

Image Processing and Computer Vision Notes

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4. Spacial Filtering

- **Spacial Filters** (or *Local Operators*) compute the **new intensity** of a pixel p **based on** the intensities of its **neighbours**.
- Useful image processing functions:
 - Sharpening (edge enhancement);
 - Denoising.
- Important subclass: **Linear Shift-Invariant** (LSI) operators, which consist in a **2D convolution** between the **input image** and the **impulse response function** (*point spread function* or *kernel*) of the **LSI operator**.

4.1. LSI Operators

- Given an input 2D signal $i(x, y)$ which is a weighted sum of two signals $i_1(x, y), i_2(x, y)$, a 2D operator $T\{\cdot\} : o(x, y) = T\{i(x, y)\}$ is said to be **linear** iff the output signal is the same weighted sum of the responses to the individual signals (*superposition of effects*):

$$T\{ai_1(x, y) + bi_2(x, y)\} = ao_1(x, y) + bo_2(x, y)$$

with $o_1(\cdot) = T\{i_1(\cdot)\}$, $o_2(\cdot) = T\{i_2(\cdot)\}$, and a, b constant.

- The operator is said to be **shift-invariant** iff the output of a displaced input signal is the displaced response to the undisplaced signal:

$$T\{i(x - x_0, y - y_0)\} = o(x - x_0, y - y_0)$$

- Assuming $i(x, y) = \sum_k w_k e_k(x - x_k, y - y_k)$ and posing $h_k(\cdot) = T\{e_k(\cdot)\}$ it follows that:

$$\begin{aligned} o(x, y) &= T\left\{\sum_k w_k e_k(x - x_k, y - y_k)\right\} = \sum_k w_k T\left\{e_k(x - x_k, y - y_k)\right\} \\ &= \sum_k w_k h_k(x - x_k, y - y_k) \end{aligned}$$

i.e. if the input signal is a weighted sum of displaced elementary functions, the output is given by the same weighted sum of the displaced responses to the elementary functions (combination of linearity and shift-invariance).

- Thus, **if the response to each elementary function is known, the output can be constructed** by combining the responses to each elementary function using input weights and

by shifting them.

- It is **always possible** to **decompose an input signal** into a **combination of** displaced and weighted simpler signals, i.e. **impulses**.

4.2. Impulse Response and Convolution

- The *Dirac Delta function* is defined as:

$$\delta(x, y) = \begin{cases} \neq 0 & \text{if } (x, y) = (0, 0) \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(x, y) dx dy = 1$$

- Any 2D signal can be expressed as an infinite weighted sum of displaced unit impulses (*Dirac delta function*):

$$i(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) \delta(x - \alpha, y - \beta) d\alpha d\beta$$

known as *sifting* property of unit impulse.

- Due to linearity and shift-invariance, the output signal can be expressed as:

$$\begin{aligned} o(x, y) &= T \left\{ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) \delta(x - \alpha, y - \beta) d\alpha d\beta \right\} \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) T \{ \delta(x - \alpha, y - \beta) \} d\alpha d\beta \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) h(x - \alpha, y - \beta) d\alpha d\beta \end{aligned}$$

where $h(x, y) = T \{ \delta(x, y) \}$ is the **impulse response** operator, i.e. the output signal when the input signal is a unit pulse.

- The above operation is called **continuous 2D convolution**.
- Thus, applying an LSI operator is about computing convolution between input signal and input response of the operator.

