HIGHER CATEGORY THEORY BY WAY OF TOPOLOGY

RILEY SHAHAR

Abstract. TODO

1. Introduction

TODO

After introducing some introductory category theory in Section 2, we construct the fundamental groupoid in Section 3. This motivates natural transformations in Section 4, in turn leading to higher category theory in Section 5.

2. Categorical Preliminaries

Many of the constructions we'll encounter are categorical, or at least are well-stated using categorical ideas, so it's worth spending some time with basic category theory. Our aim will be to develop only some essential language now, and later use our exploration of homotopy as motivation for developing higher category theory. Except where noted, I follow [Rie17, Sections 1.1-1.3].

2.1. Categories.

Definition 2.1. A *category* $\mathtt{C}$ consists of a collection of *objects* $\mathrm{Obj}(\mathtt{C})$ and a collection of *morphisms* $\mathrm{Map}(\mathtt{C})$, such that

- Each morphism f has a specific *domain* $\mathrm{dom}(f)$ and *codomain* $\mathrm{cod}(f)$; we write $f : x \to y$ or $x \xrightarrow{f} y$.
- Each object x has a specific *identity morphism* $1_x : x \to x$.
- For each pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$, there is a specific *composite morphism* $gf : x \to z$. We call f and g *composition-compatible*.

This data must be *associative* and *unital*, which is to say,

- (Associativity) For composition-compatible triplet, i.e. $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, we have $h(gf) = (hg)f$. We write merely hgf.
- (Unital) For any $f : x \to y$, we have $1_y f = f = f 1_x$.

In other words, the following diagrams commute:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & y \arrow[r, "g"] \arrow[rr, bend left, "hg"] & z \arrow[r, "h"] & w \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "1_x"] \arrow[dr, "f"'] & x \arrow[d, "f"] \\ & y \end{tikzcd} \quad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "f"'] & y \arrow[d, "1_y"] \\ & y \end{tikzcd}$$

Notation 2.2. We will use $\mathtt{C}(x, y)$ for the morphisms with domain X and codomain Y. We also choose to use multiplicative notation for composition, instead of $\circ$, to distinguish it from function composition, since in many of our categories morphism composition will be concatenation or something completely separate from function composition.

Example 2.3. Many mathematical structures can be studied as categories:

- `Set` is the category of sets and set-functions.
- `Poset` is the category of partially ordered sets and monotone functions.
- `Grp` is the category of groups and homomorphisms.
- `Top` is the category of topological spaces and continuous functions.
- For any field $\Bbbk$, $\mathtt{Vect}_{\Bbbk}$ is the category of vector spaces over k and linear maps.

These examples demonstrate the need for the word "collection" in the definition of a category: there is no set of all sets, for example. In general, we'll avoid detailing the set-theoretic issues here.

Definition 2.4. A category is *small* if its morphisms form a set. A category is *locally small* if, for each pair of objects x, y, the morphisms $\mathtt{C}(x, y)$ forms a set.

Remark 2.5. Any small category is locally small, since the morphisms between two objects form a subset of the set of morphisms. The objects in any small category form a set, since they are in bijection with the identity morphisms, which form a subset of the set of morphisms.

None of the categories in Example 2.3 is small. In fact, each of these examples takes as objects sets with certain structure and as morphisms structure-preserving functions between those sets. As the following example shows, these are far from the only kind of category.

Example 2.6. The following are also categories:

- The *empty category* is the category with no objects and no morphisms.
- The *trivial category* is the category with one object and only its identity morphism.
- A set X can be represented as a category whose objects correspond with the elements of X and which has only identity morphisms. A category with only identity morphisms is called *discrete*.
- A group G can be represented as a category with one object whose morphisms correspond with the element of G, with composition determined by multiplication. Following [Por21], we call this category $\mathbb{G}$.
- A poset P can be represented as a category whose objects correspond with the elements of P and with a single morphism $x \to y$ whenever $x \leq y$. We call this category $\mathbb{P}$.

