THE FUNDAMENTAL GROUPOID

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Abstract. We introduce basic category theory and construct the fundamental groupoid of a topological space. Using homotopy as an analogy, we define natural transformations and equivalence of categories. We conclude with the homotopy invariance of the fundamental groupoid.

1. Introduction

The fundamental groupoid is a generalization of the better-known fundamental group which captures information of the entire space, instead of just a single basepoint. It naturally emerges when one considers the categorification of a group. However, the fundamental groupoid is in some sense not a strict homotopy invariant: the disk and the point have non-isomorphic fundamental groupoids!

Instead, the correct notion of equivalence on groupoids relies on a notion called equivalence of categories. The line of thinking which leads to this definition draws heavy analogies between topological spaces and categories. Equivalence of categories turns out to be closely related to the familiar notion of homotopy equivalence.

After introducing some basic category theory in Section 2, we construct the fundamental groupoid in Section 3. This motivates natural transformations in Section 4, culminating in a proof of the homotopy invariance of the fundamental groupoid.

2. Categorical Preliminaries

Many of the constructions we'll encounter are categorical, or at least are well-stated using categorical ideas, so it's worth spending some time with basic category theory. Our aim will be to develop only some essential language now, and later use our exploration of homotopy as motivation for developing higher category theory. Except where noted, I follow [Rie17, Sections 1.1-1.3].

2.1. Categories.

Definition 2.1. A *category* $\mathtt{C}$ consists of a collection of *objects* $\mathrm{Obj}(\mathtt{C})$ and a collection of *morphisms* $\mathrm{Map}(\mathtt{C})$, such that

- Each morphism f has a specific *domain* $\mathrm{dom}(f)$ and *codomain* $\mathrm{cod}(f)$; we write $f : x \to y$ or $x \xrightarrow{f} y$.
- Each object x has a specific *identity morphism* $1_x : x \to x$.
- For each pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$, there is a specific *composite morphism* $gf : x \to z$. We call f and g *composition-compatible*.

This data must be *associative* and *unital*, which is to say,

- (Associativity) For each composition-compatible triplet, i.e. $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, we have $h(gf) = (hg)f$. We write merely hgf.
- (Unital) For any $f : x \to y$, we have $1_y f = f = f 1_x$.

In other words, the following diagrams commute:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & y \arrow[r, "g"] \arrow[rr, bend left, "hg"] & z \arrow[r, "h"] & w
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "1_x"] \arrow[dr, "f"'] & x \arrow[d, "f"] \\
& y
\end{tikzcd}
\quad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "f"'] & y \arrow[d, "1_y"] \\
& y.
\end{tikzcd}$$

Notation 2.2. We will use $\mathtt{C}(x, y)$ for the morphisms with domain X and codomain Y. We also choose to use multiplicative notation for composition, instead of $\circ$, to distinguish it from function composition, since in many of our categories morphism composition will be concatenation or something completely separate from function composition.

Example 2.3. Many mathematical structures can be studied as categories:

- `Set` is the category of sets and set-functions.
- `Poset` is the category of partially ordered sets and monotone functions.
- `Grp` is the category of groups and homomorphisms.
- `Top` is the category of topological spaces and continuous functions.
- For any field $\mathbb{k}$, $\mathtt{Vect}_{\mathbb{k}}$ is the category of vector spaces over $\mathbb{k}$ and linear maps.

These examples demonstrate the need for the word "collection" in the definition of a category: there is no set of all sets, for example. In general, we'll avoid detailing the set-theoretic issues here.

Definition 2.4. A category is *small* when its morphisms form a set. A category is *locally small* when, for each pair of objects x, y, the morphisms $\mathtt{C}(x, y)$ forms a set.

Remark 2.5. Any small category is locally small, since the morphisms between two objects form a subset of the set of morphisms. The objects in any small category form a set, since they are in bijection with the identity morphisms, which form a subset of the set of morphisms.

None of the categories in Example 2.3 is small. In fact, each of these examples takes as objects sets with certain structure and as morphisms structure-preserving functions between those sets. As the following example shows, these are far from the only kind of category.

Example 2.6. The following are also categories:

- The *empty category* is the category with no objects and no morphisms.
- The *trivial category* is the category with one object and only its identity morphism.
- A set X can be represented as a category whose objects correspond with the elements of X and which has only identity morphisms. A category with only identity morphisms is called *discrete*.

- A group G can be represented as a category with one object whose morphisms correspond with the element of G, with composition determined by multiplication. Following [Por21], we call this category $\mathbb{G}$.
- A poset P can be represented as a category whose objects correspond with the elements of P and with a single morphism $x \to y$ whenever $x \leq y$. We call this category $\mathbb{P}$.

2.2. **(Iso)morphisms.** The main philosophy of category theory is roughly that

> an object is completely determined by its relationship with other objects,
> [Bra20, p. 8]

and so to study an object, one should study its morphisms. I can't resist sharing the following example:

Example 2.7. [Rie17, Example 2.1.6(ii)] Fixing a topological space X, we can recover both the points of X and the topology on X—in other words, all the data of the space—via studying only maps involving X, as follows.

To recover the points, note that a map $f : * \to X$, where $*$ is the singleton space, consists of picking out any point $f(*) \in X$. Thus the points are in bijection with $\mathtt{Top}(*, X)$.