Properties of Convolution

- Often denoted by the symbol “*”:

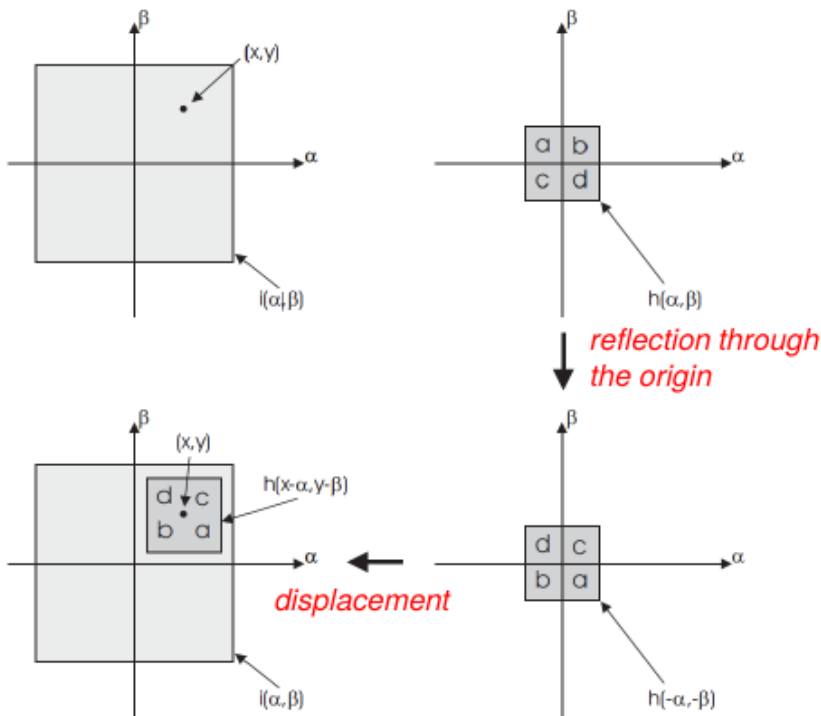
$$o(x, y) = i(x, y) * h(x, y)$$

- Properties:

1. $f * (g * h) = (f * g) * h \Rightarrow$ Associative property
2. $f * g = g * f \Rightarrow$ Commutative property
3. $f * (g + h) = f * g + f * h \Rightarrow$ Distributive property
4. $(f * g)' = f' * g = f * g' \Rightarrow$ Commutativity with Differentiation

Graphical View of Convolution

$$o(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) \delta(x - \alpha, y - \beta) d\alpha d\beta$$



Correlation

- Correlation of signal $i(x, y)$ with signal $h(x, y)$ is defined as:

$$i(x, y) \circ h(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) h(x + \alpha, y + \beta) d\alpha d\beta,$$

$$h(x, y) \circ i(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\alpha, \beta) i(x + \alpha, y + \beta) d\alpha d\beta$$

- Unlike convolution, correlation is not commutative:

$$h(x, y) \circ i(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\alpha, \beta) i(x + \alpha, y + \beta) d\alpha d\beta$$

substituting $\xi = x + \alpha, \eta = y + \beta$:

$$h(x, y) \circ i(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\xi, \eta) h(\xi - x, \eta - y) d\xi d\eta$$

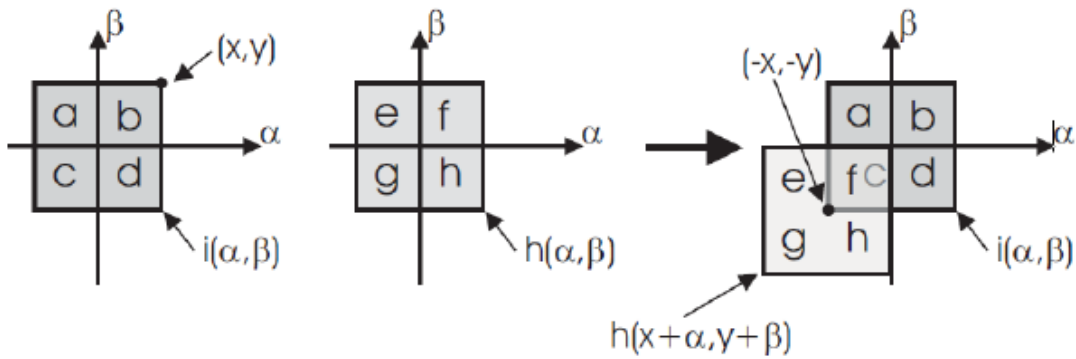
substituting again $\alpha = \xi, \beta = \eta$:

$$\begin{aligned} h(x, y) \circ i(x, y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) h(\alpha - x, \beta - y) d\alpha d\beta \\ &\neq i(x, y) \circ h(x, y) \end{aligned}$$

