Applied Categorical Structures **9:** 311–327, 2001.

A Homotopy Bigroupoid of a Topological Space

Dedicated to Horst Herrlich on his sixtieth birthday

K. A. HARDIE
Department of Mathematics, University of Cape Town, 7700 Rondebosch, South Africa

K. H. KAMPS
Fachbereich Mathematik, FernUniversität, Postfach 940, D-58084 Hagen, Germany,
e-mail: heiner.kamps@fernuni-hagen.de

R. W. KIEBOOM
Departement Wiskunde, Vrije Universiteit Brussel, Pleinlaan 2, F 10, B-1050 Brussel, Belgium

(Received: 26 February 1998; accepted: 13 January 1999)

Abstract. In this paper we give an explicit description of a homotopy bigroupoid of a topological space as a 2-dimensional structure in homotopy theory which allows one to derive some basic properties in 2-dimensional homotopical algebra using purely algebraic arguments. The main results are valid in the general setting of a bigroupoid.

Mathematics Subject Classifications (2000): 18D05, 20L05, 55Q05, 55Q35.

Key words: homotopy bigroupoid, bigroupoid, bicategory, 2-groupoid, homotopical algebra, coherence, homotopy operations.

0. Introduction

'Modern higher-dimensional algebra has its roots in the dream of finding a natural and convenient completely algebraic description of the homotopy type of a topological space' [2, p. 6090]. There are models such as simplicial groupoids that model all homotopy types completely [14]. Other models such as crossed complexes 'can either be viewed as ... complete ... for a restricted class of homotopy types or as partial models of the homotopy types of all spaces' [15, p. 222].

More recently, the interest in higher-dimensional category theory has been inspired by applications in topological quantum field theory and computer science [1, 35]. There are a number of competing definitions of higher-dimensional categories, and in order to compare them there is a great need for simple, concrete and easily understood examples.

The aim of this paper is to exhibit such an example of a 2-dimensional algebraic structure arising in homotopy theory and previously sketched in [2] (with another purpose in mind) including necessary expository detail.

To define algebraic models of n-types so as to obtain reasonably well-behaved algebraic categories, a guiding principle is to avoid identifications (as far as possible) in dimensions up to $n-1$.

For topological spaces there seems to be general agreement that convenient algebraic models of n-types for $n=0$ and $n=1$ are as follows.

In dimension 0 points p, q of X will be identified if they can be joined by a path, i.e., a continuous map $f\colon I \to X$ from the unit interval $I=[0,1]$ of real numbers such that $f(0)=p$ and $f(1)=q$. This gives rise to the set of *path-components*, $\Pi_0(X)$, of X.

In dimension 1 the points of X will be retained, but paths f, f' between fixed points p, q will be identified if there is a homotopy rel end points between them. This gives rise to the *fundamental groupoid*, $\Pi_1(X)$, of X. The class of a path will be called a 1-*track*.

Hence the most natural approach to 2-dimensional homotopical algebra of a space X is to retain points and paths between them and identify homotopies rel end points under a suitable homotopy relation. This gives rise to the notion of 2-*track*. In this way we obtain a 2-dimensional structure with points in dimension 0 (0-cells), paths in dimension 1 (1-cells) and 2-tracks in dimension 2 (2-cells). We note that 1-cells can be pasted horizontally, 2-tracks can be pasted both horizontally and vertically in the usual way by subdividing the unit interval at 1/2. Vertical pasting is well-behaved, it is strictly associative and has strict identities, whereas horizontal pasting is neither strictly associative nor do we have strict identities. However, horizontal pasting is still reasonably well-behaved in the sense that associativity does hold and strict inverses do exist up to coherent isomorphisms. Thus we obtain a bicategory, $\Pi_2(X)$, in the sense of Bénabou [5]. The bicategory $\Pi_2(X)$ has the additional feature that the 2-cells are strictly invertible with respect to vertical pasting and the 1-cells are invertible up to coherent isomorphism, i.e., $\Pi_2(X)$ is a bigroupoid which will be called the *homotopy bigroupoid* of the topological space X.

In practice, 2-tracks arise in coherent homotopy theory. They lend themselves to an extension of the methods of [20] and [18] in which 2-dimensional algebra based on the groupoid enrichment of **Top** and **Top***, the categories of topological resp. pointed topological spaces, was used to identify isomorphisms in their associated slash categories.

Moreover, since a homotopy commutative square

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowleft m_t"] & U \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

corresponds to a path in the function space X^V, the 3-dimensional algebra associated with pasting of cubical diagrams inhabited by 2-tracks converts to a 2-

dimensional algebra associated with $\Pi_2(X^V)$. This may be regarded as analogous to the 'loop space trick' whereby the *n*th homotopy group of a pointed space X is converted to the $(n-1)$th homotopy group of ΩX, the loop space of X.

It may be noted that, as for the fundamental groupoid, the construction of the homotopy bigroupoid does not depend on any choice of base points and allows one to deal with 2-tracks between nonclosed paths.

As an alternative, for a Hausdorff space, the construction of a *homotopy* 2-*groupoid* with strict inverses both in dimensions 1 and 2 has been given in [19] (see also [33]). In fact, for a Hausdorff space, there is a biequivalence in the sense of [40] between the homotopy bigroupoid constructed in this paper and the homotopy 2-groupoid in [19].

The paper is organized as follows. In Section 1 we recall the definition of a bicategory and give an explicit definition of a bigroupoid which is not easily available in the literature. Furthermore certain types of functors between bigroupoids are defined. Section 2 contains the definition of a 2-track and the construction of the homotopy bigroupoid. In Section 3 crossed module type operations inherent in the homotopy bigroupoid are defined. As a sample calculation in a bigroupoid setting the 2-endomorphism groups in the homotopy bigroupoid are identified. Further applications will be dealt with elsewhere. These include applications to function spaces, to the theory of Toda brackets and to the theory of fibrations [26]. In Section 4 we are concerned with the homotopy 2-groupoid of a Hausdorff space and its relation to the homotopy bigroupoid. It should be pointed out that the construction of the homotopy bigroupoid is quite elementary whereas the construction of the homotopy 2-groupoid is rather sophisticated. Furthermore, in contrast to the homotopy 2-groupoid, there is some hope to generalise the construction of the homotopy bigroupoid to a suitably structured category replacing the category of topological spaces (e.g., a category with a cylinder, cf. [25]). Section 5 has a discussion of some of the various approaches to 2-dimensional homotopical algebra.