To recover the topology, let $S = \{0, 1\}$ with the topology $\{\emptyset, \{1\}, X\}$. To any open set $U \subseteq X$, associate the map

$$f_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

But we can write any continuous function $f : X \to S$ in this form by setting $U = f^{-1}(\{1\})$. Thus the open sets are in bijection with $\mathtt{Top}(X, S)$.

A chiefly important kind of morphism is the *isomorphism*, which (as in most algebraic contexts) suggests a fundamental similarity between its domain and codomain.

Definition 2.8. A morphism $f : x \to y$ is an *isomorphism* when there is a morphism $g : y \to x$ such that $gf = 1_x$ and $fg = 1_y$. We call g the *inverse* of f and write $g = f^{-1}$. We say x and y are *isomorphic* and write $x \cong y$.

Example 2.9. Isomorphisms are interesting in many of our examples of categories:

- In $\mathtt{Set}$, the isomorphisms are invertible set-functions, i.e. bijections.
- In $\mathtt{Grp}$, the isomorphisms are group isomorphisms.
- In $\mathtt{Top}$, the isomorphisms are homeomorphisms.
- In $\mathtt{Vect}_{\mathbb{k}}$, the isomorphisms are vector space isomorphisms.
- For any group G, any morphism in $\mathbb{G}$ is an isomorphism; this fact corresponds to the existence of inverses in the group. (We will have more to say about this example in Section 2.4.)
- For any poset P, the isomorphisms in $\mathbb{P}$ are the identities; this fact corresponds to anti-symmetry of the relation.

Lemma 2.10. *Let $f : x \to y$ be an isomorphism with inverses g and h. Then $g = h$.*

Proof. We have $g = 1_x g = (hf)g = h(fg) = h1_y = h$. □

Notation 2.11. Lemma 2.10 justifies writing the inverse of an isomorphism f as f^{-1}, since it guarantees this specifies a unique morphism.

Further, as we expect:

Proposition 2.12. *Isomorphism forms an equivalence relation on the class of objects in a category* $\mathtt{C}$*.*

Proof. We need to show:

- Reflexivity. The identity 1_x is an isomorphism $x \to x$.
- Symmetry. For $f : x \to y$ an isomorphism, f^{-1} is an isomorphism $y \to x$, with inverse f.
- Transitivity. For $f : x \to y$ and $g : y \to z$ isomorphisms, gf is an isomorphism $x \to z$, with inverse $f^{-1}g^{-1}$. □

In a locally small category $\mathtt{C}$, where $f : x \to y$ is any morphism and z is an ambient object, we can define maps $f_* : \mathtt{C}(z, x) \to \mathtt{C}(z, y)$ and $f^* : \mathtt{C}(y, z) \to \mathtt{C}(x, z)$ via post- and pre-composition by f, respectively. In this case, Theorem 2.13 gives an important characterization of isomorphisms in terms of these maps.

Theorem 2.13. *Let* $\mathtt{C}$ *be locally small. Then the following are equivalent:*

(1) $f : x \to y$ *is an isomorphism.*
(2) *For every* $z \in \mathtt{C}$*,* f_* *is a bijection of sets.*
(3) *For every* $z \in \mathtt{C}$*,* f^* *is a bijection of sets.*

Remark 2.14. If objects are characterized by their morphisms, then Theorem 2.13 supports the idea that isomorphic objects truly look the same to the machinery of the category.

Proof of Theorem 2.13. We prove equivalence (1) ⇔ (2); the proof (1) ⇔ (3) is similar[1]. Let f be an isomorphism. We show f_*^{-1} is an inverse of f_*. In particular, for any morphisms $h : z \to x$ and $k : z \to y$, we have that

$$f_*^{-1}(f_*(h)) = f^{-1}fh = 1_X h = h \quad \text{and} \quad f_*(f_*^{-1}(k)) = ff^{-1}k = 1_Y k = k.$$

Conversely, let f_* be bijective. Letting $z = y$, by surjectivity there is some $g \in \mathtt{C}(y, x)$ such that $1_y = f_*(g) = fg$. But now letting $z = x$, we see that

$$f_*(gf) = fgf = 1_y f = f = f_*(1_x),$$

and so by injectivity $gf = 1_x$. Thus g is an inverse of f, hence f is an isomorphism. □

2.3. **Functors.** If the philosophy of category theory is to study morphisms between objects, then functors answer the obvious question: what are the morphisms between categories?

Definition 2.15. A *functor* $F : \mathtt{C} \to \mathtt{D}$ between categories $\mathtt{C}$ and $\mathtt{D}$ consists of

- For each object $x \in \mathrm{Obj}(\mathtt{C})$, an object $Fx \in \mathrm{Obj}(\mathtt{D})$.
- For each morphism $f \in \mathrm{Map}(\mathtt{C})$, a morphism $Ff \in \mathrm{Map}(\mathtt{D})$.

[1] One can also conclude the latter via studying (2) in the context of the *opposite category* of $\mathtt{C}$, but this is outside our scope. See [Rie17, Lemma 1.2.3] for such a proof.

This data must preserve the categorical structure, i.e. domains, codomains, identities, and composites, that is,

- For each $f \in \mathrm{Map}(\mathtt{C})$, $\mathrm{dom}(Ff) = F(\mathrm{dom}(f))$ and $\mathrm{cod}(Ff) = F(\mathrm{cod}(f))$.
- For each composition-compatible pair $f, g \in \mathrm{Map}(\mathtt{C})$, $(Fg)(Ff) = F(gf)$.
- For each $x \in \mathrm{Obj}(\mathtt{C})$, $F(1_x) = 1_{Fx}$.

