Dynamic Equilibrium Control of a Ball-on-Beam System: Newtonian Modeling and Stabilization Techniques

1Niha Singhania, 1*Vidya S.Rao, Bipin Krishna, Mukund Kumar Menon
Instrumentation & Control Engineering, Manipal Academy of Higher Education, Manipal, India
*rao.vidya@manipal.edu

Abstract— A pedagogical instrument was created for feedback courses, based on the classic ball-on beam problem. The design of this instrument focused on a combination of aesthetic appeal and cost-effectiveness, utilizing affordable materials to create kits. This thesis outlines the apparatus's design and the creation of two control loops: one to manage the motor's angle and the other to regulate the position of the ball along the beam.
In each control loop, a lead compensator was employed, and an extra integrator was added to the motor loop to guarantee the beam remains level when supporting the ball.

I. INTRODUCTION

Introduction:

The control and stabilization of dynamic systems have long been at the forefront of engineering research, with applications ranging from robotics to aerospace. Among these systems, the ball-on-beam configuration serves as a fundamental benchmark for testing control algorithms and evaluating the efficacy of various stabilization techniques. This paper delves into the intricate dynamics and control strategies associated with the ball-on-beam system, with a particular focus on employing Newtonian modeling principles to achieve dynamic equilibrium. The ball-on-beam system consists of a ball constrained to move along a beam, typically subjected to gravitational forces and external disturbances. This system's inherent complexity arises from the nonlinear dynamics governing the ball's motion and the beam's response to control inputs. Consequently, designing effective control algorithms necessitates a comprehensive understanding of both the system's physical characteristics and its dynamic behavior. Recent advancements in control theory and computational techniques have spurred significant progress in the development of novel control strategies for ball-on-beam systems. These advancements have enabled researchers to explore advanced control schemes, such as model predictive control, adaptive control, and optimal control, aiming to enhance stability, tracking performance, and robustness against disturbances. Additionally, the integration of machine learning and artificial intelligence techniques has opened new avenues for achieving autonomous and adaptive control in real-time scenarios.

This paper aims to provide a comprehensive overview of the dynamic equilibrium control of ball-on-beam systems, drawing upon insights from both recent developments and conventional control schemes. Through a synthesis of theoretical analysis, simulation studies, and experimental validations, we aim to elucidate the underlying principles governing the system's behavior and showcase the effectiveness of various control strategies in achieving stable and precise control of the ball's position on the beam. By bridging the gap between theoretical advancements and practical applications, this research contributes to the broader goal of advancing control theory and its practical implementation in dynamic systems.

II. Theoretical Model

II.I MODELLING FROM NEWTONIAN MECHANICS

To derive the control model for the ball-and-beam system, we'll utilize Newtonian mechanics. We'll define the x-axis along the beam and acknowledge the need to consider time derivatives of unit vectors when calculating velocities and accelerations. This is because the beam's rotation introduces a time-varying aspect to the coordinate system.

$$a_a = \omega \text{ x } \mathbf{r} + \omega \text{ x } (\omega \text{ x } \mathbf{r}) + 2\ \omega = v_{rel} + a_{rel} \quad (1)$$

Consider a body moving within a rotating reference frame, like the axes in Figure 1 that spin with angular velocity ω [4]. To describe this motion, we need to distinguish between two key measures:

Relative velocity (vrel): This reflects the body's velocity relative to the rotating axes, independent of the overall rotation.
Relative acceleration (arel): This captures the total acceleration the body experiences relative to the rotating frame. It considers both the body's linear acceleration and the effect of the frame's rotation (governed by ω).

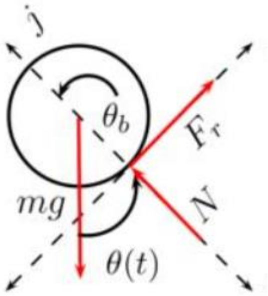

FIGURE 1. FREE BODY DIAGRAM

Imagine a reference frame that's constantly rotating with an angular speed of ω. Within this rotating frame, we have a point identified by its position vector r. The velocity of this point relative to the rotating frame is denoted by vrel, and its acceleration relative to the rotating frame is represented by arel.

$$\mathbf{R} = p\mathbf{i} \quad \mathbf{(2)}$$

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}\,\mathbf{k} \quad \mathbf{(3)}$$

$$\mathbf{v_{rel}} = \dot{\mathbf{p}}\mathbf{I} \quad \mathbf{(4)}$$

$$\mathbf{a_{rel}} = \ddot{\mathbf{p}}\mathbf{I} \quad \mathbf{(5)}$$

Let's now move forward with the vector operations required to determine the absolute acceleration.

$$\dot{\omega} \text{ x } r = \ddot{\theta}k \text{ x } pi = p\ddot{\theta}j \quad (6)$$
$$\omega \text{ x } (\omega \text{ x } r) = \dot{\theta}k \text{ x } p\dot{\theta}i = p\dot{\theta}j \quad (7)$$
$$\omega\ x\ (\omega\ \times\ r) = \dot{\theta}k\ \times p\dot{\theta}j = -p\dot{\theta}^2 i \quad (8)$$
$$2\omega\ \times v_{rel} = 2\dot{\theta}k\ \times\ \dot{p}i = 2\dot{\theta}\dot{p}j \quad (9)$$

