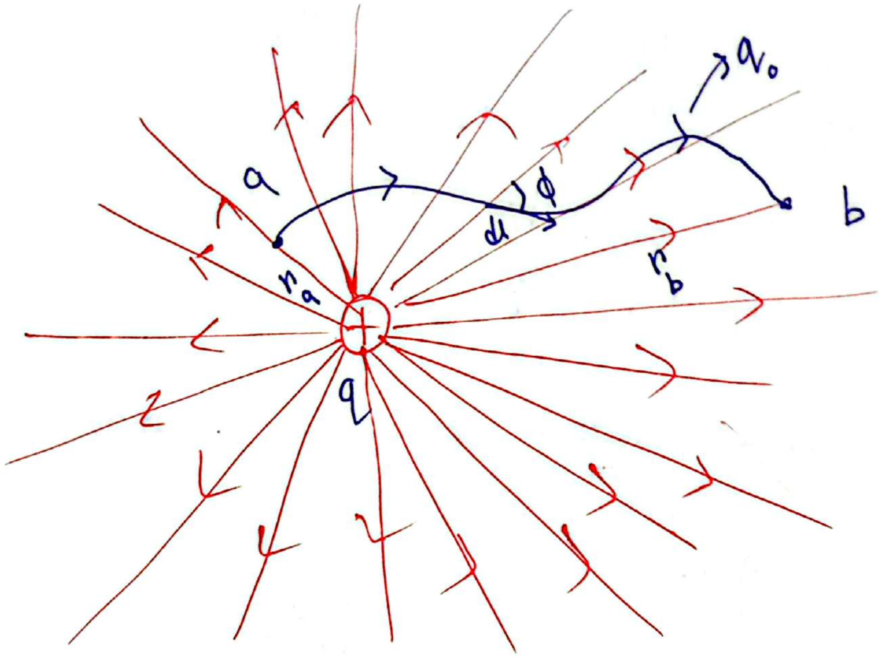
Electric Potential

Electric Potential Energy of two point charges:



Here we are assuming q_0 charge is moving from point a to b in the electric field of q.

$$W_{a\to b} = \int_{r_a}^{b} F\cos\phi\, dl = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\cos\phi\, dl$$

$$= \frac{1}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{qq_0}{r^2}\,dr$$

$$= \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_a} - \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_b}$$

So electric potential energy of two point charges,

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r}$$

Potential energy associated with q_0

$$U = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots\right)$$

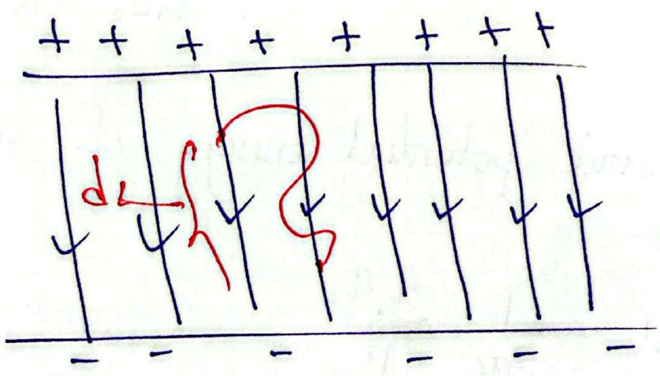
$$= \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

Total potential energy of any arrangement of point charges:

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$$

* Potential energy in uniform field is analogus to gravitational potential energy.

$$U_g = mgh \qquad U_E = qEd$$



Electric Potential:

⇒ Potential is potential energy per unit charge.

$$V = \frac{U}{q_0}$$ Unit JC^{-1}. or V (volt)

Voltage

$$\frac{W_{a\to b}}{q_0} = -\frac{\Delta U}{q_0} = -(V_b - V_a) = V_a - V_b = V_{ab}.$$

So for a point charge, $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$ [Scalar field]

⇒ If there are multiple charges $V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$.