1. Bigroupoids

A *bigroupoid* is a bicategory in the sense of Bénabou [5] such that the 2-cells are strictly invertible and the 1-cells are invertible up to coherent isomorphism. For the convenience of the reader we include a formal definition, since, although it is implicit in a lot of work, we do not know an explicit reference. To fix notation we recall some terminology for bicategories.

DEFINITION 1.1 (cf. [5, (1.1)(i)–(vi)]). A *bicategory* $\mathbf{S}$ is given by the following data:

(i) A set Ob($\mathbf{S}$), called set of *objects* (0-*cells*) of $\mathbf{S}$.
(ii) For each pair (p, q) of objects of $\mathbf{S}$, a category $\mathbf{S}(p, q)$. An object f of $\mathbf{S}(p, q)$ is called an *arrow* or 1-*cell* of $\mathbf{S}$, and written

$$f\colon\ p \longrightarrow q.$$

A morphism β from f to f' in $\mathbf{S}(p,q)$ will be called a 2-*cell* and written

$$\beta\colon f \Longrightarrow f'.$$

Composition of $\beta\colon f \Rightarrow f'$ and $\beta'\colon f' \Rightarrow f''$ will be denoted additively to give

$$\beta + \beta'\colon f \Longrightarrow f''.$$

The identity element at a 1-cell f will be written $0_f\colon f \Rightarrow f$.

(iii) For each triple (p,q,r) of objects of $\mathbf{S}$, a *composition functor*

$$\mathbf{S}(p,q) \times \mathbf{S}(q,r) \xrightarrow{\bullet} \mathbf{S}(p,r)$$

denoted $(f,g) \mapsto g \bullet f$, $(\beta,\gamma) \mapsto \gamma \bullet \beta$ on objects resp. morphisms.

(iv) For each object p of $\mathbf{S}$ an object c_p of $\mathbf{S}(p,p)$, called *identity arrow* of $\mathbf{S}$. The identity 2-cell $0_{c_p}\colon c_p \Rightarrow c_p$ will be denoted by 0_p.

(v) For each quadruple (p,q,r,s) of objects of $\mathbf{S}$, natural isomorphisms, called *associativity isomorphisms*,

$$\alpha\colon h \bullet (g \bullet f) \Longrightarrow (h \bullet g) \bullet f,$$

where $f\colon p \to q$, $g\colon q \to r$, $h\colon r \to s$ are 1-cells.

(vi) For each pair (p,q) of objects of $\mathbf{S}$, two natural isomorphisms, called *left* and *right identities*,

$$\lambda\colon c_q \bullet f \Longrightarrow f, \qquad \rho\colon f \bullet c_p \Longrightarrow f,$$

where $f\colon p \to q$ is a 1-cell.

The isomorphisms α, λ and ρ are required to satisfy the following axioms (AC) and (IC).

(AC) *Associativity coherence*: Let

$$f\colon p \longrightarrow p', \qquad g\colon p' \longrightarrow p'', \qquad h\colon p'' \longrightarrow p''',$$
$$k\colon p''' \longrightarrow p''''$$

be 1-cells of $\mathbf{S}$. Then the following diagram commutes.

$$\begin{tikzcd}
k \bullet (h \bullet (g \bullet f)) \arrow[rr, Rightarrow, "0_k \bullet \alpha"] \arrow[d, Rightarrow, "\alpha"'] & & k \bullet ((h \bullet g) \bullet f) \arrow[d, Rightarrow, "\alpha"] \\
(k \bullet h) \bullet (g \bullet f) \arrow[dr, Rightarrow, "\alpha"'] & & (k \bullet (h \bullet g)) \bullet f \arrow[dl, Rightarrow, "\alpha \bullet 0_f"] \\
& ((k \bullet h) \bullet g) \bullet f &
\end{tikzcd}$$

(IC) *Identity coherence*: Let $f\colon p \to q$ and $g\colon q \to r$ be 1-cells of $\mathbf{S}$. Then the following diagram commutes.

$$
\begin{tikzcd}
g \bullet (c_q \bullet f) \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "0_g \bullet \lambda"'] & & (g \bullet c_q) \bullet f \arrow[dl, Rightarrow, "\rho \bullet 0_f"] \\
& g \bullet f &
\end{tikzcd}
$$

DEFINITION 1.2. A *bigroupoid* is a bicategory **S** such that the following holds:

(1) For each pair (p, q) of objects of **S**, the category $\mathbf{S}(p, q)$ is a groupoid, i.e., any 2-cell is invertible.

(2) **S** is equipped with the following additional data:

(vii) For each pair (p, q) of objects of **S**, a (covariant) functor

$$^{-1}\colon \mathbf{S}(p, q) \longrightarrow \mathbf{S}(q, p).$$

(viii) For each pair (p, q) of objects of **S**, two natural isomorphisms, called *cancellation isomorphisms*,

$$\iota\colon f^{-1} \bullet f \Longrightarrow c_p, \qquad \iota'\colon f \bullet f^{-1} \Longrightarrow c_q,$$

where $f\colon p \to q$ is a 1-cell.

These data are required to satisfy the following coherence axiom (CC).

(CC) *Cancellation coherence*: Let $f\colon p \to q$ be a 1-cell of **S**. Then the following diagram (and a similar one involving (f^{-1}, f) instead of (f, f^{-1})) commutes.

$$
\begin{tikzcd}
f \bullet (f^{-1} \bullet f) \arrow[rr, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "0_f \bullet \iota"'] & & (f \bullet f^{-1}) \bullet f \arrow[d, Rightarrow, "\iota' \bullet 0_f"] \\
f \bullet c_p \arrow[dr, Rightarrow, "\rho"'] & & c_q \bullet f \arrow[dl, Rightarrow, "\lambda"] \\
& f &
\end{tikzcd}
$$

Special case. A 2-*groupoid* is a bigroupoid such that composition (iii) is strictly associative, identities (iv) and inverses (vii) are strict and the data α, λ, ρ, ι, ι' are all given by identity 2-cells.

REMARKS. (i) Property (CC) has been known from the special case of what has been called a *category with group structure* (cf. [30, 42]). In fact, a category with group structure is just a bigroupoid with one object.

