Capacitance

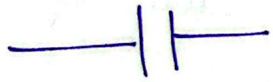
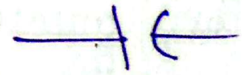
* Any two conductors seperated by ~~two insulators~~ an insulator form a capacitor.

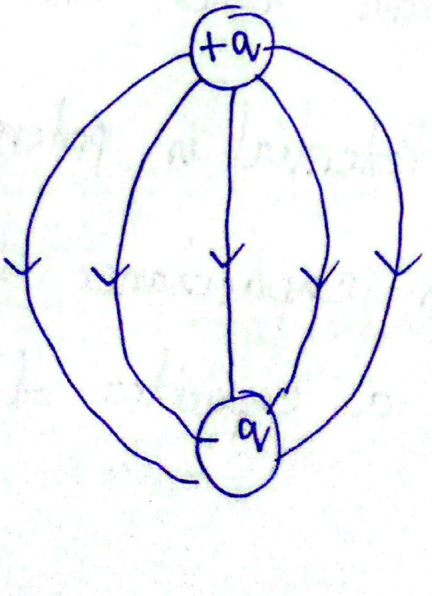
* Initially, the conductors has zero charge and electrons are transferred from one conductor to the other; this is called charging the capacitor.

* Net charge of the capacitor is zero.

* +q charge is stored in one ~~ca~~ conductor and -q charge is stored in ~~at~~ another conductor.

* We say, q charge is stored in the capacitor.

* Symbol  or .



Once the charges are established on the conductors, the battery is disconnected. This gives a fixed ~~potel~~ potential difference V between the conductors.

* The charge q and potential difference V is proportional to each other.

$$q \propto V$$

$$\Rightarrow q = CV$$

↳ This proportionally constant is called the capacitance of the capacitor.

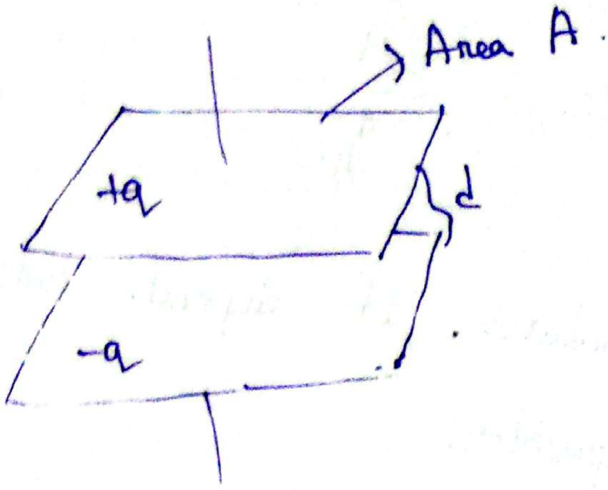
$$\Rightarrow C = \frac{q}{V}$$

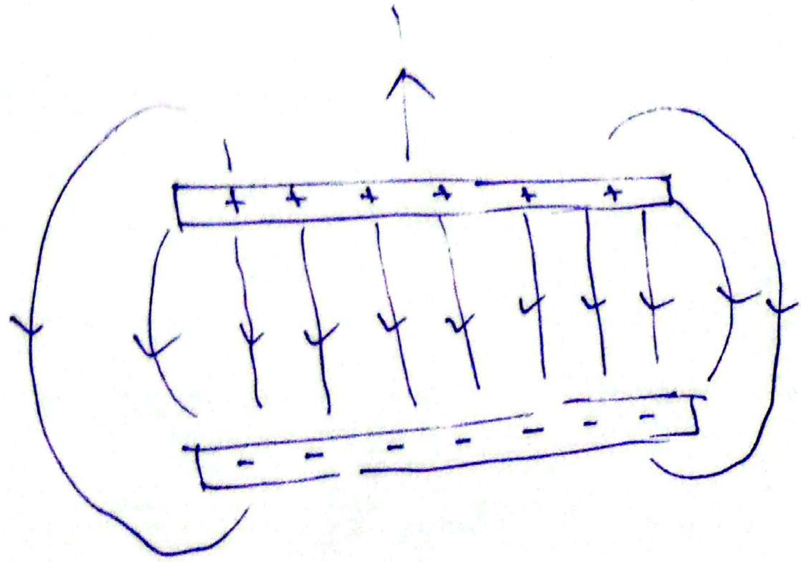
* Unit CV^{-1} or F (Farad).