⇒ For continuous distribution of charge, $V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$.

Equipotential Surface:

⇒ Adjacent points that have the same electric potential form an equipotential surface.

⇒ This surface can be an imaginary surface or a real, physical surface.

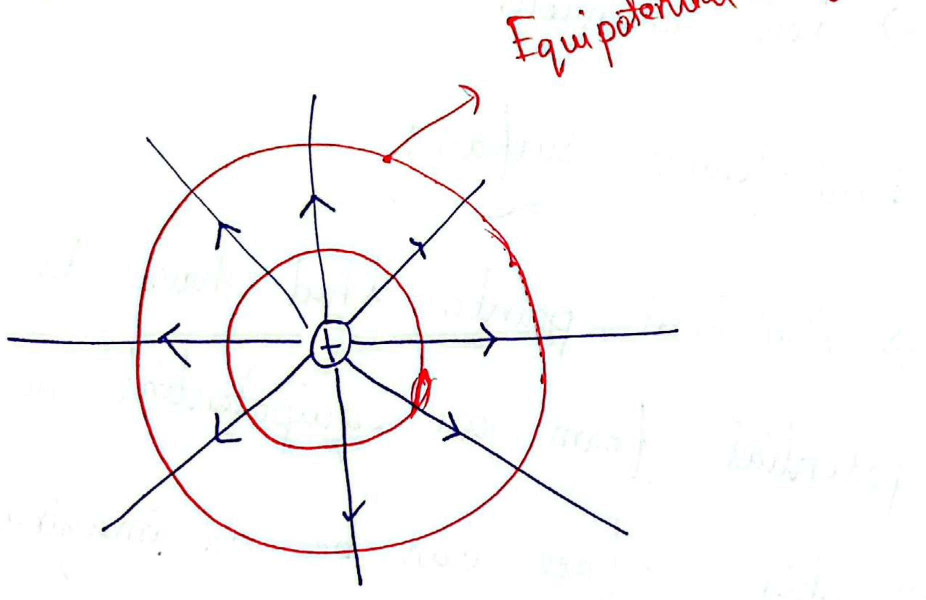
⇒ No work done on charge if ~~they~~ they move over an equipotential surface. So the force must be perpendicular with the surface.

Electric potential from electric field:

$$W_{a\to b} = \int_a^b \vec{F}.\vec{dl} = \int_a^b q_0 \vec{E}.\vec{dl}$$

$$\Rightarrow \frac{W_{a\to b}}{q_0} = \underbrace{V_{ab}}_{\text{Voltage}} = \int_a^b \vec{E}.\vec{dl} = -\int_b^a \vec{E}.\vec{dl}.$$

⇒ Field lines and equipotential surfaces are mutually perpendicular.



Potential Gradient:

$$V_a - V_b = \int_b^a dV = -\int_b^a \vec{E}\cdot\vec{dl}.$$

$$\Rightarrow \quad -dV = \vec{E}\cdot\vec{dl}$$

$$\Rightarrow \quad -dV = E_x dx + E_y dy + E_z dz$$

$$\therefore E_x = \frac{-\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}.$$

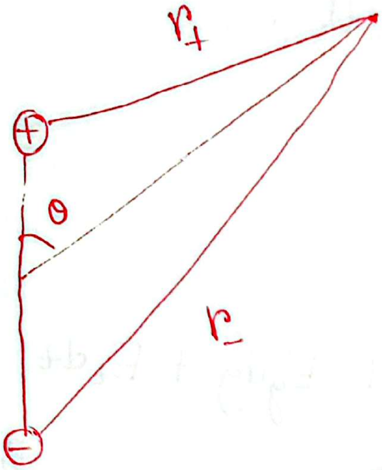
$$\therefore \vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$

