NMST — HOME ASSIGNMENT-6

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① a) We know that, Newton's forward difference interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots \quad (1)$$

where, $u = \frac{x-a}{h}$ ... (2)

differentiating eqn. ① wrt. u :

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \dots \quad (3)$$

differentiating eqn. (2) wrt. x :

$$\frac{du}{dx} = \frac{1}{h}$$

So, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{h}\left(\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6} + \dots\right)$

at $x = a$,

$u = 0$ from eqn (2).

So, $\frac{dy}{dx} = \frac{1}{h}\left(\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \dots\right)$ Ans

b) $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \dots\right)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx} \times \frac{du}{dx}\right)$

$$= \frac{d}{du}\left[\frac{1}{h}\left(\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \cdots\right]\times\frac{1}{h}\right.$$

$$= \frac{1}{h}\left[\Delta^2 y_0 + (u-1)\Delta^2 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \cdots\right]\times\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^2 y_0 + \cdots\right]$$

at $x = a$,
$u = 0$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$ Ans

c) $$\left(\frac{d^3y}{dx^3}\right)_{x=a} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d\left(\frac{d^2y}{dx^2}\right)}{du} \times \frac{du}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{(12u-18)}{12}\Delta^4 y_0 + \cdots\right]\times\frac{1}{h}$$

$$= \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{2u-3}{2}\Delta^4 y_0 + \cdots\right]$$

at $x = a$,
$u = 0$.

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0\right]$$ Ans

2) a) We know that, Newton's backward difference formula is :-

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{(u)(u+1)(u+2)}{3!}\Delta^3 y_n \quad \text{--- (1)}$$

$$u = \frac{x - x_n}{u} \quad \text{--- (2)}$$

differenciating eqn (1) wrt. u, we get...

$$\frac{dy}{du} = \nabla y_n + \frac{(2u+1)}{2!}\nabla^2 y_n + \frac{(3u^2+6u+2)}{3!}\Delta^3 y_n.$$

$$= \nabla y_n + \frac{(2u+1)}{2}\nabla^2 y_n + \frac{(3u^2+6u+2)}{3!}\Delta^3 y_n + \cdots$$

$$\frac{du}{dx} = \frac{1}{h}$$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$.

$$\frac{dy}{dx} = \frac{1}{h}\left(\nabla y_n + \nabla^2 y_n + \frac{(2u+1)}{2} + \frac{(3u^2+6u+2)}{6}\nabla^3 y_n + \cdots\right]$$

at $x = x_n$, we have $u = 0$.

$\therefore$ putting $u = 0$, we get:—

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

Ans.

b) Again diff. eqn. (1), we get:—

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}.$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \left(\frac{6u^2+18u+1}{12}\right)\nabla^2 y_n + \dots\right.$$

at $x = x_n$, $u = 0$, we get:—

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right]$$

Ans.

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c) $$\because \frac{d^3y}{dx^3} = \frac{d}{du}\left(\frac{d^2y}{dx^2}\right)\frac{du}{dx}.$$

$$= \frac{1}{h^3}\left[\frac{d}{du}(\nabla^2 y_n) + \frac{d}{du}(u+1)\nabla^3 y_n + \frac{d}{du}\frac{(6u^2+18u+11)}{12}\nabla^4 y_n + \dots\right]\frac{1}{h}$$

$$\because \left(\frac{du}{dx} = \frac{1}{h}\right)$$

$$\therefore \frac{d^3y}{dx^3} = \frac{1}{h^3}\left(0 + \nabla^3 y_n + \left(u+\frac{3}{2}\right)\nabla^2 y_n + \dots\right)$$

putting $u = 0$ at $x = x_n$, we get:—

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \dots\right]$$

Ans

3) a) Stirling central difference formula:—

$$y = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}(\nabla^2 y_{-1}) + \frac{u(u^2-1)}{3!}$$

$$\left(\frac{\nabla^3 y_{-1} + \nabla^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1^2)}{4!}\nabla^4 y_{-2} + \cdots \rightarrow (1).$$

where $u = \frac{x-a}{h} \rightarrow (2)$

Differentiating eqn (1) wrt. u, we get:—

$$\frac{dy}{du} = \frac{\Delta y_0 + \Delta y_{-1}}{2} + u.\Delta^2 y_{-1} + \frac{(3u^2-1)}{6}\left(\frac{\nabla^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots - (2)$$

