Reasoning Under Uncertainty

With slides from Dan Klein and Pieter Abbeel

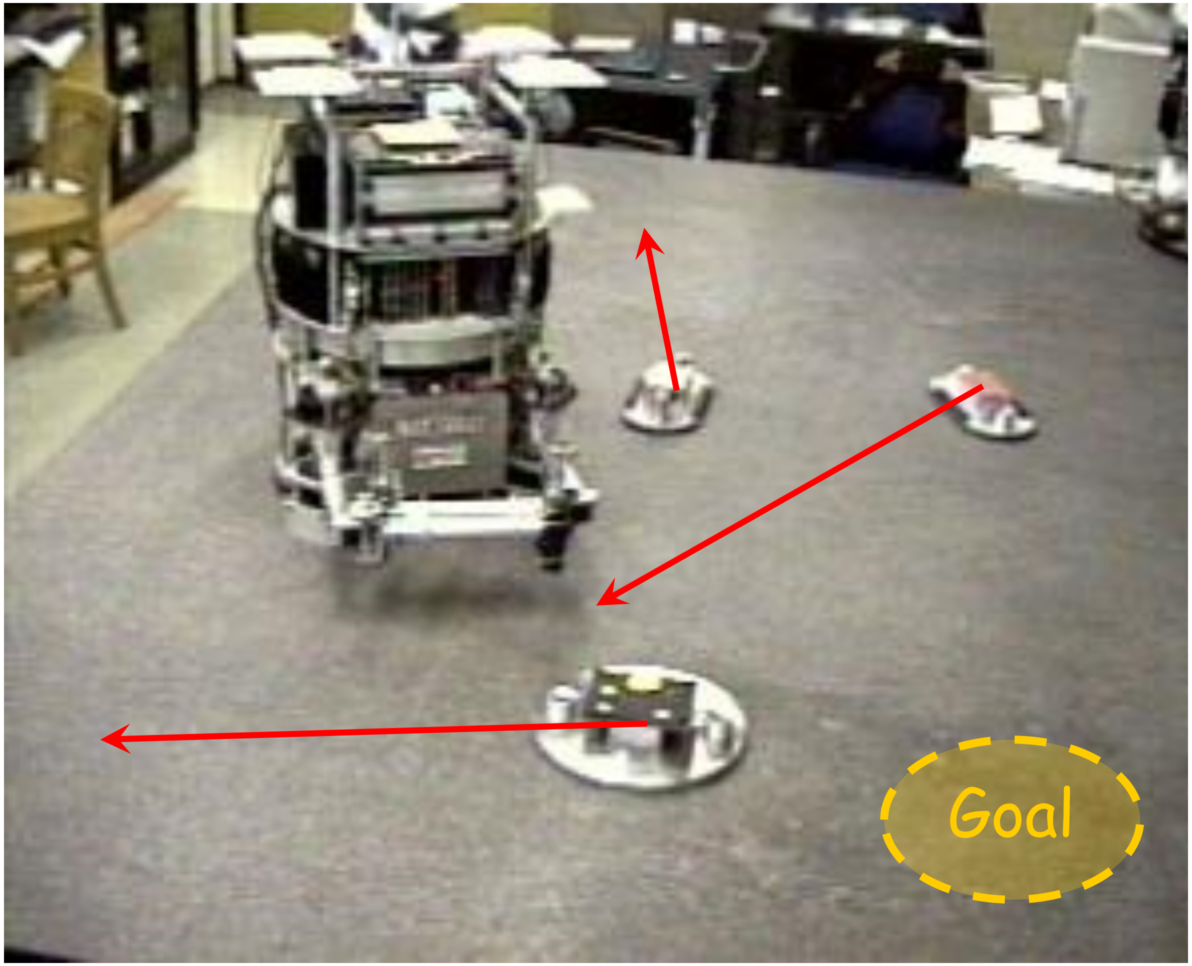
Rough Plan (Next 2 weeks)

- Acting under uncertainty
- **Today:** Probabilistic models (single decision)
  - Inference
  - Begin Bayesian reasoning
- **Next week:** uncertainty+time = sequential decisions
  - Bayesian reasoning
  - Approximate inference
  - Hidden Markov Models (HMMs)
- **Project 4:** Ghost Busters!

Uncertainty

- Uncertain input (sensors):
  https://www.youtube.com/watch?v=9OgPAyRUI3I
- Uncertainty in action (outcome):
  https://www.youtube.com/watch?v=g0TaYhjpOfo
- Dynamic environment (sensors + actions)
  https://www.youtube.com/watch?v=HacG_FWWPOw