$$= -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}.$$

$$\Rightarrow \vec{E} = -\vec{\nabla} V.$$

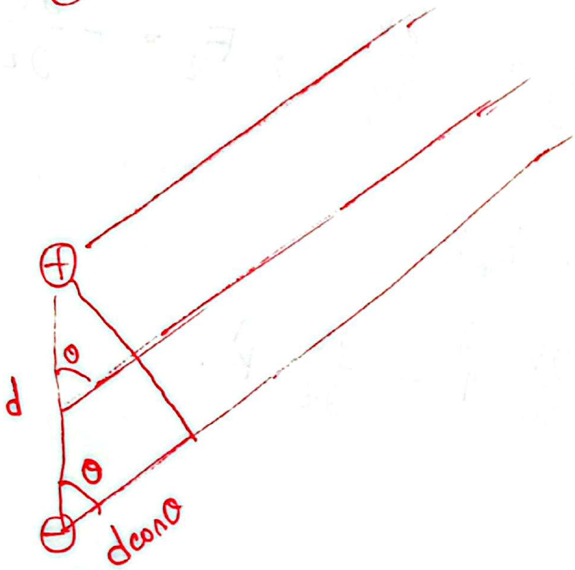
Electric field is equal to negative gradient of potential.

Potential due to a dipole:



$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r_+} + \frac{1}{4\pi\epsilon_0}\frac{-q}{r_-}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{r_- - r_+}{r_+ r_-}$$



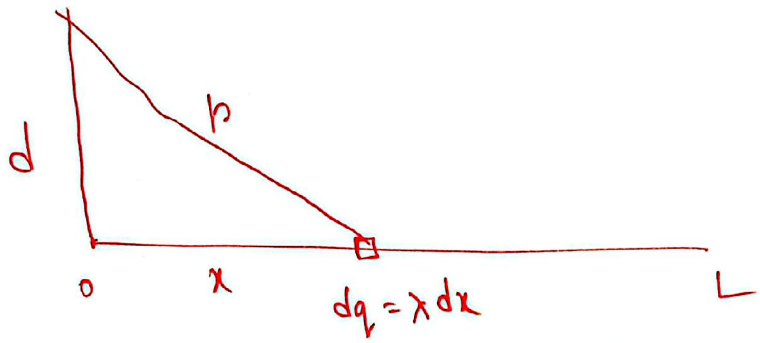
$$\simeq \frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}.$$

(Far from dipole).

Potential due to continuous charge distribution:

Line of charge:



$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{(x^2+d^2)^{1/2}}$$

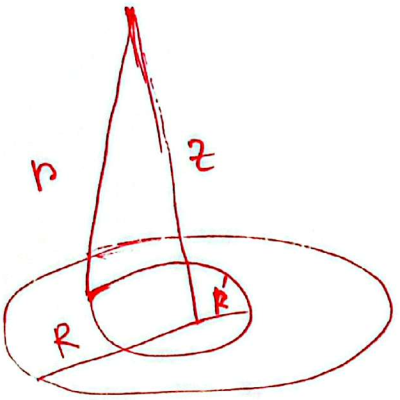
$$V = \int dV = \int_0^L \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{(x^2+d^2)^{1/2}}.$$

$$= \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{dx}{(x^2+d^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[\ln \left(x + (x^2+d^2)^{1/2} \right) \right]_0^L.$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{L + (L^2+d^2)^{1/2}}{d} \right]$$

Charged Disk:



$$\sigma = \frac{q}{\pi R^2}$$

$$dq = \cancel{\sigma(2\pi R')dr}\ \sigma(2\pi R')dR'$$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R' dR')}{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R' dR')}{\sqrt{z^2+R^2}}$$

$$V = \int dV = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{R' dR'}{\sqrt{z^2+R^2}}$$