2.2. **(Iso)morphisms.** The main philosophy of category theory is roughly that

> an object is completely determined by its relationship with other objects, [Bra20, p. 8]

and so to study an object, one should study its morphisms. I can't resist sharing the following example:

Example 2.7. [Rie17, Example 2.1.6(ii)] Fixing a topological space X, we can recover both the points of X and the topology on X—in other words, all the data of the space—via studying only maps involving X, as follows.

To recover the points, note that a map $f : * \to X$, where $*$ is the singleton space, consists of picking out any point $f(*) \in X$. Thus the points are in bijection with $\mathtt{Top}(*, X)$.

To recover the topology, let $S = \{0, 1\}$ with the topology $\{\emptyset, \{1\}, X\}$. To any open set $U \subseteq X$, associate the map

$$f_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

But we can write any continuous function $f : X \to S$ in this form by setting $U = f^{-1}(\{1\})$. Thus the open sets are in bijection with $\mathtt{Top}(X, S)$.

A chiefly important kind of morphism is the *isomorphism*, which (as in most algebric contexts) suggests a fundamental similarity between its domain and codomain.

Definition 2.8. A morphism $f : x \to y$ is an *isomorphism* if there is a morphism $g : y \to x$ such that $gf = 1_x$ and $fg = 1_y$. We call g the *inverse* of f and write $g = f^{-1}$. We say x and y are *isomorphic* and write $x \cong y$.

Example 2.9. Isomorphisms are interesting in many of our examples of categories:

- In $\mathtt{Set}$, the isomorphisms are invertible set-functions, i.e. bijections.
- In $\mathtt{Grp}$, the isomorphisms are group isomorphisms.
- In $\mathtt{Top}$, the isomorphisms are homeomorphisms.
- In $\mathtt{Vect}_{\Bbbk}$, the isomorphisms are vector space isomorphisms.
- For any group G, any morphism in $\mathbb{G}$ is an isomorphism; this fact corresponds to the existence of inverses in the group. (We will have more to say about this example in Section 2.4.)
- For any poset P, the isomorphisms in $\mathbb{P}$ are the identities; this fact corresponds to anti-symmetry of the relation.

In a locally small category $\mathtt{C}$, where $f : x \to y$ is any morphism and z is an ambient object, we can define maps $f_* : \mathtt{C}(z, x) \to \mathtt{C}(z, y)$ and $f^* : \mathtt{C}(y, z) \to \mathtt{C}(x, z)$ via post- and pre-composition by f, respectively. In this case, Theorem 2.10 gives an important characterization of isomorphisms in terms of these maps.

Theorem 2.10. *Let $\mathtt{C}$ be locally small. Then the following are equivalent:*

1. *$f : x \to y$ is an isomorphism.*
2. *For every $z \in \mathtt{C}$, f_* is a bijection of sets.*
3. *For every $z \in \mathtt{C}$, f^* is a bijection of sets.*

Remark 2.11. If objects are characterized by their morphisms, then Theorem 2.10 supports the idea that isomorphic objects truly look the same to the machinery of the category.

Proof of Theorem 2.10. We prove equivalence (1) $\Leftrightarrow$ (2); the proof (1) $\Leftrightarrow$ (3) is similar[1]. Let f be an isomorphism with inverse g. We show g_* is an inverse of f_*. In particular, for any morphisms $h : z \to x$ and $k : z \to y$, we have that

$$g_*(f_*(h)) = gfh = 1_X h = h \quad \text{ and } \quad f_*(g_*(k)) = fgk = 1_Y k = k.$$

Conversely, let f_* be bijective. Letting $z = y$, by surjectivity there is some $g \in \mathtt{C}(y, x)$ such that $1_y = f_*(g) = fg$. But now letting $z = x$, we see that

$$f_*(gf) = fgf = 1_y f = f = f_*(1_x),$$

and so by injectivity $gf = 1_x$. Thus g is an inverse of f, hence f is an isomorphism. $\square$

2.3. **Functors.** If the philosophy of category theory is to study morphisms between objects, then functors answer the obvious question: what are the morphisms between categories?