differentiating eqn. (2) wrt. x ;—

$$\frac{du}{dx} = \frac{1}{h} \rightarrow (4)$$

$$\therefore \cancel{\frac{dy}{}} \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{h}\left[\frac{\nabla y_0 + \Delta y_{-1}}{2} + u.\Delta^2 y_{-1} + \frac{(3u^2-1)}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$$

$$+ \left(\frac{4u^2-2u}{4!}\right)\nabla^4 y_{-2} + \left(\frac{5u^4-15u^2+4}{5!}\right)$$

$$\left.\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \cdots\right] - (5).$$

when $x = a$, $u = 0$.

$$\therefore \left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + \left(\frac{-1}{6}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots\right]$$

$\therefore$ Proved. Ans

b) Differentiating ⑤ wrt. x, :-

$$\frac{d^2 y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}.$$

$$= \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{(6u^2-1)}{12}\Delta^4 y_{-2} + \frac{(2u^3-3u)}{12}\frac{(\Delta^5 y_{-2} + \Delta^5 y_{-3})}{2} + \dots\right] ⑥$$

putting $u=0$, at $x=a$ in equation ⑥,

$$\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \dots\right)$$

Ans.

Bessel's central difference formula is :-

④ a) Bessel's central difference formula is :-

$$y = \frac{(y_0 - y_1)}{2} + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} + \dots \rightarrow ①$$

where, $u = \frac{x-a}{h} \rightarrow ②$.

Differentiating eq. ① wrt. u, we get :-

$$\frac{dy}{dx} = \Delta y_0 + \frac{(2u-1)}{2!}\frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} + \left(\frac{3u^2 - 3u + 1/2}{3!}\right)\nabla^3 y_{-1} + \dots - ③$$

Differentiating with eqn ② wrt. x :-

$$\frac{du}{dx} = \frac{1}{h}$$

$$\because \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{3u^2 - 3u + 1/2}{3!}\right)\right.$$
$$\left.\Delta^3 y_{-1} + \left(\frac{4u^3 - 6u^2 - 2u + 2}{4!}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots\right] \quad (5)$$

putting, $u = 0$, at $x = a$, in eqn. (5) :—

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{-1}{2}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\nabla^3 y_{-1} + \right.$$
$$\left. + \frac{1}{12}\left(\frac{\nabla^4 y_{-2} + \nabla^2 y_{-1}}{2}\right) - \frac{1}{120}\nabla^5 y_{-2} + \cdots\right] \to 6$$

$\therefore$ Proved Ans.

b) Differentiating eqn. (6) wrt x, we get :—

~~Δx^2~~

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\frac{du}{dx}.$$

$$= \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{2u-1}{2}\right)\Delta^3 y_{-1} + \left(\frac{6u^2 - 6u - 1}{12}\right)\right.$$
$$\left.\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \left(\frac{4u^3 + 6u^2 + 1}{24}\right)\nabla^5 y_{-2} + \cdots\right] \quad (7)$$

putting $u = 0$, at $x = a$, in eqn (7), we get :—

$$\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\right.$$
$$\left. + \frac{1}{924}\Delta^5 y_{-2} + \frac{1}{90}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots\right]$$

$\therefore$ Hence Proved. Ans.

⑤ The difference table:-

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

Acceleration at $t=0$, is given by:-

$$\left(\frac{dv}{dy}\right)_{t=0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^2 v_0\right]$$

$$= \frac{1}{5}\left[3 \times \frac{1}{2} \times 8 + \frac{1}{3} \times 36 - \frac{1}{4} \times 24\right]$$

$$= \frac{1}{5}\left[3 - 4 + 12 - 6\right] = \frac{1}{5} \times (15 - 10)$$

$= 1.$ Ans

$\therefore$ Acceleration $= 1\, m/s^2$.

