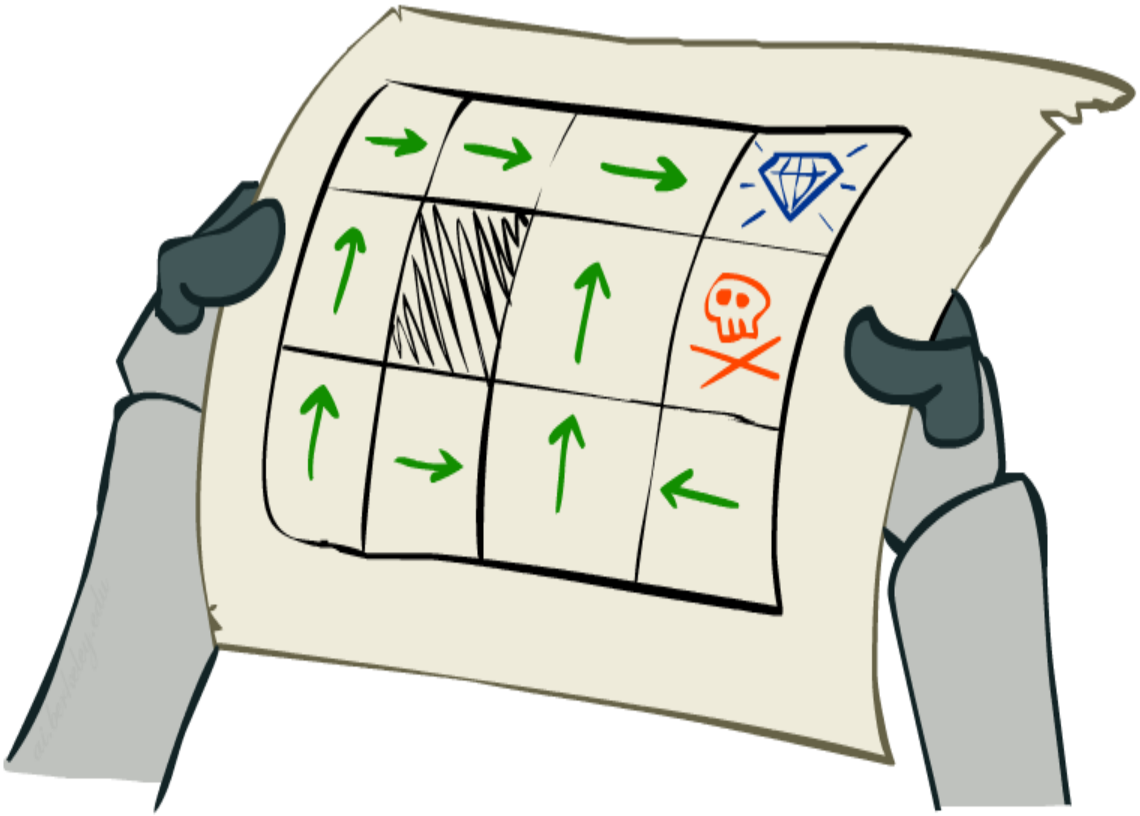
Recap: Defining MDPs

- Markov decision processes:
  - Set of states S
  - Start state s_0
  - Set of actions A
  - Transitions $P(s'|s,a)$ (or $T(s,a,s')$)
  - Rewards $R(s,a,s')$ (and discount γ)



- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility = sum of (discounted) rewards

Solving MDPs



Optimal Quantities

- The value (utility) of a state s:

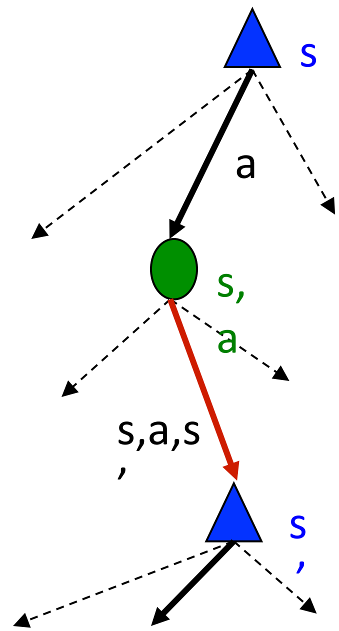
  $V^*(s)$ = expected utility starting in s and acting optimally

- The value (utility) of a q-state (s,a):

  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally

- The optimal policy:

  $\pi^*(s)$ = optimal action from state s

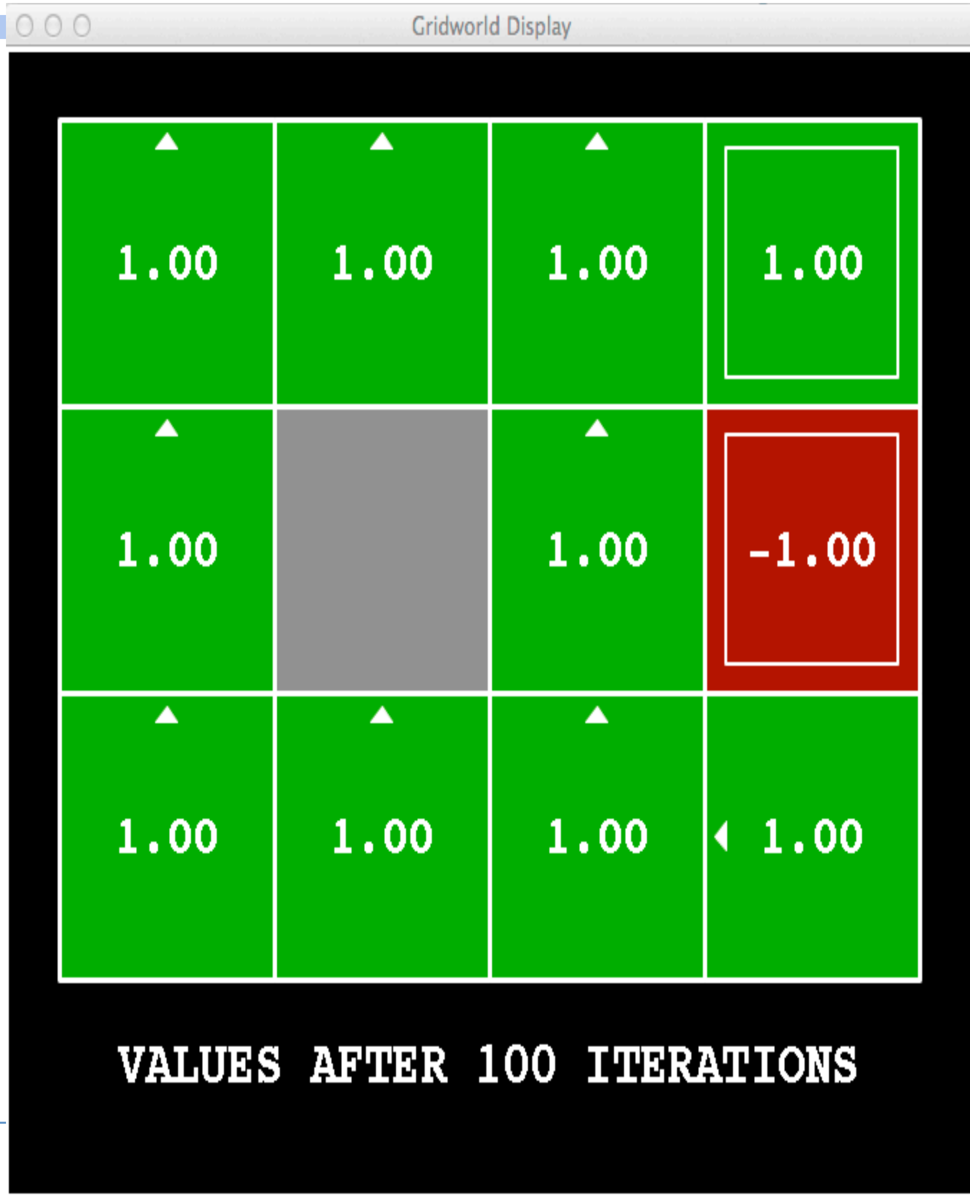


s is a *state*

(s, a) is a *q-state*

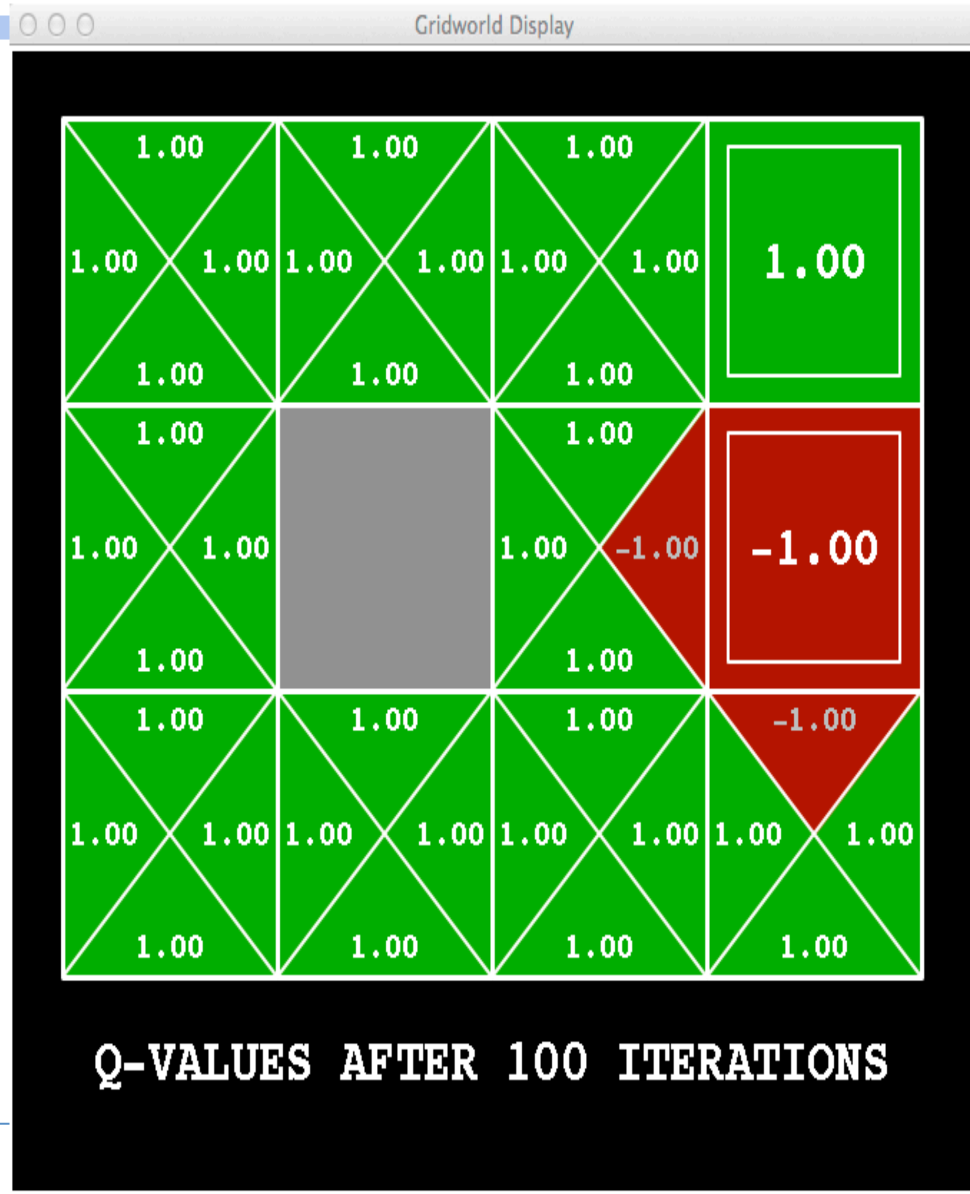
(s,a,s’) is a *transition*

Gridworld V Values



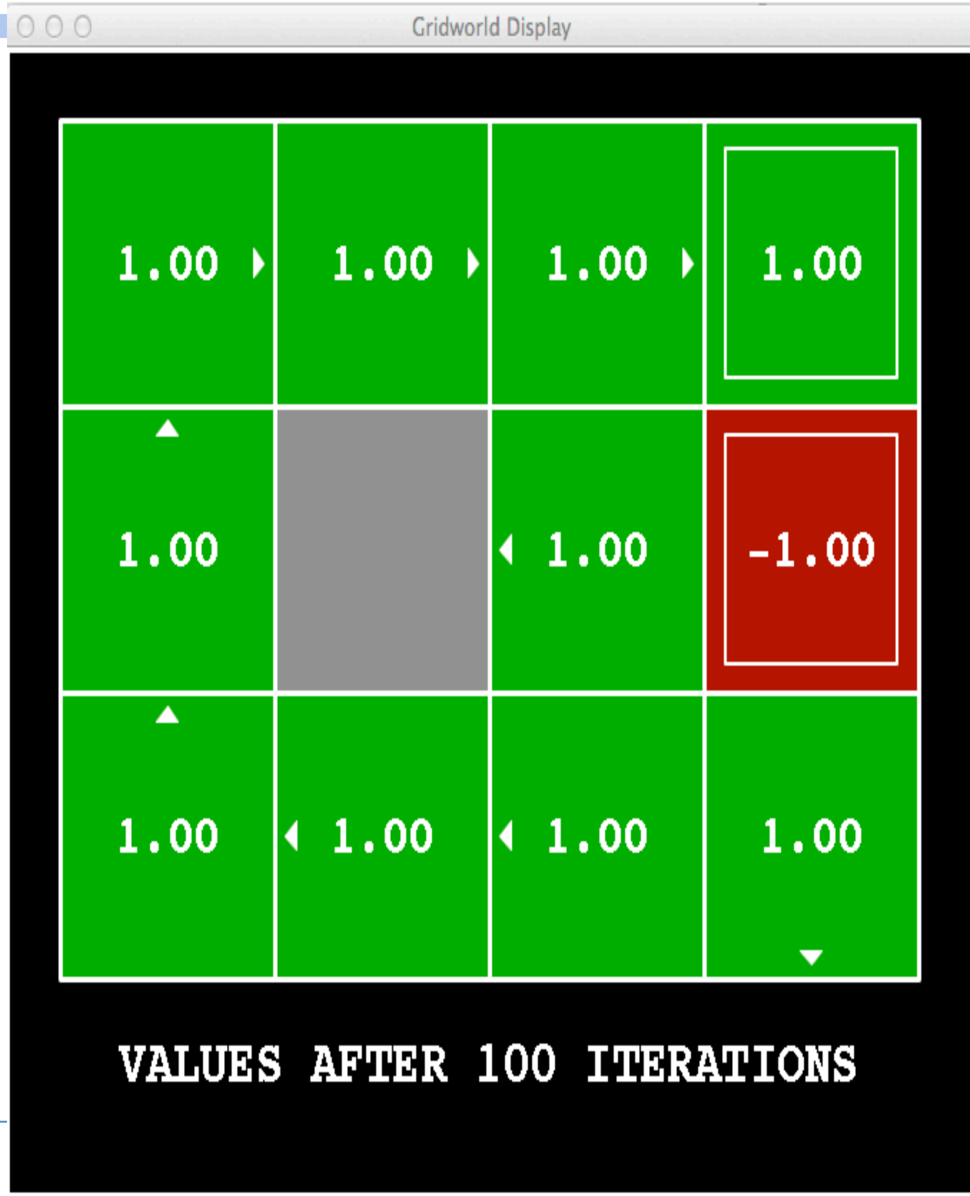
Noise = 0
Discount = 1
Living reward = 0

Gridworld Q Values



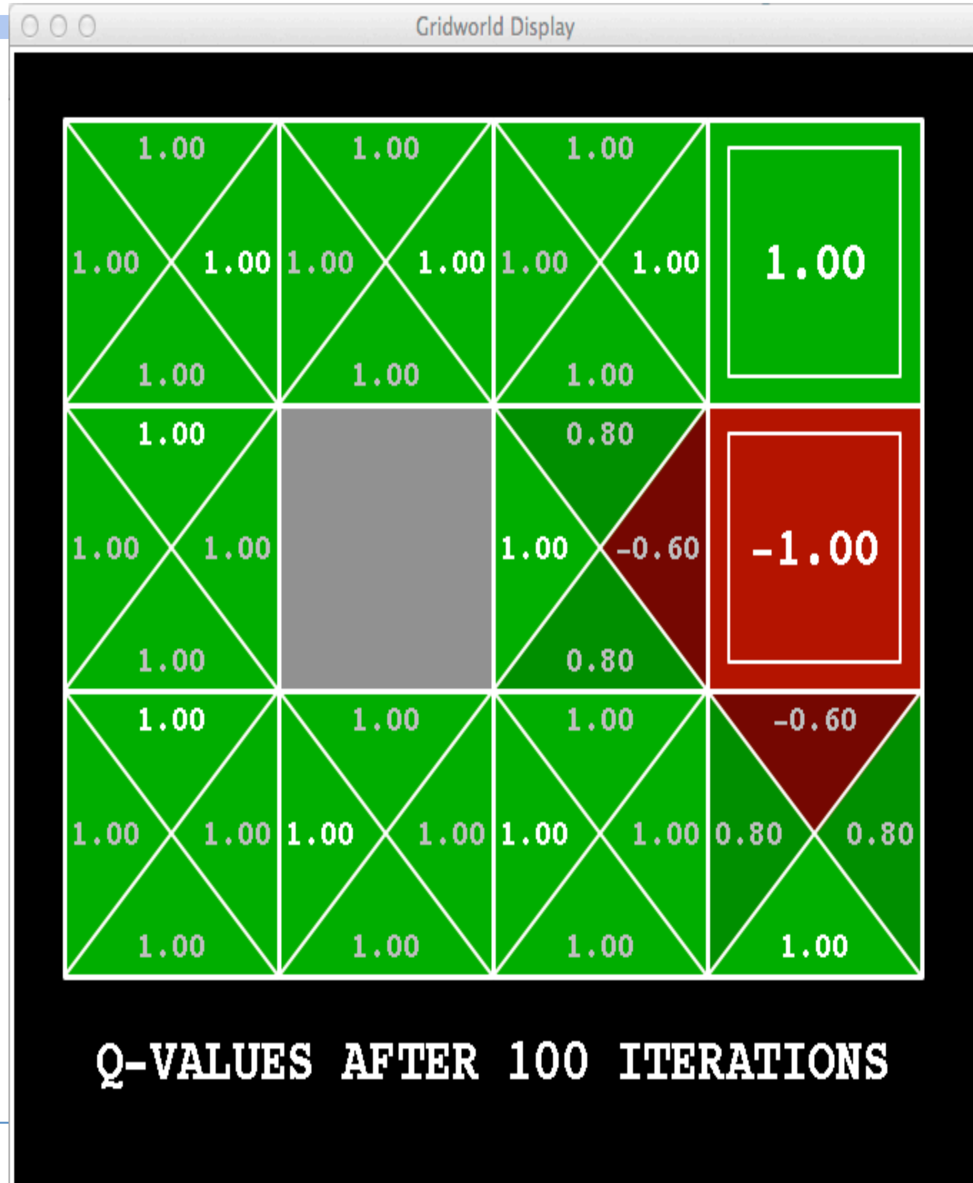
Noise = 0
Discount = 1
Living reward = 0