* The greater the value of C, the greater the magnitude q of charge on either conductor for a given potential difference V and hence the greater the amount of stored energy. (Potential is potential energy per unit charge). Thus capacitance is a measure of the ability of a capacitor to store energy.

Parallel Plate Capacitor:

* The simplest form of capacitor consists of two parallel conducting plates, each with area A, seperated by a distance d that is small in comparison with their dimensions.





* So the electric field is essentially uniform between the plates.

* This arrangement is called parallel plate capacitor.

* $E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} = \frac{q}{A\epsilon_0}$

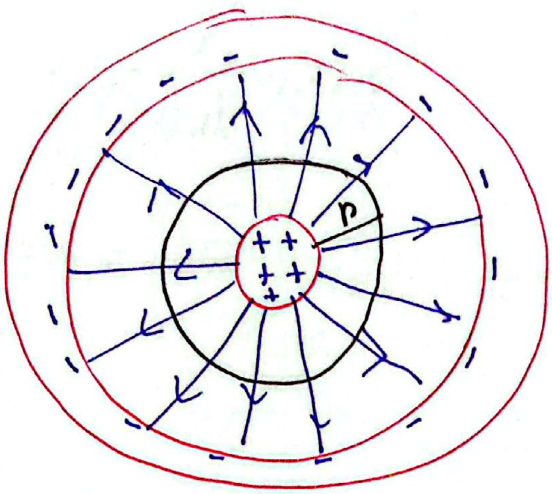
$$V = \int_0^d \vec{E}\cdot\vec{dl} = Ed = \frac{qd}{A\epsilon_0}$$

$$\boxed{\therefore C = \frac{q}{V} = \frac{q}{qd/A\epsilon_0} = \frac{\epsilon_0 A}{d}}$$

C is constant. It depends only on the geometry of the capacitor.

A Cylindrical Capacitor:

* Two cocentric cylindrical conductor shell form cylindrical capacitor.



Let's say, the inner conductor has radius a and the outer shell has radius b.

Capacitance $C = \frac{q}{V}$. where, $V = -\int_{-}^{+} \vec{E} \cdot \vec{dl}$.

E can be determined from gauss's law,

$$\oint \vec{E} \cdot \vec{dA} = \cancel{E\, 4\pi r^2}\ E\, 2\pi r L = \frac{q_{enc}}{\epsilon_0}$$

$$\therefore E = \frac{q}{2\pi \epsilon_0 L r}$$

[L is the length of the cylinder. Let's imagine the ~~gaussian~~ cylindrical gaussian surface of radius r.

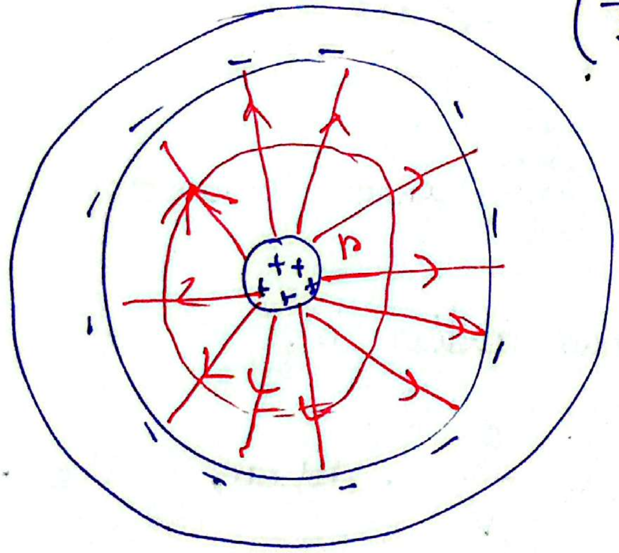
$$\therefore V = \int_{-}^{+} \vec{E}\cdot\vec{dl}$$