⑥

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.2	0.9182					
		-0.0207				
0.3	0.8975		0.0105			
		-0.0102		-0.0014		

0.4 0.8873 0.0091 0.0007

-0.0011 -0.0007 -0.0006

0.5 0.8862 0.0078

0.0151

0.7 0.9086

Using Newton's forward difference formula of differentiation, we get:-

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \dots$$ (8)

For maxima or minima,

$\frac{dy}{du} = 0$, for maxima, $\frac{d^2y}{du^2} = -ve$.

for minima, $\frac{d^2y}{du^2} = +ve$.

$$\frac{dy}{du} = -0.0207 + \frac{(2u-1)}{2} \times 0.0105 + \frac{(3u^2-6u+2) \times (-0.0014)}{3!}$$

$$= 207 + \frac{(2u-1)}{2} \times 105 + \frac{(3u^2-6u+2)}{3!} \times -14 = 0.$$

$$+ (-899) = 0.$$

$$= 42u^2 - 714u$$

$$u_1 = 18.178, \quad u_2 = -1.1775.$$

$$\frac{d^2y}{du^2} = 105 + (u-1)(-14)$$

$$\left(\frac{d^2y}{du^2}\right)_{u=18.178} = \underline{-ve.}$$

$$\left(\frac{d^2y}{du^2}\right)_{u=-1.1775} = \underline{+ve.}$$

$\therefore$ y is minimum at $u = -1.1775$.

$\Rightarrow x = 0.2 - 1.1775 \times 0.1$.

$\therefore x = 0.08225$.

$\Rightarrow y = y_0 + x\Delta y_0 + \frac{x_0(x_0-1)}{2!}\Delta^2 y_0 + \frac{x_0(x_0-1)(x_0-2)\Delta^3 y_0}{3!}$

$\boxed{\therefore y_{min} = 0.9157.}$ Ans

(7) Let, $I = \int_a^b y.dx$, where y takes the values $y_0, y_1, y_2 \ldots y_n$ for $x = x_0$, $x_1, x_2 \ldots x_n$.

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width h.

where, $h = \frac{b-a}{n}$, so that

$x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, –

$x_n = x_0 + nh = b$.

$\Rightarrow I = \int_{x_0}^{x_0+nh} f(x).dx$.

$\therefore$ any x is given by $x = x_0 + rh$ & $dx = h.dr$

$\therefore I = \int_0^n f(x_0 + rh)\, dr$.

$$I = h\int_0^n \left(y_0 + r\Delta y_0 + \frac{r(r-1)}{2}\Delta^2 y_0 + \frac{(r(r-1)(r-2))}{3}\right.$$

$$\left.\Delta^3 y_0 + \cdots\right] dr.$$

(by Newton - forward interpolation formula).

$$= h\left[r y_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^3}{2}\right)\Delta^2 y_0 + \right.$$

$$\left.\frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \cdots\right]^n.$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)\Delta^2 y_0}{12}\right.$$

$$\left. + \frac{\cancel{k} n(n-2)^2\Delta^3 y_0}{24} + \cdots\right] \rightarrow (1).$$

This is ~~qua~~ general quadratic formula & is known as Ne~~wton~~-cote's quadratic formula.

a) putting $n=1$ in eqn. (1) & taking the curve through (x_0, y_0) & (x_1, y_1) as a polynomi-al of degree one, so that differences of an order higher than one vanishes.

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

$$= \frac{h}{2}\left(2y_0 + (y_1 - y_0)\right)$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly, for the next sub interval (x_0+h, x_0+2h)

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1+y_2)$$

$$\vdots \qquad\qquad \vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1}+y_n).$$

Adding all the above equations;—

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[y_0+2(y_1+y_2+y_3+\cdots y_{n-1}) + y_n\right]$$

∴ This is Trapezoidal rule.

b) Putting $n=2$ in formula ① and taking the curve through (x_0, y_0), (x_1, y_1) & (x_2, y_2) as a polynomial of degree two so that the differences ~~Δy_s~~ of order higher than two vanishes, we get;—

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0+\Delta y_0+\frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{2h}{2} \left[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0) \right]$$

$$= \frac{h}{3} \left[y_0 + 4y_1 + y_2 \right]$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\, dx = \frac{h}{3} \left[y_2 + 4y_3 + y_4 \right]$$

$$\vdots$$

$$\int_{x+(n-2)h}^{x+nh} f(x)\, dx = \frac{h}{3} \left[y_{n-2} + 4y_{n-1} + y_n \right]$$

Adding all the above ~~&~~ eqns, :—

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \frac{h}{3} \left[(y_0 + y_n) + 4(y_1 + y_3 + \dots y_{n-1}) + 2(y_2 + y_4 + \dots y_{n-2}) \right]$$