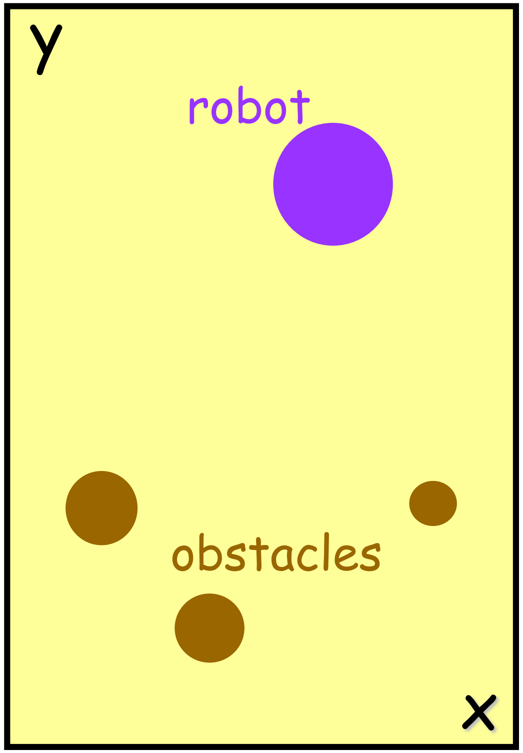
More Detailed Example: Robot Motion



A robot with imperfect sensing must reach a goal location among moving obstacles (dynamic world)

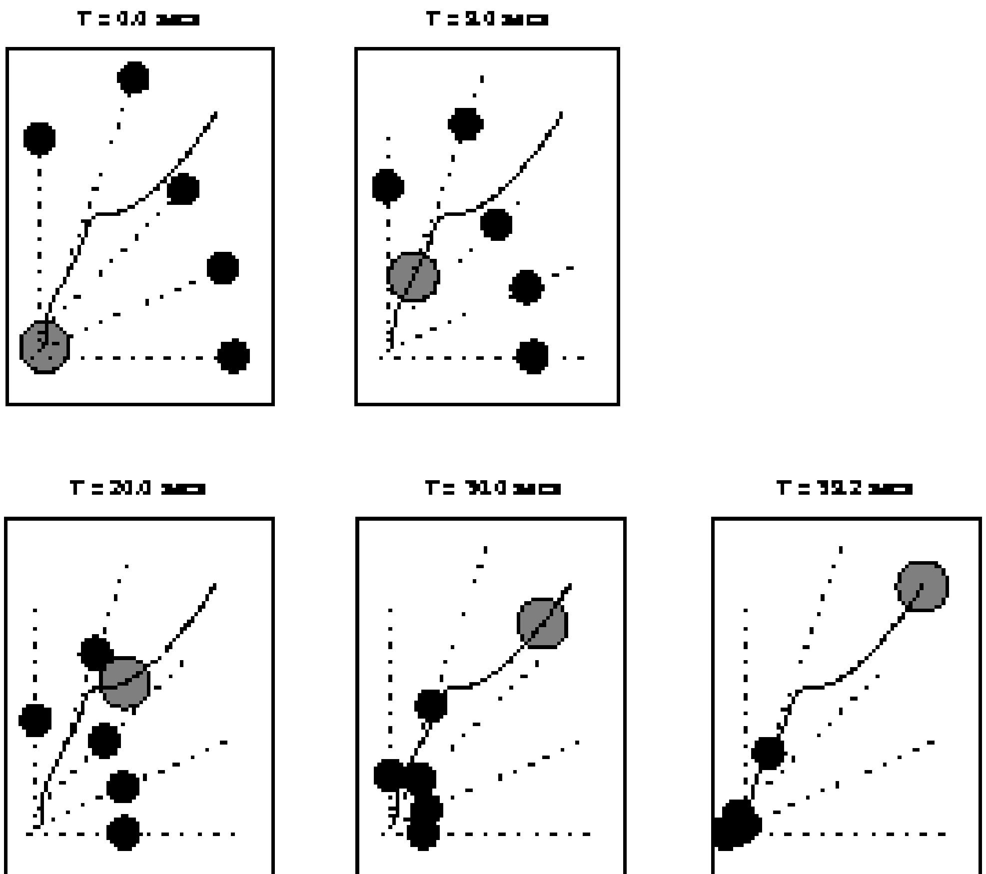
Model, Sensing, and Control

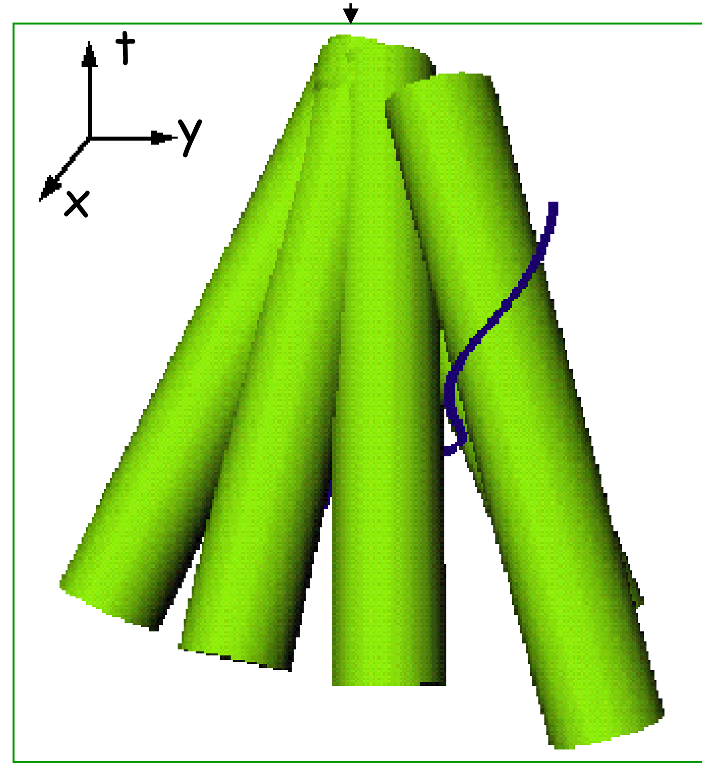
- The robot and the obstacles are represented as disks moving in the plane
- The position and velocity of each disc are measured by an overhead camera every 1/30 sec



