Game Comonads in Finite Model Theory

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Chapter 1

Introduction

Given a new mathematical theory T, model theory studies the relationship between the formal sentences needed to express T and the mathematical structures that satisfy T. Category theory, on the other hand, studies the structure-preserving relationships between two or more structures that satisfy T. The fact that both of these fields abstract notions that are employed in every mathematical field, mean they occupy a high-level, even philosophical[9][], place in mathematics. Despite their abstract nature, both model theory and category theory mainfest concretely in the field of logic applied to computer science. In the case of model theory, restricting attention to to finite mathematical structures, i.e. finite model theory, provides new avenues in complexity theory [7] and database theory [5]. In the case of category theory, many constuctions, in particular monads and comonads, are used in the semantics of programming languages [2]. This dissertation draws a bridge between the applications of these two fields. The bridge consists of capturing games, e.g. Ehrenfreuct-Fraïssé, pebbling, bisimulation, which are used to prove when two structures are indistinguisable from a logic as comonadic constructions. This categorical reformulation "internalizes" the games between two structures as a constructions within the category of structures itself. Not only does this internalization of games provide easy proofs of common results, the category of coalgebras give a categorical definition of combinatorial parameters, like tree-depth and tree-width. The inspiration for this dissertation was the paper [1] which constructed a family of comonads $\{\mathbb{T}_k\}$ that interalizes pebble games. With this dissertation we construct and prove analogous results for a family of comonads $\{\mathbb{E}_k\}$ that interalizes Ehrenfreucht-Fraïssé games. We also show that the unfolding comonad for modal $\{\mathbb{M}_k\}$ and guarded $\{\mathbb{G}_k\}$ logics detailed in [3], share similar properties and internalize bisimulation games.

All results are proven by author, unless explictly cited or stated otherwise.

1.1 History

1.1.1 Finite Model Theory and Games

Model theory, unlike most mathematical fields which develop from other mathematics, developed from philosophical preoccupations with the langauge and concepts employed by mathematicians. Early results and constructions such as Gödel's Completeness Theorem, Löwenheim-Skolem Theorems, and Robinson's hyperreals explicitly address the role of logic and infinity in the language of mathematics. Moreover, the definition of model-theoretic satisfaction used today is argued to have stemmed form Tarski's deflationary definitions of truth [9]. Nevertheless, in the last several decades, model theory has grown into a mature mathematical field generalizing and providing insight into algebraic geometry, number theory , mathematical physics, and computer science . For a long time, model theory developed with no restriction on the cardinality of the structures in question and with no particular interest in the theory of finite structures. However, Fagin's Theorem discovered in Ronald Fagin's 1973 doctoral thesis demonstrated that existential second-order logic on finite structures captured the complexity class **NP**. This spawned a new interest in the model theory of finite and recursive structures. Another reason why finite model theory has developed as its own independent study, rather than occupy a section in (unrestricted) model theory, is the different techniques used. Many of the classical results in model theory (e.g. Compactness

Theorem, Lyndon's Theorem, Lowenheim-Skolem theorems, Tarski-Vaught test) only work on or apply to infinite models. Moreover, the common constructions (e.g. Henkin models, ultraproducts) produce infinite structures. One technique that works both on finite and infinite structures are back-and-forth games. This technique, phrased as systems for partial isomorphisms, was developed in Fraïssé's 1953 thesis. Arguably, this work originates from generalizing Cantor's back-and-forth argument which showed two countable dense linear orders are isomorphic. Ehrenfreuct in 1961 phrased these back-and-forth systems perspeciously as a two-player game. Since then, there have been many such games developed such as pebble games [6] for finite-variable logics, bisimulation games [3] for modal and guarded logics, and even fragments of second-order logics like the bijection game [4] for counting-quantifier logic and extended pebble game [8] for monadic second-order logic.

1.1.2 Category Theory and Comonads

Category theory, finds its orgin the 1945 paper in *General Theory of Natural Equivalences* by Eilenberg and MacLane. This paper develops the general theory of functors and natural transformations. The notions of functor, isomorphism, and natural isomorphism are used in many mathematical fields for the purpose of classifying and distinguishing structures. Given that the back-and-forth games used in model theory are equivalent to distinguishability in a logic, category theory is natural setting to formalize these games. Moreover, the algebraic constructions of monad and comonad are used to provide semantics for functional programming languages [10] [2]. This paper is part of the general class of research programs that seek to "categorify" the notions of equivalence and classification in a wide-range of mathematical fields.

Chapter 2

Background

The primary purpose of this chapter is to recall the basic definitions, constructions and results that will be used in the subsequent chapters. This chapter also serves to fix the notation for the standard constructions employed in finite model theory and category theory. For a category-theorist, the first section is an introduction to finite model theory. For a finite model-theorist, the second section is an introduction to category theory and in particular, comonads.