∴ This is Simpson's one-third rule

c) Putting $n=3$ in formula ① & taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2) & (x_3, y_3) as a polynomial of degree so that differences of order higher than three vanishes, we get :—

$$\int_{x_0}^{x_0+3h} f(x) \cdot dx = 3h \left(y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0 \right)$$

$$= \frac{3}{8}h\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\right)$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right].$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$\vdots \qquad \vdots$$

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Adding the above integration,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_3 + y_4 + \cdots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right].$$

$\therefore$ This is Simpson's three eight rule.

d) Putting $n = 4$, in formula ① and neglecting all differences of order higher than 4, we get :-

$$\int_{x_0}^{x_0+4h} f(x)\,dx = h \int_0^4 \Big[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)\Delta^3 y_0}{3!} + \frac{r(r-1)(r-2)(r-3)}{4!} \,..\, \Delta^4 y_0 \Big] dr .$$

By newton's forward interpolation formula :-

$$= 4h \Big[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(8n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \Big(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\Big)\frac{\Delta^4 y_0}{4!} \Big]_0^4 .$$

$$= 4h \Big[y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{3}{2}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0 \Big]$$

$$= \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4 \right].$$

Similarly,

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}\left[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8 \right].$$

$\vdots \qquad\qquad \vdots$

Adding all the above equations, :—

$$\int_{x_0}^{x_0+4h} f(x)\cdot dx = \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + \cdots\right]$$

$\therefore$ This is Bode's Rule.

e) Putting $n=6$, in formula ① and neglecting all differences of order higher than six :—

$$\int_{x_0}^{x_0+6h} f(x)\,.dx = h\int_0^6 \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0\right]$$

$$= h\left[r y_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \cdots + \frac{1}{720}\left(\frac{r^7}{7} - \frac{5r^6}{2} + \frac{17r^5}{2} - \frac{225}{4}r^4 + \frac{274}{3}r^3 - 60r^3\right)\Delta^6 y_0\right]_0^6$$

$$= \frac{6h}{20}\left[20y_0 + 60\Delta y_0 + 96\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 + \frac{41}{42}\Delta^6 y_0\right]$$

$$= \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} f(x)\,.dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

$$\int_{x_0+(n-6)h}^{x_0+nh} f(x)\,.dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right].$$

Adding all the above equations:-

$$\int_{x_0}^{x_0+nh} f(x)\,.dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + \cdots\right].$$

$\therefore$ This is Weddle's rule.

Here, n must be the multiple of 6.

(8) let $h = 0.2$, $a = 4$, $b = 5.6$, $n = 6$.

$\therefore 5.2 = 4 + 6h$.

$h = \underline{0.2}$.

Now, $x_n = x_0 + nh$ & $y_n = \log x_n$.

x	y
4	1.3862
4.2	1.4350845
4.4	1.4816045
4.6	1.5260563
4.8	1.5686159
5.0	1.6094379
5.2	1.6486586

a) By using Trapezoidal rule:—

$$\int_4^{5.2} (\log x)\, dx = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}\left[3.334 + 2 \times 7.620799\right]$$

$$= 0.1 \times 18.276551$$

$$\int_4^{5.2} (\ln x)\, dx = 1.8276551.$$

b) ~~(i)~~ By Weddle's rule :–

$$\int_4^{5.2} (\ln x)\, dx = \frac{3h}{10}\left[y_0 + y_n + 5(y_1 + y_5) + y_2 + y_4 + 6y_3\right]$$

$= \frac{0.6 \times 30.464123}{10}$

$\therefore \int_4^{5.2} (\ln x)\, dx = \underline{1.827874}.$

(9) ∵ while using the simpson's one third formula, the given interval of integration must be divided into an even number of sub-intervals.

$\therefore \int_0^4 e^x dx = \frac{h}{3} [y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots y_{n-2})].$

where $h = 1$.

$\therefore \int_0^4 e^x dx = \frac{1}{3} [(1 + 54.6) + 4(2.72 + 20.09) + (2 \times 7.39)].$

$= \frac{1}{3} \times 161.62$

$\therefore \int_0^4 e^x dx = 53.87333.$

By actual integration value it :-

$\int_0^4 e^x dx = e^x \Big]_0^4 = e^4 - 1.$

$= \underline{53.6}.$

$\therefore$ Magnitude of error in simpson's one third rule $= |53.6 - 53.873331|.$

$= \underline{0.27333}$ Ans

(10) Taking, $n = 6$ we get $h = 1$,

$x_0 = 0$, $x_6 = 6$., $y = \frac{1}{1+x^2}$.

