The Fundamental Groupoid

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Categories

Categories

$x \qquad y$

$\qquad z$

- Objects

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$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "h"'] & y \arrow[d, "g"] \\ & z \end{tikzcd}$$

- Objects
- Morphisms

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$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "gf=h"'] & y \arrow[d, "g"] \\ & z \end{tikzcd}$$

- Objects
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Categories

$$
\begin{tikzcd}
x \arrow[loop above, "1_x"] \arrow[r, "f"] \arrow[dr, swap, "gf=h"] & y \arrow[loop above, "1_y"] \arrow[d, "g"] \\
& z \arrow[loop right, "1_z"]
\end{tikzcd}
$$

- Objects
- Morphisms
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- Identities

Category Axioms

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$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & y \arrow[r, "g"] \arrow[rr, bend left, "hg"] & z \arrow[r, "h"] & w \end{tikzcd}$$

Associative

Category Axioms

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & y \arrow[r, "g"] \arrow[rr, bend left, "hg"] & z \arrow[r, "h"] & w \end{tikzcd}$$

Associative

$$\begin{tikzcd} x \arrow[r, "1_x"] \arrow[dr, "f"'] & x \arrow[d, "f"] \\ & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "f"'] & y \arrow[d, "1_y"] \\ & y \end{tikzcd}$$

Unital

Examples

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- `Set`: Sets and Functions

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- $\mathtt{Set}$: Sets and Functions
- $\mathtt{Grp}$: Groups and Homomorphisms

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- `Set`: Sets and Functions
- `Grp`: Groups and Homomorphisms
- `Top`: Spaces and Continuous Functions

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Moral: not all categories have structured sets for objects and structure-preserving functions for morphisms.

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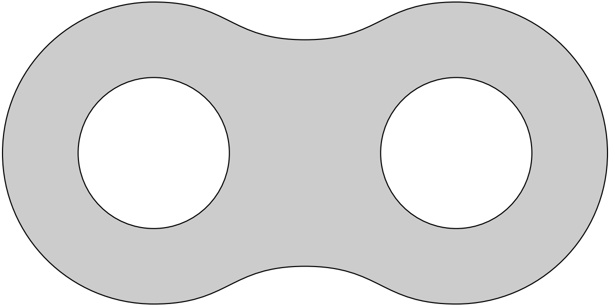

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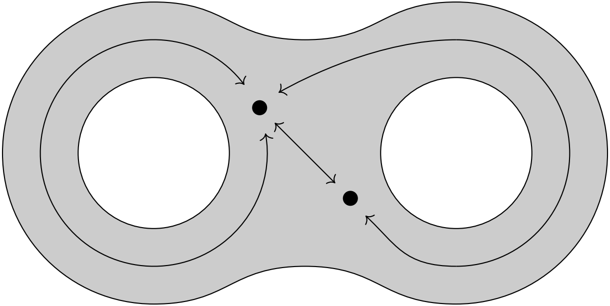

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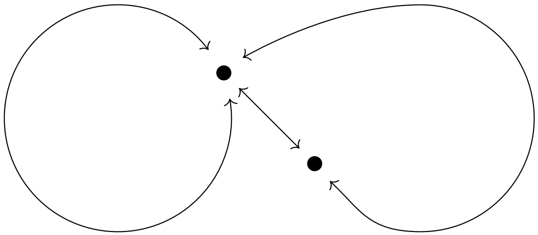

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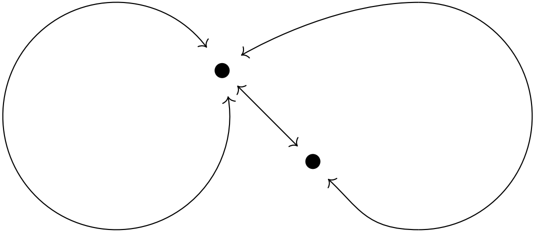

If you relax the requirement that paths in the fundamental group are loops, you get *the fundamental groupoid* Π_1.

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- The naive notion of isomorphism is too strong: we need a kind of weak equivalence, much like homotopy for topological spaces, for categories.
- This line of thought naturally leads to "higher categories", with morphisms between morphisms, etc.