$$= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - z\right)$$

Potential Problem Solution

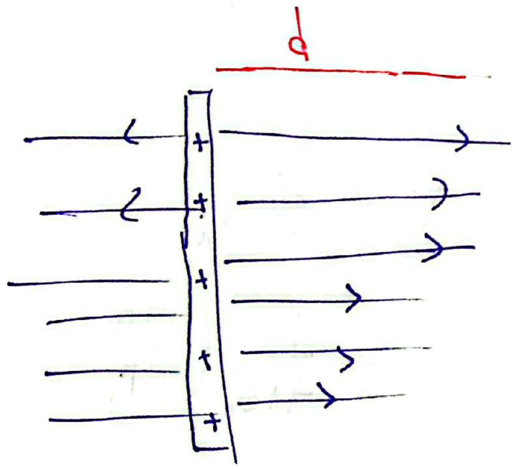
1)

$$U = U_{12} + U_{13} + U_{23}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{d} + \frac{q_1 q_3}{d} + \frac{q_2 q_3}{d}\right)$$

$$= -17 \text{ mJ}.$$

2)



a)

$$W_{a\to b} = \int_a^b \vec{F}\cdot\vec{dr}$$

$$= \int_a^b q\vec{E}\cdot\vec{dr}$$

$$= \int_0^d qE\,dr$$ [as the displacement is in the direction of $\vec{E}$.

$$= qE\int_0^d dr$$

$$= \frac{q\sigma}{2\epsilon_0}d$$

[$E = \frac{\sigma}{2\epsilon_0}$ for infinite sheet of charge]

$$= 1.87\times10^{-21}\text{ J}.$$

b)

$$V_{ab} = \frac{W_{a\to b}}{q}$$

$$\Rightarrow V_a - V_b = \frac{\sigma d}{2\epsilon_0}$$

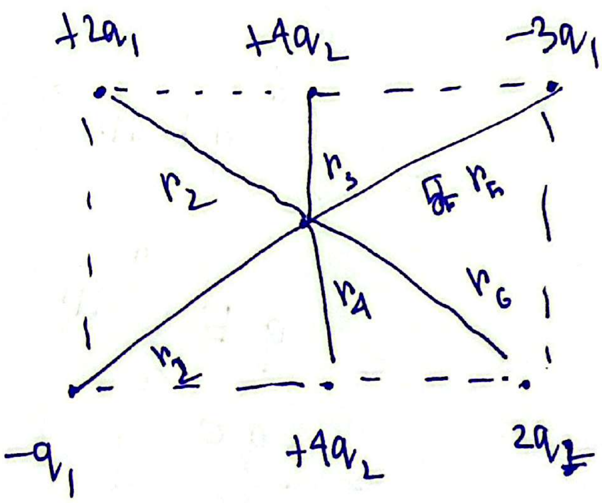
$$\Rightarrow 0 - V_b = \frac{\sigma d}{2\epsilon_0} = -1.17 \times 10^{-2}\ V.$$

3)

$$V = \sum V_i$$

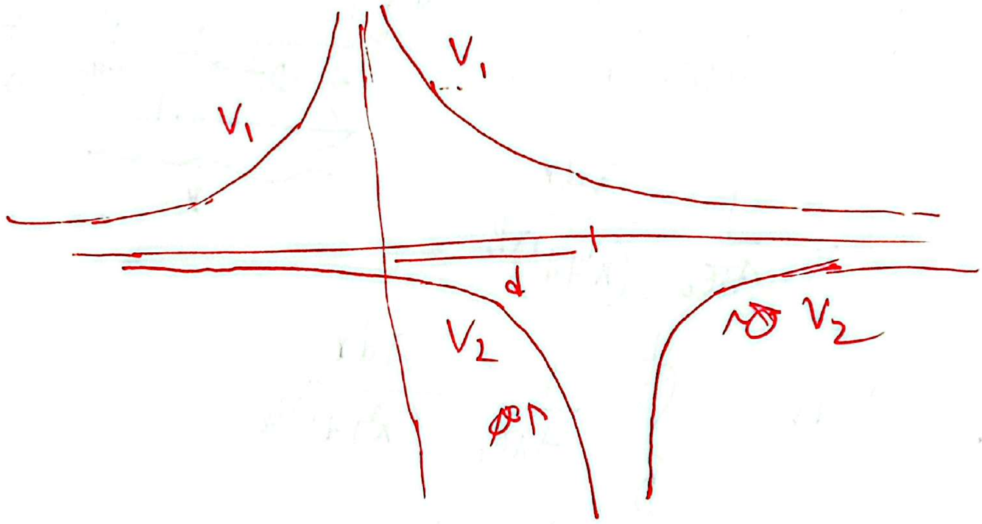
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{2q_1}{r_1} + \frac{-q_1}{r_2} + \frac{4q_2}{r_3} + \frac{4q_2}{r_4} + \frac{-3q_1}{r_5} + \frac{2q_1}{r_6}\right)$$