Example 2.16. Many common constructions are functors:

- On any category $\mathtt{C}$ there is an *identity functor* $1_{\mathtt{C}}$.
- The power set defines a functor $\mathcal{P} : \mathtt{Set} \to \mathtt{Set}$ which takes a morphism f to its action via images, i.e. $\mathcal{P}(f)(A) = f(A)$.
- On a category $\mathtt{C}$ like those in Example 2.3, the *forgetful functor* $\mathtt{C} \to \mathtt{Set}$ sends an object to its underlying set and a morphism to its underlying set-function, "forgetting" the algebraic structure.
- The free group defines a functor $\mathtt{Set} \to \mathtt{Grp}$ which sends a set to its free group and a map to its letter-wise action on words.
- A group action of a group G on a set A can be regarded as a functor $\mathbb{G} \to \mathtt{Set}$ which sends the single object of $\mathbb{G}$ to A and a morphism to the endofunction defined by that element's action on A. In fact, the funtoriality axioms correspond precisely to the axioms of group action.
- The construction f_* from Theorem 2.13 gives a functor $\mathtt{C}(z, -) : \mathtt{C} \to \mathtt{Set}$, called the *covariant functor represented by* z, which sends any object x to $\mathtt{C}(z, x)$[2].

Theorem 2.17. *Let $F : \mathtt{C} \to \mathtt{D}$ be a functor and $f : x \to y$ an isomorphism in $\mathtt{D}$. Then Ff is an isomorphism between Fx and Fy in $\mathtt{D}$.*

Proof. We have
$$(Ff^{-1})(Ff) = F(f^{-1}f) = F1_x = 1_{Fx},$$
and the same works on the other side. ☐

There is a category of categories, $\mathtt{Cat}$, whose objects are categories and whose morphisms are functors[3]. Composition of functors is defined in the obvious way. This allows us to define isomorphic categories straightforwardly:

Definition 2.18. Two categories $\mathtt{C}$ and $\mathtt{D}$ are *isomorphic* when there is a functor $F : \mathtt{C} \to \mathtt{D}$ which is an isomorphism in $\mathtt{Cat}$.

We will see later, in Remark 3.7, that this definition is generally too restrictive. That will motivate our discussion of *natural transformations* in Section 4, where we will define a more useful notion of *equivalence of categories*. One hint at the restrictiveness is the following:

Theorem 2.19. [Bro07, Theorem 6.4.3] *A functor $F : \mathtt{C} \to \mathtt{D}$ between small categories is an isomorphism if and only if it is a bijection on both objects and morphisms.*

[2] The analogous functor for f^* requires a little more machinery in the form of *contravariant functors*, outside our scope. These representation functors are critically important for tools central category-theoretic tools such as *universal properties* and the *Yoneda lemma*. See [Rie17, Chapter 2].

[3] Categories in $\mathtt{Cat}$ must be locally small, for set-theoretic reasons. It is common, e.g. in [Rie17], to call this category $\mathtt{CAT}$, to distinguish it from the category of small categories, which in turn is called $\mathtt{Cat}$. As we ignore these details, we use the more convenient notation.

Proof. Let F be an isomorphism with inverse G. For any $x \in \mathrm{Obj}(\mathtt{C})$, $(GF)x = 1_{\mathtt{C}}x = x$; this shows F is invertible, hence bijective, as a function on $\mathrm{Obj}(\mathtt{C})$. The same works for morphisms.

Conversely, let F be a bijection on both objects and morphisms, with inverse G. We need to check that G is a functor and an inverse of F. The latter is easy, since it is an inverse on both objects and morphisms.

To check functoriality, it suffices to check images of F, by surjectivity.

- Let $x \in \mathrm{Obj}(\mathtt{C})$. By functoriality of F, $G1_{Fx} = 1_x$.
- Let $f, g \in \mathrm{Map}(\mathtt{C})$ be composition-compatible. Then

$$G(FgFf) = G(Fgf) = gf = GFg \cdot GFf,$$

as desired. □

2.4. **Groupoids.** According to Example 2.6 and Example 2.9, any group can be represented by a specific kind of category, one with a single object and only isomorphisms. Indeed, any such category assembles into a group:

- Elements are given by morphisms, with multiplication given by composition.
- Multiplication is well-defined because any two morphisms have the same domain and codomain (the single object), and hence are composition-compatible.
- Multiplication is associative because composition is associative.
- The group identity is the identity morphism on the single object.
- Morphisms have inverses since they are isomorphisms.

It turns out that the requirement of a single object isn't necessary for the algebraic structure we recover to be interesting. Indeed, relaxing this requirement gives our fundamental object of study:

Definition 2.20. A *groupoid* is a category in which every morphism is an isomorphism.

The second bullet above suggests that a groupoid will have to sacrifice well-definideness of multiplication. Indeed, there is a purely algebraic picture of groupoids defined in terms of partial functions, but the categorical one will be sufficient for our needs. For a more complete discussion, see [Bro07].

Definition 2.21. We can define several common kinds of groupoids:

- A groupoid with one element is a *group*. Per the above discussion, we can indeed take this as a definition.
- A groupoid with only identity morphisms is a *discrete groupoid*. Such groupoids correspond bijectively to sets; see Example 2.3.
- A groupoid with at least one morphism between any two objects is *connected*.
- A groupoid with precisely one morphism between any two objects is a *tree groupoid*.