Motion Planning

The robot plans its trajectories in configuration×time space using a probabilistic roadmap (PRM) method





Obstacle map to cylinders in configuration×time space

But executing this trajectory is likely to fail ...

1) The measured velocities of the obstacles are inaccurate
2) Tiny particles of dust on the table affect trajectories and contribute further to deviation

   $\rightarrow$ Obstacles are likely to deviate from their expected trajectories

3) Planning takes time, and during this time, obstacles keep moving

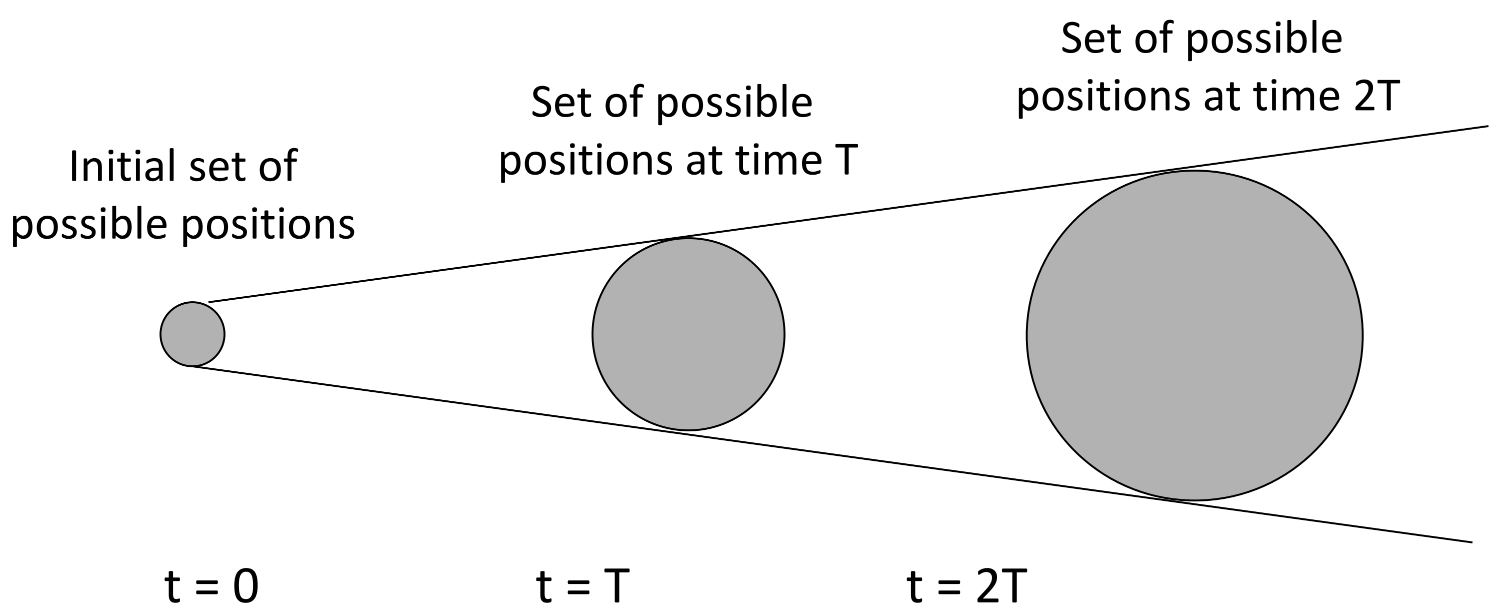
   $\rightarrow$ The computed robot trajectory is not properly synchronized with those of the obstacles

$\Rightarrow$ Planning must take both uncertainty in world state and time constraints into account

Dealing with Uncertainty

- The robot can handle uncertainty in an obstacle position by representing the set of all positions of the obstacle that the robot think possible at each time (belief state)

- For example, this set can be a disc whose radius grows linearly with time

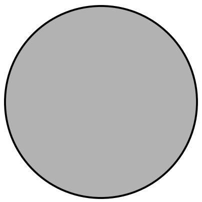


Dealing with Uncertainty

- The robot can handle uncertainty in an obstacle position by representing the set of all positions of the obstacle that the robot think possible at each time (belief state)