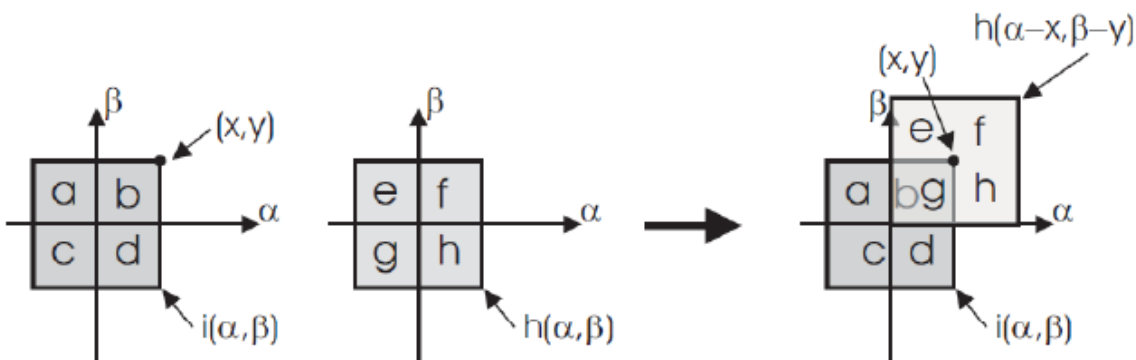
- Correlation is more about finding patterns in the image.

Graphical View of Correlation

$$i(x, y) \circ h(x, y)$$



$$h(x, y) \circ i(x, y)$$



Convolution and Correlation

- The correlation of h and i is similar to convolution: the product of the two signals is integrated after displacing h **without reflection**.
- Hence, if h is an even (symmetric about origin) function ($h(x, y) = h(-x, -y)$) then convolution between i and h is the same as correlation between h and i (only one direction, not both):

$$\begin{aligned}
 i(x, y) * h(x, y) &= h(x, y) * i(x, y) \\
 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) h(x - \alpha, y - \beta) d\alpha d\beta \\
 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(\alpha, \beta) h(\alpha - x, \beta - y) d\alpha d\beta \\
 &= h(x, y) \circ i(x, y)
 \end{aligned}$$

- Correlation is never commutative, even if h is symmetric about origin.
- To recap:
 1. Convolution is commutative: $i * h = h * i$
 2. Correlation is not commutative: $i \circ h \neq h \circ i$
 3. If h is symmetric about origin: $i * h = h * i = h \circ i$

Discrete Convolution

- Consider a discrete 2D LSI operator, $T\{\}$, whose response to the 2D discrete unit impulse (*Kronecker Delta function*) is denoted as $H(i, j)$:

$$H(i, j) = T\{\delta(i, j)\}, \delta(i, j) = \begin{cases} 1 & \text{if } (i, j) = (0, 0) \\ 0 & \text{elsewhere} \end{cases}$$

- Given a discrete 2D input signal $I(i, j)$, the output signal $O(i, j)$ is given by the **discrete 2D convolution** between $I(i, j)$ and $H(i, j)$:

$$O(i, j) = T\{I(i, j)\} = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} I(m, n)H(i - m, j - n)$$

- Analogously to continuous signals, discrete convolution consists in summing the product of the two signals where one has been reflected about the origin and displaced.
- The properties of continuous convolution hold for discrete convolution too.