2.1 Finite Model Theory

Model theory studies the relationship between sentences in a logic $\mathcal{L}$ and classes of mathematical structures $\mathcal{C}$ that satisfy these sentences. Sentences are built from non-logical symbols (specified by a signature) and the familar logical symbols (e.g. $\neg, \vee, \wedge, \exists, \forall$). Structures are built from sets that specifiy a domain of objects and interpretations for the symbols in a signature.

2.1.1 Definitions

Definition 2.1.1. A *signature* or *vocabulary*, denoted usually with lowercase greek letter σ, is a set of constant symbols $c_1, c_2, \ldots,$ function symbols $f_1, f_2, \ldots,$ and relation symbols $R_1, R_2, \ldots$ where every function symbol and relation symbol has an associated arity.

Example 2.1.1. The signature for groups can be $\sigma = \{*, ^{-1}, e\}$ where $*$ is the binary group operation, $^{-1}$ is the unary inverse operation, and e is the constant denoting the idenity in the group.

Definition 2.1.2. A signature σ is a *relational signature* if it only has relation symbols.

Example 2.1.2. A coloured graph has relational signature $\sigma = \{E, R, G, B\}$ where E is the symbol for the binary edge relation. The R, G, and B are unary relations symbols picking out the red, green, and blue (respectively) coloured vertices.

Example 2.1.3. A m-uniform hypergraph has signature $\sigma = \{E\}$ where E is an m-ary relation symbol for hyperedges.

Definition 2.1.3. Given a signature σ, a *σ-structure*

$$A = \langle |A|, \{c_i^A\}, \{f_i^A\}, \{R_i^A\} \rangle$$

consists of a set $|A|$ called the *universe* of A together with an interpretation of:

- each constant symbol c_i from σ is an element $c_i^A \in |A|$
- each function symbol f_i with arity k from σ is a function $f_i^A : |A|^k \longrightarrow |A|$
- each relation symbol R_i with arity k from σ is a k-ary relation R_i^A on $|A|$ (i.e. $R_i^A \subseteq |A|^k$)

As observed in many mathematical fields, a class of structures comes with a corresponding notion of structure-preserving function, or homomorphism.

Definition 2.1.4. Given a signature σ, and two σ-structures, A, B. function $g : |A| \longrightarrow |B|$ is a *σ-morphism* if:

- for constant symbol $c_i \in \sigma$, $g(c_i^A) = c_i^B$
- for function symbol $f_i \in \sigma$ with arity k, $g(f_i^A(a_1, \ldots, a_k)) = f_i^B(g(a_1), \ldots, g(a_k))$
- for relation symbol $R_i \in \sigma$ with arity k, $(a_1, \ldots, a_k) \in R_i^A \Rightarrow (g(a_1), \ldots, g(a_k)) \in R_i^B$

This also leads to a very fine notion of equivalence between two σ-structures.

Definition 2.1.5. A bijective σ-morphism $f : A \longrightarrow B$ whose inverse $g : B \longrightarrow A$ is also a σ-morphism is called an *isomorphism.* Two σ-structures A, B are *isomorphic*, denoted $A \cong B$, if there exists an isomorphism between them.

A common simplification in finite model theory texts is to only consider relational signatures. This is a resonable assumption since any function symbol f of arity k can be replaced with relation symbol R_f of arity $k+1$ whose intrepretation R_f^A is the graph of the intrepretation f^A. Any constant symbol c can be replaced with relation symbol P_c of arity 1 whose interpretation is the singleton $\{c^A\}$. Hence, for the rest of this paper we will consider σ to be a relational signature. It should be noted that translations to a relational signature do not preserve measures of formula complexity, such as quantifier rank or number of variables.

The logics we will consider will have countably many variables. Namely, for every $j \in \omega$, there is symbol x_j representing a variable. For clarity of presentation, we may use lowercase latin letters (possibly with subscripts) $z, y, w, \ldots$ also as variables in our logic, but these can always be replaced with the more formal x_j for some $j \in \omega$. Define $[n] := \{1, \ldots, n\}$, i.e. the finite segment of the natural numbers.

Definition 2.1.6. A *formula* $\phi(\mathbf{x})$ with free variables among $\mathbf{x} = (x_1, \ldots, x_n)$ is recursively defined as:

$$\phi(\mathbf{x}) ::= x_i = x_j \mid R_z(x_{i_1}, \ldots, x_{i_k}) \mid \neg\phi(\mathbf{x}) \mid \bigvee_{j\in J} \phi_j(\mathbf{x}) \mid \bigwedge_{j\in J} \phi_j(\mathbf{x}) \mid \exists y\phi(y, \mathbf{x}) \mid \forall y\phi(y, \mathbf{x})$$

with $\{R_z\}$ the relation symbols in signature σ, $i, j, i_1, \ldots, i_k \in [n]$, and J a (possibly infinite) indexing set.