- For example, this set can be a disc whose radius grows linearly with time

The robot must plan to be outside this disc at time t = T

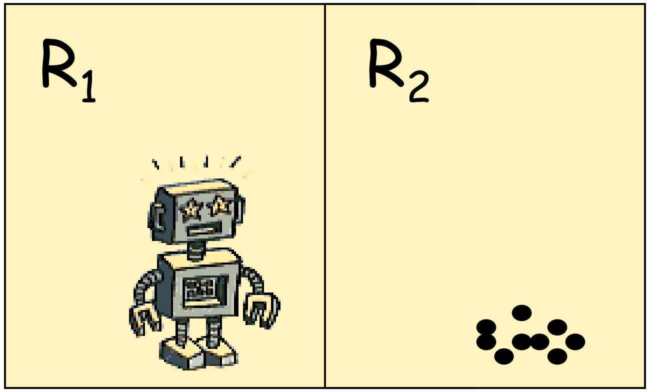


t = T

Imperfect Observation of the World

Observation of the world can be:

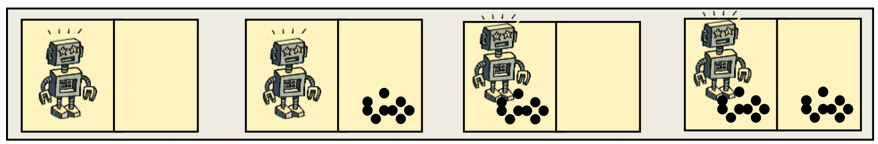
- Partial, e.g., a vision sensor can't see through obstacles (lack of percepts)



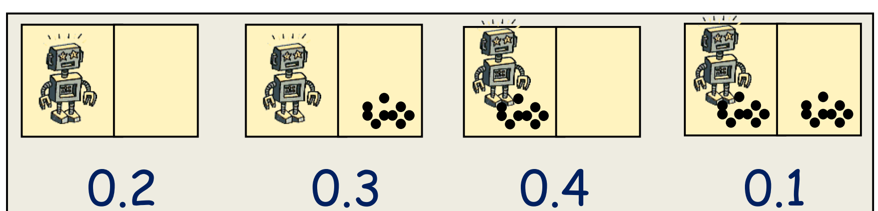
The robot may not know whether there is dust in room R2

Definition: Belief State

- In the presence of non-deterministic **sensory uncertainty**, an agent belief state represents all the states of the world that it thinks are possible at a given time or at a given stage of reasoning

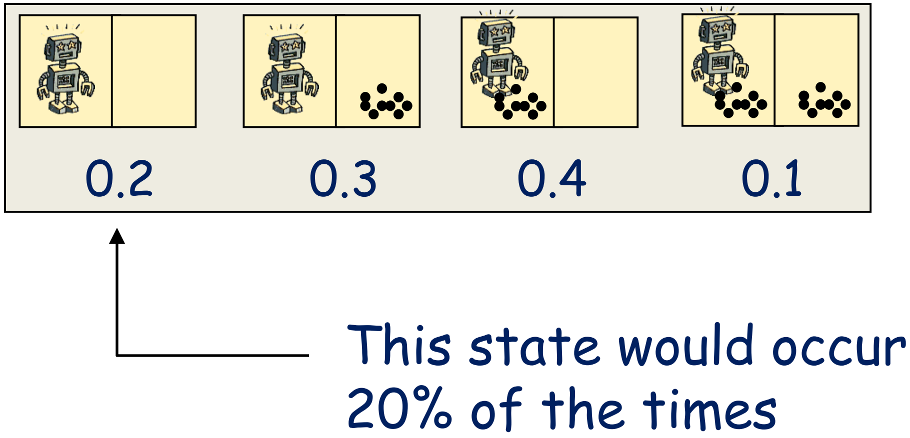


- In the probabilistic model of uncertainty, a probability is associated with each state to measure its likelihood to be the actual state



What do probabilities mean?

- Probabilities have a natural frequency interpretation
- The agent believes that if it was able to return many times to a situation where it has the same belief state, then the actual states in this situation would occur at a relative frequency defined by the probabilistic distribution



Belief State: Example

- Consider a world where a dentist agent D meets a new patient P
- D is interested in only one thing: whether P has a cavity, which D models using the proposition Cavity
- Before making any observation, D's belief state is:

Cavity	$\neg$ Cavity
p	1-p

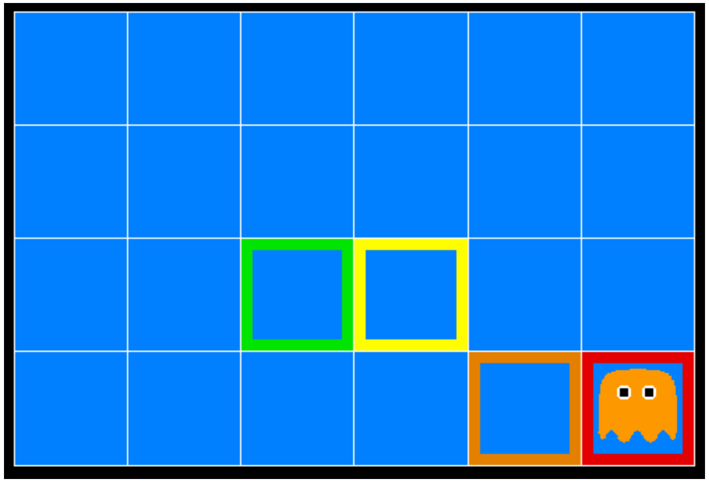
- This means that D believes that a fraction p of patients have cavities

Where do probabilities come from?