Implementation

- In image processing both the input signal (image I) and the impulse response (kernel H) are stored into matrices of given sizes:

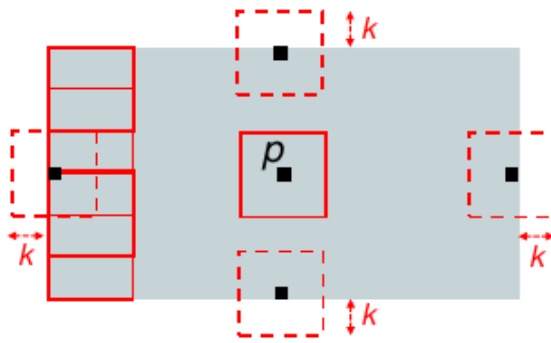
$$\begin{pmatrix} & \vdots & \\ \cdots & I(i, j) & \cdots \\ & \vdots & \end{pmatrix} * \begin{pmatrix} H(-k, -k) & \cdots & H(-k, 0) & \cdots & H(-k, k) \\ H(0, -k) & \cdots & H(0, 0) & \cdots & H(0, k) \\ H(k, -k) & \cdots & H(k, 0) & \cdots & H(k, k) \end{pmatrix}$$

- Conceptually, the kernel slides across the whole image to compute the new intensity at each pixel (without overwriting the input matrix).

```
\* I: M*N pixels, H: (2k+1)*(2k+1) coefficients *\
```

```
for (i = k; i < M-k; i++)
  for (j = k; j < N-k; j++) {
    temp = 0;
    for (m = -k; m <= k; m++)
      for (n = -k; n <= k; n++)
        temp += I[i-m, j-n]*h[m+k, n+k];
    O(i, j) = temp;
  }
```

- At the borders of the image convolution cannot be computed, since those pixels does not have neighbours:
 - the image is cropped;
 - k columns of zeroes are added on the left and right sides, and k rows of zeroes are added on the top and bottom sides (padding).



4.3. Mean Filter

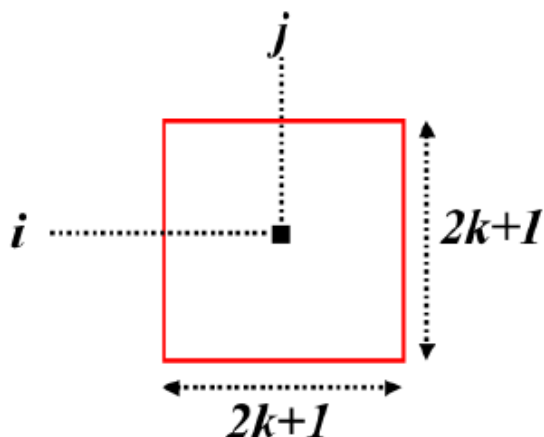
- Mean filtering is the simplest and fastest way to carry out an **image smoothing** (i.e. low-pass filtering) operation, useful for **denoising** and to **cancel out small-sized unwanted details** that might hinder the image analysis task.
- Smoothing is also the key to create the so-called **scale-space**, which endows feature-based algorithms with **scale invariance**.
- **Assuming** that **noise** has **zero mean** and that it's **equally distributed** across the image, it's possible to reduce it using a **filter**: each **pixel intensity** is **replaced** by the **average intensity of its neighbours**.
- It is a LSI operator, as it can be defined through a kernel (e.g. 3×3):

$$\begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Inherently fast because multiplications are not needed; moreover, it can be **implemented very efficiently by incremental calculation schemes** (*box-filtering*) \Rightarrow computation complexity **does not depend** on the **size of the matrix**.