(ii) Properties related to (CC) arise in the theory of compact closed categories (cf. [27, 28]) and in the theory of autonomous categories (cf. [16, 31]).

(iii) R. Street has pointed out to us that the theory of Azumaya algebras and Morita equivalences gives rise to a bigroupoid in algebra (cf. [17]).

We are now concerned with morphisms of bigroupoids. When dealing with homotopy bigroupoids of spaces, we shall encounter the following type of morphism between bigroupoids which corresponds to the notion of 2-functor between bicategories in the sense of [40, p. 567].

DEFINITION 1.3. Let $\mathbf{S}$ and $\overline{\mathbf{S}}$ be bigroupoids. Then a 2-*functor* $F\colon \mathbf{S} \to \overline{\mathbf{S}}$ is given by the following data.

(i) A map $F\colon \mathrm{Ob}(\mathbf{S}) \to \mathrm{Ob}(\overline{\mathbf{S}})$.
(ii) For each pair (p, q) of objects of $\mathbf{S}$, a functor

$$F\colon \mathbf{S}(p, q) \longrightarrow \overline{\mathbf{S}}(Fp, Fq).$$

These data are required to commute with all the constraints (iii)–(viii) in the obvious way.

For completeness we include the more general notion of a pseudo functor of bigroupoids corresponding to the notion of pseudo functor of bicategories in the sense of [40, pp. 566–567].

DEFINITION 1.4. Let $\mathbf{S}$ and $\overline{\mathbf{S}}$ be bigroupoids. Then a *pseudo functor* $F\colon \mathbf{S} \to \overline{\mathbf{S}}$ is given by the following data.

(i) A map $F\colon \mathrm{Ob}(\mathbf{S}) \to \mathrm{Ob}(\overline{\mathbf{S}})$.
(ii) For each pair (p, q) of objects of $\mathbf{S}$, a functor

$$F\colon \mathbf{S}(p, q) \longrightarrow \overline{\mathbf{S}}(Fp, Fq).$$

(iii) For each object p of $\mathbf{S}$, an isomorphism (2-cell) of $\overline{\mathbf{S}}$

$$\varphi\colon c_{Fp} \Longrightarrow Fc_p.$$

(iv) For each triple (p, q, r) of objects of $\mathbf{S}$ natural isomorphisms (2-cells) of $\overline{\mathbf{S}}$

$$\psi\colon Fg \bullet Ff \Longrightarrow F(g \bullet f),$$

where $f\colon p \to q$, $g\colon q \to r$ are 1-cells of $\mathbf{S}$.
(v) For each pair (p, q) of objects of $\mathbf{S}$, natural isomorphisms (2-cells) of $\overline{\mathbf{S}}$

$$\chi\colon (Ff)^{-1} \Longrightarrow F(f^{-1})$$

where $f\colon p \to q$ is a 1-cell of $\mathbf{S}$.

These data are required to satisfy the following coherence axioms.

(PF 1) Let $f\colon p \to q$, $g\colon q \to r$, $h\colon r \to s$ be 1-cells of $\mathbf{S}$. Then the following diagram commutes.

$$\begin{tikzcd}
& Fh \bullet (Fg \bullet Ff) \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow, "0_{Fh}\bullet\psi"] & \\
(Fh \bullet Fg) \bullet Ff \arrow[d, Rightarrow, "\psi\bullet 0_{Ff}"'] & & Fh \bullet F(g \bullet f) \arrow[d, Rightarrow, "\psi"] \\
F(h \bullet g) \bullet Ff \arrow[dr, Rightarrow, "\psi"'] & & F(h \bullet (g \bullet f)) \arrow[dl, Rightarrow, "F\alpha"] \\
& F((h \bullet g) \bullet f) &
\end{tikzcd}$$

(PF 2) Let $f\colon\ p \to q$ be a 1-cell of $\mathbf{S}$. Then the following diagram (and a similar one involving ρ instead of λ) commutes.

$$\begin{tikzcd}
c_{Fq} \bullet Ff \arrow[r, Rightarrow, "\varphi\bullet 0_{Ff}"] \arrow[d, Rightarrow, "\lambda"'] & Fc_q \bullet Ff \arrow[d, Rightarrow, "\psi"] \\
Ff & F(c_q \bullet f) \arrow[l, Rightarrow, "F\lambda"]
\end{tikzcd}$$

(PF 3) Let $f\colon\ p \to q$ be a 1-cell of $\mathbf{S}$. Then the following diagram (and a similar one involving ι' instead of ι) commutes.

$$\begin{tikzcd}
(Ff)^{-1} \bullet Ff \arrow[rr, Rightarrow, "\chi\bullet 0_{Ff}"] \arrow[d, Rightarrow, "\iota"'] & & F(f^{-1}) \bullet Ff \arrow[d, Rightarrow, "\psi"] \\
c_{Fp} \arrow[dr, Rightarrow, "\varphi"'] & & F(f^{-1} \bullet f) \arrow[dl, Rightarrow, "F\iota"] \\
& Fc_p &
\end{tikzcd}$$

Finally, the notion of biequivalence between bigroupoids can be defined as follows (cf. [40, p. 570]).

DEFINITION 1.5. Let $F\colon\ \mathbf{S} \to \overline{\mathbf{S}}$ be a pseudo functor between bigroupoids. Then F is called a *biequivalence* if each of the functors $F\colon\ \mathbf{S}(p, q) \to \overline{\mathbf{S}}(Fp, Fq)$ is an equivalence of groupoids, and if for each object x of $\overline{\mathbf{S}}$ there exists an object p of $\mathbf{S}$ and a 1-cell $Fp \to x$ in $\overline{\mathbf{S}}$.

2. The Homotopy Bigroupoid of a Space

The aim of this section is to describe the structure of a bigroupoid, $\Pi_2(X)$, for a topological space X. We recall first some basic notions of homotopy theory and define the notion of a 2-track.

