Outline for Direct Proof

Proposition If P, then Q.

Proof. Suppose P.

$\vdots$

Therefore Q. ■

Use the method of direct proof to prove the following statements.

1. If x is an even integer, then x^2 is even.

2. If x is an odd integer, then x^3 is odd.

3. If a is an odd integer, then a^2+3a+5 is odd.

4. Suppose $x, y \in \mathbb{Z}$. If x and y are odd, then xy is odd.

16. If two integers have the same parity, then their sum is even. (Try cases.)

17. If two integers have opposite parity, then their product is even.

18. Suppose x and y are positive real numbers. If $x < y$, then $x^2 < y^2$.

Outline for Contrapositive Proof

Proposition If P, then Q.

Proof. Suppose $\sim Q$.

$\vdots$

Therefore $\sim P$. ■

A. Use the method of contrapositive proof to prove the following statements. (In each case you should also think about how a direct proof would work. You will find in most cases that contrapositive is easier.)

1. Suppose $n \in \mathbb{Z}$. If n^2 is even, then n is even.

2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

3. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2-2b)$ is odd, then a and b are odd.

4. Suppose $a, b, c \in \mathbb{Z}$. If a does not divide bc, then a does not divide b.

5. Suppose $x \in \mathbb{R}$. If $x^2+5x < 0$ then $x < 0$.

6. Suppose $x \in \mathbb{R}$. If $x^3-x > 0$ then $x > -1$.

B. Prove the following statements using either direct or contrapositive proof. Sometimes one approach will be much easier than the other.

14. If $a, b \in \mathbb{Z}$ and a and b have the same parity, then $3a+7$ and $7b-4$ do not.

15. Suppose $x \in \mathbb{Z}$. If x^3-1 is even, then x is odd.

16. Suppose $x \in \mathbb{Z}$. If $x+y$ is even, then x and y have the same parity.

17. If n is odd, then $8 \mid (n^2-1)$.

Outline for Proof by Contradiction

> **Proposition** P.
>
> *Proof.* Suppose $\sim P$.
>
> $\vdots$
>
> Therefore $C \wedge \sim C$. ■

Outline for Proving a Conditional Statement with Contradiction

> **Proposition** If P, then Q.
>
> *Proof.* Suppose P and $\sim Q$.
>
> $\vdots$
>
> Therefore $C \wedge \sim C$. ■

A. Use the method of proof by contradiction to prove the following statements. (In each case, you should also think about how a direct or contrapositive proof would work. You will find in most cases that proof by contradiction is easier.)

1. Suppose $n \in \mathbb{Z}$. If n is odd, then n^2 is odd.
2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.
3. Prove that $\sqrt[3]{2}$ is irrational.
4. Prove that $\sqrt{6}$ is irrational.
5. Prove that $\sqrt{3}$ is irrational.
6. If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.
7. If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.
8. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Outline for If-and-Only-If Proof

> **Proposition** P if and only if Q.
>
> *Proof.*
> [Prove $P \Rightarrow Q$ using direct, contrapositive or contradiction proof.]
> [Prove $Q \Rightarrow P$ using direct, contrapositive or contradiction proof.] ■

Prove the following statements. These exercises are cumulative, covering all techniques addressed in Chapters 4–7.

1. Suppose $x \in \mathbb{Z}$. Then x is even if and only if $3x+5$ is odd.
2. Suppose $x \in \mathbb{Z}$. Then x is odd if and only if $3x+6$ is odd.
3. Given an integer a, then $a^3 + a^2 + a$ is even if and only if a is even.
4. Given an integer a, then $a^2 + 4a + 5$ is odd if and only if a is even.
5. An integer a is odd if and only if a^3 is odd.
6. Suppose $x, y \in \mathbb{R}$. Then $x^3 + x^2 y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$.
7. Suppose $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.