$$= -\frac{q}{2\pi\epsilon_0 L}\int_b^a \frac{dr}{r}$$

$$= \frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$\therefore C = \frac{q}{V} = 2\pi\epsilon_0 \frac{L}{\ln(b/a)}.$$

A spherical Capacitor: (~~Two spheres~~) One solid sphere and a spherical shell).



(Inner sphere has radius a and outer sphere has radius b)

To calculate the electric field let's imagine a spherical gaussian surface of radius r.

$$\oint \vec{E}\cdot\vec{dA} = E\,4\pi r^2 = q/\epsilon_0$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

$$V = \int_{-}^{+} \vec{E}.\vec{dl} = -\frac{q}{4\pi\epsilon_0}\int_b^a \frac{dr}{r^2}.$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\boxed{C = \frac{q}{V}} \qquad = \frac{q}{4\pi\epsilon_0}\frac{b-a}{ab}.$$

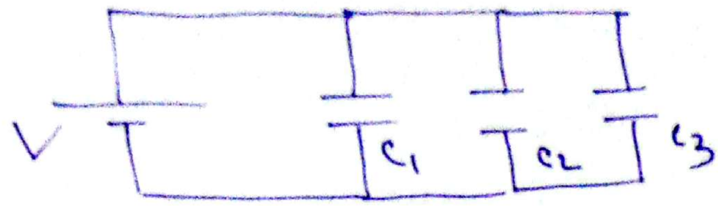
$$C = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

Isolate sphere:

$$C = 4\pi\epsilon_0 \frac{a}{1-a/b}$$

$= 4\pi\epsilon_0 a$ (Capacitance of a single sphere of radius r).

Capacitors in parallel:



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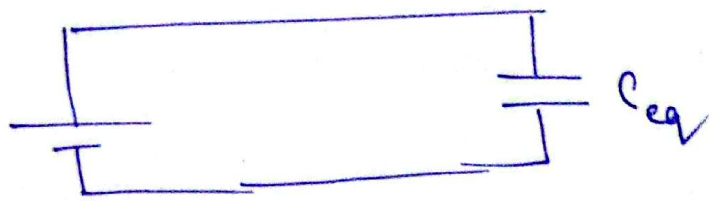


$q_1 = C_1 V$, $q_2 = C_2 V$, $q_3 = C_3 V$. and $q = C_{eq} V$.

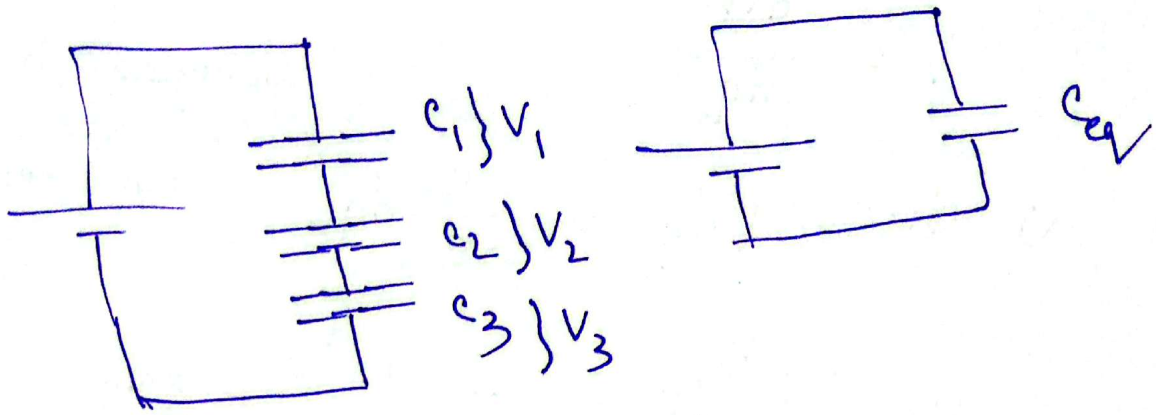
Now, $q = q_1 + q_2 + q_3$

$\Rightarrow C_{eq} V = C_1 V + C_2 V + C_3 V$

$\Rightarrow C_{eq} = C_1 + C_2 + C_3$.

