

The Fundamental Groupoid

Riley Shahar

Categories

Categories

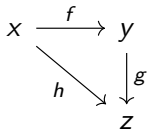
► Objects

x

y

z

Categories



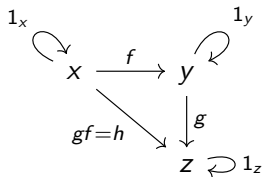
- Objects
- Morphisms

Categories

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ & \searrow gf=h & \downarrow g \\ & & z \end{array}$$

- Objects
- Morphisms
- Composition

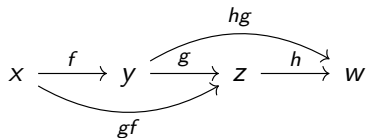
Categories



- Objects
- Morphisms
- Composition
- Identities

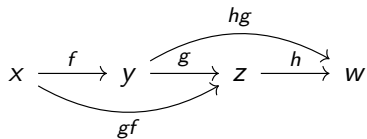
Category Axioms

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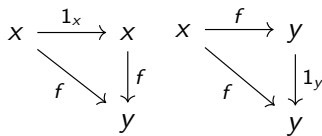


Associative

Category Axioms



Associative



Unital

Examples

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- ▶ Set: Sets and Functions

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- ▶ Set: Sets and Functions
- ▶ Grp: Groups and Homomorphisms
- ▶ Top: Spaces and Continuous Functions

Groups as Categories

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Moral: not all categories have structured sets for objects and structure-preserving functions for morphisms.

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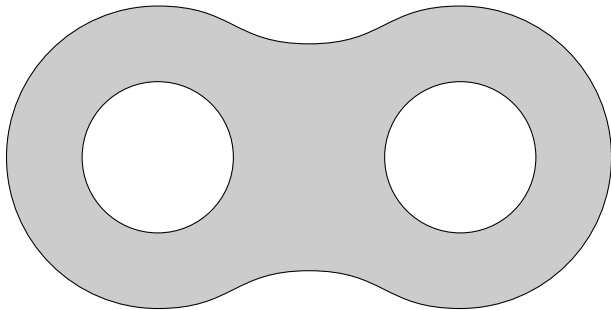
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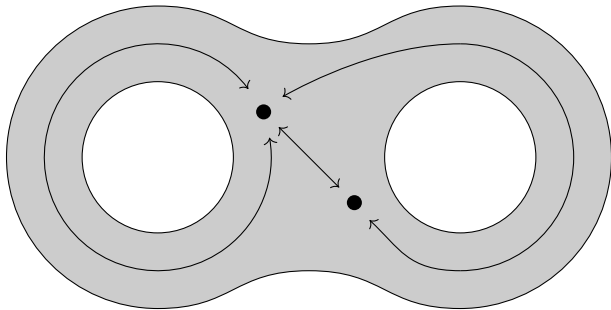
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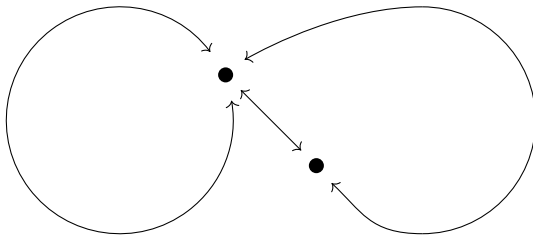
Topology??



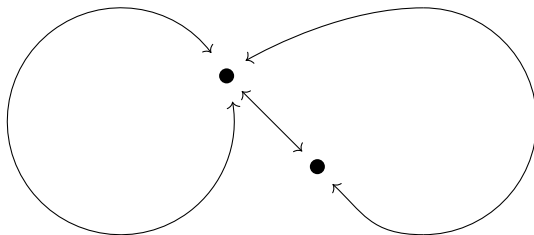
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If you relax the requirement that paths in the fundamental group are loops, you get *the fundamental groupoid* Π_1 .

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- ▶ The naive notion of isomorphism is too strong: we need a kind of weak equivalence, much like homotopy for topological spaces, for categories.
- ▶ This line of thought naturally leads to "higher categories", with morphisms between morphisms, etc.