Since groupoids are defined as categories, the morphisms between them are functors:

Definition 2.22. [Bro07, Section 6.4] A *groupoid morphism* is a functor between groupoids.

Unsurprisingly, there is a category, `Grpd`, of groupoids and groupoid morphisms.

3. The Fundamental Groupoid

A central idea of algebraic topology is that topological notions are encoded in algebraic invariants of a space. The first such invariant we study is the *fundamental groupoid*, in some sense a more natural object than the fundamental group, constructed from the homotopy classes of paths in a space. Except where noted, I follow [Bra20, Chapter 6].

3.1. Homotopy.

3.1. Homotopy. Recall that a *homotopy* between continuous functions $f, g : X \to Y$ is a continuous function $H : X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$; in other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y, & \end{tikzcd}$$

where i_0 and i_1 are the inclusions at $(-,0)$ and $(-,1)$.

This homotopy is further a *path homotopy*[4] when $X = I$ (so f and g are paths) and when $H(0,-)$ and $H(1,-)$ are constant.

Two functions are *homotopic* when there exists a homotopy between them. We write $f \simeq g$. When the functions are paths and the homotopy is a path homotopy, the paths are *path homotopic* and we write $f \simeq_p g$. Both are equivalence relations on the class of functions with appropriate domain and codomain.

Two spaces X and Y are *homotopy equivalent* when there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$. This is an equivalence relation on the spaces; we write $X \simeq Y$.

We can form the category `hTop` by taking spaces as objects and homotopy classes of maps as morphisms. One needs to check that composition is well-defined in this category; this was done in class.

Remark 3.1. Two spaces are homotopy equivalent precisely they are isomorphic in `hTop`. There is a functor `Top` $\to$ `hTop` that is the identity on objects and sends a continuous function to its path homotopy class. We therefore call functors out of `hTop` *homotopy invariants*, since by Theorem 2.17 they homotopy equivalent spaces to isomorphic objects.

3.2. Construction. Before defining the fundamental groupoid, there is a nice geometric intuition. Fix a topological space X and pick some points and paths between those points. When you “erase” the other information of the underlying space, you get several points and a bunch of (double-headed) arrows.

[4]When the context is unclear, we will call a general homotopy a *free homotopy*, and a homotopy with fixed endpoints a *path homotopy*.

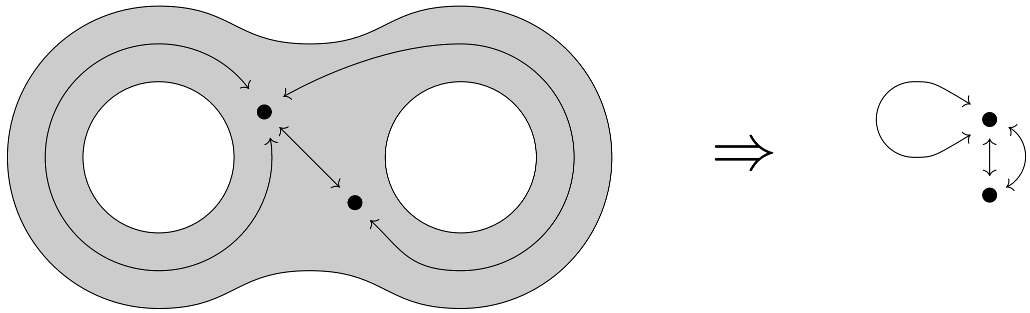


Figure 1. Some points and paths on a two-holed disk.

This looks suspiciously like a groupoid—in particular, each of the arrows are double-headed, and hence "invertible". The issue is that we don't necessarily see the inverses or composites when we just take some points and some paths.

We'll return to this intuition of the fundamental groupoid "forgetting" some of the underlying geometry of the space. More immediately, however, this view strongly suggests that when we take all the path classes from our underlying space, we should get back a groupoid.

Definition 3.2. The *fundamental groupoid* $\Pi_1 X$ of a space X is the category whose objects are the points of X and whose morphisms are the path homotopy classes of paths in X.

Specifically, let $x, y, z \in X$, f a path from x to y, and g a path from y to z. Then,

- A path's domain is its startpoint: $\mathrm{dom}([f]) = x$.
- A path's codomain is its endpoint: $\mathrm{cod}([f]) = y$.
- The identity is the constant map: $1_x = [c_x]$.
- Composition is concatenation: $[g][f] = [f * g]$.

This construction is well-defined specifically because we are working with path homotopies. For example, in general two paths with different start points may be free homotopic, meaning without restricting to path homotopy we could not even write down the domain and codomain of our morphisms.

Example 3.3. [Bro07, p. 213] We can immediately compute a few fundamental groupoids.

- The fundamental groupoid of a convex subspace of $\mathbb{R}^n$ is a tree groupoid (see Definition 2.21). This corresponds to the fact that any two paths with the same endpoints in such a space are homotopic via the straight line homotopy.
- The fundamental groupoid of a totally disconnected space is a discrete groupoid. This corresponds to the fact that the only paths in such spaces are the constant paths.

3.3. **Categorical Properties.** We can learn a lot about the fundamental groupoid by studying it categorically. First, we should confirm it is what we claim it is:

Proposition 3.4. *The fundamental groupoid is a groupoid.*

Proof. All of this work was already done in class for the fundamental group. We restate the results here for groupoids.