Substituting (7) – (8) into (9) we obtain the acceleration relative to the rotating axes

$$a_a = p\ddot{\theta}j - p\dot{\theta}^2 i + 2\dot{\theta}\dot{p}j + \ddot{p}i \quad (10)$$
$$a_a = \left(\ddot{p} - p\dot{\theta}^2\right)i + (p\ddot{\theta} + 2\dot{\theta}\dot{p})j \quad (11)$$

Substituting (10) into (11), we arrive at our first equation of motion

$$J_b \ddot{\theta}_b = F_r \cdot r \tag{12}$$

$$\theta_b = -\frac{p}{r} \tag{13}$$

Take a look at the free body diagram in Figure 2. The result of adding torques about the ball's rotational axis is

$$F_r = -\frac{J^b}{r^2}\ddot{p} \tag{14}$$

The angle of rotation of the ball about its center, denoted as θb, is determined by the moment of inertia of the ball, Jb

$$F_r - mg\sin\theta = m(\ddot{p} - p\dot{\theta}^2) \tag{15}$$

By substituting equation (14) into equation (15) and subsequently solving for the variables, we can proceed

$$\left(\frac{J_b}{r^2} + m\right)\ddot{p} + mg\sin\theta - mp\dot{\theta}^2 \tag{16}$$

Next, we proceed to sum the forces acting on the ballin the i-direction by utilizing the i component of the relative acceleration vector.

$$N = m\big(p\ddot{\theta} + 2\dot{\theta}\dot{p}\big) + mg\cos\theta \tag{17}$$

To derive the second equation of motion, our initial step involves calculating the normal force, denoted as N, as depicted in Figure 2. Summing the forces actingon the ball in the j-direction results in the following expression:[3]

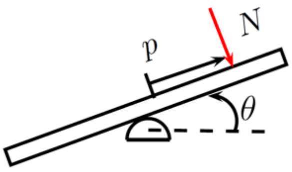

Figure 2. Free Body Diagram of the Beam

Summing torques acting on the beam yields

$$\tau - Np = J\ddot{\theta} \tag{18}$$

$$(mp^2 + J)\ddot{\theta} + 2mp\dot{p}\dot{\theta} + mgp\cos\theta = \tau \tag{19}$$

II.b. NONLINEAR AND LINEAR STATEVARIABLE REPRESENTATION

To analyze the Ball and Beam system using state-space methods, we need to define a state vector.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} p(t) \\ \dot{p}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \tag{20}$$

The state vector is made up of the bare minimum of variables required to forecast the future response of the system given the input and its current state. The following equations of motion (constant ball position I and zero velocity) can therefore be written in terms of these state variables.

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} \frac{m(x_1 x_4{}^2 - g \sin x_3)}{\frac{J}{r^2} b + m} \\ \frac{-2mx_1x_2x_4 - mgx_1 \cos x_3 + \tau}{(mx_1{}^2 + J)} \end{bmatrix} = \mathrm{f(x,r)} \tag{21}$$

In addition, we determine an operational point that is equivalent to a stationary ball position 'I' moving at zero speed. The beam's angular velocity and angle are both zero at this operational point. The following is a description of the particular operating state:

$$x_0 = \begin{bmatrix} p_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

Furthermore, we ascertain the operating input essential for preserving this equilibrium by equating it to zero and assessing it at the specified operating point. This input represents the torque needed to keep the ball stationary at position 'I.' The Jacobian of the righthand side of the equation, concerning the state vector 'x,' results in[3]

$$u_r = mgp_0 \tag{23}$$

When evaluated at the operating point, this procedure leads to the determination of the A matrix in the state variable representation. The Jacobian with respect to the input τ yields

$$\frac{\partial f}{\partial x}(x, r) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{mx_4{}^2}{\frac{J_b}{r^2} + m} & 0 & \frac{-mg \cos x_3}{\frac{J_b}{r^2} + m} & \frac{2mx_1x_4}{\frac{J_b}{r^2} + m} \\ 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial x_1} & \frac{-2mx_1x_4}{mx^2{}_1 + J} & \frac{mgx_1 \sin x_3}{mx^2{}_1 +} & \frac{-2mx_1x_2}{mx^2{}_1 +} \end{bmatrix} \tag{24}$$

$$\frac{\partial f_4}{\partial x_1} = \frac{(-2mx_1x_4 - mg \cos x_3)(mx^2{}_1 + J) - (-2mx_1x_2x_4 - mgx_1 \cos x_3 + \tau)}{(mx^2{}_1 + J)^2} \tag{25}$$

Setting the analysis around a specific operating point defines the A matrix in the state-space representation

$$A = \frac{\partial f}{\partial x}(x_0, \tau_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{\frac{J_b}{r^2} + m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-mg}{mp_0^2 + J} & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

Upon evaluation at the operating point, we obtain the B matrix in the state-variable representation.

$$B = \frac{\partial f}{\partial x}(x_0, \tau_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{mp_0^2 + J} \end{bmatrix} \tag{27}$$