- Frequencies observed in the past, e.g., by the agent, its designer, or others
- Symmetries, e.g.:
  - If I roll a dice, each of the 6 outcomes has probability 1/6
- Subjectivism, e.g.:
  - If I drive on Highway 280 at 120mph, I will get a speeding ticket with probability 0.6
  - Principle of indifference: If there is no knowledge to consider one possibility more probable than another, give them the same probability

Inference in Ghostbusters

- A ghost is in the grid somewhere
- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green

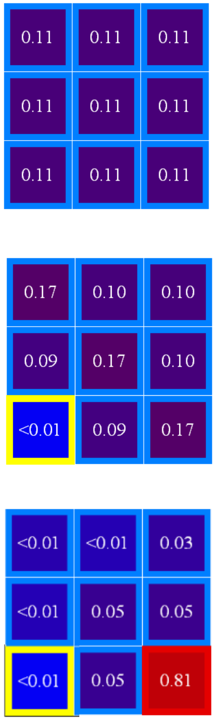


- Sensors are noisy, but we know P(Color | Distance)

P(red | 3)	P(orange | 3)	P(yellow | 3)	P(green | 3)
0.05	0.15	0.5	0.3

Uncertainty

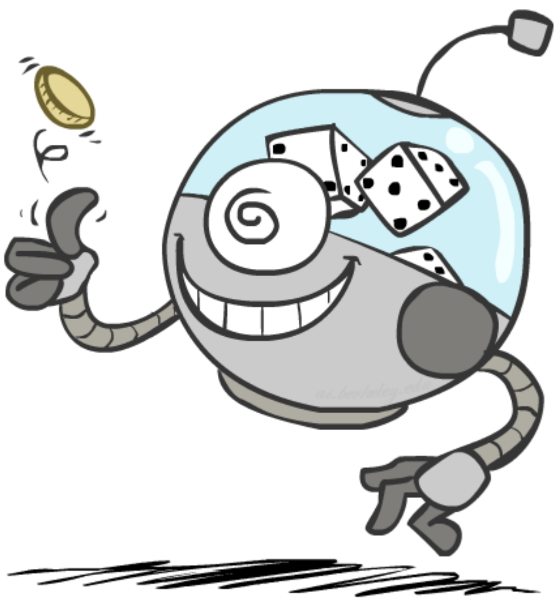
- General situation:
  - **Observed variables (evidence)**: Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
  - **Unobserved variables**: Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
  - **Model**: Agent knows something about how the known variables relate to the unknown variables



- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge

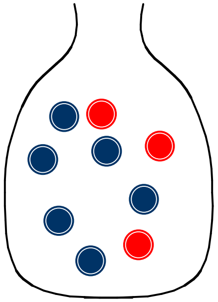
Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters
- Like variables in a CSP, random variables have domains
  - R in {true, false}   (often write as {+r, -r})
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), …}



Probability Review

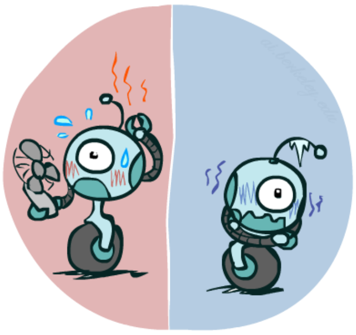
- Bag with 10 marbles: 3 red, 7 blue



  - Reach in, take one, put it back
  - Repeat lots of times.
  - What fraction red? About .3
  - P(red) = .3

Probability Distributions

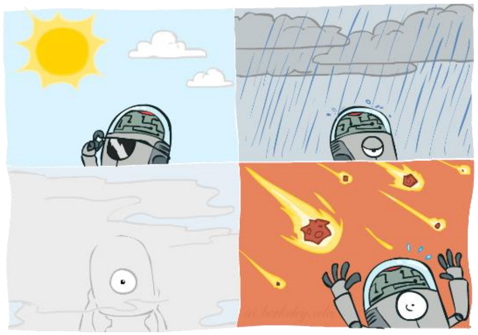
- Associate a probability with each value
  - Temperature:



$P(T)$

T	P
hot	0.5
cold	0.5

- Weather:



$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distribution

- The probability for each value of a random variable

  if color = (red, blue)

  P(color) = (.3, .7)