Gridworld V Values



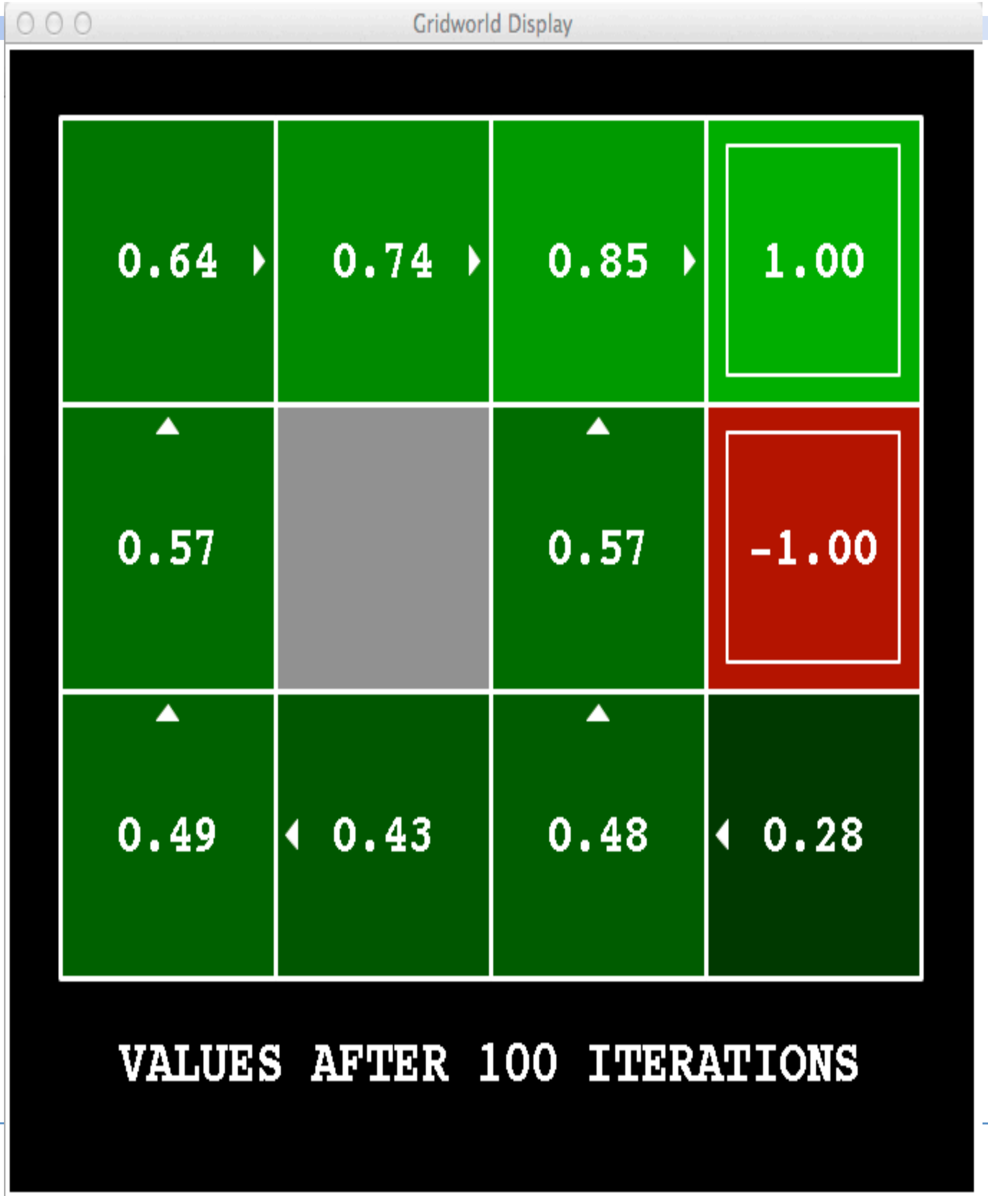
Noise = 0.2
Discount = 1
Living reward = 0

Gridworld Q Values



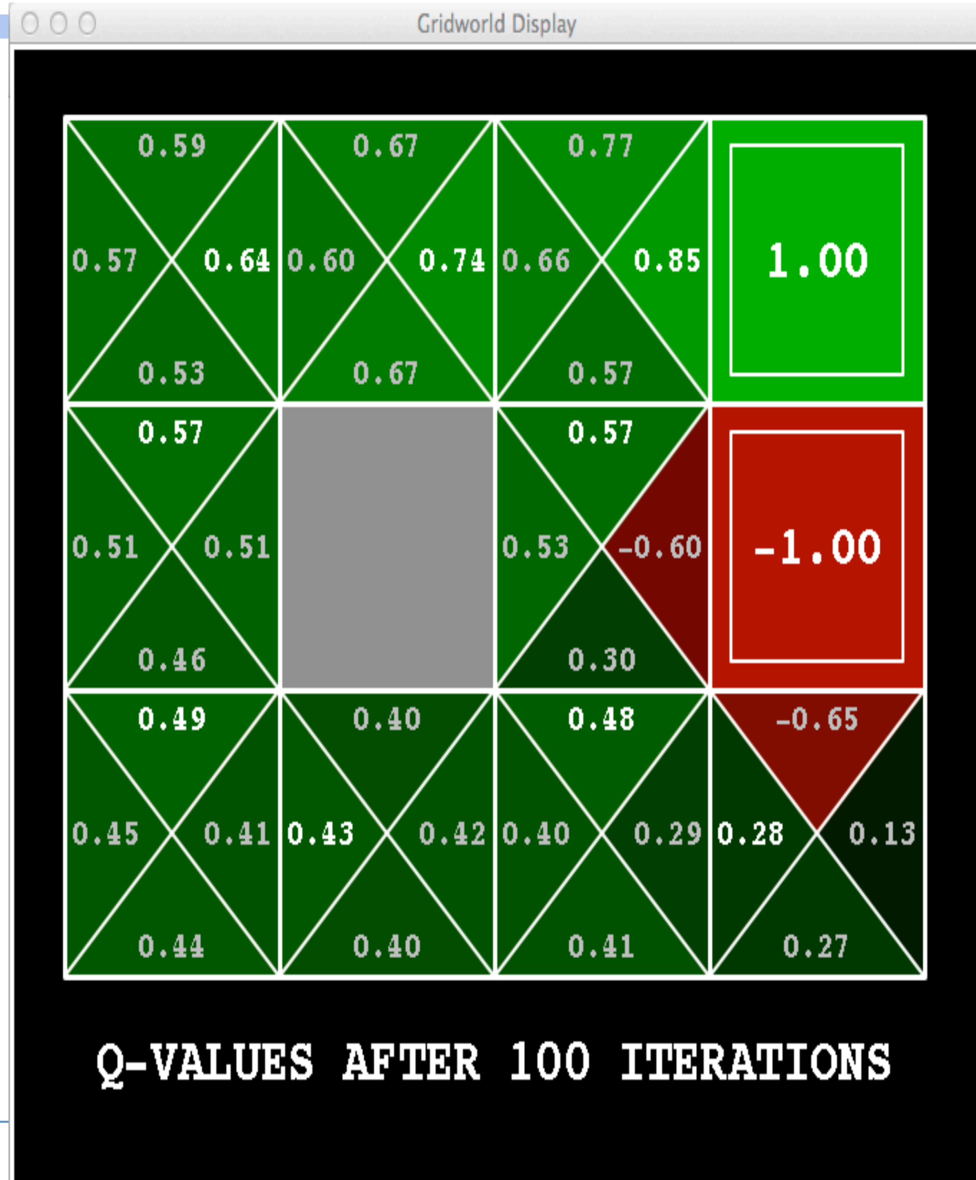
Noise = 0.2
Discount = 1
Living reward = 0

Gridworld V Values



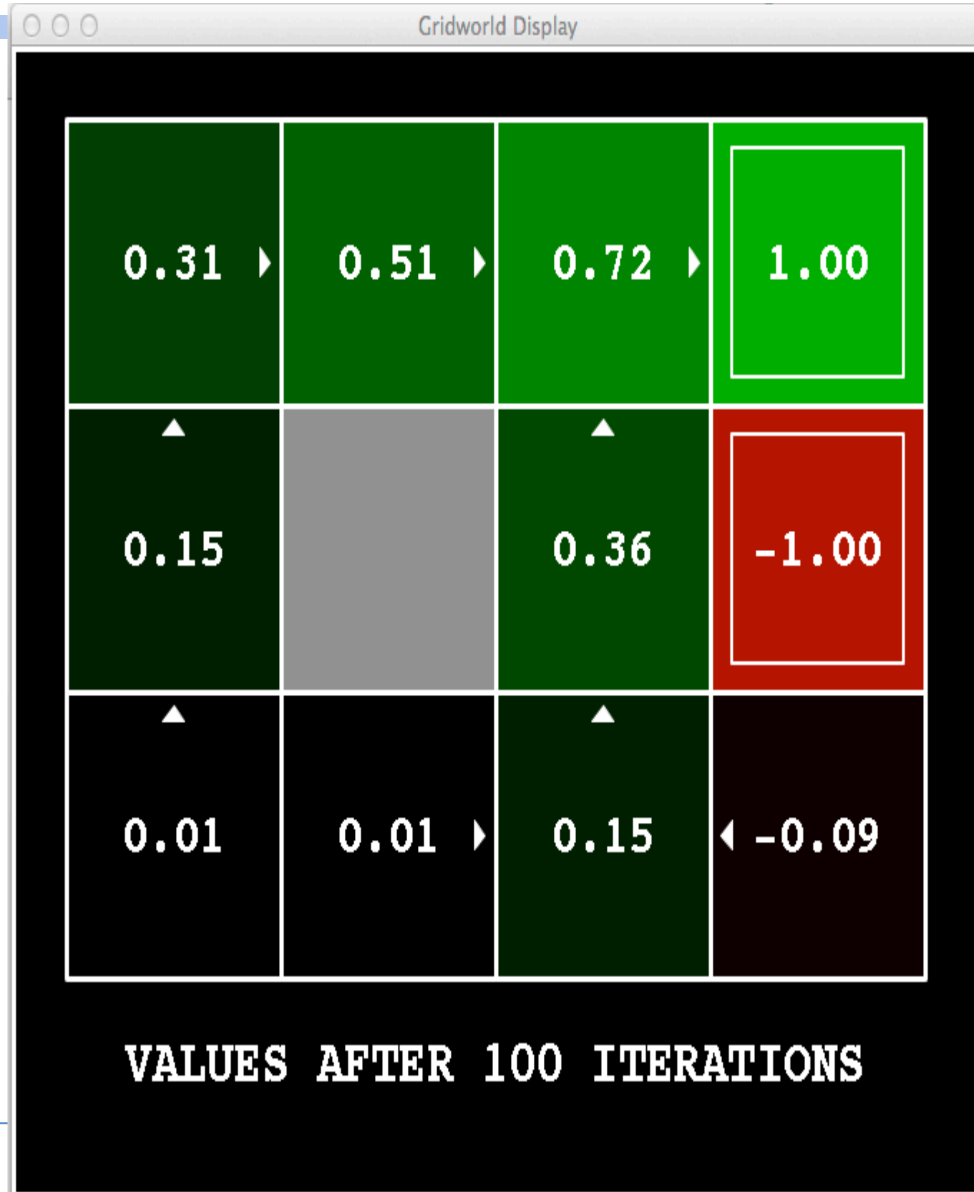
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld Q Values