Defining the output of the system to be the ball position yields the *C* matrix of the state variable representation

$$C = [1 \quad 0 \quad 0 \quad 0] \tag{28}$$

$$x\dot{(t)} = Ax(t) + Bu(t) \tag{29}$$

$$y(t) = Cx(t) \tag{30}$$

Defining the output of the system to be the ball position yields the C matrix of the state variable representation, m is mass, g is acceleration due to gravity, J is the moment of inertia of the beam and Jb is the moment of inertia of the ball about its center.

$$m = 0.11kg$$
$$r = 0.015m$$
$$g = 9.81\frac{m}{s^2}$$
$$J = 19 \times 10^{-3} kg \cdot m^2$$
$$J_b = 9.99 \times 10^{-6} kg \cdot m^2$$

$$p_0 = 0 \tag{31}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 \\ -56.8 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 52.6 \end{bmatrix} \tau(t) \tag{32}$$

$$\boldsymbol{y(t) = [1 \quad 0 \quad 0 \quad 0]x(t)} \tag{33}$$

Analyzing this state space representation enables the examination of the system, facilitating the design of a controller.

III. Stability analysis using Lyapunov method

We lay the groundwork for developing a continuous, stabilizing feedback control method. This strategy ensures the asymptotic stability of the system's motion under the condition that the variable q2 remains within the range (pi/2, -pi/2). In mechanical terms, the challenge entails guiding both the ball and the beam to the unstable equilibrium point simultaneously while ensuring the ball remains positioned above the beam. This objective is pursued with the specific goal of attaining asymptotic stability. The balancing control issue is addressed using the Lyapunov direct method [1].

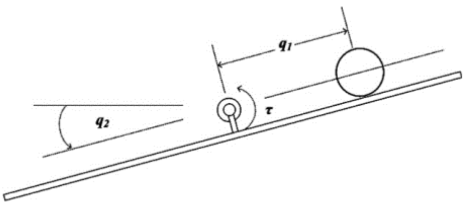

Figure 5. The ball on the actuated beam.

Analyze a simplified version of the Ball and Beam System (BBS), depicted in the Figure, derived using Lagrange equations. This system is described by the resulting normalized differential equations:

$$\begin{gathered} \ddot{q}_1 - q_1\dot{q}_2^2 + \sin q_2 + \delta\dot{q}_1 = 0, \\ (q_1^2 + I)\ddot{q}_2 + 2q_1\dot{q}_1\dot{q}_2 + q_1\cos q_2 = \tau, \end{gathered} \tag{60}$$

(61)

In this context, q1 signifies the position of the ball along the beam, q2 denotes the angle of the beam, θ represents the normalized torque acting as the system's control input, ζ indicates the damping force coefficient acting directly on the non-actuated coordinate q1, and I represent the normalized beam inertia. After applying the subsequent feedback.

$$\tau = u(q_1^2 + I) + 2q_1\dot{q}_1\dot{q}_2 + q_1\cos q_2 \tag{62}$$

$$\begin{gathered} \ddot{q}_1 = q_1\dot{q}_2^2 - \sin q_2 - \delta\dot{q}_1 \\ \ddot{q}_2 = u, \end{gathered} \tag{63}$$

where u directly influences the actuated coordinate q2.

The aim of control is to stabilize system near its unstable equilibrium position, while ensuring that the beam angle remains within the interval (pi/2, pi/2). To facilitate the development of the requisite Lyapunov function, adjustments in coordinates are introduced for simplification.

$$\dot{q}_1 = p_1 \dot{q}_2 = \frac{p_2}{\eta(q_1)} \tag{64}$$

The specific choice of the smooth variable δ(q1) will be determined subsequently. Accordingly, a suitable transformation is applied to u.

$$f = u\eta(q_1) + \frac{\eta'(q_1)}{\eta(q_1)} p_1 p_2 \tag{65}$$

Clearly, f can be viewed as the updated controller affecting the actuated coordinate p2. Hence, with both previously established transformations in place, system (2) can be interpreted as:

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{q}_2 &= \frac{p_2}{\eta(q_1)} \\ \dot{p}_1 &= \frac{q_1 p_2^2}{\eta^2(q_1)} - \sin q_2 - \delta p_1 \\ \dot{p}_2 &= f \end{aligned} \tag{66}$$

(67)

Typically, the physical damping component is omitted from the model to streamline the stabilization task. Nevertheless, the presence of dissipation force could render the closed-loop system unstable.

III.b. The direct Lyapunov method

This section outlines a method for designing a continuous feedback control strategy. The goal is twofold:
Asymptotic Stability: Ensure the system described by equation (5) reaches a stable equilibrium point and remains there over time.

Ball Position Constraint: Maintain the ball's position (variable q2) consistently above the beam within a specific range (θ/2 on either side of the center).

In simpler terms, we aim to simultaneously stabilize both the ball and the beam at an unstable equilibrium point, while keeping the ball constantly positioned above the beam. To achieve this control objective, we'll leverage the Lyapunov direct method [5].