Definition 2.12. A *functor* $F : \mathtt{C} \to \mathtt{D}$ between categories $\mathtt{C}$ and $\mathtt{D}$ consists of

- For each object $x \in \mathrm{Obj}(\mathtt{C})$, an object $Fx \in \mathrm{Obj}(\mathtt{D})$.
- For each morphism $f \in \mathrm{Map}(\mathtt{C})$, a morphism $Ff \in \mathrm{Map}(\mathtt{D})$.

This data must preserve the categorical structure, i.e. domains, codomains, identities, and composites, that is,

- For each $f \in \mathrm{Map}(\mathtt{C})$, $\mathrm{dom}(Ff) = F(\mathrm{dom}(f))$ and $\mathrm{cod}(Ff) = F(\mathrm{cod}(f))$.
- For each composition-compatible pair $f, g \in \mathrm{Map}(\mathtt{C})$, $Fg \cdot Ff = F(g \cdot f)$.
- For each $x \in \mathrm{Obj}(\mathtt{C})$, $F(1_x) = 1_{Fx}$.

Example 2.13. Many common constructions are functors:

- The power set defines a functor $\mathcal{P} : \mathtt{Set} \to \mathtt{Set}$ which takes a morphism f to its action via images, i.e. $\mathcal{P}(f)(A) = f(A)$.
- On a category $\mathtt{C}$ like those in Example 2.3, say $\mathtt{Grp}$, the *forgetful functor* $\mathtt{C} \to \mathtt{Set}$ sends an object to its underlying set and a morphism to its underlying set-function, ”forgetting” the algebraic structure.
- The free group defines a functor $\mathtt{Set} \to \mathtt{Grp}$ which sends a set to its free group and a map to its letter-wise action on words.
- A group action of a group G on a set A can be regarded as a functor $\mathbb{G} \to \mathtt{Set}$ wich sends the single object of $\mathbb{G}$ to A and a morphism to the endofunction defined by that element's action on A.

Theorem 2.14. *Let $F : \mathtt{C} \to \mathtt{D}$ be a functor and $f : x \to y$ an isomorphism in $\mathtt{D}$. Then Ff is an isomorphism between Fx and Fy in $\mathtt{D}$.*

[1] One can also conclude the latter via studying (2) in the context of the *opposite category* of $\mathtt{C}$, but this is outside our scope. See [Rie17, Lemma 1.2.3] for such a proof.

Proof. Let $g : y \to x$ invert f. Then

$$(Fg)(Ff) = F(gf) = F1_x = 1_{Fx},$$

and the same works on the other side. $\square$

There is a category of categories, $\mathtt{Cat}$, whose objects are categories and whose morphisms are functors[2]. This allows us to define isomorphic categories straightforwardly:

Definition 2.15. Two categories $\mathtt{C}$ and $\mathtt{D}$ are *isomorphic* if there is a functor $F : \mathtt{C} \to \mathtt{D}$ which is an isomorphism in $\mathtt{Cat}$.

We will see later, in Remark 3.7, that this definition is generally too restrictive. That will motivate our discussion of *natural transformations* in Section 4, where we will define a more useful notion of *equivalence of categories*.

2.4. **Groupoids.** According to Example 2.6 and Example 2.9, any group can be represented by a specific kind of category, one with a single object and only isomorphisms. Indeed, any such category assembles into a group:

- Elements are given by morphisms, with multiplication given by composition.
- Multiplication is well-defined because any two morphisms have the same domain and codomain (the single object), and hence are composition-compatible.
- Multiplication is associative because composition is associative.
- The group identity is the identity morphism on the single object.
- Morphisms have inverses since they are isomorphisms.

It turns out that the requirement of a single object isn't necessary for the algebraic structure we recover to be interesting. Indeed, relaxing this requirement gives our fundamental object of study:

Definition 2.16. A *groupoid* is a category in which every morphism is an isomorphism.