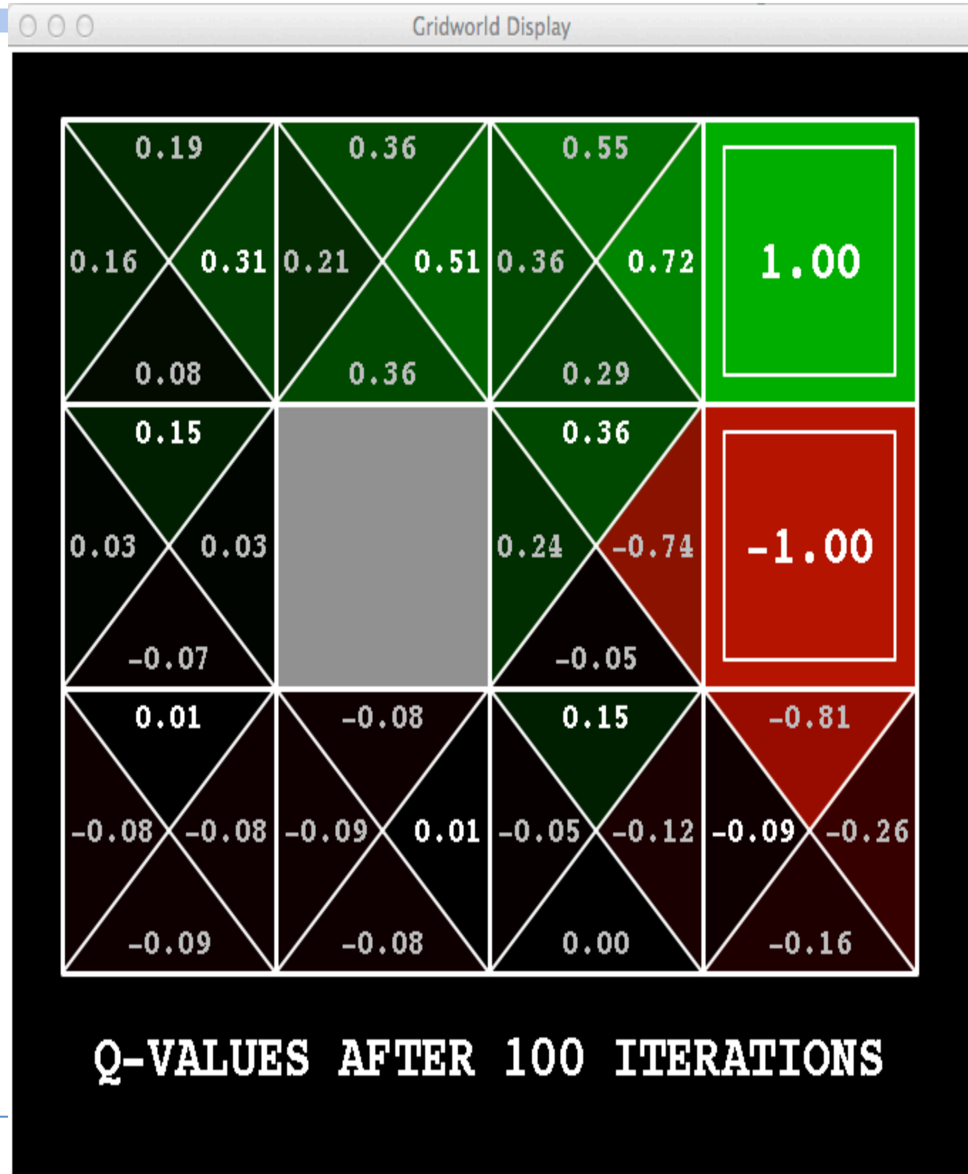
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld V Values



Noise = 0.2
Discount = 0.9
Living reward = -0.1

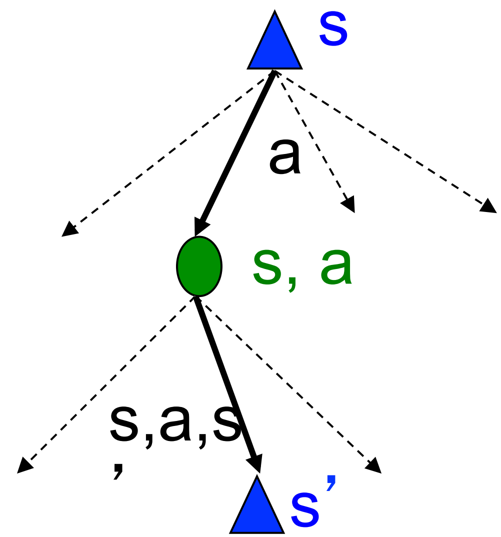
Gridworld Q Values



Noise = 0.2
Discount = 0.9
Living reward = -0.1

Values of States

- Fundamental operation: compute the (expectimax) value of a state
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is what expectimax computed!



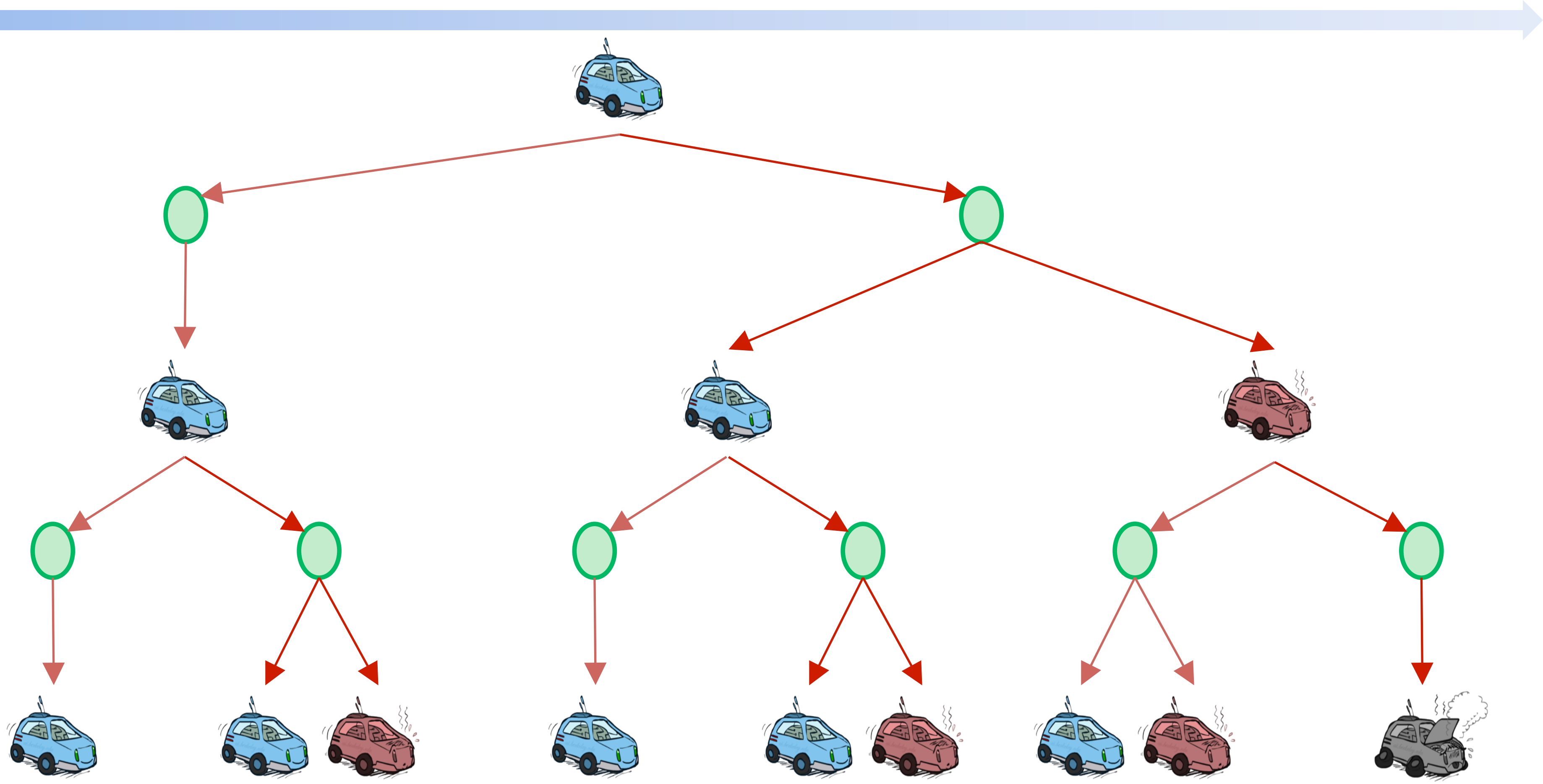
- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

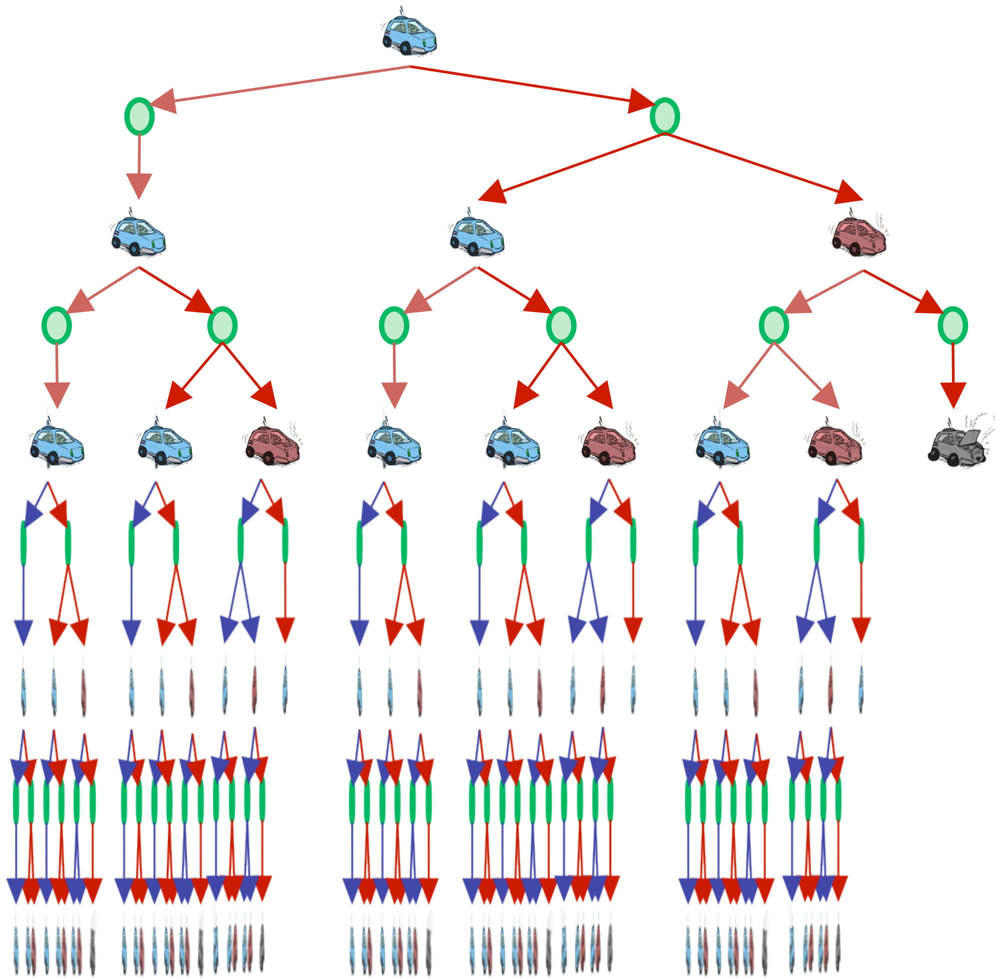
$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