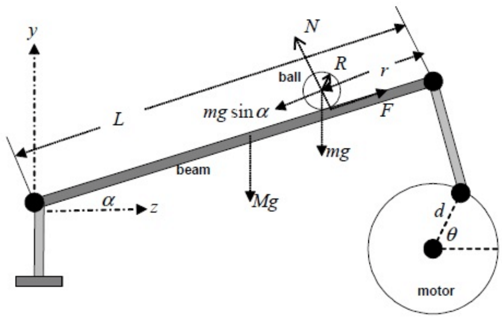

Figure 6. Full FBD of ball and beam system

To formulate the required Lyapunov function, we initially divide the controller f into components.

$$f = -\kappa(q) - c_1(q,p)p_1 - c_2(q,p)p_2 + u_n - \delta_1 p_1 - \delta_2 p_2 \tag{68}$$

The first part allows for the adjustment of the closed-loop potential energy, while the following parts enable the modification of the closed-loop kinetic energy. On the contrary, to counteract any undesired damping effect resulting from the p1 term, if present, we introduce the linear term ipi, where i is either 1 or 2. This term is disregarded when ζ = 0.

$$\dot{q} = M(q)p \quad (69)$$

$$\dot{p} = -C_n(q,p)p - G_n(q) - D_n p + F u_n \quad (70)$$

where FT ¼ [0, 1], M(q) ¼ diag{1, 1(q1) and

$$C_n(q,p) = \begin{bmatrix} 0 & -\frac{p_2}{\eta^2(q_1)} \\ c_1(q,p) & c_2(q,p) \end{bmatrix}; D_n = \begin{bmatrix} \delta & 0 \\ \delta_1 & \delta_2 \end{bmatrix}; \quad (71)$$

$$G_n(q) = \begin{bmatrix} \sin q_2 \\ \kappa(q) \end{bmatrix}. \quad (72)$$

Now, we propose a candidate energy function, as

$$V(q,p) = \Phi(q) + \frac{1}{2} p^T K(q_1) p \quad (73)$$

The function K(q) is selected to fulfill the condition K(q) = KT(q)401, and the function ψ(q) needs to be positive with a local minimum at the origin. For simplicity, ψ is chosen so that rq(0) = 0 and r2 q(0) > 0. As a result, ψ is strictly convex around the origin. Next, we calculate the time derivative of the candidate Lyapunov function along the trajectories of system (7) as follows:

$$\dot{V} = \dot{q}^T \left[\nabla_q \Phi(q) + \frac{1}{2} \nabla_q (p^T K(q) p) \right] \quad (74)$$

$$+ p^T K(q) [-C_n(q,p)p - G_n(q) - D_n p + F u_n] \quad (75)$$

By substituting the equalities q_ = M(q)p and r_q(x^TK(q)x) = [r_q(K(q)x)] ^Tx into the left side of (10), we obtain:

$$\dot{V} = p^T W(q,p) p + p^T Z(q) - p^T K(q) D p + p^T K(q) F u_n, \quad (76)$$

Where,

$$W(q,p) = \frac{1}{2} M(q) \left[\nabla_q (K(q)p) \right]^T - K(q) C_n(q,p) \quad (77)$$

And

$$Z(q) = M(q) \nabla_q \Phi(q) - K(q) G_n(q) \quad (78)$$

Note that the time derivative of V is readily given by:

$$\dot{V} = -k p^T K(q) F F^T K(q) p - p^T K(q) D p, \quad (79)$$

if the following conditions are fulfilled:

(i) un ¼ kFTK(q)p, with k40; (ii) K and are selected such that W(q, p) ¼ WT(q, p) and Z(q) ¼ 0;

(ii)

(iii) 1 and 2 are selected such that pTK(q)Dp 0.

In the Lyapunov sense, the stability of the closed-loop system is guaranteed, if V is chosen to be positive (proper on its sub-levels), has a local minimum at the origin, and acts as a non-increasing function. This decision ensures that q and p remain bounded within a specified compact set, indicating stability. In the following discussions, we will clarify the asymptotic convergence of the closed-loop system.

IV.a. Controller Design

The ball-on-beam system utilizes a cascaded control approach with two independent loops depicted in Figure 5.1. The inner loop prioritizes maintaining the desired motor position based on sensor feedback, while the outer loop focuses on regulating the ball's location along the beam. Despite the potential benefits of an additional beam angle measurement loop for enhanced control, it's excluded from the current design due to limitations in available sensors.

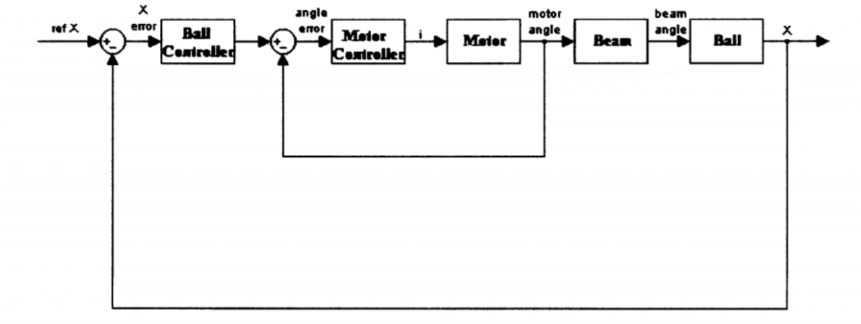

Figure 7. The Closed Loop Ball-on-Beam System features an inner loop governing the motor position and an outer loop regulating the ball's position along the beam.