$\Rightarrow C_{eq} = \sum_{i=1}^{N} C_i$

Capacitors in series



$$V = V_1 + V_2 + V_3$$

$$\Rightarrow \frac{q}{C_{eq}} = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}.$$

$$\therefore \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

$$\frac{1}{C_{eq}} = \sum_{j=1}^{N} \frac{1}{C_j}$$

Energy stored in Capacitor:

Suppose you are charging a capacitor from zero charge to Q charge

$$V = \frac{dW}{dq}$$

$$\Rightarrow dW = V dq = \frac{q}{c} dq$$

$$\therefore W = \int_0^W dW = \frac{1}{c}\int_0^Q q\,dq$$

$$= \frac{Q^2}{2c}$$

$$\therefore U = \frac{Q^2}{2c} = \frac{1}{2}cV^2 = \frac{1}{2}QV.$$

Energy density: $\frac{U}{V}$ $= \frac{\frac{1}{2}cV^2}{Ad}$

$V \to$ volume

$$= \frac{1}{2}\frac{\frac{\epsilon_0 A}{d} E^2 d^2}{Ad}$$

$$= \frac{1}{2}\epsilon_0 E^2.$$

Capacitors with a diaelectric:

* Dielectric is a special insulator ~~that~~ can be polarised by an applied electric field.

* There are two kinds of dielectries.

i) Polar: Consists of small dipoles arranged in random directions.

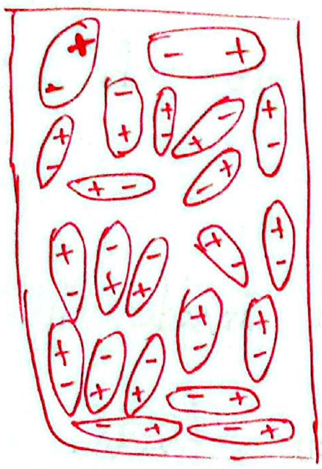
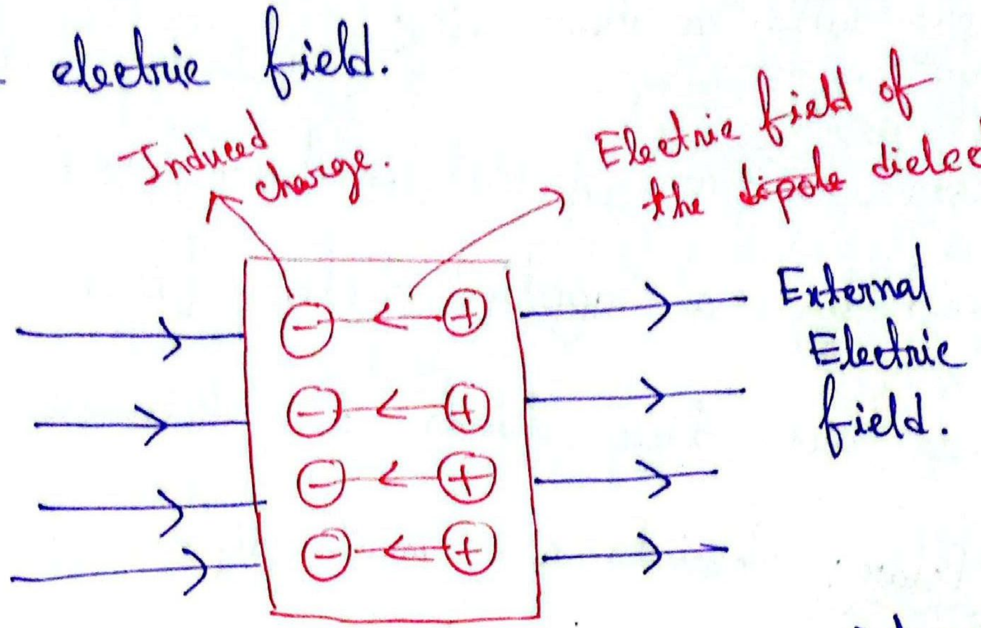