Racing Search Tree

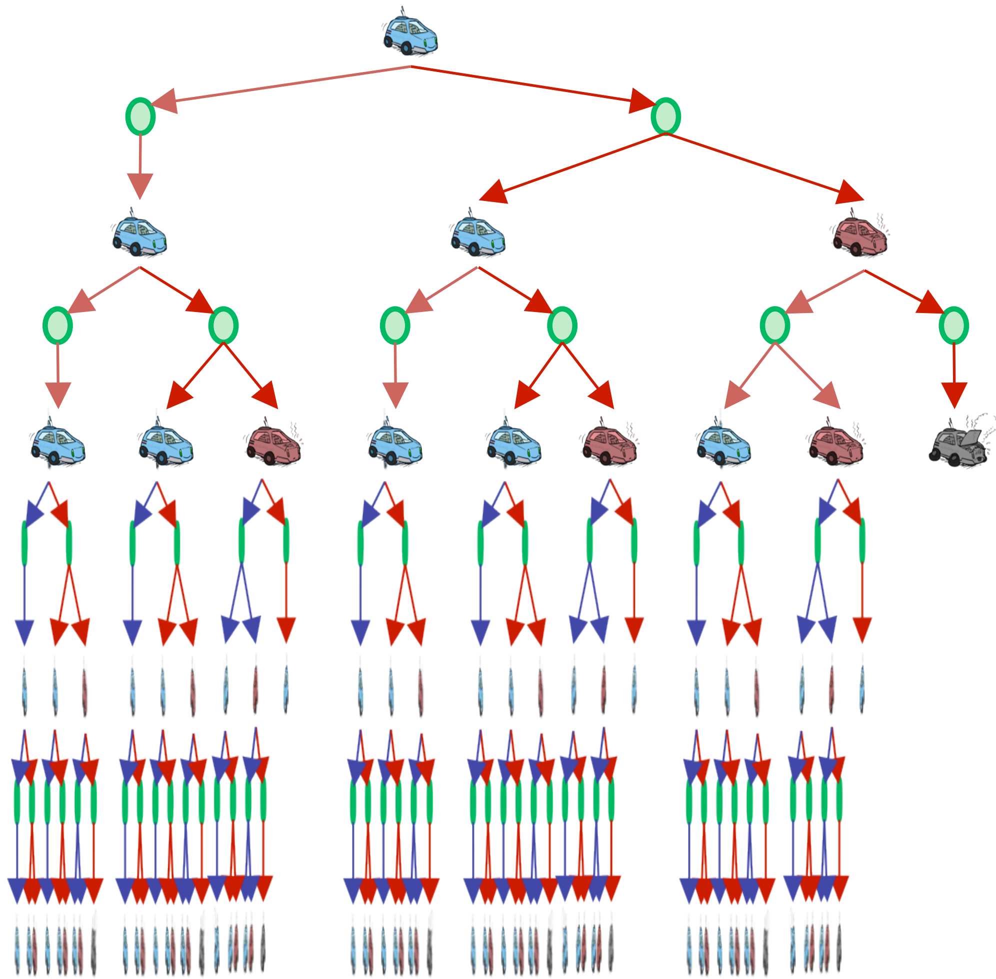


Racing Search Tree

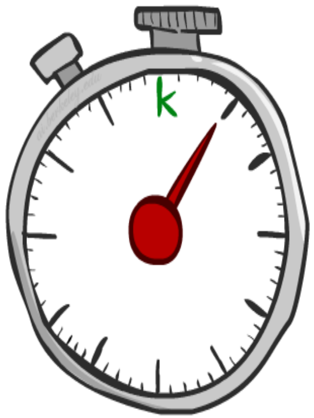


Racing Search Tree

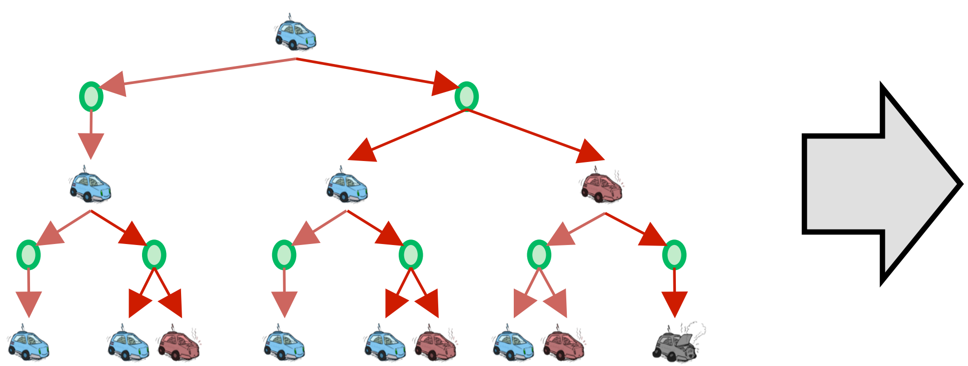
- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$