IV.b. Motor Controller Design

The controller settings were fine-tuned to match the ball-on-beam system's real-world needs. This involved using special filters (lead compensators) to counteract the system's natural response (multiple poles) and make it react quicker (improve transient response). Additionally, an integrator was added to the motor control to eliminate any lingering position errors and handle external disturbances. To effectively manage the ball control, the motor control system needed to react much faster (higher bandwidth). This resulted in a higher operating range (crossover frequency) of around 25 Hz for the motor control. Finally, a custom filter (Gm) designed specifically for the motor was implemented to achieve these optimized settings.

$$\frac{(0.5s+1)(s+50)}{0.5s(s+500)} \tag{80}$$

In addition to the lead compensator, an integrator with a time constant of 0.5 seconds was incorporated in cascade. The transfer function of the integrator can be represented as follows.

$$\frac{(0.5s+1)}{0.5s} \tag{81}$$

The motor controller was formed by combining the integrator with the motor lead compensator.

$$G_{bol} = \frac{g(0.5s+1)(s+50)(s+2)k_t}{s^2\left(1+\frac{2}{5}\left(\frac{a}{a'}\right)^2\right)[0.5s(s+500)/s^2+(0.5s+1)(s+50)k_t](s+20)}. \tag{82}$$

Figures 5.2 and 5.3 visually represent the open-loop behavior (Bode plots) of the controller (including the lead compensator, integrator, and integrated motor controller) and the plant described by Equation 2.10. By combining the controller and motor dynamics, we obtain the forward transfer function, which essentially describes how control signals are translated into motor movement. This motor transfer function is then provided for further analysis.

IV.c. Ball Controller Design

The ball controller significantly influences the system's overall behavior, as defined by the transfer function in Equation 5.5. Figure 5.5 highlights how integrating the inner motor control loop improves the system's controllability. However, achieving a high bandwidth (around 25 Hz) for the ball control loop itself is unrealistic due to the ball's dynamics. As a result, the ball's dynamics have minimal impact on the motor's behavior. To manage the ball's position, a lead compensator was implemented, similar to the motor control loop, aiming for a more modest bandwidth of 1 Hz (6 rad/s) with the chosen lead controller design.

$$\frac{(-368.2)}{(s^4+1.7764e^{-15}s^3-3.9080e^{-14}s^2)} \tag{83}$$

Since the inner motor control loop already incorporates an integrator to eliminate position errors, including another integrator in the ball control loop becomes redundant. Merging the ball controller with the pre-integrated motor loop establishes the complete open-loop system. This combined open-loop system represents the overall dynamics before feedback is introduced for closed-loop control.

$$G_m = \frac{(s+2)}{(s+20)} \tag{84}$$

The complete closed-loop transfer function for the entire ball-on-beam system is formulated as:

$$G_{sys} \quad = \frac{g(0.5s+1)(s+50)(s+2)k_t}{s^2\left(1+\frac{2}{5}\left(\frac{a}{a'}\right)^2\right)[0.5s(s+500)Js^2+(0.5s+1)(s+50)k_t](s+20)+g(0.5s+1)(s+50)(s+2)k_t}. \tag{85}$$

IV.d. Control Design: nonlinear system

We'll build on the derived equations of motion to design a controller. First, we'll create a simplified model by linearizing the system around an equilibrium state. This linearized model will then be used to develop a stabilizing controller that utilizes angular position measurements for feedback

The equations of motion are:

$$\begin{gathered}(I_b + m_s \cdot r_b^2) \cdot \ddot{\theta}_b + 2 \cdot m_z \cdot r_b \cdot \dot{r}_b \cdot \dot{\theta}_b + m_s \cdot g \cdot r_b \cdot \cos\theta_b = \tau \\ \ddot{\imath}_b + \frac{5}{7} \cdot \left(g \cdot \sin\theta_b - r_b \cdot \dot{\theta}_b^2\right) = 0\end{gathered} \tag{86}$$

The nonlinear states consist of θb, θ̇b, r, and ṙ, and the corresponding state equations are:

$$\dot{x} = A \cdot x + B \cdot u \tag{87}$$

$$y = C \cdot x \tag{88}$$

To establish a desired equilibrium in: we thus define the states a,

$$\theta_e = 0 \qquad \dot{\theta}_e \tag{89}$$

$$r_e \qquad \dot{r_e} \tag{90}$$

To obtain a linearized state model suitable for controller design, we substitute the revised values into the state equations and consider the equilibrium point (x_1=x_2=x_3=x_4=u=0). This model expresses the system's dynamics using state variables (represented by the vector x), an input (scalar u), and an output (scalar y).

$$x_1 = \theta - \theta_e = \theta \tag{91}$$

$$x_2 = \dot{\theta} - \dot{\theta}_e = \dot{\theta} \tag{92}$$

$$x_3 = r - r_e = r - 1 \tag{93}$$

$$x_4 = \dot{r} - \dot{r_e} = \dot{r} \tag{94}$$

The matrices A, B, and C define the connections between the state, input, and output variables. In our system, these matrices are expressed as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-m_s \cdot g}{I_b + m_s} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-5}{7} \cdot g & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 \\ \frac{1}{I_b + m_z} \\ 0 \\ 0 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{95}$$

$$\alpha_0(s) = (s+5)^4 = s^4 + 20s^3 + 150s^2 + 500s + 625$$
$$\alpha_0(A) = A^4 + 20A^3 + 150A^2 + 5004A + 625I$$