Fig: Polar dielectrics.

ii) Non-polar: Does not have dipoles.

* When placed in an electric field the dipoles of the polar dielectric rearrange according to the electric field. and the non-polar dielectries rearrange their positive and negative ions ~~aco~~ according to

the electric field.

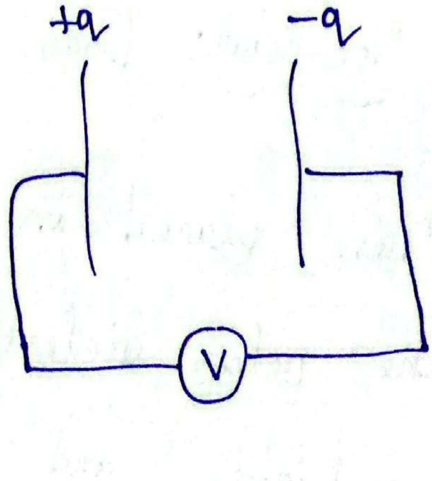


* As a result, there is an ~~external~~ internal electric field inside the dielectric opposite to the external electric field.

* So the net electric field inside the dielectric is less than the external electric field.

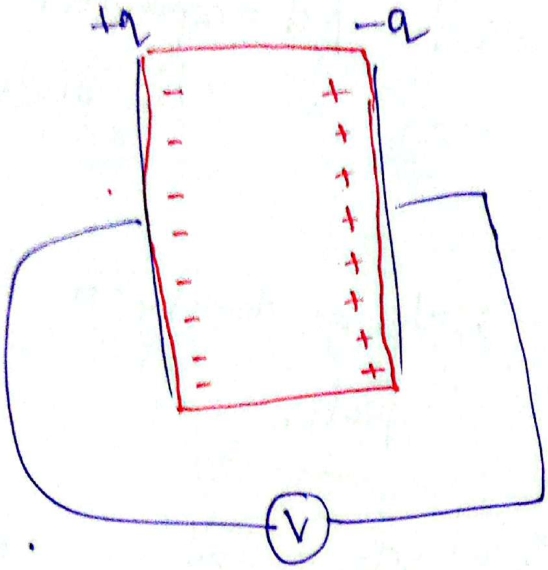
* Let's say we have a capacitor of capacitance

$C_0 = \frac{q}{V_0}$.



* We are measuring the voltage using a voltmeter.

* What will happen if we put a dielectric in the space between the two ~~pal~~ plates of the capacitor?



* We will see the voltage reading of the voltmeter ~~is decreasing~~ has decreased. Let's say the new voltage is V.

Now, $V < V_0$. So, $C = \frac{q}{V}$ and $C_0 = \frac{q}{V_0}$

as $V < V_0$ $\quad C > C_0$

$$\frac{C}{C_0} = \frac{V_0}{V} = k$$ [dielectric constant of the material.]

* The relationship betⁿ ~~two~~ electric field and voltage without and with dielectric is respectively,

$E_0 = V_0 d$ and $E = V d$ [d = separation of the plates]

$\therefore \dfrac{E_0}{E} = \dfrac{V_0}{V} = K.$

We know, $E_0 = \dfrac{\sigma}{\epsilon_0}$ $\xrightarrow{}$ σ → Charge density on the plates.