- Composition is well-defined, since concatenation preserves homotopy class.
- Composition is associative, since concatenation is associative up to homotopy.
- Every object x has $[c_x]$ as an identity, where c_x is the constant path at x.
- Every morphism $[f]$ has $[\bar{f}]$ as an inverse, where $\bar{f}$ is f traversed in reverse.

The first three say that $\Pi_1 X$ is a category, and the last says that it is a groupoid. $\square$

The construction of the fundamental groupoid naturally gives rise to a functor

$$\Pi_1 : \texttt{Top} \to \texttt{Grpd}.$$

In particular, let $f : X \to Y$ be a continuous function. We can view f as acting on paths via composition. Accordingly, we define

$$\begin{aligned} \Pi_1 f \colon \Pi_1 X &\to \Pi_1 Y \\ [\gamma] &\mapsto [f \circ \gamma]. \end{aligned}$$

This mapping is well-defined because composition preserves homotopy equivalence.

Proposition 3.5. *The fundamental groupoid is a functor.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be spaces and continuous maps. Let γ be a path in X. We have

$$\Pi_1 g \Pi_1 f([\gamma]) = \Pi_1 g([f \circ \gamma]) = [g \circ f \circ \gamma] = \Pi_1(gf)[\gamma]$$

and

$$\Pi_1 1_X([\gamma]) = [1_X \circ \gamma] = [\gamma] = 1_{\Pi_1 X}([\gamma]).$$

Thus Π_1 is compatible with composition and identities, as desired. $\square$

This result is an improvement over the fundamental group, where we needed a functor out of based spaces for the definition to make sense. This is a first hint that the fundamental groupoid in some sense captures more of the structure of a space than the fundamental group does.

Corollary 3.6. *The fundamental groupoid is a topological invariant. More precisely, if $X \cong Y$, then $\Pi_1 X \cong \Pi_1 Y$.*

Proof. This follows from Theorem 2.17 and Proposition 3.5. $\square$

Remark 3.7. As defined, the fundamental groupoid is a *not* a homotopy invariant. For instance, since objects of the fundamental groupoid are in bijection with elements of the underlying space, the fundamental groupoid of D^n has uncountably many objects, whereas the fundamental groupoid of the point has only one, hence they are not isomorphic by Theorem 2.19; nevertheless, D^n is contractible.

This unfortunate fact suggests an issue with our notion of isomorphism of categories: it requires objects to be in bijection, which is far too strong to express homotopy invariants. There is a more natural notion, equivalence of categories, somewhat analogous to homotopy equivalence, which requires some additional machinery to develop.

4. Equivalence of Categories

If we want our categorical constructions to play nicely with homotopy, it will help to define an analogue to homotopy on categories. It turns out the correct notion is a *natural transformation*, a way to relate functors analogous to a homotopy. This notion will allow us to define *equivalence of categories*. Here I follow [Rie17, Sections 1.4-1.7].

4.1. **Natural Transformations as Categorical Homotopy.** Let H be a homotopy between continuous maps $f, g : X \to Y$. We can think of $H(x, -)$ as morphing the point $f(x)$ into $g(x)$. In the same way, we want to define a transformation between functors $F, G : \mathtt{C} \to \mathtt{D}$ which we can think of as morphing the object Fx into the point Gx. Of course, the right thing to do this morphing is exactly a morphism in $\mathtt{D}$. That motivates the following definition:

Definition 4.1 (Natural Transformation)**.** A *natural transformation* $\alpha : F \Rightarrow G$ between functors $F, G : \mathtt{C} \to \mathtt{D}$ consists of, for each object $x \in \mathrm{Obj}(\mathtt{C})$, a morphism $\alpha_x : F_x \to G_x \in \mathtt{D}$, called the *components* of α.

This data must be compatible with morphisms in $\mathtt{C}$, in the sense that for any morphism $f : x \to y \in \mathtt{C}$, the following square must commute:

$$\begin{tikzcd} Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\alpha_y"'] & Gy. \end{tikzcd}$$

Thinking of natural transformations as categorical analogues of homotopy will be critical for what is to come. The commutative square above is analogous to the requirement that a homotopy behaves as f at $t = 0$ and as g at $t = 1$; we require that the transformation behaves as F "before" the transformation and as G "after" the transformation.

Example 4.2. The following are examples of natural transformations:

- For any functor $F : \mathtt{C} \to \mathtt{D}$, there is an *identity natural transformation* $1_F : F \Rightarrow F$ whose components $(1_F)_x$ are each the identity 1_{Fx}.
- There is a natural transformation $\eta : 1_{\mathtt{Set}} \Rightarrow \mathcal{P}$, where $\mathcal{P}$ is the power set functor, where for any set X we have $\eta_X(x) = \{x\}$.
- Let G be a group acting on sets X and Y. Recall from Example 2.16 that this can be thought of as a pair of functors $\mathbb{G} \to \mathtt{Set}$. Since there is only one object of $\mathbb{G}$, a natural transformation between these functors consists of a single function $X \to Y$. Naturality implies that this function preserves the structure of the action: algebraically, this is called *G-equivariance*.