If X is a topological space then a *path*, $f : p \simeq q$, in X from p to q, where p and q are points of X, is a map f from the unit interval $I = [0, 1]$ into X such that $f(0) = p$ and $f(1) = q$. If $f : p \simeq q$ and $g : q \simeq r$ are paths in X we denote by $g \bullet f : p \simeq r$ their *concatenation*, i.e.,

$$(g \bullet f)(s) = f(2s), \quad 0 \leq s \leq 1/2;$$
$$(g \bullet f)(s) = g(2s - 1), \quad 1/2 \leq s \leq 1.$$

The *constant path* at $p \in X$ will be denoted c_p. If f is a path in X, we denote by f^{-1} the path *reverse* to f, i.e., $f^{-1}(s) = f(1 - s)$.

Let $f, f' : p \simeq q$ be paths. A *relative homotopy* $f_t : f \simeq f' : p \simeq q$ is a homotopy $f_t : f \simeq f'$ such that the initial and final points remain fixed during the homotopy.

Relative homotopies can be concatenated (pasted) vertically and horizontally. More precisely, let $f, f', f'' : p \simeq q$ be paths, let $f_t : f \simeq f'$, $f'_t : f' \simeq f''$ be relative homotopies we define the *vertical pasting* $f_t + f'_t : f \simeq f'' : p \simeq q$ to be the relative homotopy h_t such that

$$h_t = f_{2t}, \quad 0 \leq t \leq 1/2; \qquad h_t = f'_{2t-1}, \quad 1/2 \leq t \leq 1.$$

If $f_t : f \simeq f' : p \simeq q$ and $g_t : g \simeq g' : q \simeq r$ are relative homotopies, then we obtain the *horizontal pasting* $g_t \bullet f_t : g \bullet f \simeq g' \bullet f' : p \simeq r$ by concatenation of the respective paths at each stage of the homotopy.

Vertical and horizontal pasting are interrelated by the following *interchange formula*. The proof is a simple verification.

PROPOSITION 2.1. *Let*

$$f_t : f \simeq f' : p \simeq q, \qquad f'_t : f' \simeq f'' : p \simeq q, \qquad g_t : g \simeq g' : q \simeq r,$$
$$g'_t : g' \simeq g'' : q \simeq r$$

be relative homotopies. Then the following equality holds:

$$(g_t + g'_t) \bullet (f_t + f'_t) = (g_t \bullet f_t) + (g'_t \bullet f'_t) : g \bullet f \simeq g'' \bullet f'' : p \simeq r.$$

Let $f_t, f'_t : f \simeq f' : p \simeq q$ be two relative homotopies. We may consider f_t and f'_t themselves to be *relatively homotopic*, if they are homotopic via a homotopy $I \times I \times I \to X$ which is constant on the boundary of $I \times I$. The relative homotopy class $\{f_t\}$ of $f_t : f \simeq f' : p \simeq q$ will be called a 2-*track*. In that case we will use the notation

$$\{f_t\} : f \Longrightarrow f' : p \simeq q \text{ or simply } \{f_t\} : f \Longrightarrow f'.$$

The corresponding set will be denoted by $\Pi_2(X)(p, q)(f, f')$.

Vertical resp. horizontal pasting of relative homotopies gives rise to vertical resp. horizontal composition of 2-tracks in the obvious way. In order to fix notation, we give the details. If $\{f_t\} : f \Rightarrow f' : p \simeq q$ and $\{f'_t\} : f' \Rightarrow f'' : p \simeq q$ are 2-tracks, then

$$\{f_t\} + \{f'_t\} : f \Longrightarrow f'' : p \simeq q$$

is given by $\{f_t\} + \{f'_t\} = \{f_t + f'_t\}$.

Similarly, if $\{f_t\} : f \Rightarrow f' : p \simeq q$ and $\{g_t\} : g \Rightarrow g' : q \simeq r$ are 2-tracks, then

$$\{g_t\} \bullet \{f_t\} : g \bullet f \Longrightarrow g' \bullet f' : p \simeq r$$

is given by $\{g_t\} \bullet \{f_t\} = \{g_t \bullet f_t\}$.

We note that the interchange formula (2.1) for relative homotopies gives rise to an *interchange law* for 2-tracks.

We are now in a position to describe the structure of a bigroupoid, $\Pi_2(X)$, for a topological space X. The data are as follows.

(i) The objects (0-cells) of $\Pi_2(X)$ are the points p, q etc. of X.

(ii) For each pair (p, q) of points of X we have a category $\Pi_2(X)(p, q)$. The objects (arrows, 1-cells) are the paths f, f' etc. in X from p to q. The morphisms (2-cells) are the 2-tracks $\{f_t\} : f \Rightarrow f' : p \simeq q$, where $f_t : f \simeq f' : p \simeq q$ is a relative homotopy. Composition in $\Pi_2(X)(p, q)$ is vertical composition of 2-tracks. The identity element $0_f\colon f \Rightarrow f$ is the 2-track of the constant homotopy at f. Note that $\Pi_2(X)(p, q)$ is a groupoid since relative homotopies are reversible.

(iii) For each triple of points (p, q, r) of X we have a composition functor

$$\Pi_2(X)(p, q) \times \Pi_2(X)(q, r) \xrightarrow{\bullet} \Pi_2(X)(p, r)$$

given on objects by concatenation of paths and on morphisms by horizontal composition of 2-tracks.

Note that functoriality follows from the interchange law for 2-tracks corresponding to Proposition 2.1.

(iv) For each point p of X the constant path at p, c_p, serves as identity arrow of $\Pi_2(X)(p, p)$.

(vii) For each pair of points (p, q) of X the *reverse path functor*

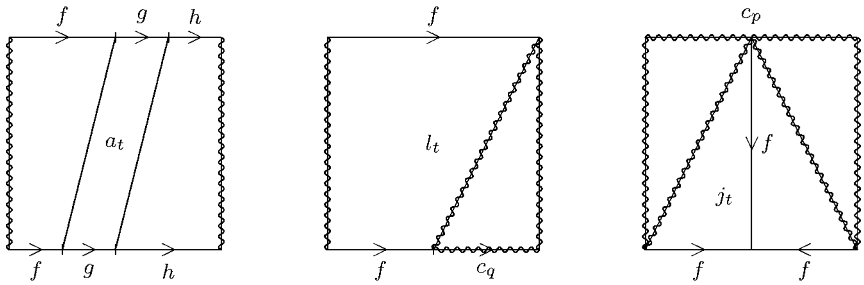
$$^{-1}\colon \Pi_2(X)(p, q) \longrightarrow \Pi_2(X)(q, p)$$

assigns to a path $f : p \simeq q$ the path $f^{-1} : q \simeq p$ reverse to f and to a 2-track $\{f_t\} : f \Rightarrow f' : p \simeq q$ the 2-track $\{f_t^{-1}\}$ obtained by reversing f_t at each level t.