Box-Filtering

- Consider the following scenario:



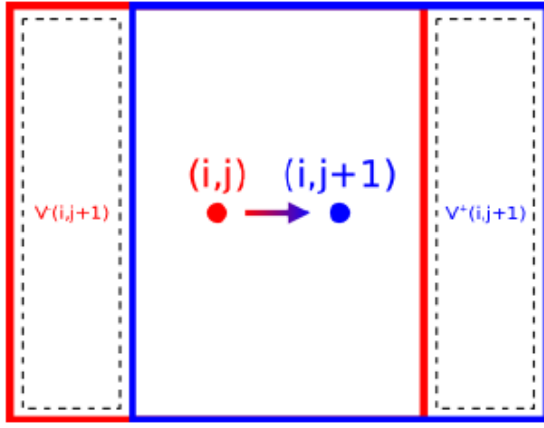
where the intensity of pixel (i, j) in the output image is determined by the average intensity of the pixels belonging to the sliding window of size $(2k + 1) \times (2k + 1)$ centered on that pixel.

- The result of this operation is:

$$\mu(i, j) = \frac{\sum_{m=-k}^k \sum_{n=-k}^k I(i+m, j+n)}{(2k+1)^2} = \frac{s(i, j)}{(2k+1)^2}$$

whose computational effort is due to $s(i, j)$.

- If $s(i, j)$ is known, instead of computing $s(i, j+1)$ from scratch it is possible to readjust the previous sum:
 - by adding $V^+(i, j+1)$, that is the sum of the pixels of the new column now included in the sliding window;
 - by subtracting $V^-(i, j+1)$, that is the sum of the pixels of the old column now excluded by the sliding window.



- Thus, the sum will be:

$$s(i, j+1) = s(i, j) + V^+(i, j+1) - V^-(i, j+1)$$

or more simply:

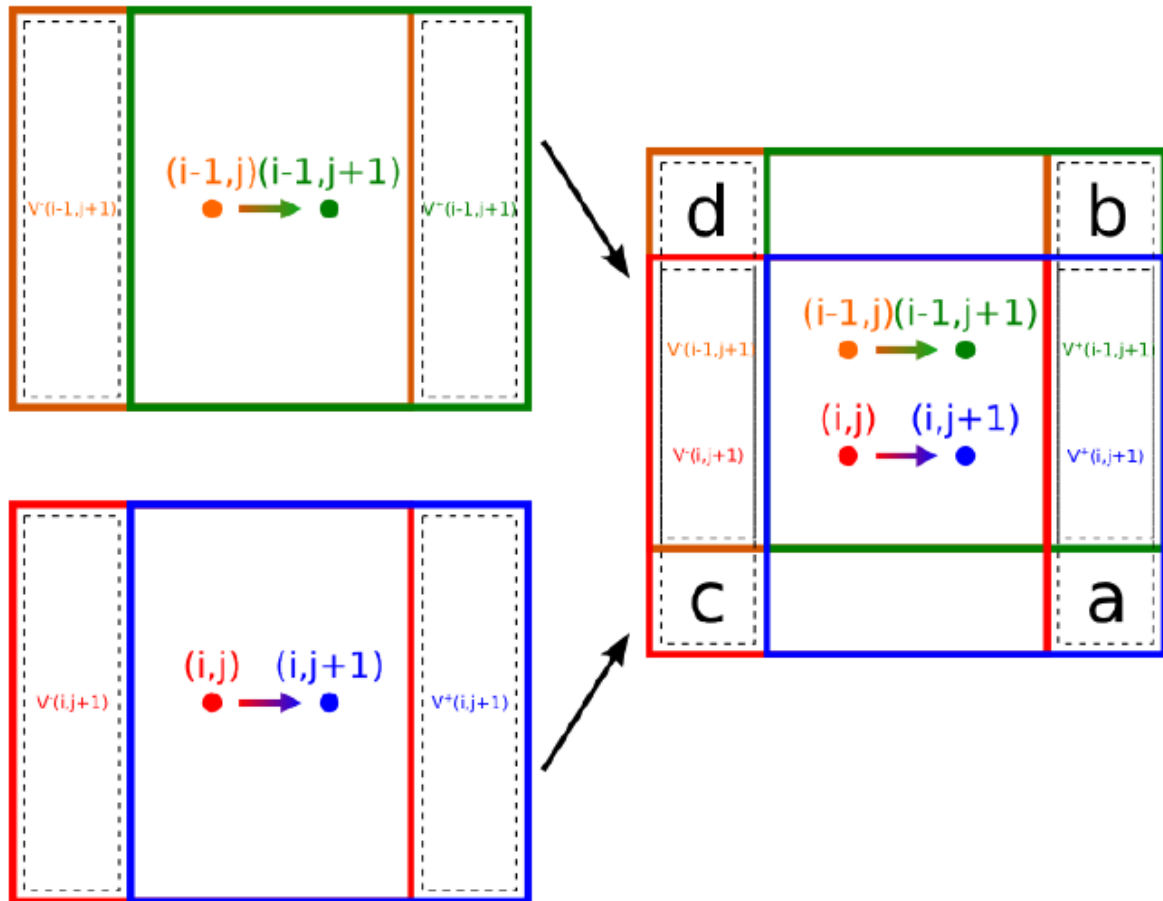
$$s(i, j+1) = s(i, j) + \Delta(i, j+1)$$