An alternative definition makes the translation even clearer. Let $\mathbb{2}$ be the category with two objects, 0 and 1, and a single non-identity morphism $0 \to 1$. Defining the product category in the obvious way, with component-wise composition, a natural transformation between F and G corresponds bijectively to a functor $H : \mathtt{C} \times \mathbb{2} \to \mathtt{D}$ such that the following diagram commutes:

$$
\begin{tikzcd}
\mathtt{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathtt{C} \times \mathbb{2} \arrow[d, "H"] & \mathtt{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\
& \mathtt{D}. &
\end{tikzcd}
$$

Here i_0 and i_1 are the obvious inclusion functors. For a proof, see [Rie17, Lemma 1.5.1].

We can compose natural transformation in two ways.

Lemma 4.3. *Let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be natural transformations with components α_x and β_x, respectively. Then the components $\beta_x \alpha_x$ assemble into the $\beta\alpha : F \Rightarrow H$.*

Proof. Gluing together the commutative diagrams from Definition 4.1 for α and β respectively, we obtain:

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"] & Hx \arrow[d, "Hf"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy.
\end{tikzcd}
$$

The outer rectangle is exactly the required result for $\beta\alpha$. □

Lemma 4.4. *Let $F, F' : \mathtt{C} \to \mathtt{D}$ and $G, G' : \mathtt{D} \to \mathtt{E}$, with $\alpha : F \Rightarrow F'$ and $\beta : G \Rightarrow G'$. Then there is a natural transformation $\beta * \alpha : GF \Rightarrow G'F'$ with components given by the following commutative square:*

$$
\begin{tikzcd}
GFx \arrow[r, "G\alpha_x"] \arrow[d, "\beta_{Fx}"'] \arrow[dr, dashed, "(\beta*\alpha)_x"] & GF'x \arrow[d, "\beta_{F'x}"] \\
G'Fx \arrow[r, "G'\alpha_x"'] & G'F'x.
\end{tikzcd}
$$

Proof. Commutativity of this square is just naturality of β, hence the components are well-defined. Noting that functors preserve composites and hence commutative diagrams, and gluing together the commutative diagrams for $G\alpha$ and β respectively, we obtain:

$$
\begin{tikzcd}
GFx \arrow[r, "G\alpha_x"] \arrow[d, "GFf"'] & GF'x \arrow[r, "\beta_{F'x}"] \arrow[d, "GF'f"] & G'F'x \arrow[d, "G'F'f"] \\
GFy \arrow[r, "G\alpha_y"'] & GF'y \arrow[r, "\beta_{F'y}"'] & G'F'y.
\end{tikzcd}
$$

The outer rectangle is exactly the required result for $\beta * \alpha$. □

Notation 4.5. We call $\beta\alpha$ of Lemma 4.3 the *vertical composite* and $\beta * \alpha$ of Lemma 4.4 the *horizontal composite*.

Remark 4.6. There are two ways to compose homotopies: when they share a common function, and when they are between composition-compatible functions. There are two ways to compose natural transformations: when they share a common functor, and when they are between composition-compatible functors.

4.2. **Equivalence of Categories.** One key difference between homotopy and natural transformations is that homotopies are always invertible, in the sense that for a homotopy H from f to g, $H(-, 1-t)$ is a homotopy from g to f. This key fact means homotopies can be used to define an equivalence relation on continuous maps. Since we are primarily concerned with the special case of groupoids, in which "everything" is invertible, this ought to be a non-issue.

Definition 4.7. A *natural isomorphism* is a natural transformation whose components are each isomorphisms. Two functors F and G are *naturally isomorphic* when there exists a natural isomorphism $\alpha : F \Rightarrow G$. We write $F \cong G$.

Remark 4.8. Any natural transformation between functors into a groupoid is a natural isomorphism.

Of course, as we expect:

Proposition 4.9. *Natural isomorphism forms an equivalence relation on the class of functors* $\mathtt{C} \to \mathtt{D}$.

Proof. The result follows immediately by applying Proposition 2.12 to each of the components. $\square$

With our new machinery in mind, we can define a better equivalence on categories than our old notion of isomorphism. Again, we want to work by analogy to homotopy equivalence, defined in Section 3.1. As such:

Definition 4.10. Two categories $\mathtt{C}$ and $\mathtt{D}$ are *equivalent* when there exist functors $F : \mathtt{C} \to \mathtt{D}$ and $G : \mathtt{D} \to \mathtt{C}$ such that $GF \cong 1_{\mathtt{C}}$ and $FG \cong 1_{\mathtt{D}}$. We write $\mathtt{C} \simeq \mathtt{D}$.

Proposition 4.11. *Equivalence of categories forms an equivalence relation on the class of categories.*

Proof. We need to show:

- Reflexivity. The natural transformation $1_{1_{\mathtt{C}}} : 1_{\mathtt{C}} \Rightarrow 1_{\mathtt{C}}$ is an equivalence.
- Symmetry. The definition is symmetric: the functors witness both equivalences.
- Transitivity. Given natural isomorphisms $GF \Rightarrow 1_{\mathtt{C}}$ and $G'F' \Rightarrow 1_{\mathtt{D}}$, apply horizontal composition to produce an isomorphism $GG'F'F \Rightarrow G1_{\mathtt{D}}F = GF \Rightarrow 1_{\mathtt{C}}$. $\square$

Proposition 4.12. *Suppose* $\mathtt{C}$ *and* $\mathtt{D}$ *are small*[5] *and each have one object. If* $\mathtt{C} \simeq \mathtt{D}$*, then* $\mathtt{C} \cong \mathtt{D}$.