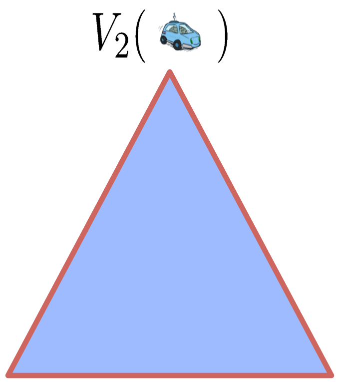


Time-Limited Values

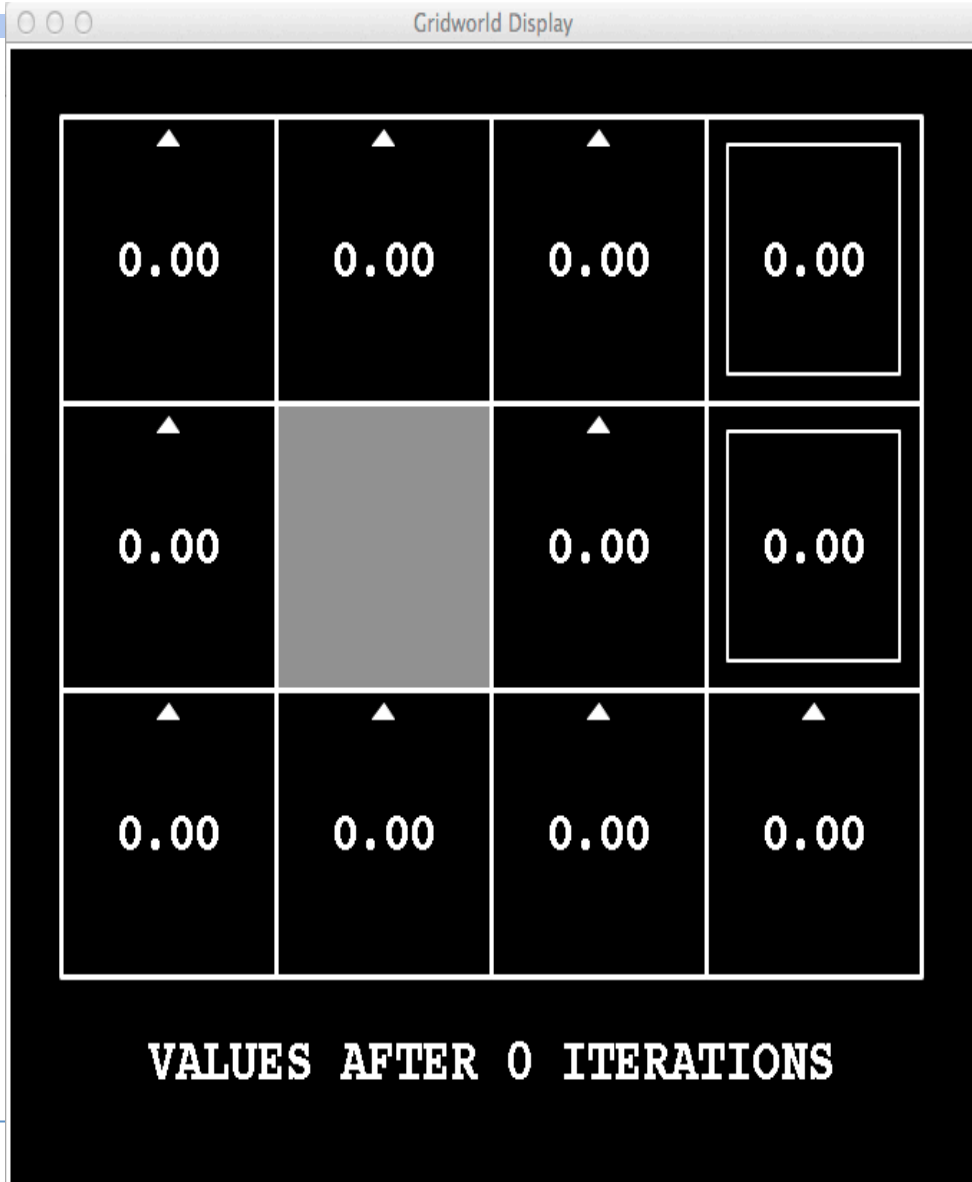


- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s





k=0

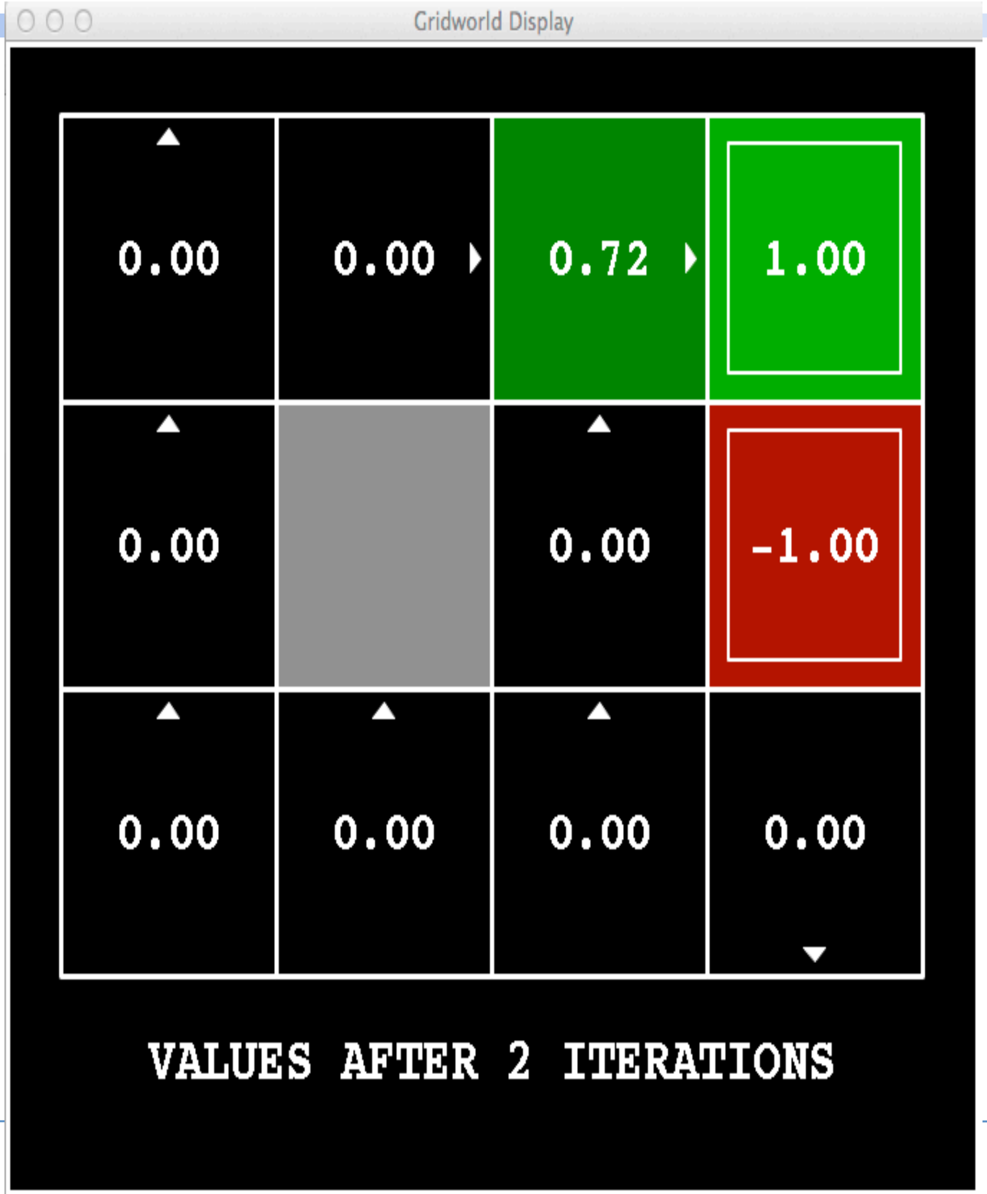


Noise = 0.2
Discount = 0.9
Living reward = 0

k=1

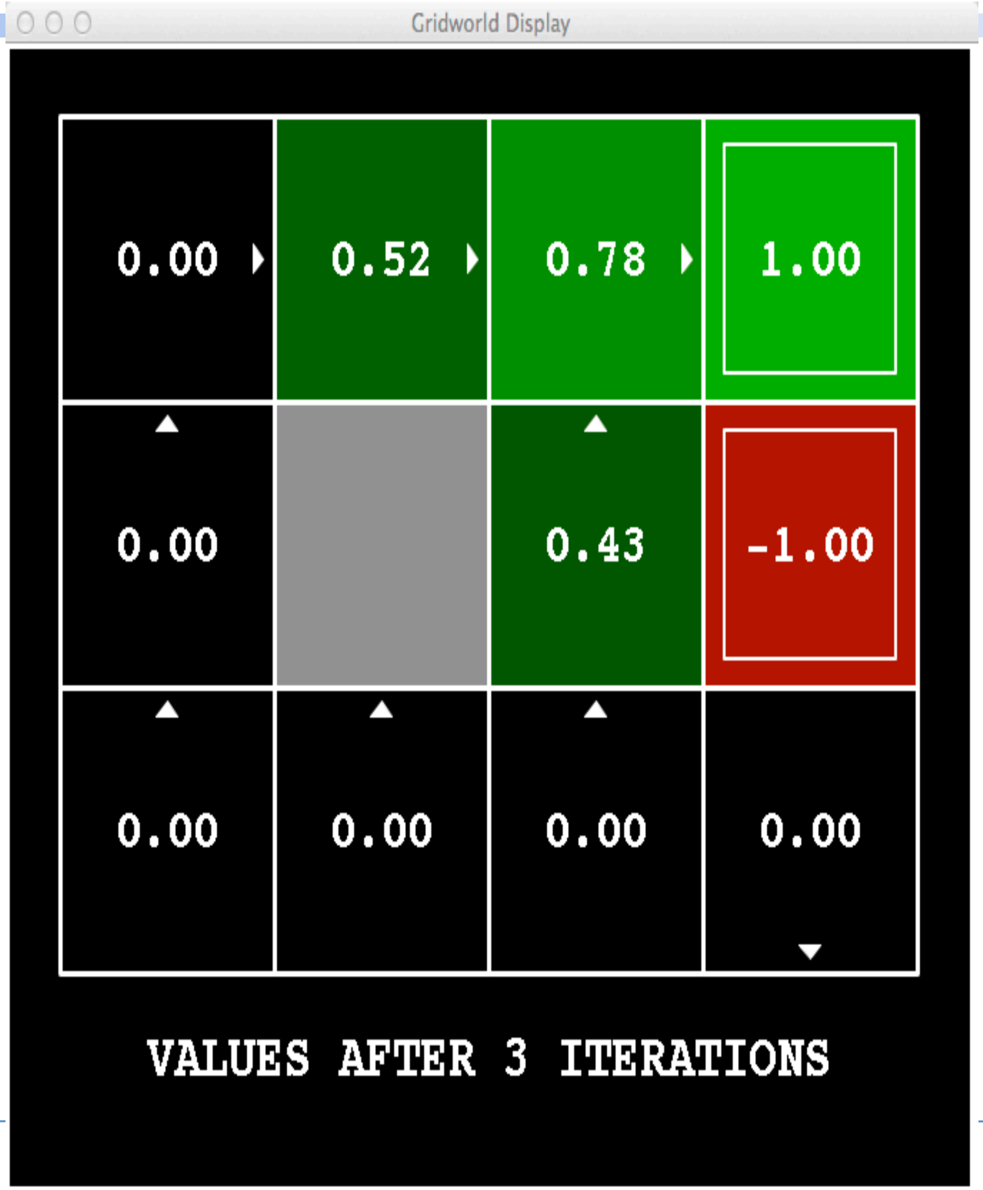
Noise = 0.2
Discount = 0.9
Living reward = 0

k=2



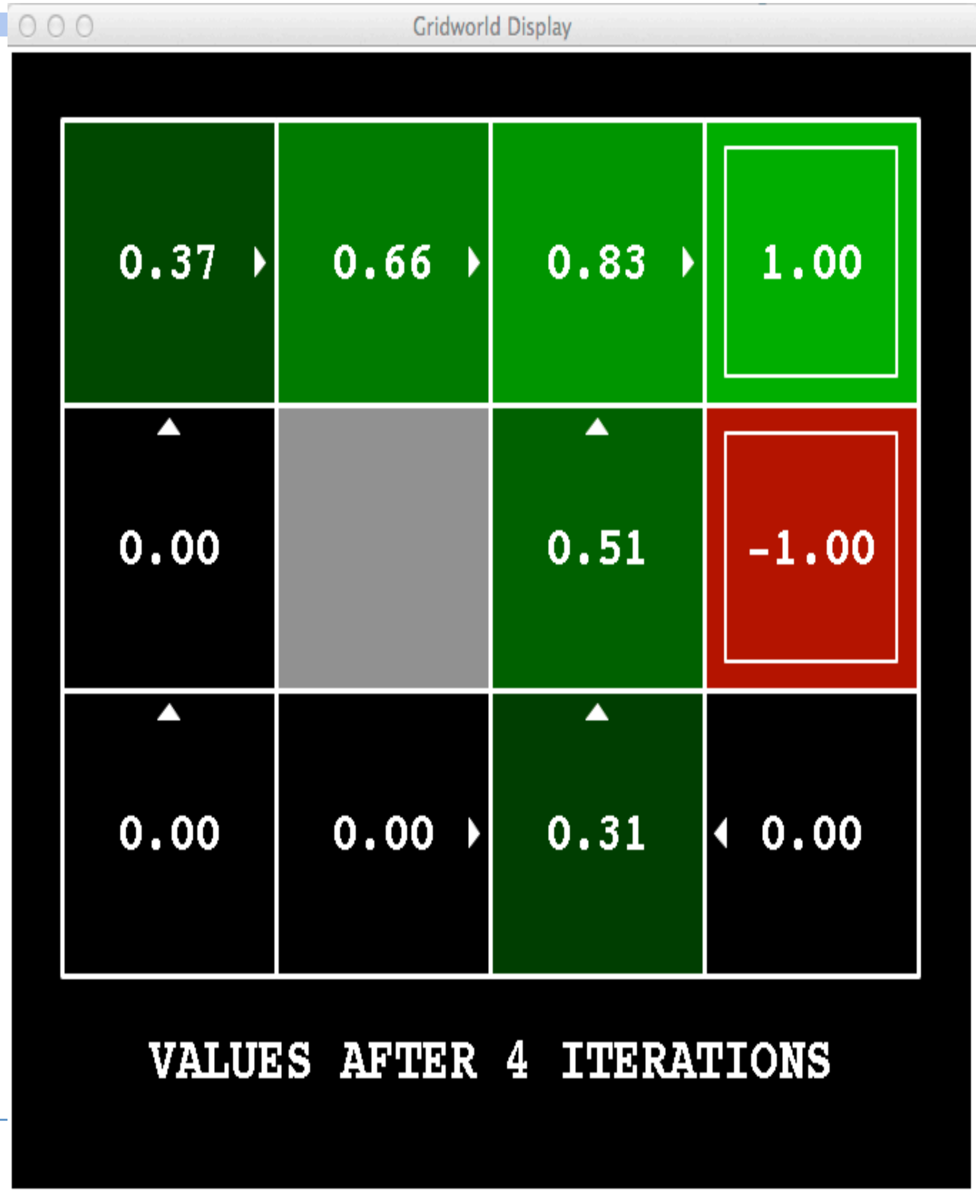
Noise = 0.2
Discount = 0.9
Living reward = 0