The second bullet above suggests that a groupoid will have to sacrifice well-definideness of multiplication. Indeed, there is a purely algebraic picture of groupoids defined in terms of partial functions, but the categorical one will be sufficient for our needs. For a more complete discussion, see [Bro07].

Example 2.17. The following are examples of groupoids:

- Per the above discussion, a group is a groupoid with one element.
- The categorification of a set, from Example 2.3, is a groupoid.
- The empty and trivial categories, from Example 2.6, are groupoids.
- A groupoid with only identity morphisms is a *discrete groupoid*.
- A groupoid with at least one morphism between any two objects is *connected*.
- A groupoid with precisely one morphism between any two objects is a *tree groupoid*.

Since groupoids are defined as categories, unsurprisingly, the morphisms between them are functors:

[2]Categories in $\mathtt{Cat}$ must be locally small, for set-theoretic reasons. It is common, e.g. in [Rie17], to call this category $\mathtt{CAT}$, to distinguish it from the category of small categories, which in turn is called $\mathtt{Cat}$. As we ignore these details, we use the more convenient notation.

Definition 2.18. [Bro07, Section 6.4] A *groupoid morphism* is a functor between groupoids.

Unsurprisingly, there is a category, `Grpd`, of groupoids and groupoid morphisms.

3. The Fundamental Groupoid

A central idea of algebraic topology is that topological notions are encoded in algebraic invariants of a space. The first such invariant we study is the *fundamental groupoid*, in some sense a more natural object than the fundamental group, constructed from the homotopy classes of paths in a space. Except where noted, I follow [Bra20, Chapter 6].

3.1. **Homotopy.** Recall that a *homotopy* between continuous functions $f : X \to Y$ and $g : X \to Y$ is a continuous function $H : X \times I \to Y$ such that $H(-, 0) = f$ and $H(-, 1) = g$. This homotopy is further a *path homotopy*[3] when $X = I$ (so f and g are paths) and when $H(0, -)$ and $H(1, -)$ are constant.

Two functions are *free homotopic* when there exists a homotopy between them. This forms an equivalence relation on the class of continuous functions $X \to Y$; we write $f \simeq g$. When the functions are paths and the homotopy is a path homotopy, the paths are *path homotopic* and we write $f \simeq_p g$.

Two spaces X and Y are *homotopy equivalent* when there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$.

We can form the category `hTop` by taking spaces as objects and homotopy classes of maps as morphisms. One needs to check that composition "works" in this category; this was done in class.

Remark 3.1. Two spaces are homotopy equivalent precisely they are isomorphic in `hTop`. There is a functor `Top` $\to$ `hTop` that is the identity on objects and sends a continuous function to its path homotopy class. We therefore call functors out of `hTop` *homotopy invariants*, since by Theorem 2.14 they homotopy equivalent spaces to isomorphic objects.

3.2. **Construction.** Before defining the fundamental groupoid, there is a nice geometric picture to tell. Fix a topological space X and pick some points and paths between those points. When you "erase" the other information of the underlying space, you get several points and a bunch of (double-headed) arrows.

[3] When the context is unclear, we will call a general homotopy a *free homotopy*, and a homotopy with fixed endpoints a *path homotopy*.

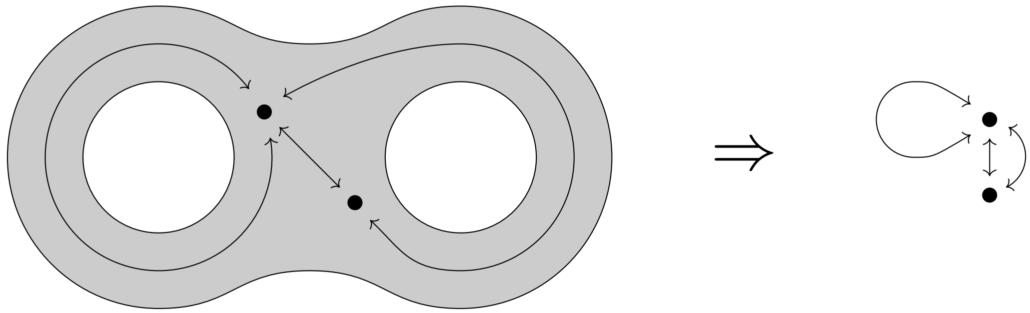

FIGURE 1. Some points and paths on a two-holed disk.