Proof. Let c and d be the respective objects. Let $F : \mathtt{C} \to \mathtt{D}$ and $G : \mathtt{D} \to \mathtt{C}$ be equivalences, with $\alpha : GF \Rightarrow 1_{\mathtt{C}}$ a natural isomorphism. Then $Fc = d$ and $Gd = c$; thus F and G are

[5]The result is true for larger categories, but the only proof I've seen requires some machinery outside our scope.

bijective on objects. By Theorem 2.19, it remains to show that F and G are bijective on morphisms.

The data of α is an isomorphism $\alpha_c : c \to c$. Naturality asserts that for each $f \in \mathtt{C}(c, c)$, $GFf = \alpha_c f \alpha_c^{-1}$. By Theorem 2.13, precomposition by α_c and postcomposition by α_c^{-1} induce bijections on $\mathtt{C}(c, c)$; therefore GF also induces such a bijection. Therefore F is injective on morphisms and G is surjective on morphisms. Treating the other natural isomorphism identically gives the symmetric result. ☐

4.3. **Groupoids, Redux.** Finally, we can apply our new technology to show the isomorphism of path-connected fundamental groups and the homotopy invariance of the fundamental groupoid. Here I largely follow [Bro07, Section 6.5].

Definition 4.13. Let $\mathtt{C}$ be a category and $x \in \mathrm{Obj}(\mathtt{C})$. The *automorphism group of* x, $\mathrm{Aut}(g)$, is the subcategory of $\mathtt{C}$ whose single object is x and whose morphisms are precisely the isomorphisms $x \to x$.

Observe that we still have categorical structure on $\mathrm{Aut}(g)$, since the composition of isormorphisms is an isormophism, by Proposition 2.12. By Definition 2.21, this category is indeed a group. Note that if $\mathtt{C}$ is a groupoid, the automorphism group of x is just the set $\mathtt{C}(x, x)$ together with the composition structure.

Theorem 4.14. *Let* $\mathtt{G}$ *be a connected groupoid with* $g \in Obj(\mathtt{G})$. *Then* $Aut(g) \simeq \mathtt{G}$.

Proof. Let $i : \mathrm{Aut}(g) \hookrightarrow \mathtt{G}$ be the inclusion of $\mathrm{Aut}(g)$ into $\mathtt{G}$.

For each object $x \in \mathtt{G}$, fix[6] an isormorphism $\alpha_x : g \to x$, which exists by connectedness; ensure[7] that $\alpha_g = 1_g$. Let $r : \mathtt{G} \to \mathrm{Aut}(g)$ take each object to g and take each (iso)morphism $f : x \to y$ to $\alpha_y^{-1} f \alpha_x$. Since $\alpha_g = 1_g$, we have $ri = 1_{\mathrm{Aut}(g)}$.

To see $ir \cong 1_{\mathtt{G}}$, collect the α_x into a natural transformation $\alpha : ir \Rightarrow 1_{\mathtt{G}}$. Naturality consists of the following square:

$$\begin{tikzcd} g \arrow[r, "\alpha_x"] \arrow[d, "\alpha_y^{-1} f \alpha_x"'] & x \arrow[d, "f"] \\ g \arrow[r, "\alpha_y"'] & y, \end{tikzcd}$$

which trivially commutes. ☐

Corollary 4.15. *Let* g *and* h *be objects in the small connected groupoid* $\mathtt{G}$. *Then their automorphism groups are isomorphic.*

Proof. Noting groups have one object, the result follows from Theorem 4.14, Proposition 4.11, and Proposition 4.12. ☐

Corollary 4.16. *Let* x *and* y *be points in the path-connected space* X. *Then their fundamental groups are isomorphic.*

[6]Here we assume the axiom of choice.

[7]This caveat gives us a deformation retraction, instead of just an equivalence.

Proof. Observe that the fundamental groupoid of a path-connected space is connected, and that paths in any space form a set, and apply Corollary 4.15. $\square$

With a more general equivalence on categories, hence also groupoids, in hand, our final objective is the homotopy invariance of the fundamental groupoid up to equivalence.

Lemma 4.17. *Let $f, g : X \to Y$ be continuous. If $f \simeq g$, then $\Pi_1 f \cong \Pi_1 g$.*

Proof. [Hig71, Proposition 13] Let $H : X \times I \to Y$ be a homotopy. By Remark 4.8, it suffices to give a natural transformation $\alpha : \Pi_1 f \Rightarrow \Pi_1 g$, whose components, α_x, are morphisms in $\Pi_1 Y$, hence path classes in Y, with start and endpoints at $f(x)$ and $g(x)$, respectively. We will set $\alpha_x = [H(x, -)]$.