(96)

Once the linearized state model has been obtained, we must evaluate the controllability and observability of the system. These characteristics dictate the degree to which we can modify the behaviour of the system (controllability) and reconstruct its internal state from the output (observability). If all states of a system can be independently changed with the input and the controllability matrix has a rank of 4, the system is said to be fully controllable. Similar to this, full observability necessitates a rank of 4 in the observability matrix, meaning that every state can be uniquely ascertained from the system's output alone.

$$Q_c = [B \quad AB \quad A^2B \quad A^3B] \tag{97}$$

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \tag{98}$$

$$\begin{cases} \frac{d\theta}{dt} = \dot{\theta} = f_1 \\ \frac{d\dot{\theta}}{dt} = \ddot{\theta} = -\left(\frac{2 \cdot m_s \cdot r_b}{I_b + m_s \cdot r_b^2}\right) \cdot \dot{\theta} \cdot \dot{r}_b - \left(\frac{m_s \cdot g \cdot r_b \cdot \cos\theta_b}{I_b + m_s \cdot r_b^2}\right) + \left(\frac{1}{I_b + m_z \cdot r_b^2}\right) \cdot \tau = f_2 \\ \frac{dr}{dt} = \dot{r} = f_3 \\ \frac{d\dot{r}}{dt} = \ddot{r} = \frac{5}{7} \cdot r_b \cdot \dot{\theta}_b^2 - \frac{5}{7} \cdot g \cdot \sin\theta_b = f_4 \end{cases} \tag{99}$$

$$\tau_e = m_s \cdot g \cdot r_e \tag{100}$$

Ensuring a stable closed-loop system is vital for controller design. This requires the characteristic polynomial of the closed-loop system to have all its roots (poles) located strictly in the left half of the complex plane. To achieve this, we'll determine a state-feedback gain (k) that places the closed-loop system's poles at a desired location, such as all four poles at s = -1. The characteristic polynomial, which defines the overall system behavior, can be expressed as... (the original equation for the polynomial can be inserted here).

$$\alpha_c(s) = (s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1$$
$$\alpha_c(A) = A^4 + 4A^3 + 6A^2 + 4A + I \tag{101}$$

Hence, by utilizing Ackermann's formula with the controllability matrix (Q_c) as defined earlier, we position the observer poles at (s = -5), which is five times higher than our controller poles. This selection results in enhancing the system's speed.

$$k = [0\ 0\ 0\ 1] \cdot Q_c^{-1} \cdot \alpha_c(A) \tag{102}$$

(Qo) represents the observability matrix defined earlier, and based on this, we construct the observer feedback controller as follows:

$$L = \alpha_0(A) \cdot Q_0^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{103}$$

$$\frac{d\hat{x}}{dt} = (A - B \cdot k - L \cdot C) \cdot \hat{x} + L \cdot y$$
$$u = -k \cdot \hat{x} \tag{104}$$

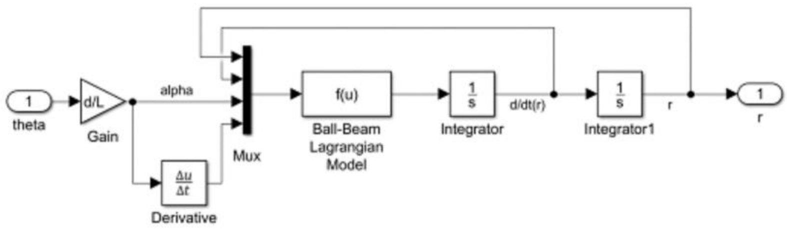

Figure 8. Controller of linearized Lagrangian model in Simulink

V. Pole Placement Design

Our system form is

$$\dot{x} = Ax + bu \qquad (105)$$

The system's stability and open-loop behavior depend heavily on the eigenvalues of its matrix (A). These eigenvalues represent the inherent dynamic characteristics of the system. The state vector (x) describes the system's state, while the input signal (u) can be a force or torque depending on the system (e.g., pushing/pulling force for pendulums or cranes, torque for ball-and-beam systems). In real-world applications, electric motors generate these forces/torques. However, for simplicity, lab models often exclude explicit actuator modeling. Here, control is achieved using pole placement, where the control signal (u) is a weighted sum of the state variables as shown in Figure 6. The specific weights or gains that influence this control signal are represented by the controller parameters$u = -k^T x = -k_1x_1 - k_2x_2 - k_3x_3 - k_4x_4$ (106)

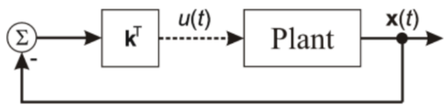

Figure 9. The pole placement based control system

The eigenvalues of \(A\) can be determined by solving the characteristic equation.