This looks suspiciously like a groupoid—in particular, each of the arrows are double-headed, and hence "invertible". The issue is that we don't necessarily see the inverses or composites when we just take some points and some paths.

We'll return to this intuition of the fundamental groupoid "forgetting" some of the underlying geometry of the space. More immediately, however, this view strongly suggests that when we take all the path classes from our underlying space, we should get back a groupoid.

Definition 3.2 (Fundamental Groupoid)**.** The *fundamental groupoid* $\Pi_1 X$ of a space X is the category whose objects are points of X and whose morphisms are path homotopy classes of paths in X.

Specifically, let $x, y, z \in X$, f a path from x to y, and g a path from y to z. Then,

- A path's domain is its startpoint: $\mathrm{dom}([f]) = x$.
- A path's codomain is its endpoint: $\mathrm{cod}([f]) = y$.
- The identity is the constant map: $1_x = [c_x]$.
- Composition is concatenation: $[g][f] = [f * g]$.

This construction is well-defined specifically because we are working with path homotopies. For example, in general two paths with different start points may be free homotopic, meaning without restricting to path homotopy we could not even write down the domain and codomain of our morphisms.

Example 3.3. [Bro07, p. 213] We can immediately compute a few fundamental groupoids.

- The fundamental groupoid of a convex space is a tree groupoid (defined in Example 2.17). This corresponds to the fact that any two paths with the same endpoints in such a space are homotopic via the straight line homotopy.
- The fundamental groupoid of a totally disconnected space is a discrete groupoid. This corresponds to the fact that the only paths in such spaces are the constant paths.

3.3. **Categorical Properties.** We can learn a lot about the fundamental groupoid by studying it categorically. First, we should confirm it is what we claim it is:

Proposition 3.4. *The fundamental groupoid is a groupoid.*

Proof. All of this work was already done in class for the fundamental group. We restate the results here for groupoids.

- Composition is well-defined, since concatenation preserves homotopy class.
- Composition is associative, since concatenation is associative up to homotopy.
- Every object x has $[c_x]$ as an identity.
- Every morphism $[f]$ has $[\bar{f}]$ as an inverse.

The first three say that $\Pi_1 X$ is a category, and the last says that it is a groupoid. $\square$

The construction of the fundamental groupoid naturally gives rise to a functor

$$\Pi_1 : \texttt{Top} \to \texttt{Grpd}.$$

In particular, let $f : X \to Y$ be a continuous function. We can view f as acting on paths via composition. Accordingly, we define

$$\begin{aligned} \Pi_1 f \colon \Pi_1 X &\to \Pi_1 Y \\ [\gamma] &\mapsto [f \circ \gamma]. \end{aligned}$$

This mapping is well-defined because composition preserves homotopy equivalence.

Proposition 3.5. Π_1 *is a functor.*

Proof. Again, much of this work was done in class.

- Π_1 respects composition, since composition is associative.
- Π_1 respects identities, since composing the identity fixes homotopy classes. $\square$

This result is an improvement over the fundamental group, where we needed a functor out of based spaces for the definition to make sense. This is a first hint that the fundamental groupoid in some sense captures more of the structure of a space than the fundamental group does.

Corollary 3.6. *The fundamental groupoid is a topological invariant. More precisely, if* $X \cong Y$, *then* $\Pi_1 X \cong \Pi_1 Y$.

Proof. This follows from Theorem 2.14 and Proposition 3.5. $\square$

Remark 3.7. As defined, the fundamental groupoid is a *not* a homotopy invariant. For instance, since objects of the fundamental groupoid are in bijection with elements of the underlying space, the fundamental groupoid of D^n has uncountably many objects, whereas the fundamental groupoid of the point has only one.