$x_n = x_0 + nh$

$y_n = \frac{1}{1+x_n^2}$

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	1/17	1/26	1/37

ii) By simpson's one third rule:—

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+1/37)+4(0.5+0.1+1/26)+2(0.2+1/17)\right]$$

$$= \int_0^6 \frac{dx}{1+x^2}$$

$$= \underline{1.366173413}.$$

i) By Trapezoid ruler

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left((1+1/37)+2(0.5+0.2+0.1+1/17+1/26)\right)$$

$$\therefore \int_0^6 \frac{dx}{1+x^2} = \underline{1.410798581}. \text{ Ans}$$

By actual integration:—

$$\int_0^6 \frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^6 = \tan^{-1}6 - \tan^{-1}0$$

$$= 80.5376^\circ$$

$$= \underline{1.40564765 \text{ rad}}. \text{ Ans}$$

Magnitude of error in Trapezoidal rule:—

$= |1.40564765 - 1.41079858|$

$= 0.0051509$

Magnitude of error in simpson's one-third rule:—

$= |1.40564765 - 1.36617341|$

$= \underline{0.039474}$ Ans.

(11) using simpson's one-third formula,

when $n = 4$.

$$\int_{x_0}^{x_0+hn} f(x)\,dx = \frac{h}{3}\left[(y_0 - y_n) + 4(y_1 - y_3) + 2y_2\right]$$

$x_n = x_0 + nh$, $y_n = \frac{1}{5+3x}$, $h = 0.25$.

x	1	1.25	1.50	1.75	2.00
y	0.125	0.1142	0.1052	0.097	0.0909

$$\Rightarrow \int_1^2 \frac{dx}{5+3x} = \frac{1/4}{3}\left[(0.125 + 0.0909) + 4(0.1142 + 0.097) + 2 \times 0.1052\right]$$

$$= \frac{1}{12} \times (1.2715)$$

$$\therefore \int_1^2 \frac{dx}{5+3x} = \underline{0.1059}$$

b) when, $n = 8$

$$h = \frac{2-1}{8} = 1/8$$

$x_n = x_0 + nh$, $y_n = \frac{1}{5+3x}$

x	1	1.125	1.25	1.375	1.5	1.625	1.75
y	0.125	0.1194	0.1142	0.1095	0.1052	0.1012	0.097

x	1.875	2
y	0.0941	0.0909

$$\therefore \int_1^2 \frac{dx}{5+3x} = \frac{1/8}{3}\left[y_0 + y_8 + 4(y_1+y_3+y_5+y_7) + 2(y_2+y_4+y_6)\right]$$

$$= \frac{1}{24} \times 2.5455$$

$$\therefore \int_1^2 \frac{dx}{(5+3x)} = \underline{0.1060} \text{ Ans.}$$

By Actual Integration,

$$I = \int_1^2 \frac{dx}{(5+3x)} = \frac{1}{3}\left(\ln(5+3x)\right)_1^2$$

$$= 1/3 (\ln 11 - \ln 8)$$

$$= 1/3 \ln(11/8)$$

$$\therefore I = \underline{0.10615} \text{ Ans.}$$

Absolute error when $n=4$,

$$= |0.10615 - 0.10591|$$

$$= \underline{0.00025}.$$

Absolute error when $n=8$,

$$= |0.10615 - 0.10601|$$

$$= \underline{0.00015} \text{ Ans}$$

(12) Simpson's 3/8 formula is:

$$\int_{x_0}^{x_0+nh} f(x)\cdot dx = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + \cdots) + 2(y_3 + y_6 + \cdots + y_{n-3})\right]$$

where, $h = \frac{b-a}{n}$, n = no of intervals.

Case-1: when, $n = 3$

$x_n = x_0 + nh$, $y_n = \frac{1}{5+3x_n}$.