(v), (vi), (viii) Let $f : p \simeq q$, $g : q \simeq r$, $h : r \simeq s$ be paths in X. Then the constraints α, λ, ρ and ι are defined to be the 2-tracks of the respective relative homotopies

$$a_t : h \bullet (g \bullet f) \simeq (h \bullet g) \bullet f : p \simeq s \quad (\textit{rescaling})$$
$$l_t : c_q \bullet f \simeq f : p \simeq q, \quad r_t : f \bullet c_p \simeq f : p \simeq q \quad (\textit{dilation})$$
$$j_t : f^{-1} \bullet f \simeq c_p \quad (\textit{cancellation}).$$

The constraint ι' is obtained by applying ι to f^{-1} instead of f. The constructions are indicated by the following figures.



Explicit formulae are given in [37, pp. 47–48].

THEOREM 2.2. *If X is a topological space, then $\Pi_2(X)$ is a bigroupoid.*

We will call $\Pi_2(X)$ the *homotopy bigroupoid* of X.

Sketch of proof. We have to verify the axioms (AC) (associativity coherence), (IC) (identity coherence) and (CC) (cancellation coherence).

In order to prove (IC) we have to compare the relative homotopies $\psi_t = g \bullet l_t$ and $\psi'_t = a_t + (r_t \bullet f)$. We observe that ψ_t and ψ'_t can be factored

$$\psi_t = I \times I \xrightarrow{\phi_t} I \xrightarrow{g \bullet f} X, \qquad \psi'_t = I \times I \xrightarrow{\phi'_t} I \xrightarrow{g \bullet f} X$$

in such a way that ϕ_t and ϕ'_t coincide on the boundary of $I \times I$. Since the codomain of ϕ_t and ϕ'_t is a convex set, it follows that ϕ_t and ϕ'_t, hence $\psi_t = (g \bullet f)\phi_t$ and $\psi'_t = (g \bullet f)\phi'_t$ are relatively homotopic.

The proof of (AC) uses a similar factorisation argument, where the unit interval has to be subdivided into four segments.

The proof of (CC) again uses factorisation through the unit interval. The technical details are left to the reader. □

REMARKS. (i) We note that the construction of the Poincaré category in the sense of [5, (7.1)] associated to the homotopy bigroupoid, $\Pi_2(X)$, yields the *fundamental groupoid*, $\Pi_1(X)$, of X (cf. [6, 7]).

(ii) The term track seems first to have been used by S. Wylie and adopted by M. G. Barratt [3].

(iii) The homotopy bigroupoid, $\Pi_2(X)$, of a topological space X has the additional feature that each reverse path functor is an involution. Furthermore, we have the equations $c_p \bullet c_p = c_p$, $0_p \bullet \beta = \beta = \beta \bullet 0_p$ where p is a point of X and β: $c_p \Rightarrow c_p$ is a 2-track from c_p to itself. Hence, in some cases, computing in the homotopy bigroupoid of a space is a little simpler than in a general bigroupoid setting (cf. Proposition 3.8).

If $u\colon X \to Y$ is a continuous map between topological spaces, then a 2-functor (cf. Definition 1.3)

$$u_*\colon \Pi_2(X) \longrightarrow \Pi_2(Y)$$

is given by the following data.

(i) On objects u_* sends $p \in X$ to $u(p) \in Y$.
(ii) For each pair (p, q) of points of X, the functor

$$u_*\colon \Pi_2(X)(p, q) \longrightarrow \Pi_2(Y)(u(p), u(q))$$

is induced by postcomposition with u in the obvious way:

$$f : p \simeq q \longmapsto uf : u(p) \simeq u(q)$$
$$\{f_t\} : f \Longrightarrow f' \longmapsto \{uf_t\} : uf \Longrightarrow uf'.$$

3. Operations

In any 2-groupoid the structure of a crossed module over a groupoid is inherent (cf. [34]). This means, in particular, that an operation of the 1-cells of the 2-groupoid on the 2-cells is available. In this section we describe the analogue in the setting of the homotopy bigroupoid. As a result we deduce some basic algebraic properties of 2-dimensional homotopy theory.

We would like to point out that the following constructions are available in an arbitrary bigroupoid and the results of this section are valid in the general setting as far as they make sense.

Let $f, f' : p \simeq q$ and $g, g' : q \simeq r$ be paths in X. Then we define

$$g_*\colon \Pi_2(X)(p, q)(f, f') \longrightarrow \Pi_2(X)(p, r)(g \bullet f, g \bullet f'), \tag{3.1}$$

$$f^*\colon \Pi_2(X)(q, r)(g, g') \longrightarrow \Pi_2(X)(p, r)(g \bullet f, g' \bullet f) \tag{3.2}$$

by

$$g_*(\beta) = 0_g \bullet \beta, \quad f^*(\gamma) = \gamma \bullet 0_f,$$

where $\beta : f \Rightarrow f' : p \simeq q$ and $\gamma : g \Rightarrow g' : q \simeq r$.

We describe some of the algebraic properties of the $*$-operations.

First we note that, by the interchange law, the $*$-operations respect vertical composition of 2-tracks.

By naturality of the associativity isomorphism α we have

PROPOSITION 3.3. *Let $f, f' : p \simeq q$, $g : q \simeq r$, $h : r \simeq s$ be paths in X. Then the following diagram commutes.*

$$\begin{tikzcd}
\Pi_2(X)(p,q)(f,f') \arrow[r, "g_*"] \arrow[d, "(h\bullet g)_*"'] & \Pi_2(X)(p,r)(g\bullet f, g\bullet f') \arrow[d, "h_*"] \\
\Pi_2(X)(p,s)((h\bullet g)\bullet f, (h\bullet g)\bullet f') & \Pi_2(X)(p,s)(h\bullet(g\bullet f), h\bullet(g\bullet f')) \arrow[l, "\alpha_\#"]
\end{tikzcd}$$

Here, the bijective map $\alpha_\#$ is given by the conjugation type formula

$$\alpha_\#(\beta) = -\alpha + \beta + \alpha. \tag{3.4}$$

Thus we have

$$(h \bullet g)_*(\beta) = \alpha_\# h_* g_*(\beta) = -\alpha + h_* g_*(\beta) + \alpha. \tag{3.5}$$

Similarly, if $f, f' : p \simeq q$, $h : n \simeq p$, $g : m \simeq n$ are paths in X, then, for $\beta : f \Rightarrow f' : p \simeq q$, we have

$$(h \bullet g)^*(\beta) = \alpha + g^* h^*(\beta) - \alpha. \tag{3.6}$$

As a central result we have the following Rutter type bijection (cf. [36]).