k=3



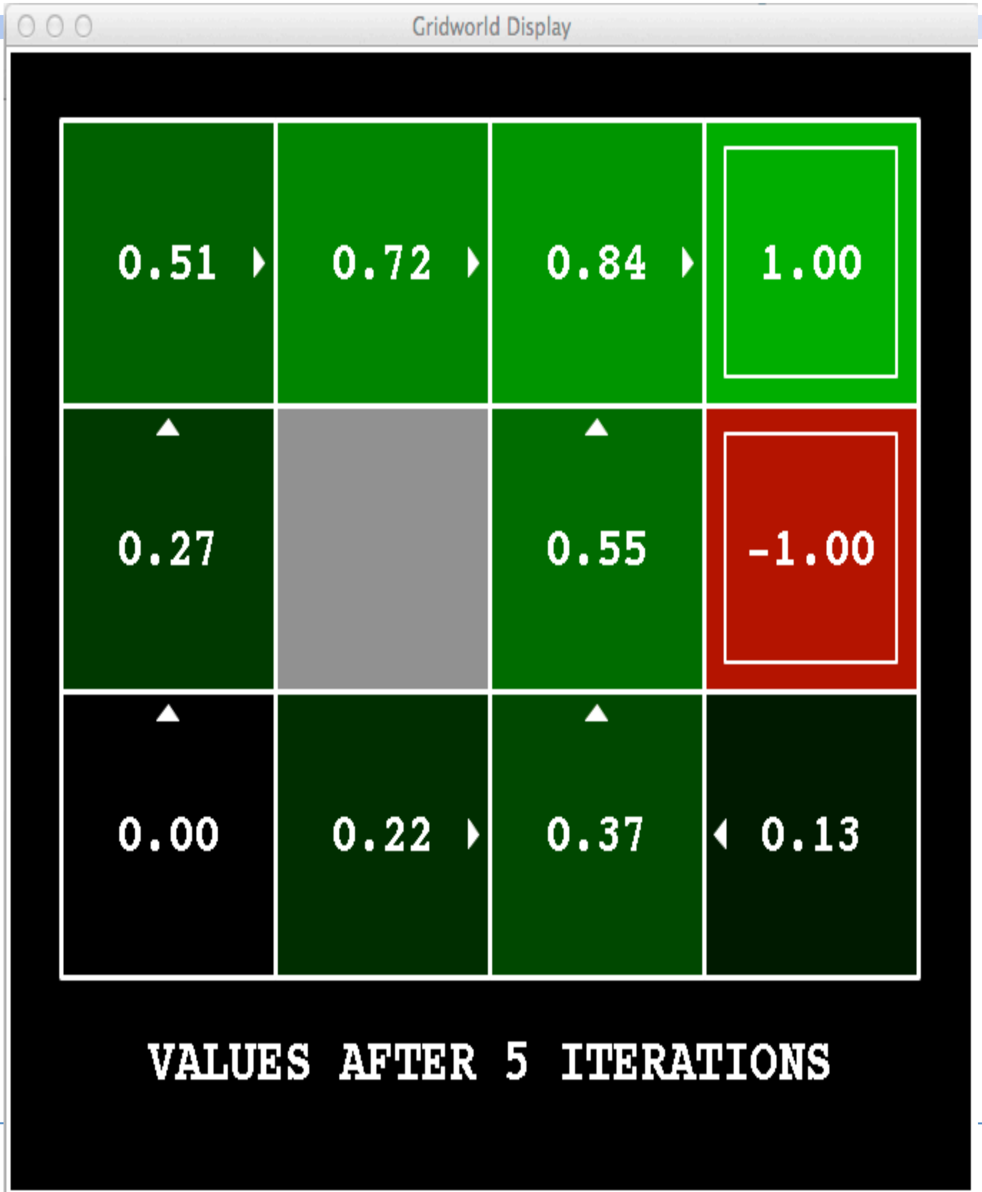
Noise = 0.2
Discount = 0.9
Living reward = 0

k=4



Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



Noise = 0.2
Discount = 0.9
Living reward = 0

k=6

Noise = 0.2
Discount = 0.9
Living reward = 0

k=7

Noise = 0.2
Discount = 0.9
Living reward = 0

k=8

Noise = 0.2
Discount = 0.9
Living reward = 0

k=9

Noise = 0.2
Discount = 0.9
Living reward = 0

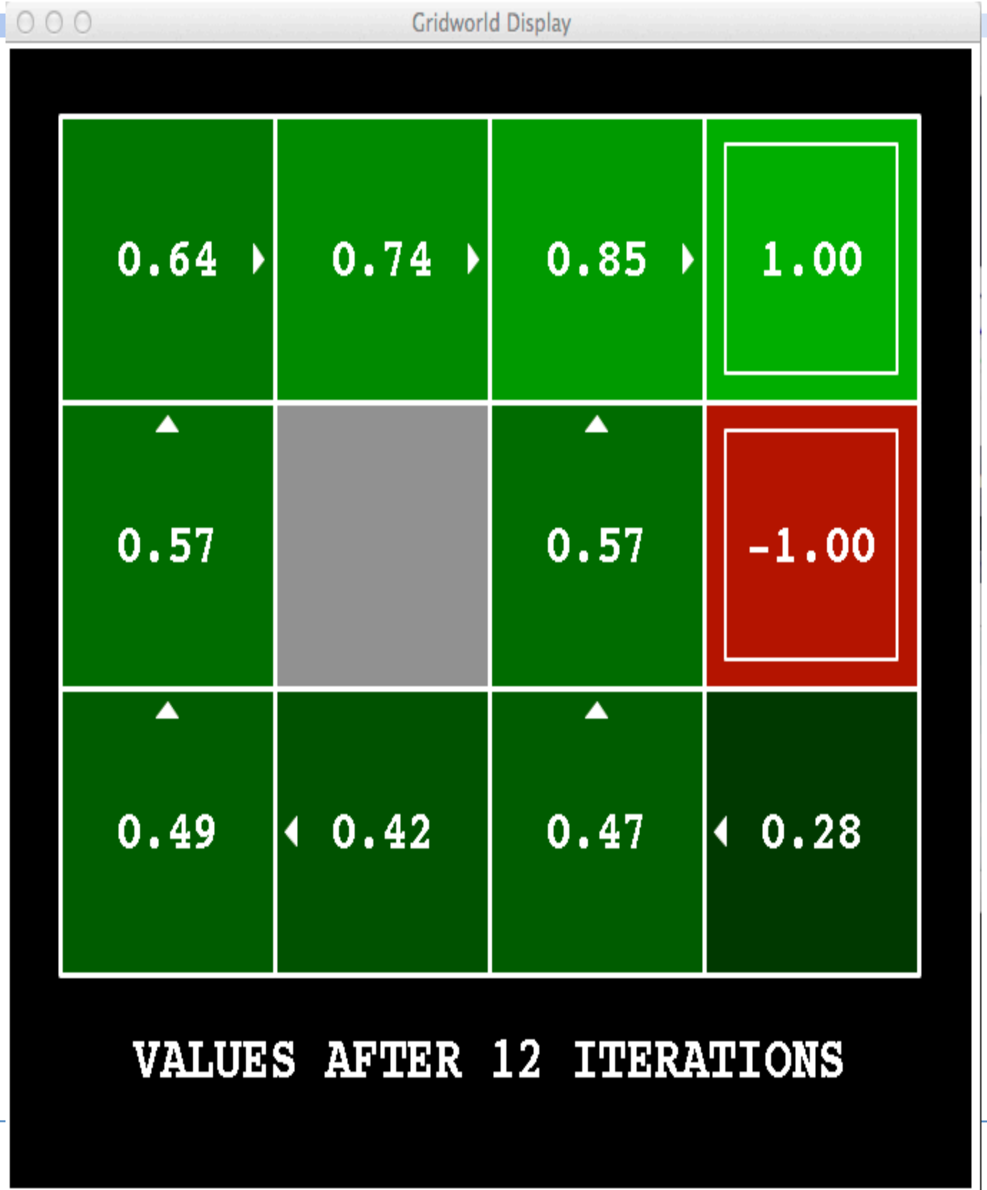
k=10

Noise = 0.2
Discount = 0.9
Living reward = 0

k=11

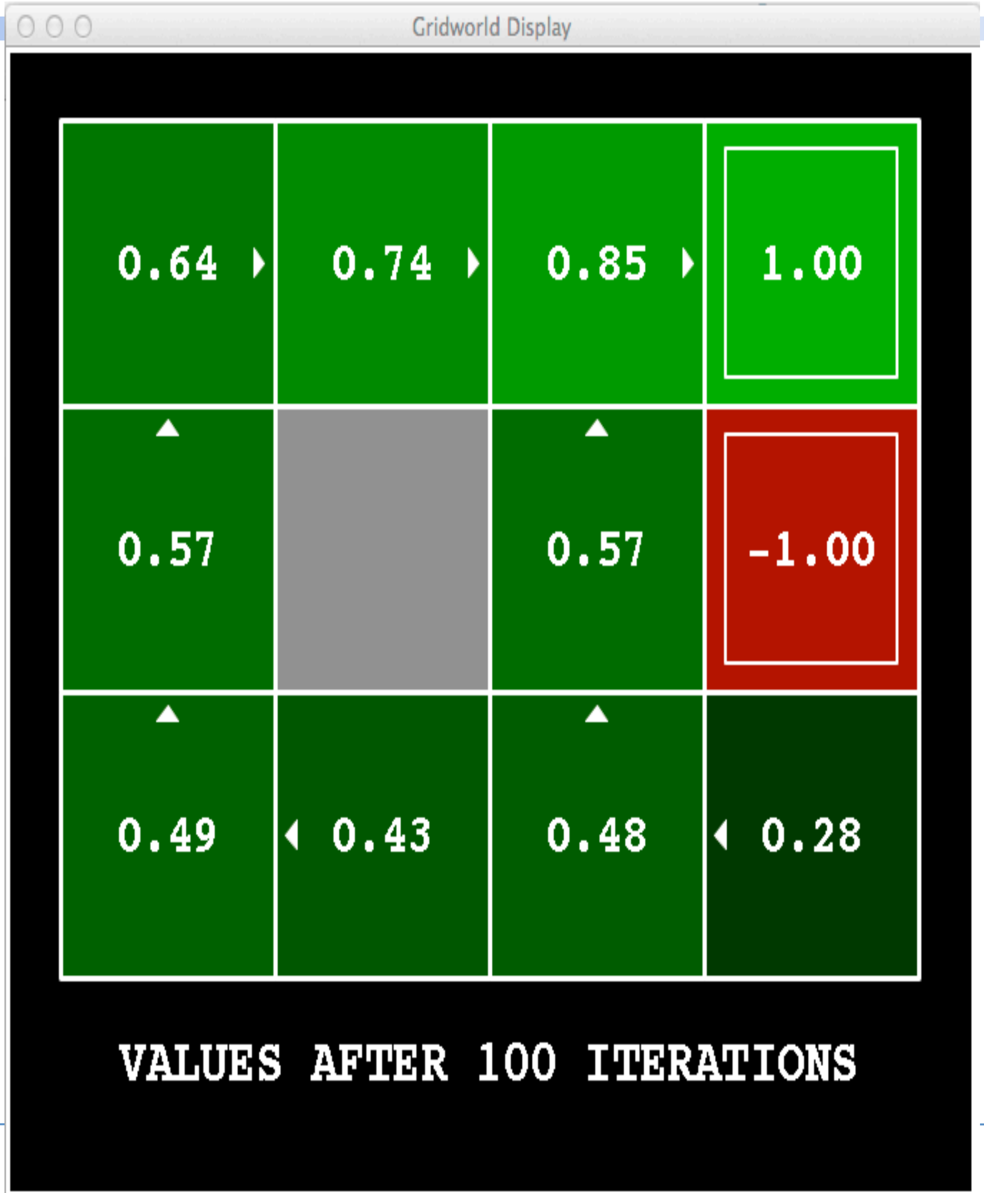
Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