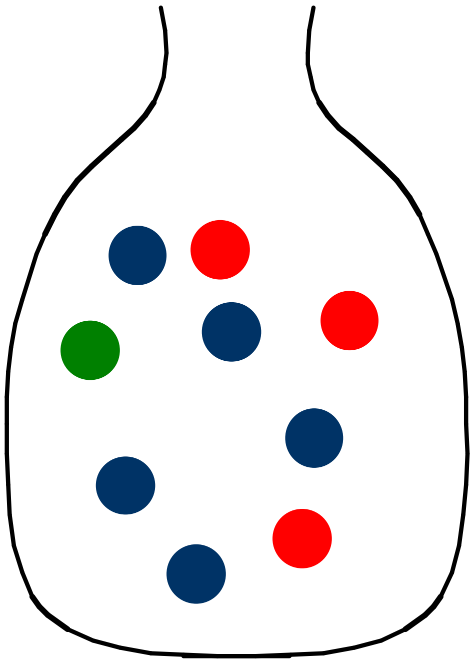
Basic Properties

- $0 \leq P(A) \leq 1$
- $P(true) = 1$

    $P(red \vee blue \vee green) = 1$

- $P(false) = 0$

    $P(black) = 0$



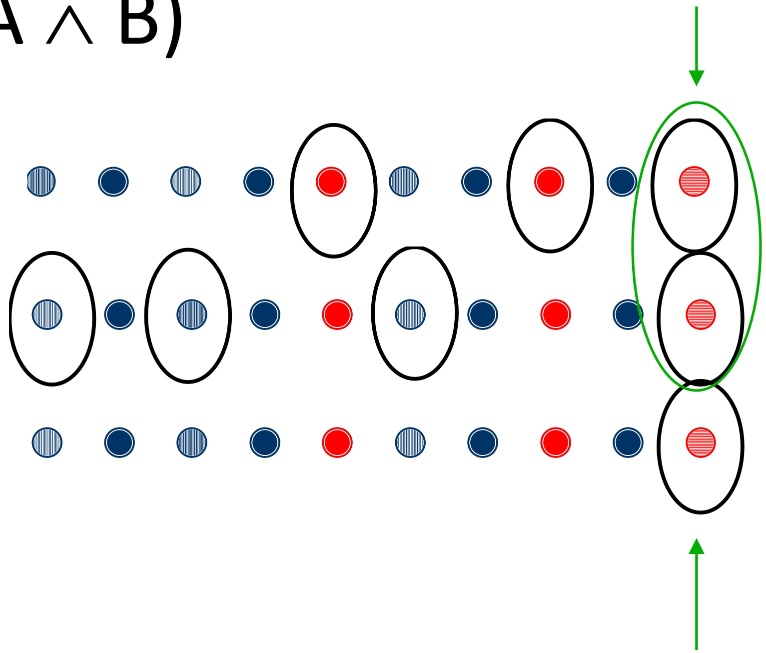
Basic Properties

- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

.3	P(red)
+ .4	P(striped)
- .1	P(red $\wedge$ striped)

P(red $\vee$ striped) = .6

Counted twice



So subtract once

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- A distribution is a TABLE of probabilities of values

- A probability (lower case value) is a single number $P(W = rain) = 0.1$

- Must have: $\forall x \;\; P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

Shorthand notation:

$$P(hot) = P(T = hot),$$
$$P(cold) = P(T = cold),$$
$$P(rain) = P(W = rain),$$
$$\ldots$$

OK *if* all domain entries are unique

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Must obey: $P(x_1, x_2, \ldots x_n) \geq 0$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

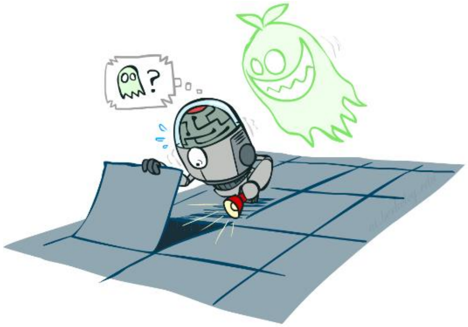
- Size of distribution of **n** variables with domain sizes **d**?
  - O(size) = ?
  - For all but the smallest distributions, impractical to write out!

Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized:* sum to 1.0
  - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: state whether assignments are possible
  - Ideally: only certain variables directly interact

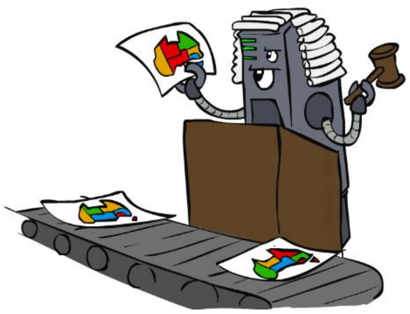
Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3



Constraint over T,W

T	W	P
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T



Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny
    P(+hot, + sun) =
  - Probability that it's hot?
    P(+hot) =
  - Probability that it's hot OR sunny?
  - **P(+hot OR +sun)=**