$$\varphi_0(\lambda) = |\lambda \mathbf{I}_{4\times 4} - \mathbf{A}| = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \qquad (107)$$

The system matrix (A) in this case is a 4x4 identity matrix. The open-loop system's transient behavior, characterized by its eigenvalues (denoted by λ), is determined by the coefficients (a_1, a_2, a_3, a_4). Feedback control allows adjustments to these dynamics [107]. In some systems, like pendulums or the ball-and-beam system, certain eigenvalues might be zero or lie on the right side of the complex plane, causing instability. Here, feedback becomes essential to shift these problematic eigenvalues to the left side for stabilization. Conversely, for inherently stable systems like cranes with slow or undesirable transient responses, feedback control can be used to design new, more favorable eigenvalues for the closed-loop system. Feedback achieves this by modifying the system matrix through a term called a dyad, which can be expressed as... (insert the original equation for the dyad here).

$$\dot{x} = Ax + bu = (A - bk^T)x \qquad (108)$$

The closed-loop system's required eigenvalues can be set up as follows:
Here, the required eigenvalues (lambda_1, lambda_2, lambda_3, lambda_4) determine the coefficients (p_1, p_2, p_3, p_4), for which the feedback gains can be created using Ackermann's formula:

$$k^t = [0\ \ 0\ \ 0\ \ 1]M_c^{-1}\varphi_{cl}(A) \qquad (109)$$

In this context, the matrix (M_c) is crucial, referred to as the controllability matrix, and is defined as shown. Its rank must be maximum, i.e.,

$$M_c = [b \quad Ab \quad A^2 b \quad A^3 b] \tag{110}$$

the inverse cannot be computed in equation (108). Systems with this property are termed controllable systems. The last term in equation (109) is computed as follows:

$$\&\varphi_{\mathrm{cl}}(\mathbf{A}) = \mathbf{A}^4 + p_1\mathbf{A}^3 + p_2\mathbf{A}^2 + p_3\mathbf{A} + p_4\mathbf{I}_{4\times4} \tag{111}$$

$$\& \quad \begin{aligned} \varphi_{\mathrm{cl}}(\lambda) = |\lambda\mathbf{I}_{4\times4} - (\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}})| &= (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4) \\ &= \lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0, \end{aligned} \tag{112}$$