Similarly, $E = \dfrac{\sigma - \sigma_i}{\epsilon_0}$

$\boxed{\sigma_i = \text{Induced charge density of the dielectric medium.}}$

$$\therefore \frac{\frac{\sigma}{\epsilon_0}}{\frac{\sigma - \sigma_i}{\epsilon_0}} = K$$

$$\Rightarrow \sigma_i = \sigma\left(1 - \frac{1}{K}\right)$$

$$\Rightarrow \frac{q_i}{A} = \frac{q}{A}\left(1 - \frac{1}{K}\right)$$

$$\Rightarrow q_i = q\left(1 - \frac{1}{K}\right).$$

Again $E = \frac{\sigma - \sigma_i}{\epsilon_0} = \frac{\sigma - \sigma(1 - \frac{1}{k})}{\epsilon_0}$

$$= \frac{\cancel{\sigma} - \cancel{\sigma} + \frac{\sigma}{k}}{\epsilon_0}$$

$$= \frac{\sigma}{k\epsilon_0}$$

$$= \frac{\sigma}{\epsilon}$$

* $\epsilon = k\epsilon_0$ is called permittivity of the dielectric.

Now, $C = KC_0 = K\epsilon_0 \frac{A}{d} = \epsilon \frac{A}{d}$

$$\cancel{u = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}}$$

$$\cancel{u = \frac{\frac{1}{2}CV^2}{Ad} = \frac{\frac{1}{2}K C_0 \frac{V_0^2}{K^2}}{Ad} = \frac{\frac{1}{2}C_0 V_0^2}{KAd}}$$

$$\Rightarrow \cancel{u = u_0/K}$$

$$\Rightarrow \cancel{u_0 = uK}$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\varepsilon \frac{A}{d}(Ed)^2$$

$$= \frac{1}{2}\varepsilon A d E^2$$

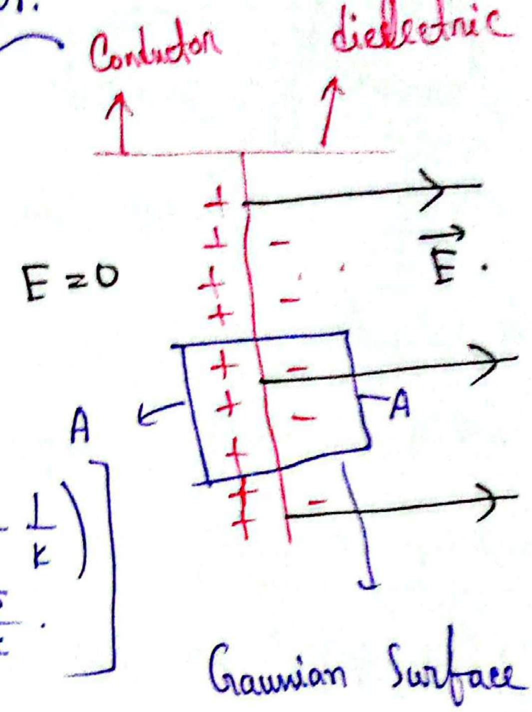
$\therefore u = \frac{U}{Ad} = \frac{1}{2}\varepsilon E^2$. [Energy density in dielectric]

Gauss's law in dielectrics:

Applyin Gauss's law

$$EA = \frac{q_{enc}}{\epsilon_0}$$

$$\Rightarrow EA = \frac{q\,(\sigma - \sigma_i)A}{\epsilon_0}$$



$$\Rightarrow EA = \frac{\sigma A}{K\epsilon_0} \quad \left[\begin{array}{l} \sigma = \sigma_i\left(1 - \frac{1}{K}\right) \\ \sigma - \sigma_i = \frac{\sigma}{K}. \end{array}\right]$$

$$\Rightarrow EA = \frac{\sigma A}{\epsilon}$$

Now we can write gauss's law in dielectric as,

$$\oint \vec{E}.\vec{dA} = \frac{q_{encl-free}}{\epsilon}$$

Charge on the capacitor plate. enclosed by the surface.