PROPOSITION 3.7. *The maps g_* and f^* in* (3.1) *resp.* (3.2) *are bijective.*

Sketch of proof. The inverse of g_* is the composite map

$$h = \lambda_\# \iota_\# \alpha_\# (g^{-1})_*.$$

Here again, each of the #-maps is defined by a conjugation type formula as in (3.4).

The proof that h is left inverse to g_*, i.e., $hg_* = \mathrm{Id}$, does not make use of the coherence axioms (AC), (IC), (CC). It is a simple application of Proposition 3.3, the interchange law and the naturality of λ.

The proof that h is right inverse to g_* is also straightforward, but technically more complicated. The first step is to express the 2-track

$$0_g \bullet \alpha\colon\ g \bullet (g^{-1} \bullet (g \bullet f)) \Longrightarrow g \bullet ((g^{-1} \bullet g) \bullet f)$$

as a composite according to axiom (AC). Then the coherence axioms (CC), (IC) and a variant of (IC) corresponding to [23, Proposition 1.1] come into the play in a canonical way. □

The algebraic properties of $\Pi_2(X)$ exhibited so far allow one to prove some basic properties in 2-dimensional homotopical algebra using purely algebraic arguments.

We introduce some notation. If $f : p \simeq q$ is a path in X, let $\Pi_2(X, f)$ denote the group under vertical composition, $+$, of 2-tracks $f \Rightarrow f$ from f to itself. If $f, f' : p \simeq q$ are paths and $\gamma\colon\ f \Rightarrow f'$ is a 2-track, let

$$\gamma_\#\colon\ \Pi_2(X, f) \longrightarrow \Pi_2(X, f')$$

denote the isomorphism of groups induced by conjugation

$$\gamma_\#(\beta) = -\gamma + \beta + \gamma.$$

Let p be a point of X. Since $c_p \bullet c_p = c_p$, horizontal composition of 2-tracks induces a second operation, $\bullet$, in the group $\Pi_2(X, c_p)$.

PROPOSITION 3.8. *Let p be a point of X. Then vertical and horizontal composition in $\Pi_2(X, c_p)$ coincide and are abelian. Furthermore, $\Pi_2(X, c_p)$ is isomorphic to the second homotopy group $\pi_2(X, p)$ of X at p.*

Proof. The first part follows from routine manipulations (known as the Eckmann–Hilton trick) based on the interchange law together with the fact that $0_p \bullet \beta = \beta = \beta \bullet 0_p$ for each $\beta \in \Pi_2(X, c_p)$.

In order to interpret $\Pi_2(X, c_p)$ as the second homotopy group $\pi_2(X, p)$ of X at p, we observe that the elements of $\Pi_2(X, c_p)$ are the relative homotopy classes of homotopies $I \times I \to X$ which map the boundary of $I \times I$ to the point p. □

COROLLARY 3.9. *Let $f : p \simeq q$ be a path. Then $\Pi_2(X, f)$ is abelian and isomorphic to the second homotopy group $\pi_2(X, p)$. The isomorphism class of $\Pi_2(X, f)$ depends only on the path component of p in X.*

Proof. By the interchange law and Proposition 3.7 we have the following isomorphisms of groups.

$$\Pi_2(X, c_p) \xrightarrow{f_*} \Pi_2(X, f \bullet c_p) \xrightarrow{\rho_\#} \Pi_2(X, f),$$
$$\Pi_2(X, c_q) \xrightarrow{f^*} \Pi_2(X, c_q \bullet f) \xrightarrow{\lambda_\#} \Pi_2(X, f).$$ □

REMARK. In order to prove the first part of Proposition 3.8 for an arbitrary bigroupoid, **S**, conjugation with $\lambda = \rho\colon\ c_p \bullet c_p \Rightarrow c_p$ has to be used.

4. The Homotopy 2-Groupoid of a Hausdorff Space

In this section we are concerned with the homotopy 2-groupoid, $G_2(X)$, of a Hausdorff space X in [19] and its relation to the homotopy bigroupoid, $\Pi_2(X)$. We recall the details of the construction of the homotopy 2-groupoid relevant to our purposes.

For the remainder of this section we assume that X is a Hausdorff space.

DEFINITION 4.1. Let $f, f' : p \simeq q$ be paths in X. Then a relative homotopy

$$\psi_t : f \simeq f' : p \simeq q$$

is *thin* if there is a factorisation

$$\psi_t\colon\ I \times I \xrightarrow{\phi_t} J \xrightarrow{p} X,$$

where J is a tree, $\phi_t : \phi \simeq \phi'$ is a relative homotopy, ϕ and ϕ' are paths in J which (i) satisfy $\phi(0) = \phi'(0)$, $\phi(1) = \phi'(1)$, (ii) are piecewise linear and (iii) satisfy $p\phi = f$, $p\phi' = f'$.

Here, by a *tree*, we mean the underlying space $|K|$ of a finite 1-connected 1-dimensional simplicial complex K. A map ϕ: $|K| \to |L|$, where K and L are (finite) simplicial complexes, is *piecewise linear* if there exist subdivisions of K and L relative to which ϕ is simplicial.

The class $\mathcal{T}$ of thin homotopies into a Hausdorff space satisfies the following properties.

- $\mathcal{T}$(i) Rescale, dilation and cancellation are thin. Any constant relative homotopy is thin.
- $\mathcal{T}$(ii) $\mathcal{T}$ is closed under vertical pasting of relative homotopies.
- $\mathcal{T}$(iii) $\mathcal{T}$ is closed under vertical reversion of relative homotopies: if $\sigma_t : f \simeq f' : p \simeq q$ is thin, then $\sigma_{1-t} : f' \simeq f : p \simeq q$ is thin.
- $\mathcal{T}$(iv) $\mathcal{T}$ is closed under horizontal pasting of relative homotopies.