$h = 1/3$.

x	1	1.33	1.66	2
y	0.1250	0.1112	0.1002	0.0909

$$\therefore \int_1^2 \frac{dx}{5+3x} = \frac{3\times 1/3}{8}\left[0.125 + 0.909 + 3(0.1112 + 0.1002)\right]$$

$$= \frac{1}{8}(0.2159 + 0.4338)$$

$$= \frac{1}{8} \times 0.8501$$

$$\Rightarrow \int_1^2 \frac{dx}{(5+3x)} = \underline{0.1062}.$$

Case-2, when $n = 6$, $h = \frac{2-1}{6} = 1/6$;

x	1	1.166	1.332	1.498	1.664	1.830	2.0
y	0.125	0.1176	0.1112	0.1053	0.1002	0.095	0.0909

$$\Rightarrow \int_1^2 \frac{dx}{5+3x} = \frac{3h}{8}\left[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3\times 1/6}{8}\left[0.125 + 0.0909 + 3[0.1176 + 0.112 + 0.1002 + 0.095] + 2\times 0.105\right]$$

$$= 1/16 \times 1.6992$$

$$\therefore \int_1^2 \frac{dx}{5+3x} = \underline{0.1062}$$

Now,

By exact integration,

$$\int_1^2 \frac{dx}{(5+3x)} = \ln(5+3x)\Big]_1^2$$

$$= -\ln 8 + \ln 11$$

$$= 0.10613.$$

when, $n=6$

$$\text{Absolute errors} = |0.10615 - 0.1062|$$

$$= \underline{0.00005}.$$

(13) Boole's formula is :-

$$\int_{x_0}^{x_4} y\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

where $h = x_1 - x_0$.

x	0.0	0.1	0.2	0.3	0.4
y	1.000	0.9975	0.99	0.9776	0.8604

Now, we have :

$x_0 = 0.000$, $x_1 = 0.1$.

$x_4 = 0.4$

$$\therefore h = (x_4 - x_0)/n$$

$$= (0.4 - 0.0)/4$$

$$h = \underline{0.1}.$$

from ①, we have:-

$$\int_{0.0}^{0.4} y\cdot dx = \frac{2\times 0.1}{45}\left(7\times 1.00 + 32\times 0.9975 + 32\times 0.9776\times 0.8604\right)$$

$$= \int_{0}^{0.4} y\cdot dx = \frac{2\times 0.1}{45}\times 88.106$$

$$= 0.39158$$

$$\boxed{\therefore \int_{0.0}^{0.4} y\cdot dx = 0.39158}$$ Ans.

⑭ $y = 1/x$, $a=1$, $b=2$, $N=8$.

$$\therefore h = \frac{b-a}{N} = \frac{2-1}{8} = 1/8$$

$\rightarrow x_n = a + n.h$, $y_n = 1/x_n$

Now,

x	1	9/8	10/8	11/8	12/8	13/8	14/8	15/8	2
y	1	8/9	8/10	8/11	8/12	8/13	8/14	8/15	1/2

Using Boole's formula,

$$\int_a^{a+nh} y\cdot dx = \frac{2h}{45}\left(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \cdots\right)$$

$$\therefore \int_1^2 y\,dx = \frac{2\times 1/8}{45}\left(7\times 1 + 32\times\frac{8}{9} + 12\times\frac{8}{10} + 14\times\frac{8}{12} + 32\times\frac{8}{13} + 12\times\frac{8}{14} + 14\times\frac{1}{2} + \cdots\right)$$

$\Rightarrow \int_1^2 y.dx = \frac{2 \times 1/8}{45} \times 128.2666$

$\boxed{\therefore \int_1^2 y.dx = 0.71259.}$ Ans.

(15) We have: $y = 1/x$, $a = 1$; $b = 2$ &

$N = 6$.

$\therefore h = \frac{b-a}{N} = \frac{1}{6}$.

Weddle's formula is:—

$$\int_a^{a+nh} y.dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + ---]$$

Here, we have:—

$x_n = a + nh$, $y_n = 1/x_n$.

x	1	7/6	8/6	9/6	10/6	11/6	2
y	1	6/7	6/8	6/9	6/10	6/11	1/2

putting the respective values of y_n in eqn ①;—

$$\int_1^2 y.dx = \frac{3 \times 1/6}{10}\left[1 + 5 \times \frac{6}{7} + \frac{6}{8} + 6 \times 6/9 + \frac{5 \times 6}{11} + 1/2 + \dots\right]$$

$\Rightarrow \int_1^2 y.dx = \frac{1}{20} \times 13.863$

$\therefore \int_1^2 y \cdot dx = 0.69315$

Ans.