where $\Delta(i, j+1) = V^+(i, j+1) - V^-(i, j+1)$.

- This way, the complexity turns from quadratic to linear.
- Moreover, by the time the sliding window shifts from (i, j) to $(i, j+1)$, the previous row $i-1$ will have already been computed; in particular, the shift from $(i-1, j)$ to $(i-1, j+1)$ will have already been performed, and thus $V^+(i-1, j+1)$ and $V^-(i-1, j+1)$ are known.
- $V^+(i-1, j+1)$ and $V^-(i-1, j+1)$ differ from $V^+(i, j+1)$ and $V^-(i, j+1)$ only because of 4 pixels:

$$\begin{aligned} V^+(i, j+1) &= V^+(i-1, j+1) + a - b, \\ V^-(i, j+1) &= V^-(i-1, j+1) + c - d \end{aligned}$$

where a, b, c, d are the pixels shown in the following figure:



- Therefore, the final sum will be:

$$s(i, j+1) = s(i, j) + \Delta(i-1, j+1) + a - b - c + d$$

where $\Delta(i-1, j+1) = V^+(i-1, j+1) - V^-(i-1, j+1)$.

- In conclusion, computing the mean filter for a pixel requires **5** sums \Rightarrow **constant complexity**.
- Linear filtering can be useful to reduce Gaussian noise (though blurs the image), but is **useless** when applied to **impulse noise** (slightly reduces noise intensity, but also spreads it).

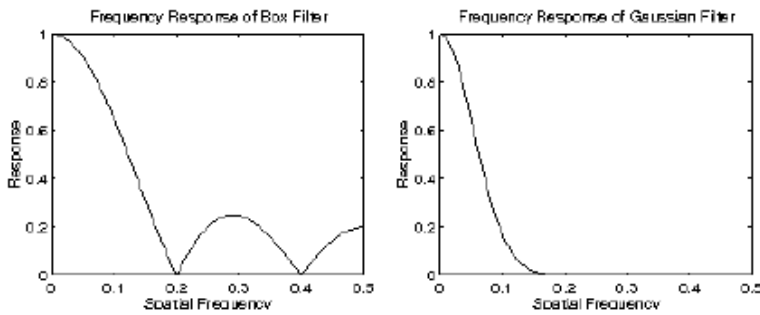
4.4. Gaussian Filter

- LSI operator whose **impulse response** is a **2D Gaussian function** (the product between two 1D Gaussian functions, one along x and one along y) with zero mean and constant diagonal covariance matrix.

$$G(x, y) = G(x)G(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

- $G(x, y)$ is circularly symmetric.
- The **higher** σ , the **stronger** the **smoothing**: in fact, as σ increases, the weights of closer points get smaller while those of farther points get larger.
- The Fourier transform of a Gaussian is a Gaussian with $\sigma_\omega = 1/\sigma$, so that the **higher** σ the **narrower** the **bandwidth** of the filter.
- The Gaussian filter is a **more effective low-pass operator than the Mean filter**, since the frequency response of the former is monotonically decreasing, while the one of the latter exhibits

significant ripples.