1]

a) $C_0 = \frac{\epsilon_0 A}{d} = 1.77 \times 10^{-10}$ F

b) $q = C_0 V_0 = 0.531$ μC

c) $C = \frac{q}{V} = 5.31 \times 10^{-10}$ F

d) $k = \frac{C}{C_0} = \frac{V}{V_0} = 3$

e) $\epsilon = k\epsilon_0 = 2.66 \times 10^{-11}$

f) $q_i = q\left(1 - \frac{1}{k}\right) = 3.54 \times 10^{-7}$ C.

g) $V_0 = E_0 d =$

$\Rightarrow E_0 = \frac{V_0}{d} = 3 \times 10^{5}$

h) $E = \frac{V}{d} = 1 \times 10^{5}$ V/m

i) $U_0 = \frac{1}{2} C_0 V_0^2 = 7.97 \times 10^{-4}$ J

$U = \frac{1}{2} C V^2 = 2.66 \times 10^{-4}$ J.

$u_0 = \frac{1}{2} \epsilon_0 E_0^2 = 0.398$ J/m^3

$u = \frac{1}{2} \epsilon E^2 = 0.133$ J/m^3.

2)

a) $C_0 = \frac{\epsilon_0 A}{d} = 8.21 \times 10^{-12}$ F

b) $q = C_0 V_0 = 7.02 \times 10^{-10}$ C.

c) In the gap, From gaussian surface I.

$$E_0 \int dA = \frac{q}{\epsilon_0}$$

$$\Rightarrow E_0 = \frac{q}{\epsilon_0 A} = 6900 \text{ V/m}.$$

d) From gaussian surface II.

$$\cancel{E \oint dA} \quad \oint \vec{E_1} \cdot \vec{dA} = \frac{q}{\epsilon}$$

$$\Rightarrow E_1 = \frac{q}{\epsilon A} = \frac{q}{K\epsilon_0 A} \quad [\epsilon = K\epsilon_0]$$

$$= 2.64 \times 10^3 \text{ N/C}$$

e) $V = \int_-^+ E ds = E_0(d-b) + E_1 b$

$$= (6900 \text{ N/C})(0.0124 - 0.00780\text{m})$$

$$+ (2640)(0.00780) = 52.3 \text{ V}.$$

f) $C = \frac{q}{V} = 1.34 \times 10^{-11}$ F.

3]

$$q_0 = C_0 V_0 = 22.365 \times 10^{-6} C.$$

$V_1 = V_2$

$q_0 = q_1 + q_2$

$\Rightarrow \frac{q_1}{C_1} = \frac{q_2}{C_2}$

$\Rightarrow q_2 = q_0 - q_1$

$$\Rightarrow \frac{q_1}{C_1} = \frac{q_0 - q_1}{C_2}$$

$\Rightarrow q_1 = 6.35 \mu C$

$\therefore q_2 = 16 \mu C.$

4)

a) $$\frac{1}{C_{35}} = \frac{1}{C_3} + \frac{1}{C_5}$$

$$C_{35} = \frac{C_3 C_5}{C_3 + C_5} = 2 \mu F$$

$$C_{P_1} = C_4 + C_{35} + C_2 = 6 \mu F$$

$$C_{P_2} = C_1 + C_6 = 6 \mu F$$

$$C_{eq} = \frac{C_{P_1} C_{P_2}}{C_{P_1} + C_{P_2}} = 3 \mu F.$$

b) $q = CV = 20 \times 3 \times 10^{-6} = 60 \mu C.$

c) $V_1 = 10 V$ as $C_{P_1} = C_{P_2}$. So votage split into half.

d) $q_1 = C_1 V_1 = 30 \mu C.$

e) $V_2 = 10 V.$

f) $q_2 = C_2 V_2 = 20 \mu C.$

g) $V_3 = 5 V$ so, $q_3 = C_3 V_3 = 20 \mu C.$