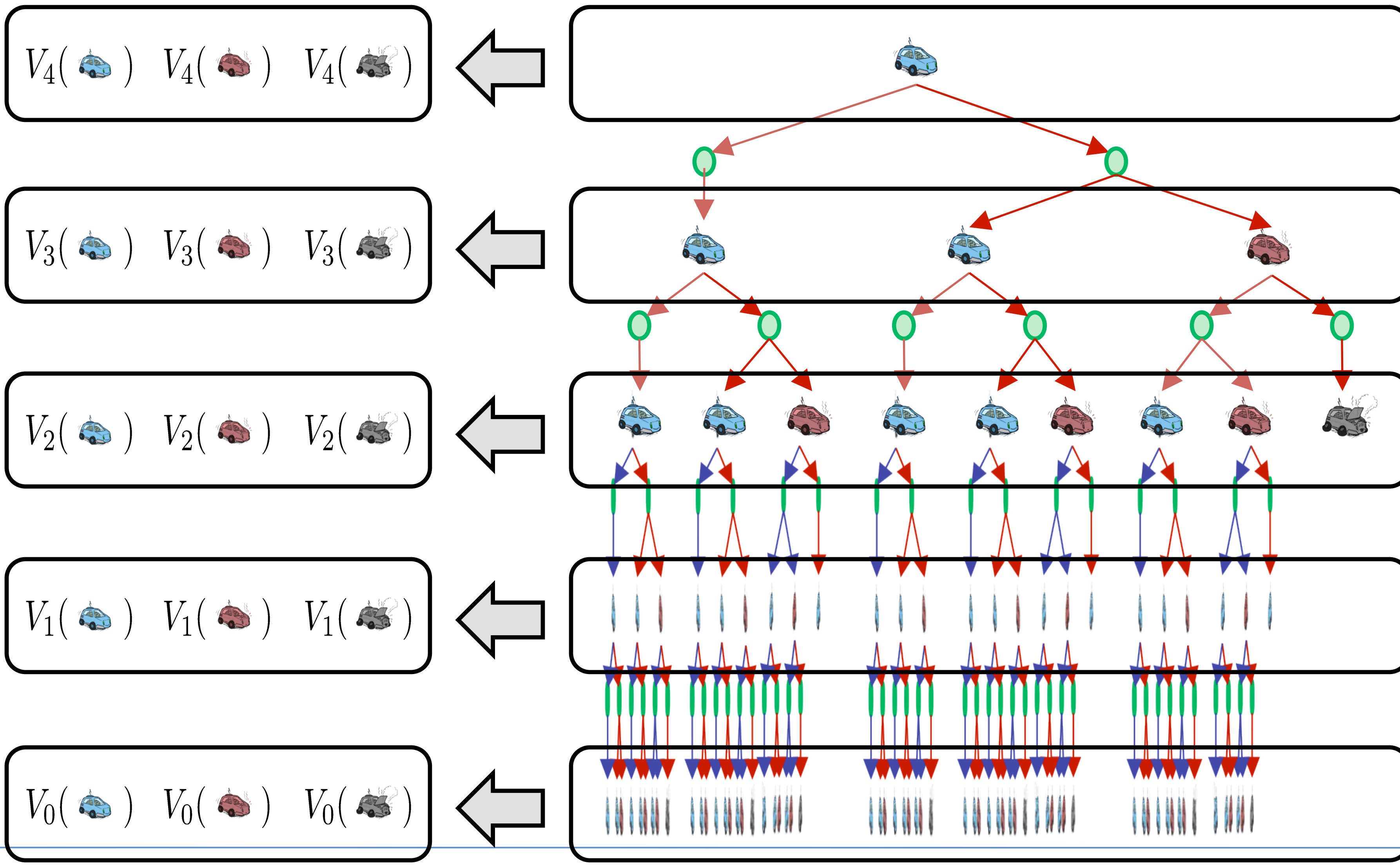
Noise = 0.2
Discount = 0.9
Living reward = 0

k=100

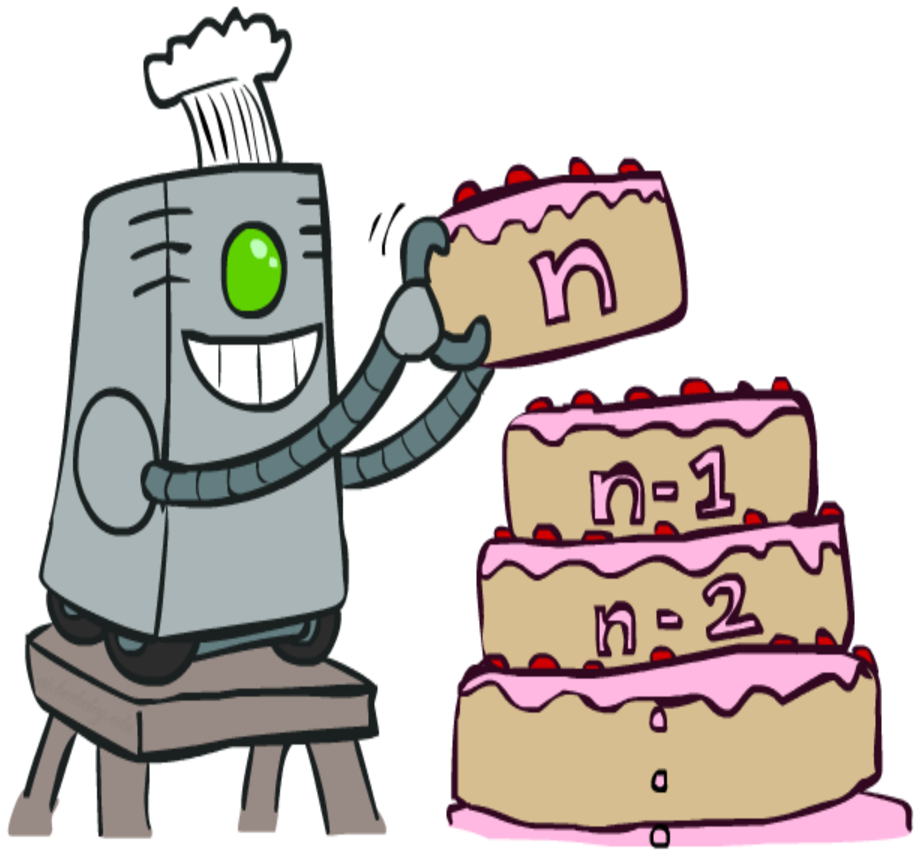


Noise = 0.2
Discount = 0.9
Living reward = 0

Computing Time-Limited Values



Value Iteration

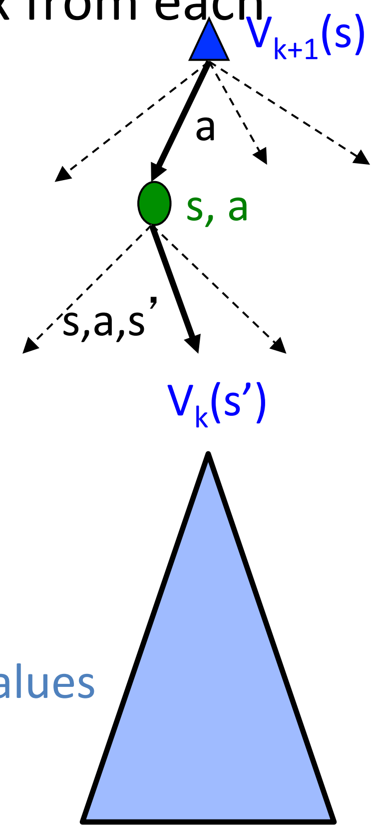


Key Algorithm: Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero

- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$



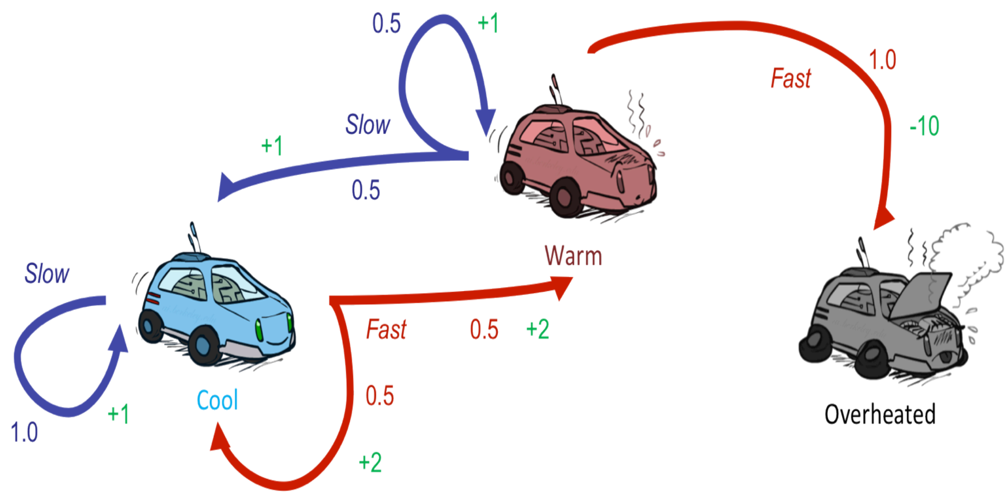
- Repeat until convergence

- Complexity of each iteration: $O(S^2A)$

- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

Example: Value Iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

https://www.cs.ubc.ca/~poole/demos/mdp/vi.html

Convergence*

- How do we know the V_k vectors are going to converge?

- Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values
  - Terminal state or depth limit (evaluation function)

- Case 2: If the discount is less than 1
  - Sketch: For any state V_k and V_{k+1} can be viewed as depth k+1 expectimax results in nearly identical search trees
  - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
  - That last layer is at best all R_{MAX}
  - It is at worst R_{MIN}
  - But everything is discounted by γ^k that far out
  - So V_k and V_{k+1} are at most $\gamma^k \max|R|$ different
  - So as k increases, the values converge