VI. SIMULATIONS

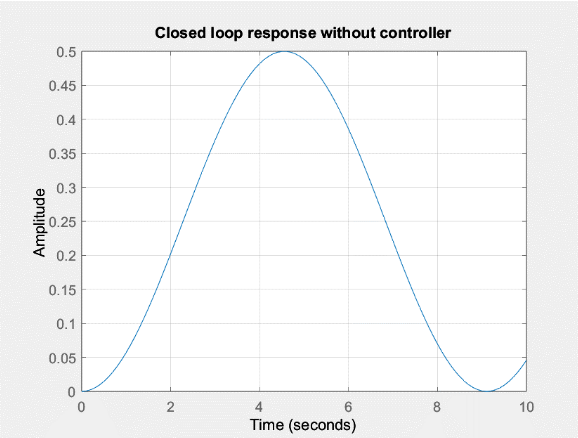

Figure 9. Closed loop response without controller

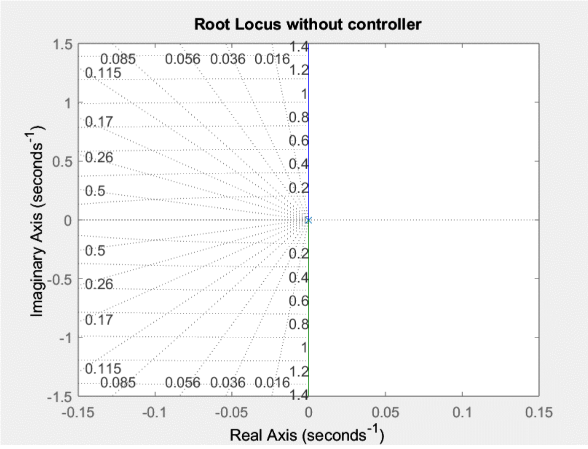

Figure 10. Root Locus without controller

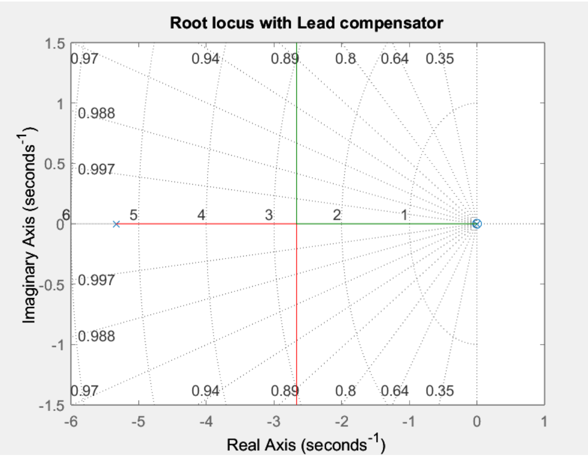

Figure 11. Root Locus with Lead Compensator

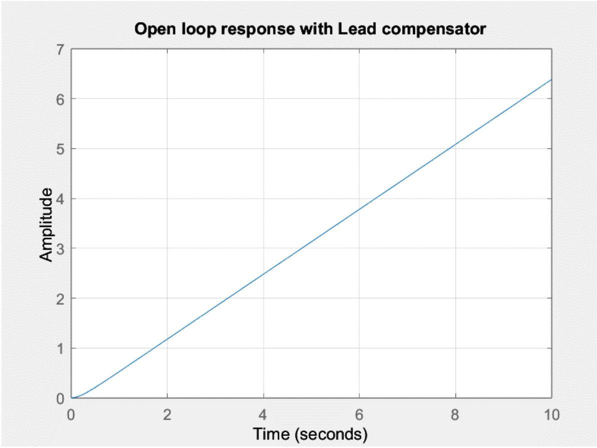

Figure 12. Open loop response with lead compensator

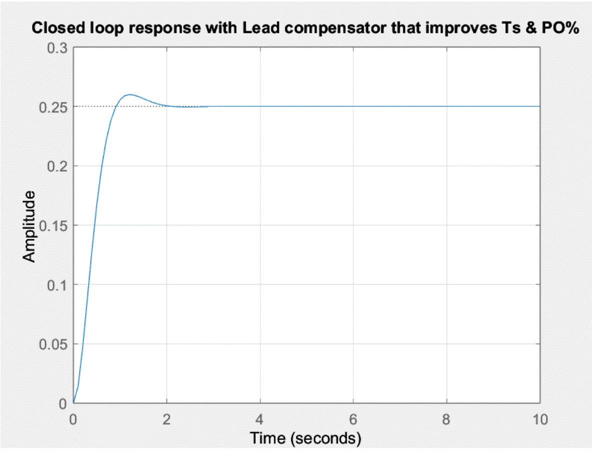

Figure 9. Closed loop response with Lead compensator that improves Ts

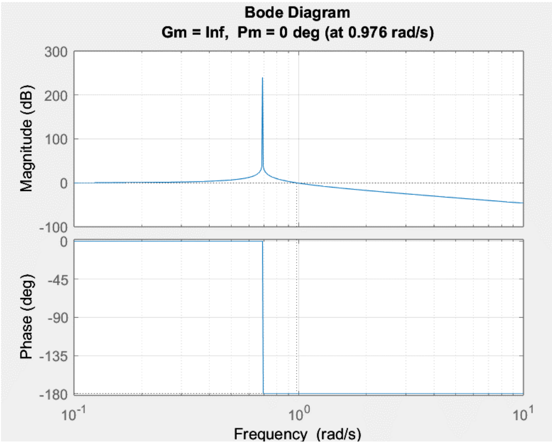

Figure 10. Bode Diagram

VII. SUMMARY

The paper investigates the dynamic behavior and control strategies for a ball-on-beam system, a classic example in the field of mechatronics and control engineering. The system involves a ball moving along a beam, and the study focuses on modeling, analysis, and control methodologies to understand and improve its performance. Key aspects include the derivation of equations of motion, stability analysis, and the design of control strategies. The paper explores techniques such as feedback control, pole placement, and state space representation to enhance the system's stability and response. Practical considerations, such as actuator modeling and torque application, are also discussed. Overall, the research contributes to the broader understanding of dynamic systems and provides insights into the control of complex mechanical structures like the ball-on-beam system

VIII. CONCLUSION

In conclusion, this paper has delved into the comprehensive analysis and control of a ball-on-beam system, shedding light on various aspects crucial for understanding and optimizing its dynamic behavior. The derivation of equations of motion provided a solid foundation, allowing us to explore the system's inherent characteristics. Through stability analysis, we identified key parameters influencing the system's response and devised strategies to enhance its stability. The implementation of control techniques, such as feedback control and pole placement, proved instrumental in achieving desired system behavior. By leveraging state-space representation, we gained a deeper insight into the system's dynamics, facilitating the design of effective control strategies. The consideration of practical factors, including actuator modeling and torque application, added a layer of realism to our study. The findings of this research contribute not only to the specific understanding of the ball-on-beam system but also to the broader field of mechatronics and control engineering. The methodologies and insights presented here can be applied to similar dynamic systems, paving the way for advancements in control strategies for complex mechanical structures. As technology continues to evolve, the knowledge gained from this study may find applications in diverse areas, from robotics to automation and beyond.

IX. REFERENCES

1. Aguilar-Ibañez (2009) presented a study on stabilizing the ball on an actuated beam using the Lyapunov direct method, published in the International Journal of Control.

2. Virseda (2004) conducted research on the modeling and control of the Ball and Beam Process at Lund Institute of Technology.

3. Rosales (2004) developed a Ball-on-Beam Project Kit as part of their Bachelor's degree requirements at the Massachusetts Institute of Technology.

4. Bolívar-Vincenty and Beauchamp-Báez (Year not provided) discussed the modeling of the Ball-and-Beam System using Newtonian Mechanics and Lagrange Methods, affiliated with the Universidad de Puerto Rico Recinto de Mayagüez.

5. Aguilar-Ibañez et al. (Year not provided) explored the stabilization of the Ball on the Beam System through the Inverse Lyapunov Approach, along with Suarez-Castanon and de Jesús Rubio.