(16) $y = \frac{x}{\sinh(x)} = \frac{2x}{e^x - e^{-x}}$.

$x_0 = 0.4$, $x_0 + nh = 1.6$, $n = 12$.

$0.4 + 12h = 1.6$

$\therefore h = 0.1$.

Now, $x_n = x_0 + nh$

$y_n = \frac{2x_n}{e^{x_n} - e^{-x_n}}$

x	y
0.4	0.97382
0.5	0.95967
0.6	0.94243
0.7	0.90079
0.8	0.90079
0.9	0.87675
1.0	0.85092
1.1	0.82357
1.2	0.79499
1.3	0.76543
1.4	0.73518
1.5	0.70416
1.6	0.67352

Weddle's formula is given by:—

$$\int_{x_0}^{x_0+nh} y \cdot dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12} + \ldots\right]$$

$$\Rightarrow \int_{0.4}^{1.6} \frac{x}{\sinh(x)} \cdot dx = \frac{3 \times 0.1}{10}\left[0.97382 + 5 \times 0.95967 + 0.9423 + 6 \times 0.92277 + 0.90079 + 5 \times 0.87675 + 2 \times 0.87092 + \ldots\right]$$

$$\Rightarrow \int_{0.4}^{1.6} \frac{x}{\sinh(x)} dx = \frac{0.3}{10} \times 33.673$$

$$\boxed{\int_{0.4}^{1.6} \frac{x}{\sinh(x)} dx = 1.0102.}$$

Ans.

(17) To find sum of fourth power of natural numbers:

$\therefore y(x) = x^4$

Now, Euler-Maclaurin formula:—

$$\int_{x_0}^{x_n} y \cdot dx = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \ldots y_n\right] - \frac{h^2}{12}\left[y_n' - y_0'\right] + \frac{h^4}{720}\left[y_n''' - y_0'''\right] \ldots$$

$$= \int_1^n x^4 \cdot dx = \frac{1}{2}\left[y_0 + 2(y_1 + y_2 + \ldots y_{n-1}) + y_n\right] \ldots$$

$\Rightarrow y_0 + y_1 + y_2 + \dots + y_n$

$= \frac{(n^5-1)}{5} + \frac{(1+n^4)}{2} + \frac{1}{3}(n^3-1) + \frac{1}{13}(n-1)$

$\Rightarrow y_0 + y_1 + y_2 \dots + y_n = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$

$= \frac{1}{30}(2n+1)(n)(n+1)(3n^2+3n-1)$

$$\boxed{\therefore \sum_{x=1}^{n} x^4 = \frac{1}{30}(n)(n+1)(2n+1)(3n^2+3n-1)}$$

Ans.

(18) To evaluate:

$$\sum_{m=0}^{\infty} \frac{1}{(m+10)^2}$$

$y(x) = \frac{1}{(10+x)^2}$, $h = 1$

Now, Euler's - Maclaurin formula gives:-

$$\int_{x_0}^{x_n} y \cdot dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n] - \frac{h^2}{12}(y'_n - y'_0) - \frac{h^4}{720}(y'''_n - y'''_0).$$

$\Rightarrow$ We can write:-

$$\frac{2h}{2}(y_0 + y_1 + y_2 + \dots + y_n) = \int_{x_0}^{x_n} y \cdot dx + \frac{h}{2}(y_0 + y_n) + \frac{h^2}{12}(y'_n - y'_0) - \frac{h^4}{720}(y'''_n - y'''_0)$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{1}{(10+m)^2} = \frac{1}{10} + \frac{1}{200} - \frac{1}{6\times 10^3} + \frac{1}{3\times 10^6} + \cdots$$

$$\boxed{\therefore \sum_{m=0}^{\infty} \frac{1}{(10+m)^2} = 0.10483}$$

Ans.

(19) Given that:

$$\int_{-h}^{h} f(x)\,dx = h\cdot\left[a.f(-h) + bf(0) + af(h) + h^2 c\{f'(-h) - f'(h)\}\right] \rightarrow (1)$$

Now, we know that:—

$$y(x_{i+1}) = y(x_i) + (x_{i+1} - x_i) f'(x_i) + \frac{(x_{i+1} - x_i)^2}{2!} f''(x_i) + \cdots$$

where, $(x_{i+1} - x_i = h)$

we can write,

$$\int_{-h}^{h} f(x)\cdot dx = \int_{-h}^{h} \left(f(x_0) + h.f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots\right) dx$$

$$= h\left[2f(0) + f(h) + f(-h) + \frac{h^2}{2}(f'(+h) - f'(-h) + \cdots\right] \rightarrow (2)$$

On comparing equation (1) and (2), we get—

$$\boxed{a = 1, \quad b = 2, \quad c = \frac{-1}{2}}$$

Ans

(20) Given that:—

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx.$$

Taking $n = 6$, we get $h = 1/6$.