Let $NX(p,q)$ denote the subgroupoid of $\Pi_2(X)(p,q)$ whose morphisms are the relative homotopy classes of thin homotopies $\psi_t : f \simeq f' : p \simeq q$. Then $NX(p,q)$ is a normal subgroupoid of $\Pi_2(X)(p,q)$ and $G_2(X)(p,q)$ is defined to be the quotient groupoid $\Pi_2(X)(p,q)/NX(p,q)$. The main result of [19] is

THEOREM 4.2. *The sets $NX(p,q)(f,f')$ are either singletons or empty.*

COROLLARY 4.3. *The obvious projections give rise to a biequivalence* (*cf. Definition* 1.5) *between the homotopy bigroupoid,* $\Pi_2(X)$, *and the homotopy* 2-*groupoid,* $G_2(X)$.

Furthermore, as a corollary we obtain the following coherence theorem for the homotopy bigroupoid.

COHERENCE THEOREM 4.4. *Let $\psi_t, \psi'_t : f \simeq f' : p \simeq q$ be relative homotopies which can be constructed from rescale, dilation, cancellation and constant relative homotopies by means of vertical pasting, vertical reversion and horizontal pasting. Then the* 2-*tracks $\{\psi_t\}$ and $\{\psi'_t\}$ coincide.*

In particular the axioms (AC), (IC) and (CC) are a consequence of the coherence theorem.

REMARKS 4.5. (i) Under projection the 2-cells in NX are mapped into identity 2-cells in $G_2(X)$. Consequently, conjugation type bijections in $\Pi_2(X)$ induced by 2-cells in NX, such as the mapping $\alpha_\#$ in (3.4), go over to identity mappings. This means, in particular, that in order to convert the formulae of Section 3 from the homotopy bigroupoid to the homotopy 2-groupoid setting, one has to omit conjugation induced by elements of NX at any occurrence.

(ii) R. Street has pointed out to us that a general coherence theorem for bigroupoids can be derived from existing coherence theorems for bicategories ([32, 17]) using a construction involving adjunctions which is related to a construction in [24]. Alternatively, a proof using cohomology of groups was given in [22]. In the special case of a category with group structure, i.e., a bigroupoid with one object, a coherence theorem has been proved in [42, 30].

(iii) Although every bigroupoid is biequivalent to a 2-groupoid, by no means does that affect the importance and utility of the structure $\Pi_2(X)$ for, after all, not every problem is a problem up to homotopy. (Analogously every groupoid is homotopy equivalent to a disjoint union of groups, nevertheless the theory of groupoids has proved fruitful.)

5. Discussion

As far as dealing with nonassociativity of composition of paths is concerned, the construction of the homotopy bigroupoid has been sketched in [2] where the construction of what is called the 'fundamental 2-groupoid' is being discussed. What we call rescaling appears in [2, p. 6090] under the name of associator. However, full details have not been given there. Moreover, cancellation isomorphisms and cancellation coherence have not been considered in [2].

As an alternative approach to the theory of paths one has the notion of a Moore path. If X is a topological space and p, q are points of X, then a *Moore path* from p to q is a pair (f, r) where $r \geq 0$ is a real number and $f\colon [0, r] \to X$ is a map such that $f(0) = p$ and $f(r) = q$. Composition of Moore paths is strictly associative and has strict identities. However, this approach does not give rise either to strict inverses. It should be possible to construct a homotopy bigroupoid of a space based on the notion of Moore path which is equivalent to our $\Pi_2(X)$. However, the notion of 2-track would be more complicated in the setting of Moore paths. Furthermore, it should be pointed out that the approach chosen in this paper gives the interchange law for 2-tracks for free.

A general construction of a weak n-groupoid of a topological space based on simplicial methods has been given by Tamsamani [41].

The introduction to [19] has a discussion of some of the various possible approaches to 2-dimensional homotopical algebra: 2-*groupoids, crossed modules over groupoids*, and *double groupoids with thin structure* which are all equivalent (cf. [8–12, 34, 39, 38]).

Cegarra and Fernández [13] define the notions of *categorical group cofibred over a groupoid* resp. *strict categorical group cofibred over a groupoid*. It is stated in [13] (without proof) that strict categorical groups cofibred over groupoids are equivalent to 2-groupoids whereas categorical groups over groupoids are equivalent to bigroupoids. If X is a space together with subspaces $B \subseteq A \subseteq X$, and a set of base points $S \subseteq B$, Cegarra and Fernández construct a cofibred categorical group $W(X, A, B, S)$ over the fundamental groupoid $\Pi_1(B, S)$ of B on the set S which they call the *Whitehead cofibred categorical group* of (X, A, B, S). As an example of the construction it is claimed in [13] that $W(X, X, X, X)$ corresponds to our homotopy bigroupoid $\Pi_2(X)$.

The referee has pointed out that it would be desirable to have a closer link between the constructions of the homotopy bigroupoid resp. the homotopy 2-groupoid and classical methods used in 2-dimensional homotopy and combinatorial group theory (cf. [21]). In particular, it would be interesting to look for a sort of 'minimal model' for $\Pi_2(X)$. In case X is a space with a specified 2-dimensional CW structure one should be able to give a presentation of a 2-groupoid G that is biequivalent to $G_2(X)$. It would also be interesting to give a version of 'pictures' (cf. [21, Chapter V]) that applies to $G_2(X)$. However, those items leading into a different direction of interest, they will not be addressed in this paper.

Acknowledgements

The authors acknowledge helpful discussions and correspondence with R. Brown, T. Porter, R. Street and J. J. C. Vermeulen, and helpful comments and suggestions by the referee which are reflected in the introduction and the structure of the paper.

The first author acknowledges support from the Foundation for Research Development and the University of Cape Town via the Topology and Category Theory Group.