This unfortunate fact suggests an issue with our notion of isomorphism of categories: it requires objects to be in bijection, which is far too strong to express homotopy invariants. There is a more natural notion, equivalence of categories, somewhat analogous to homotopy equivalence, which requires some additional machinery to develop.

4. Natural Transformations and 2-Categories

If we want our categorical constructions to play nicely with homotopy, it will help to define an analogue to homotopy on categories. It turns out the correct notion is a *natural transformation*, a way to relate functors analogous to a homotopy. This notion will allow us to define *equivalence of categories*. Here I follow [Rie17, Sections 1.4-1.7].

4.1. **Natural Transformations as Categorical Homotopy.** Let H be a homotopy between continuous maps $f, g : X \to Y$. We can think of $H(x, -)$ as morphing the point $f(x)$ into $g(x)$. In the same way, we want to define a transformation between functors $F, G : \mathtt{C} \to \mathtt{D}$ which we can think of as morphing the object Fx into the point Gx. Of course, the right thing to do this morphing is exactly a morphism in $\mathtt{D}$. That motivates the following definition:

Definition 4.1 (Natural Transformation)**.** A *natural transformation* $\alpha : F \Rightarrow G$ between functors $F, G : \mathtt{C} \to \mathtt{D}$ consists of, for each object $x \in \mathrm{Obj}(\mathtt{C})$, a morphism $\alpha_x : F_x \to G_x \in \mathtt{D}$, called the *components* of α.

This data must preserve morphisms in $\mathtt{C}$, in the sense that for any morphism $f : x \to y \in \mathtt{C}$, the following square must commute:

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\
Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

Thinking of natural transformations as categorical analogues of homotopy will be critical for what is to come. The commutative square above is analogous to the requirement that a homotopy behaves as f at $t = 0$ and as g at $t = 1$; we require that the transformation behaves as F "before" the transformation and as G "after" the transformation.

An alternative definition makes the translation even clearer. Let $\mathbb{2}$ be the category with two objects, 0 and 1, and a single non-identity morphism $0 \to 1$. Defining the product category in the obvious way, with component-wise composition, a natural transformation between F and G corresponds bijectively to a functor $H : \mathtt{C} \times \mathbb{2} \to \mathtt{D}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathtt{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathtt{C} \times \mathbb{2} \arrow[d, "H"] & \mathtt{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\
& \mathtt{D} &
\end{tikzcd}$$

Here i_0 and i_1 are the obvious inclusion functors. (TODO: proof from [Rie17, Lemma 1.5.1].)

One key difference between homotopy and natural transformations is that homotopies are always invertible, in the sense that for a homotopy H from f to g, $H(-, 1-t)$ is a homotopy from g to h. This key fact means homotopies can be used to define an equivalence relation on continuous maps. Since we are primarily concerned with the special case of groupoids, in which ”everything” is invertible, this ought to be a non-issue.

Definition 4.2. A *natural isomorphism* is a natural transformation whose components are each isomorphisms.

Remark 4.3. Any natural transformation into a groupoid is a natural isomorphism.

Theorem 4.4. *Let $F, G : \mathtt{C} \to \mathtt{D}$ be functors, and α a natural isomorphism between them. For each $x \in \mathtt{D}$, let β_x be the inverse of α_x. Then the β_x collect into a natural isomorphism, β, between* $\mathtt{D}$ *and* $\mathtt{C}$*.*

That β is an isomorphism follows because inverses of isomorphisms are isomorphisms. We need to show that β is a natural transformation, i.e. that for any $x, y \in \mathtt{C}$, the following square commutes:

$$\begin{tikzcd} Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"'] & Fx \arrow[d, "Ff"] \\ Gy \arrow[r, "\beta_y"'] & Fy \end{tikzcd}$$

(TODO: proof)

4.2. **Equivalence of Categories.** TODO.

4.3. **The 2-Category of Categories.** TODO.

5. Higher Categories and Higher Groupoids

TODO.

6. ∞-groupoids as spaces

TODO; not sure I’ll have space for this. Following [Por21].

References

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