Definition 2.1.7. A *sentence* ϕ is a formula with no free variables.

With these defintions in place, we can define the central relationship studied in model theory. Namely, the relation $\vDash$, or 'satisfies', between the structures as defined in (2.1.3) and formulas as defined in (2.1.6).

Definition 2.1.8. Given an σ-structure A with $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ and formula $\phi(\mathbf{x})$ with free variables among $\mathbf{x} = (x_1, \ldots, x_n)$, define $\vDash$ by induction on the complexity on formula ϕ:

$$\begin{aligned}
A, \mathbf{a} &\vDash x_i = x_j \Leftrightarrow a_i = a_j \\
&\vDash R_z(x_{i_1}, \ldots, x_{i_m}) \Leftrightarrow (a_{i_1}, \ldots, a_{i_m}) \in R_z^A \\
&\vDash \neg\phi(\mathbf{x}) \Leftrightarrow \text{it is not the case that } A, \mathbf{a} \vDash \phi(\mathbf{x}) \\
&\vDash \bigvee_{j\in J} \phi_j(\mathbf{x}) \Leftrightarrow \text{for some } j \in J, A, \mathbf{a} \vDash \phi_j(\mathbf{x}) \\
&\vDash \bigwedge_{j\in J} \phi_j(\mathbf{x}) \Leftrightarrow \text{for every } j \in J, A, \mathbf{a} \vDash \phi_j(\mathbf{x}) \\
&\vDash \exists y\phi(y, \mathbf{x}) \Leftrightarrow \text{for some } a \in A, A, a\mathbf{a} \vDash \phi(y, \mathbf{x}) \\
&\vDash \forall y\phi(y, \mathbf{x}) \Leftrightarrow \text{for every } a \in A, A, a\mathbf{a} \vDash \phi(y, \mathbf{x})
\end{aligned}$$

For every formula, $\mathrm{var}(\phi)$ denotes the number of distinct variables, free and bound, in ϕ. For every formula, $\mathrm{qr}(\phi)$ denotes the quantifier rank of ϕ. Inductively defined as:

$$\begin{aligned}
\mathrm{qr}(x_i = x_j) = \mathrm{qr}(R_i(x_{i_1}, \ldots, x_{i_k})) &= 0 \\
\mathrm{qr}(\bigvee_{j\in J} \phi_j(\mathbf{x})) = \mathrm{qr}(\bigwedge_{j\in J} \phi_j(\mathbf{x})) &= \sup_{j\in J} \mathrm{qr}(\phi_j(\mathbf{x})) \\
\mathrm{qr}(\exists y\phi(y, \mathbf{x})) = \mathrm{qr}(\forall y\phi(y, \mathbf{x})) &= \mathrm{qr}(\phi(y, \mathbf{x})) + 1
\end{aligned}$$

Definition 2.1.9. Given a signature σ, a *language* is the collection of formulas as defined in (2.1.6). A *logic* is collection of languages which differ only in signature.

In particular, the collection of all formulas in definition (2.1.6), ranging over every signature, gives the infinitary logic usually denoted $\mathcal{L}_{\infty,\omega}$. We will be studying expressibilty in fragments of $\mathcal{L}_{\infty,\omega}$. For this reason, we need notation to denote the different fragments.

Definition 2.1.10. For every α ordinal or $\alpha = \infty$, $z \in \omega + 1$ and $y \in \omega + 1$, $\mathcal{L}^{y}_{\alpha,z}$ denotes the logic with formulas ϕ where:

- the indexing set J (used in the $\bigvee, \bigwedge$ clauses in the definition (2.1.6)) has cardinality $< \alpha$ (or unrestricted if $\alpha = \infty$)
- $\mathrm{qr}(\phi) \leq z$ if $z \in \omega$ or arbitrary if $z = \omega$
- $\mathrm{var}(\phi) \leq y$ if $y \in \omega$ or arbitrary if $y = \omega$

If y is not specified, then the $y = \omega$, i.e. the logic $\mathcal{L}_{\alpha,z} = \mathcal{L}^{\omega}_{\alpha,z}$. Moreover, for a logic $\mathcal{L}$, the fragment without $\neg$ or $\forall$ statements is called the existential positive fragment and is denoted, $\exists^{+}\mathcal{L}$.

Hence, ordinary first-order logic is $\mathcal{L}^{\omega}_{\omega,\omega} = \mathcal{L}_{\omega,\omega}$. As an example, $\exists^{+}\mathcal{L}^{5}_{\omega,3}$ is the existential positive fragment of first-order logic with formulas using at most 5 distinct variables and quantifier rank ≤ 3. With this notation, we can define two important fragments, the finite rank and finite variable fragments.