References

1. Baez, J. C.: An introduction to n-categories, in *7th International Conference, CTCS '97, Santa Margherita Ligure, Italy, 1977,* Lecture Notes in Comput. Sci. 1290, Springer-Verlag, Berlin, 1997, pp. 1–33.
2. Baez, J. C. and Dolan, J.: Higher-dimensional algebra and topological quantum field theory, *J. Math. Phys.* **36** (1995), 6073–6105.
3. Barratt, M. G.: Track groups I, *Proc. London Math. Soc. (3)* **5** (1955), 71–106.
4. Baues, H. J.: *Combinatorial homotopy and 4-dimensional complexes*, W. de Gruyter, Berlin, New York, 1991.
5. Bénabou, J.: Introduction to bicategories, in *Reports of the Midwest Category Seminar*, Lecture Notes in Math. 47, Springer-Verlag, Berlin, 1967, pp. 1–77.
6. Brown, R.: Groupoids and Van Kampen's theorem, *Proc. London Math. Soc. (3)* **17** (1967), 385–401.
7. Brown, R.: *Elements of Modern Topology*, McGraw-Hill, Maidenhead, 1968.
8. Brown, R.: Higher dimensional group theory, in *Proceedings of the Conference on Topology in Low Dimension, Bangor 1979*, London Math. Soc. Lecture Notes Series 48, Cambridge University Press, 1982, pp. 215–238.
9. Brown, R.: Groupoids and crossed objects in algebraic topology, *Lectures Given at the Summer School on the Foundations of Algebraic Topology*, Grenoble, 1997.
10. Brown, R. and Higgins, P. J.: On the connection between the second relative homotopy groups of some related spaces, *Proc. London Math. Soc. (3)* **36** (1978), 193–212.
11. Brown, R. and Higgins, P. J.: On the algebra of cubes, *J. Pure Appl. Algebra* **21** (1981), 233–260.
12. Brown, R. and Spencer, C. B.: Double groupoids and crossed modules, *Cahiers Topologie Géom. Différentielle Catég.* **17** (1976), 343–362.
13. Cegarra, A. M. and Fernández, L.: Cohomology of cofibred categorical groups, Preprint, Granada, August 1997.
14. Dwyer, W. and Kan, D. M.: Homotopy theory and simplicial groupoids, *Proc. Konink. Neder. Akad.* **87** (1987), 379–389.

15. Ehlers, P. J. and Porter, T.: Varieties of simplicial groupoids I : Crossed complexes, *J. Pure Appl. Algebra* **120** (1997), 221–233.
16. Freyd, P. and Yetter, D. N.: Coherence theorems via knot theory, *J. Pure Appl. Algebra* **78** (1992), 49–76.
17. Gordon, R., Power, A. J. and Street, R.: *Coherence for Tricategories*, Memoirs AMS 117, No. 558, 1995.
18. Hardie, K. A. and Kamps, K. H.: Coherent homotopy over a fixed space, in *Handbook of Algebraic Topology*, Elsevier, Amsterdam, 1995, pp. 195–211.
19. Hardie, K. A., Kamps, K. H. and Kieboom, R. W.: A homotopy 2-groupoid of a Hausdorff space, *Appl. Categ. Structures* **8** (2000), 209–234.
20. Hardie, K. A., Kamps, K. H. and Porter, T.: The coherent homotopy category over a fixed space is a category of fractions, *Topology Appl.* **40** (1991), 265–274.
21. Hog-Angeloni, C., Metzler, W. and Sieradski, A. J. (eds.): *Two-dimensional Homotopy and Combinatorial Group Theory*, London Math. Soc. Lecture Notes Series 197, Cambridge University Press, 1993.
22. Joyal, A. and Street, R.: *Braided Monoidal Categories*, Macquarie Math. Reports # 860081, Nov. 1986.
23. Joyal, A. and Street, R.: Braided tensor categories, *Adv. Math.* **102** (1993), 20–78.
24. Joyal, A. and Street, R.: The geometry of tensor calculus II, *Synthese Lib.* **259**, 29–68.
25. Kamps, K. H. and Porter, T.: *Abstract Homotopy and Simple Homotopy Theory*, World Scientific, Singapore, 1997.
26. Kamps, K. H. and Porter, T.: A homotopy 2-groupoid from a fibration, under submission.
27. Kelly, G. M.: Many-variable functorial calculus I, in *Coherence in Categories*, Lecture Notes in Math. 281, Springer-Verlag, Berlin, 1972, pp. 66–105.
28. Kelly, G. M. and Laplaza, M. L.: Coherence for compact closed categories, *J. Pure Appl. Algebra* **19** (1980), 193–213.
29. Kelly, G. M. and Street, R.: Review of the elements of 2-categories, in *Category Seminar Sydney 1972/73*, Lecture Notes in Math. 420, Springer-Verlag, Berlin, 1974, pp. 75–103.
30. Laplaza, M. L.: Coherence for categories with group structure: an alternative approach, *J. Algebra* **84** (1983), 305–323.
31. Lyubashenko, V.: Tangles and Hopf algebras in braided categories, *J. Pure Appl. Algebra* **98** (1995), 245–278.
32. MacLane, S. and Paré, R.: Coherence for bicategories and indexed categories, *J. Pure Appl. Algebra* **37** (1985), 59–80.
33. Moerdijk, I. and Svensson, J.-A.: Algebraic classification of equivariant homotopy 2-types, I, *J. Pure Appl. Algebra* **89** (1993), 187–216.
34. Porter, T.: Crossed modules in Cat and a Brown-Spencer theorem for 2-categories, *Cahiers Topologie Géom. Différentielle Catég.* **26** (1985), 381–388.
35. Porter, T.: Review of [1], *Zbl. Math.* 886.18005 (1998).
36. Rutter, J. W.: A homotopy classification of maps into an induced fibre space, *Topology* **6** (1967), 379–403.
37. Spanier, E. H.: *Algebraic Topology*, McGraw-Hill, New York, 1966.
38. Spencer, C. B.: An abstract setting for homotopy pushouts and pullbacks, *Cahiers Topologie Géom. Différentielle Catég.* **18** (1977), 409–430.
39. Spencer, C. B. and Wong, Y. L.: Pullback and pushout squares in a special double category with connection, *Cahiers Topologie Géom. Différentielle Catég.* **24** (1983), 161–192.
40. Street, R.: Categorical structures, in *Handbook of Algebra*, Elsevier, Amsterdam, 1996, pp. 529–577.
41. Tamsamani, Z.: Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux, Thèse, Toulouse, 1996.
42. Ulbrich, K.-H.: Kohärenz in Kategorien mit Gruppenstruktur, *J. Algebra* **72** (1981), 279–295.