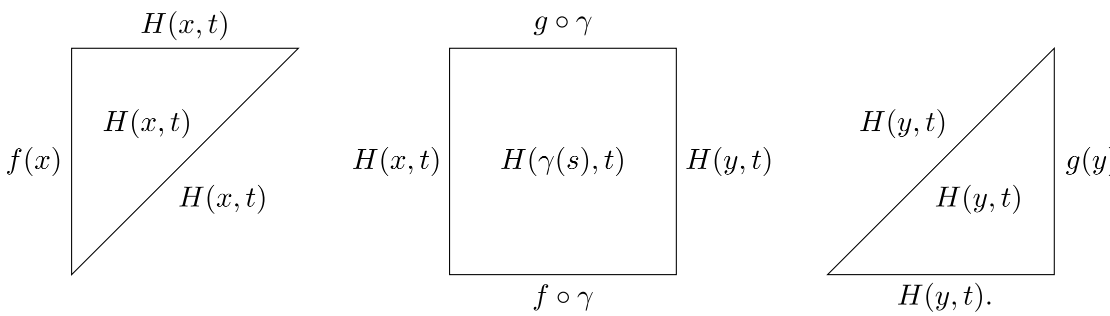
Let γ be a path in X. Naturality asserts commutativity of the following square:

$$\begin{tikzcd} f(x) \arrow[r, "{[H(x,-)]}"] \arrow[d, "{[f\circ\gamma]}"'] & g(x) \arrow[d, "{[g\circ\gamma]}"] \\ f(y) \arrow[r, "{[H(y,-)]}"'] & g(y). \end{tikzcd}$$

This amounts to giving a path homotopy

$$(f \circ \gamma) * H(y, -) \simeq_p H(x, -) * (g \circ \gamma).$$

To construct our homotopy, we glue together the following 2-cells:



This gives precisely the desired homotopy. $\square$

Theorem 4.18. *Let $X \simeq Y$. Then $\Pi_1 X \simeq \Pi_1 Y$.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be homotopy equivalences, so that $gf \simeq 1_X$ and $fg \simeq 1_Y$. The fundamental groupoid induces groupoid morphisms

$$\Pi_1 f : \Pi_1 X \to \Pi_1 Y \quad \text{and} \quad \Pi_1 g : \Pi_1 Y \to \Pi_1 X.$$

We show these form equivalences $\Pi_1 X \simeq \Pi_1 Y$. By the lemma, we have that

$$\Pi_1 g \Pi_1 f = \Pi_1 gf \cong \Pi_1 1_X = 1_{\Pi_1 X};$$

the other direction follows by symmetry of the situation. $\square$

5. Higher Categories, Briefly

The analogy we've drawn between homotopies and natural transformation can be made rigorous in several ways. Here we sketch a few, and discuss advantages and disadvantages of each. The main objective is to generalize categories to allow morphisms between morphisms, like homotopies in $\mathtt{Top}$ and natural transformations in $\mathtt{Cat}$. Here I largely follow [Lei04].

Our general issue will be that homotopies only associate up to homotopy. Our concern, as we will quickly see, will be to find a convenient weakening of associativity which accommodates homotopy.

5.1. **Strict n-categories.**

Probably the most straightforward way to accomplish this is strict 2-categories. Note that in an ordinary (locally small) category, the morphisms $\mathtt{C}(x, y)$ form a set for any pair of objects. In many settings, we can say something stronger: recall, for instance, that the set of linear transformations $\mathcal{L}(V, W)$ between any two vector spaces is itself a vector space.

Definition 5.1. Let $\mathtt{K}$ be a category. A *category enriched over* $\mathtt{K}$ is a category such that for any two objects $x, y \in \mathrm{Obj}(\mathtt{C})$, $\mathtt{C}(x, y)$ is an object in $\mathtt{K}$.

We also need to require an additional structure on composition, essentially so that it is compatible with $\mathtt{K}$. To do this we need a way to product two objects in $\mathtt{K}$, so we can define composition as a morphism $\mathtt{C}(x, y) \otimes \mathtt{C}(y, z) \to \mathtt{C}(x, z)$; for example, one uses the tensor product in $\mathtt{Vect}_{\Bbbk}$. The general way to do this is to insist that $\mathtt{K}$ be *monoidal*.

Definition 5.2. A *strict 2-category* is a category enriched over $\mathtt{Cat}$.

One can show that there is a 2-category of 1-categories, $\mathtt{1Cat}$, where the categorical structure on morphisms is given by vertical composition. In that case, horizontal composition together with the ordinary product category is exactly the right gadget needed to make the monoidal structure work.

One can keep going:

Definition 5.3. A *0-category* is a set. An *(n+1)-category* is a category enriched over $\mathtt{nCat}$, the category of all n-categories.

What to do about $\mathtt{Top}$? We could take homotopy classes of homotopies as our 2-cells. In fact for spaces X and Y, there is a category of continuous functions $X \to Y$ and homotopy classes of homotopies between those functions.

This approach will break down quickly. Informally, either our 3-cell structure will have to be trivial, since we would be asking for homotopies between homotopy classes, or we will have to sacrifice associativity for 2-cells and therefore no longer be a strict n-category.

5.2. **Bicategories.**

A bicategory is probably the easiest way to define a weak 2-category.

Definition 5.4. A *bicategory* consists of *0-cells*, *1-cells*, and *2-cells*, with the appropriate domains, codomains,

References

[Bra20] Tai-Danae Bradley. *Topology*. MIT Press, London, England, August 2020.

[Bro07] Ronald Brown. *Topology and Groupoids*. BookSurge, Scotts Valley, CA, December 2007.

[Hig71] Philip J Higgins. *Notes on categories and groupoids*. 1971.

[Lei04] Tom Leinster. *Higher operads, higher categories*. London Mathematical Society lecture note series. Cambridge University Press, Cambridge, England, July 2004.

[Por21] Timothy Porter. Spaces as infinity-groupoids. In *New Spaces in Mathematics*, pages 258–321. Cambridge University Press, April 2021.

[Rie17] Emily Riehl. *Category theory in context*. Courier Dover Publications, March 2017.