- Typically, the events we care about are *partial assignments*, like P(T=hot)

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Events

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  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?
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$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

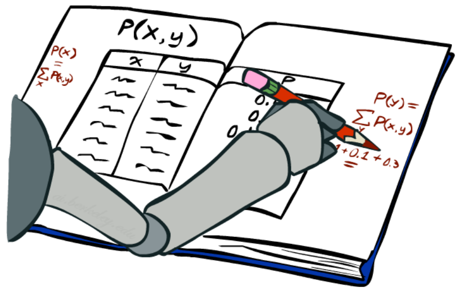
Quiz 1: Events (work in pairs)

- P(+x, +y) ?
- P(+x) ?
- P(-y OR +x) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions



- Marginal distributions are <u>sub-tables</u> which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t, s)$$

$P(T)$

T	P
hot	?
cold	?

$$P(s) = \sum_t P(t, s)$$

$P(W)$

W	P
sun	?
rain	?

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Marginal Distributions

- Marginal distributions are <u>sub-tables</u> which eliminate variables
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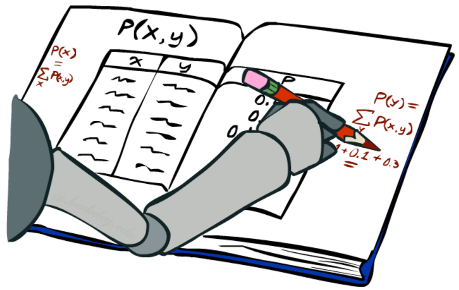


$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t,s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$$P(s) = \sum_t P(t,s)$$

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz P2: Marginal Distributions

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x,y)$$

$$P(y) = \sum_x P(x,y)$$

$P(X)$

X	P
+x	
-x	

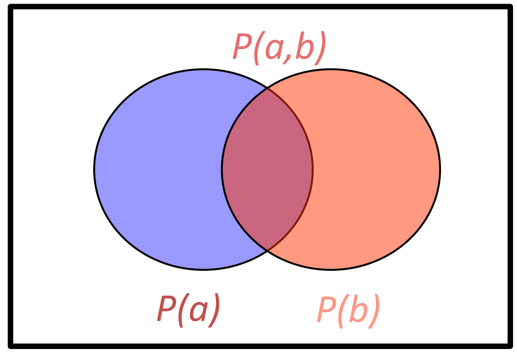
$P(Y)$

Y	P
+y	
-y	

Conditional Probabilities

- Relates joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$\boxed{P(a|b) = \frac{P(a,b)}{P(b)}}$$



$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$
$$= 0.2 + 0.3 = 0.5$$

P(hot|sun) = ?

P(cold|rain) = ?

Quiz P3: Conditional Probabilities

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W|T)$

$P(W|T = hot)$

W	P
sun	0.8
rain	0.2

$P(W|T = cold)$

W	P
sun	0.4
rain	0.6

Joint Distribution

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned} P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\ &= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.2}{0.2+0.3} = 0.4 \end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned} P(W=r|T=c) &= \frac{P(W=r,T=c)}{P(T=c)} \\ &= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.3}{0.2+0.3} = 0.6 \end{aligned}$$

$$P(W|T=c)$$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence →

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one) →

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W|T = c)$

W	P
sun	0.4
rain	0.6

- Why does this work? Sum of selection is P(evidence)! (P(T=c), here)

$$P(x_1|x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}$$

Quiz: Normalization Trick

- P(X | Y=-y) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

NORMALIZE the selection (make it sum to one)

To Normalize

- (Dictionary) To bring or restore to a normal condition

All entries sum to ONE

- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

Normalize

Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize

Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's *beliefs* given the eviden

- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes *beliefs to be update*

Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \ldots E_k = e_1 \ldots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \ldots H_r$

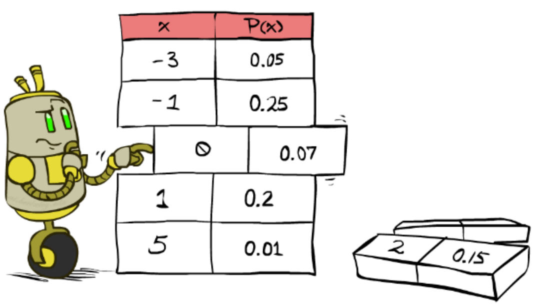
$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ $X_1, X_2, \ldots X_n$ *All variables*

- We want:

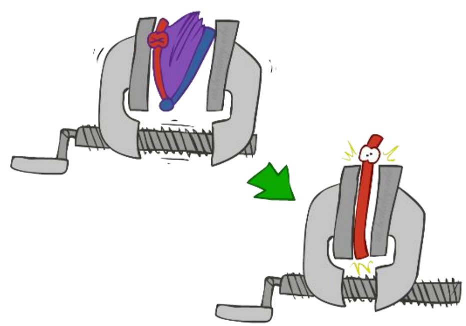
** Works fine with multiple query variables, too*

$$P(Q|e_1 \ldots e_k)$$

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out H to get joint prob of Query and evidence



$$P(Q, e_1 \ldots e_k) = \sum_{h_1 \ldots h_r} \underbrace{P(Q, h_1 \ldots h_r, e_1 \ldots e_k)}_{X_1, X_2, \ldots X_n}$$

- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration

- P(W)?