$$= 2.21\ V.$$



4)
$$V = \sum V_i$$
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{d} + \frac{q}{d} - \frac{q}{d} - \frac{q}{2d}\right)$$
$$= 5.62 \times 10^{-4}\ V.$$

5)



a) $$V(x) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{x} - \frac{3}{d-x}\right) = 0$$
$$\Rightarrow x = \frac{d}{4}$$
$$\Rightarrow x = 6\ cm.$$

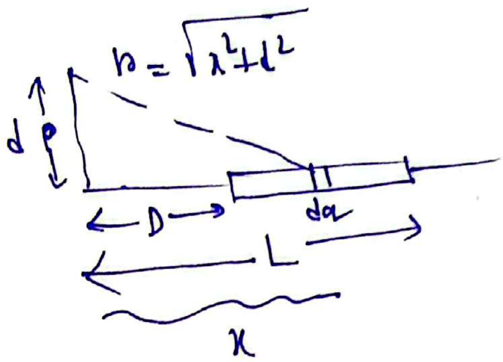
b) $$V(x) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{x} - \frac{3}{d+x}\right) = 0$$
$$\Rightarrow x = 12\ cm.$$

6)

$$V_p = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r}$$

$$= \frac{KQ}{R} = 6.21 \text{ V.}$$

7)



$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{r}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dx}{(x^2+d^2)^{1/2}}$$

$$V = \int dV = \int_D^L \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dx}{(x^2+d^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \int_D^L \frac{dx}{(x^2+d^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \Big[\ln \left(x + (x^2+d^2)^{1/2} \right) \Big]_D^L$$

$$= \frac{\lambda}{4\pi\epsilon_0} \Big[\ln \left(L + (L^2+d^2)^{1/2} \right) - \ln \left(D + (D^2+d^2)^{1/2} \right) \Big]$$

$$= \frac{\lambda}{4\pi\epsilon_0} \Big[\ln \left(4d + (16d^2+d^2)^{1/2} \right) - \ln \left(d + (2d^2)^{1/2} \right) \Big]$$

$$\Rightarrow \frac{\lambda}{4\pi\epsilon_0}\left\{\ln\left(4d+\sqrt{17}d\right) - \ln \cancel{2d}\left(\sqrt{2}+1\right)d\right\}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \frac{\cancel{d}(4+\sqrt{17})}{\cancel{2d}\,\cancel{d}(\sqrt{2}+1)}$$

$$= \frac{2\times 10^{-3}}{4\times\pi\times\epsilon_0} \ln\left(\frac{4+\sqrt{17}}{\sqrt{2}+1}\right)$$

$$= 2.18\times 10^{6}\ \text{V}.$$

8)

$$\vec{E} = -\vec{\nabla} V$$

$$= -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$= -\left(2yz^2\hat{i} + 2xz^2\hat{j} + 4xyz\,\hat{k}\right)$$

$$= -\left(2.(-2).4^2\hat{i} + 2.3.4^2\hat{j} + 4.3.(-2).4\hat{k}\right)$$

$$= \left(64\hat{i} - 96\hat{j} + 96\hat{k}\right)\ \text{N/C}$$