- Intuitively, **Gaussian filter** gives **more importance to central pixel and its closer neighbours**, and less importance to far away pixels; on the contrary, **Mean filter** gives the **same importance to every pixel** of the kernel \Rightarrow Gaussian filter is better, since it smooths pixels likely to belong to same surface.
- **Mean filter** with box-filtering is **much faster**, since no multiplications are performed.
- Gaussian filter can be also used to carry out a so-called **multiscale image analysis**, that is to be able to focus on different scales; in fact, as σ increases small details disappear and only the large structure remains.

Implementation

- **Discrete Gaussian kernel** can be obtained by **sampling the corresponding continuous function**, which is however of infinite extent.
- Therefore a finite size must be properly chosen:
 - **larger size** \Rightarrow **more accurate discrete approximation** of ideal continuous filter;
 - **computational cost** grows with filter size;
 - Gaussian gets **smaller and smaller away from the origin**.
- Therefore, for filters with **high σ larger sizes** are required, whereas for filters with **small σ smaller sizes** can be used
- As the interval $[-3\sigma, +3\sigma]$ captures **99%** of the area (“energy”) of the Gaussian area, a possible choice is to take a $(2k + 1) \times (2k + 1)$ kernel with $k = 3\sigma$.
- To **speed-up filtering operation**, it may be convenient to convolve the image by an **integer kernel** rather than a floating point one.
- An integer Gaussian kernel can be obtained by dividing all coefficients by smallest one, rounding to nearest integer and normalizing by sum of integer coefficients:

$$k \rightarrow k_1 = \frac{k}{k_{min}} \rightarrow k_2 = \text{round}(k_1) \rightarrow k_3 = \frac{k_2}{\text{sum}(k_2)}$$

Separability

To further speed-up the filtering operation, since 2D Gaussian is the product between two 1D Gaussians, it is possible to split the original 2D convolution into the chain of two 1D convolutions (**separability property**):

$$\begin{aligned} I(x, y) * G(x, y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(\alpha, \beta) G(x - \alpha, y - \beta) d\alpha d\beta, \\ G(x, y) &= G(x)G(y) \\ \Rightarrow I(x, y) * G(x, y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(\alpha, \beta) G(x - \alpha) G(y - \beta) d\alpha d\beta \\ &= \int_{-\infty}^{+\infty} G(y - \beta) \left(\int_{-\infty}^{+\infty} I(\alpha, \beta) G(x - \alpha) d\alpha \right) d\beta \\ &= (I(x, y) * G(x)) * G(y) = (I(x, y) * G(y)) * G(x) \end{aligned}$$

which results in a speed-up of $S = \frac{(2k+1)^2}{2(2k+1)} = k + \frac{1}{2} = 3\sigma + \frac{1}{2}$ (complexity turns **from quadratic to linear**).

4.5. Median Filter

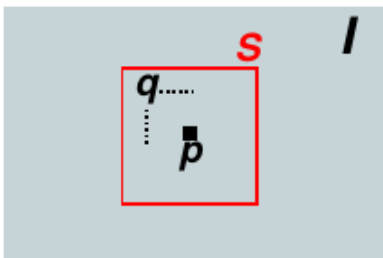
- **Non-linear** filter whereby each pixel intensity is replaced by the **median** over a given neighborhood (the median is the value falling half-way in the sorted set of intensities).

$$\text{median}[A(x) + B(y)] \neq \text{median}[A(x)] + \text{median}[B(y)]$$

- Median filtering **counteracts impulse noise** effectively, as *outliers* (i.e. noisy pixels) tend to fall at either the top or the bottom end of sorted intensities.
- Median filtering tends to keep **sharper edges** than linear filtering.
- It is **not effective** against **Gaussian noise** such as sensor noise, thus after applying it to counteract impulse noise, **linear filters** can **also** be **applied** to deal with Gaussian-like noise.