- P(W | winter)?

- P(W | winter, hot)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

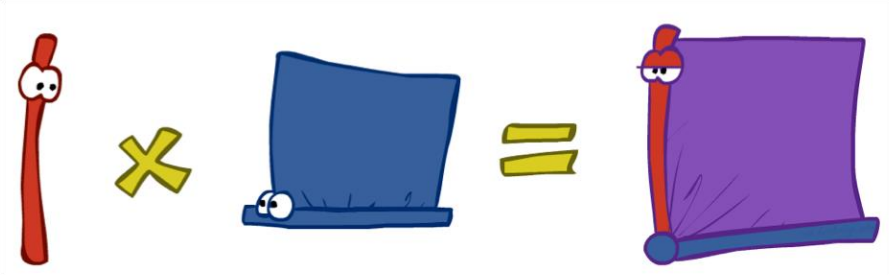
Inference by Enumeration: Issues

- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

More Efficient Inference: <u>The Product Rule</u>

- Sometimes **given** <u>conditional</u> distributions but **want** the <u>joint</u>

$$P(y)P(x|y) = P(x, y) \quad \Longleftrightarrow \quad P(x|y) = \frac{P(x,y)}{P(y)}$$



The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this true?
  - Recursive decomposition using product rule

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

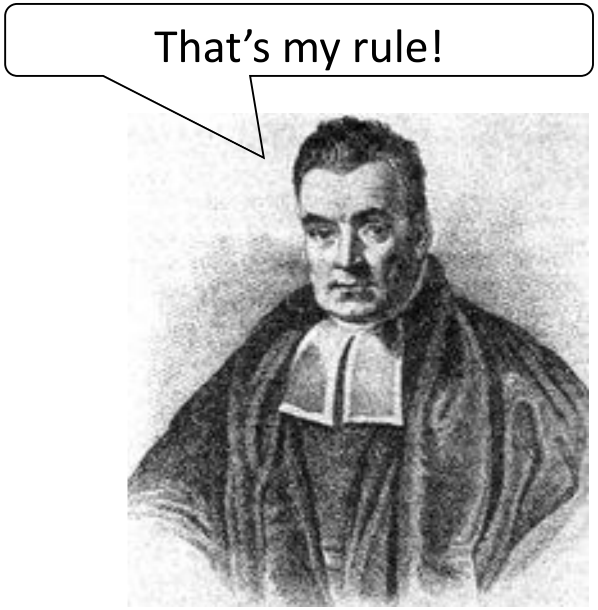
$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later (e.g. ASR, MT)

- **In the running for most important AI, ML, DM equation!**



Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

- Example: $P(\text{cause}|\text{effect}) = \dfrac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example } \textbf{givens}$$

$$P(+m|+s) =$$

Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

- Example: $P(\text{cause}|\text{effect}) = \dfrac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example } \textbf{givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

  - Note: posterior probability of meningitis still very small
  - Note: you should still get stiff necks checked out! Why?

Quiz P4: Inference with Bayes' Rule

- Given:

$P(W)$

W	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

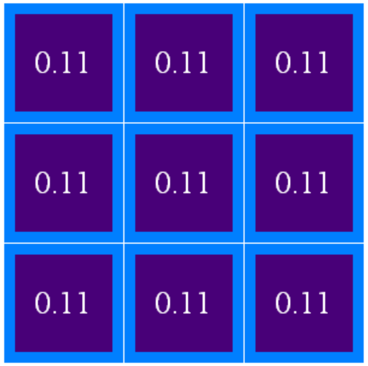
W	P
sun	
Rain	

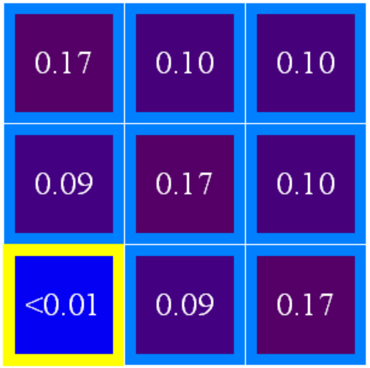
Ghostbusters, Revisited

- Let's say we have two distributions:
  - Prior distribution over ghost location: P(G)
    - Let's say this is uniform
  - Sensor reading model: P(R | G)
    - Given: we know what our sensors do
    - R = reading color measured at (1,1)
    - E.g. P(R = yellow | G=(1,1)) = 0.1

- We can calculate the posterior distribution P(G|r) over ghost locations given a reading using Bayes' rule:

$$P(g|r) \propto P(r|g)P(g)$$





Next Time: Bayes Nets

- Tuesday: P4 (Ghostbusters) formally assigned
- Will post preview tomorrow