$\Rightarrow x_n = x_0 + nh, \quad y_n = \frac{1}{1+x_n^2}$

x	y
0	0.97287
1/6	0.90072
2/6	0.80000
3/6	0.69209
4/6	0.59036
5/6	0.50000
1	

Now, Euler-MacLaurin's formula is given by:—

$$\int_{x_0}^{x_0+nh} y.dx = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \cdots + y_n\right] - \frac{h^2}{12}\left[y_n' - y_0'\right] + \frac{h^4}{720}\left[y_n''' - y_0'''\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/6}{2}\left[1 + 2(0.97287 + 90072 + 0.8 + 0.69209 + 0.59036 + 0.5\right] - \frac{h^2}{12}\left[-1/2 - 0\right] + \cdots$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{12} \times 9.4121 + \frac{1}{864} + \cdots$$

$$\pi = \frac{9.4121}{3} + \frac{1}{216} + \cdots \approx \underline{3.1437}$$

Hence, the value of π we get from above formula is 3.1437.

---

(21) To evaluate:—

$$I = \int_0^2 \frac{(1+x)^2}{1+(x+1)^4}\, dx .$$

Reducing the interval $[0,2]$ to $[-1,1]$ to apply the Gauss rule:—

$x = at + b$

$0 = -a + b \rightarrow (1)$

$2 = a + b \rightarrow (2)$

$\Rightarrow$ on solving, we get:—

$b = 1,\ a = 1.$

$\therefore\ x = t + 1,\ dx = dt$

Now, integral becomes:—

$$I = \int_{-1}^{1} \frac{(2+t)^2}{1+(2+t)^4}\, dt .$$

$$f(t) = \frac{(2+t)^2}{1+(2+t)^4}$$

Using formula, of Gauss three point

$$I = \frac{1}{9}\left[5f\left(-\frac{\sqrt{3}}{5}\right) + 8f(0) + 5f(\sqrt{3}/5)\right]$$

$$I = 1/9\,[5\times 0.481348 + 8\times 0.235294 + 5\times 0.127742]$$

$$I = \frac{4.8278}{9} = \underline{0.586422} \text{ Ans.}$$

where, $f(t) = \frac{(2+t)^2}{1+(2+t)^4}$.

---

(22) $J = \int_0^2 \frac{dx}{3+4x}$

for exact solution,

$$J = \left[\frac{\ln(3+4x)}{4}\right]_0^2 = \frac{\ln(11) - \ln(3)}{4}$$

$$\boxed{J = 0.32482}$$

Now, reducing the interval $[0,2]$ to $[-1,1]$ to apply Gauss rule:—

$x = at + b$

$0 = -a + b \rightarrow$ (i).

$2 = a + b \longrightarrow$ (ii)

solving (i) & (ii), we get:—

$a = 1$, $b = 1$

$\therefore\ x = t + 1$, $dx = dt$.

Now, integral becomes:—

$$J = \int_{-1}^{1} \frac{dt}{(4t+7)}, \quad f(t) = \frac{1}{4t+7}$$

Using Gauss two point rule, we get:-

$J = f(-1/\sqrt{3}) + f(1/\sqrt{3})$

$= 0.21319 + 0.10742$

$\boxed{\therefore J = 0.32061}$

Using Gauss ~~tree p~~ three point rule, we get:-

$J = \frac{1}{9}\left[5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})\right]$

$J = \frac{1}{9}[5 \times 0.25630 + 8 \times 0.14286 + 5 \times 0.09903]$

$J = \frac{2.9195}{9}$

$\therefore \boxed{J = 0.32439}$

Magnitude of error in two point rule:-

$|0.32482 - 0.32061|$

$= \underline{0.00421}$.

Magnitude of error in three point rule:-

$\cancel{\Delta}|0.32452 - 0.32439|$

$= \underline{\underline{0.00043}}$. Ans.