4.6. Bilateral Filter

- Advanced **non-linear filter** to **accomplish denoising of Gaussian-like noise without blurring** the image (*edge-preserving smoothing*):



- Given S sliding window, p pixel considered, q running pixel (generic pixel belonging to S), I_p, I_q intensity of p, q :

$$O(p) = \sum_{q \in S} H(p, q) \cdot I_q$$

- H must be chosen s.t. it's output is large when the two pixels are close and their intensity are similar \Rightarrow **product of two Gaussians**, one based on **spacial distance** and the other based on **intensity difference**:

$$H(p, q) = \frac{1}{W(p, q)} \cdot G_{\sigma_s}(d_s(p, q)) \cdot G_{\sigma_r}(d_r(I_p, I_q))$$

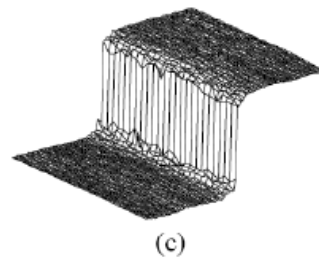
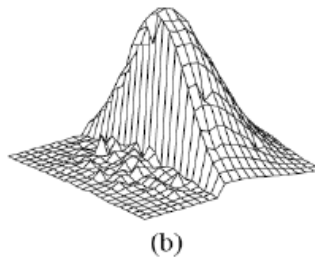
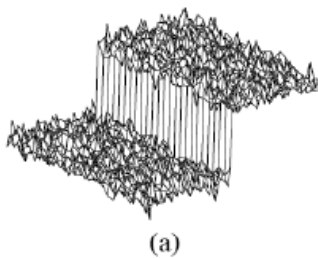
where:

- $d_s(p, q) = \|p - q\| = \sqrt{(u_p - u_q)^2 + (v_p - v_q)^2} \Rightarrow$ Spacial (Euclidean) Distance
- $d_r(p, q) = |I_p - I_q| \Rightarrow$ Range (Intensity) Distance
- $W(p, q) = \sum_{q \in S} G_{\sigma_s}(d_s(p, q)) \cdot G_{\sigma_r}(d_r(I_p, I_q)) \Rightarrow$ Normalization Factor (Unity Gain)

Step-edge as wide as 100 gray-levels

$H(p, q)$ at a pixel just across the edge in the brighter region

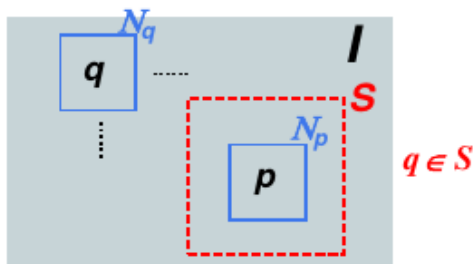
Output provided by the filter $\sigma_s = 5, \sigma_r = 50$



- Given supporting neighborhood, neighbours take larger weight since they are **closer** and **more similar** to central pixel, whereas pixels on other side of edges will be too different to contribute significantly (small weight) to output value.

4.7. Non-local Means Filter

- *Edge-preserving* smoothing filter, based on the idea that the similarity among patches spread over the image can be deployed to achieve denoising.



- Given S sliding window, p pixel considered, q running pixel (generic pixel outside S), N_p, N_q patches around p, q of the same size:

$$O(p) = \sum_{q \in I} w(p, q) I(q)$$

where $w(p, q)$ must be high when N_p looks similar to N_q (measured using norm):

$$w(p, q) = \frac{1}{Z(p)} e^{-\frac{\|N_p - N_q\|^2}{h^2}}$$

where h is patch size and $Z(p) = \sum_{q \in I} e^{-\frac{\|N_p - N_q\|^2}{h^2}}$ is the normalization.

- It gives better results than Mean and Gaussian filters.