Higher Homotopy Groupoids

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1 Preliminaries: Categories

1.1 Basics

Definition 1.1 (Category)**.** A *category* $\mathcal{C}$ consists of a collection of *objects* $\mathrm{Obj}(\mathcal{C})$ and a collection of *morphisms* $\mathrm{Hom}(\mathcal{C})$, such that

- Each morphism f has a specific *domain* $\mathrm{dom}(f)$ and *codomain* $\mathrm{cod}(f)$; we write $f : X \to Y$ or $X \xrightarrow{f} Y$.
- Each object X has a specific *identity morphism* $1_X : X \to X$.
- For each pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a specific *composite morphism* $gf : X \to Z$.

This data must be *associative* and *unital*, which is to say,

- (Associativity) For any triplet of composition-compatible morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, we have $h(gf) = (hg)f$. We write merely hgf.
- (Unital) For any $f : X \to Y$, we have $1_Y f = f = f 1_X$.

In other words, the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] \arrow[rr, bend left, "hg"] & Z \arrow[r, "h"] & W \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[dr, "f"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, "1_Y"] \\ & Y \end{tikzcd}$$

Notation. I will use $\mathcal{C}(X, Y)$ for the collection of morphisms with domain X and codomain Y. I also choose to use multiplicative notation for composition, instead of $\circ$, to distinguish it from composition of continuous functions, since in many of our categories morphism composition will be concatenation, not function composition.

There are numerous special kinds of morphisms. We define some now.

Definition 1.2 (Isomorphism)**.** A morphism $f : X \to Y$ is an *isomorphism* if there is a morphism $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. We say X and Y are isomorphic and write $X \cong Y$.

Theorem 1.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $f : X \to Y$ an isomorphism in $\mathcal{D}$. Then $F(f)$ is an isomorphism between $F(X)$ and $F(y)$ in $\mathcal{D}$.*

Proof. Let f^{-1} be the inverse of f. Then

$$F(f^{-1})F(f) = F(f^{-1}f) = F(1_x) = 1_{F(x)},$$

and the same works on the other side. $\square$

1.2 Groups and Groupoids

% TODO

2 The Fundamental Groupoid

Notation. When the context is unclear, we will call a general homotopy a *free homotopy*, and a homotopy with fixed endpoints a *path homotopy*.

Before defining the fundamental groupoid, there is a nice geometric picture to tell about the fundamental group. Fix a topological space X and a point x_0. Draw a representative of each of the non-identity homotopy classes of loops at x_0, and note that each arrow is double-sided, since paths can be traversed in either direction. When you "erase" the other information of the underlying space, you get a single point and a bunch of double-headed arrows: exactly the "categories-as-dots-and-arrows" picture of a group!

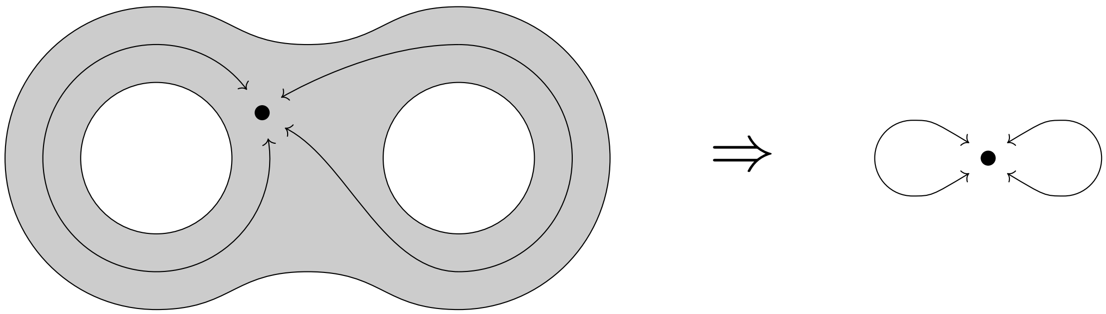

Figure 1: Generators of the fundamental group on a two-holed disk.

We'll keep returning to this intuition of homotopy groups "erasing" some of the underlying geometry of the space. More immediately, however, this view strongly suggests that when we take multiple points from our underlying space, we should get back a group with multiple objects—a groupoid—from the construction.

Definition 2.1 (Fundamental Groupoid)**.** The *fundamental groupoid* $\Pi_1(X)$ of a space X is the category whose objects are points of X and whose morphisms are path homotopy classes of paths in X.

Specifically, let $x, y, z \in X$, f a path from x to y, and g a path from y to z. Then,

- A path's domain is its startpoint: $\mathrm{dom}([f]) = x$.
- A path's codomain is its endpoint: $\mathrm{cod}([f]) = y$.
- The identity is the constant map: $1_x = [c_x]$.
- Composition is concatenation: $[g][f] = [f * g]$.

This construction is well-defined specifically because we are working with path homotopies. For example, in general two paths with different start points may be free homotopic, meaning without restricting to path homotopy we could not even write down the domain and codomain of our morphisms.

Example. We can immediately compute a few fundamental groupoids. [Brown p. 213]

- The fundamental groupoid of a convex space is a *tree groupoid*, i.e. a groupoid with precisely one morphism between any two objects. This corresponds to the fact that any two paths with the same endpoints in such a space are homotopic via the straight line homotopy.
- The fundamental groupoid of a totally disconnected space is a *discrete groupoid*, i.e. a groupoid with only identity morphisms. This corresponds to the fact that the only paths in such spaces are the constant paths.

Proposition 2.1. *The fundamental groupoid is a groupoid.*

Proof. All of this work was already done in class for the fundamental group. We restate the results here for groupoids.

- Composition is well-defined, since concatenation preserves homotopy equivalence.
- Composition is associative, since concatenation is associative up to homotopy.

- Every object x has $[c_x]$ as an identity.
- Every morphism $[f]$ has $[\bar{f}]$ as an inverse.

The first three say that $\Pi_1(X)$ is a category, and the last says that it is a groupoid. $\square$

The construction of the fundamental groupoid naturally gives rise to a functor

$$\Pi_1 : \mathrm{Top} \to \mathrm{Grpd}.$$

In particular, let $f : X \to Y$ be a continuous function. We can view f as acting on paths via composition. Accordingly, we define

$$\begin{aligned} \Pi_1(f) \colon \Pi_1(X) &\to \Pi_1(Y) \\ [\gamma] &\mapsto [f \circ \gamma]. \end{aligned}$$

% TODO: picture

This mapping is well-defined because composition preserves homotopy equivalence.

% TODO: output is a homomorphism

Proposition 2.2. Π_1 *is a functor.*

Proof. Again, much of this work was done in class.

- Π_1 respects composition, since composition is associative.
- Π_1 respects the identity, since composition by the identity fixes homotopy classes. $\square$

This result is an improvement over the fundamental group, where we needed a functor out of based spaces for the definition to make sense. This is a first hint that the fundamental groupoid in some sense captures more of the structure of a space than the fundamental group does.

Corollary 2.2.1. *The fundamental groupoid is a topological invariant. More precisely, if $X \cong Y$, then $\Pi_1(X) \cong \Pi_1(Y)$.*

Proof. This follows from Theorem 1.1 and Proposition 2.2. $\square$

Theorem 2.3. *The fundamental groupoid is a homotopy invariant. More precisely, if $X \simeq Y$, then $\Pi_1(X) \cong \Pi_1(Y)$.*

This theorem is harder than topological invariance, because it tells us something specific about the functor Π_1.

% TODO: what does the correct result look like? e.g. Dn and 1 have different % groupoids (because of cardinality), but maybe the same up to some kind of % natural isomorphism of categories

Proof. $\square$