Definition 2.1.11. Given a $k \in \omega$, the *k-rank logic* is $\mathcal{L}_{\omega,k}$.

Ostensibly, it may seem odd to study the finite rank fragments of first-order logic, rather than the more general infinitary logic. However, the k-rank fragment of first order logic is in fact equivalent to the k-rank fragment of infinitary logic (from the perspective of finite model theory). This clearly follows from a standard result in finite model theory:

Proposition 2.1.1 ([8, Lemma 3.13])**.** *Given a finite signature σ, then up to logical equivalence, there are only finite many formulae ϕ over σ with $qr(\phi) \leq k$ and m free variables.*

Definition 2.1.12. Given a $k \in \omega$, the *k-variable logic* is $\mathcal{L}^{k}_{\infty,\omega}$.

Note that, unlike with the case of k-rank logic, there are in general, infinitely many formulas, up to logical equivalence, that use k distinct variables. Two other fragments that we will be studying are the modal and guarded fragments of first-order logic. These fragments are obtained by altering the $\exists$ and $\forall$ clauses in definition (2.1.6) to be guarded by a relation $R_w \in \sigma$ with arity $\neq 1$. More precisely,

Definition 2.1.13. A formula $\phi(\mathbf{x})$ with free variables among $\mathbf{x} = (x_1, \ldots, x_n)$ in the *guarded fragment* is recursively defined just as in (2.1.6) but with the $\exists$ and $\forall$ clauses altered to:

$$\exists \mathbf{y}(R_w(\mathbf{x}, \mathbf{y}) \wedge \phi(\mathbf{y}, \mathbf{x})) \mid \forall \mathbf{y}(R_w(\mathbf{x}, \mathbf{y}) \rightarrow \phi(\mathbf{y}, \mathbf{x}))$$

$\{R_w\}$ the relation symbols in signature σ with arity $\neq 1$, $i_1, \ldots, i_k \in [n]$. The $\rightarrow$ is an abbreviation, i.e. $\phi \rightarrow \psi := \neg\phi \vee \psi$ and represents 'implication'. The notation $\exists \mathbf{y}$ abbreviates $\exists y_l \exists y_{l-1} \ldots \exists y_1$ for $\mathbf{y} = (y_1, \ldots, y_l)$.

For the guarded fragment, we will a the notion of complexity, similar to quantifier rank, called guarded depth. For a guarded formula $\phi(\mathbf{x})$, $\mathrm{gd}(\phi)$ denotes the guarded depth of ϕ. Inductively defined as:

$$\begin{aligned} \mathrm{gd}(x_i = x_j) = \mathrm{qd}(R_i(x_{i_1}, \ldots, x_{i_k})) &= 0 \\ \mathrm{gd}(\bigvee_{j \in J} \phi_j(\mathbf{x})) = \mathrm{gd}(\bigwedge_{j \in J} \phi_j(\mathbf{x})) &= \sup_{j \in J} \mathrm{gd}(\phi_j(\mathbf{x})) \\ \mathrm{gd}(\exists \mathbf{y}(R_w(\mathbf{x}, \mathbf{y}) \wedge \phi(\mathbf{y}, \mathbf{x}))) = \mathrm{gd}(\forall \mathbf{y}(R_w(\mathbf{x}, \mathbf{y}) \rightarrow \phi(\mathbf{y}, \mathbf{x}))) &= \mathrm{gd}(\phi(\mathbf{y}, \mathbf{x})) + 1 \end{aligned}$$

Note that guarded depth ϕ is less than the quantifier rank of ϕ since a guarded quantification allows for quantification over full tuples $\mathbf{y} = (y_1, \ldots, y_l)$. We will denote the guarded fragment of depth k of $\mathcal{L}_{\infty,\omega}$ as $\mathcal{G}^{k}_{\infty,\omega}$.

Definition 2.1.14. The *modal fragment* is the guarded fragment restricted to signatures σ with relations that have arity at most 2.

We will denote the modal fragment of guarded depth k of $\mathcal{L}_{\infty,\omega}$ as $\mathcal{M}^{k}_{\infty,\omega}$.

2.1.2 Inexpressiblity and Games

The primary question, once a logic $\mathcal{L}$ is defined, is to understand what properties of structures can be expressed in $\mathcal{L}$ and what properties are impossible to express in $\mathcal{L}$. In particular, finite model theory is concerned with what properties of *finite* structures can be expressed in a given logic. Although mathematicians have intuitive idea of what 'property' means, we will be more explicit:

Definition 2.1.15. Given a signature σ, a *property* P of σ-structures, is a (possibly, proper class) function:

$$P : \mathcal{C} \longrightarrow \{0, 1\}$$

such that for $A, B \in \mathcal{C}$:

$$A \cong B \Rightarrow P(A) = P(B)$$

where $\mathcal{C}$ is a class of all finite σ-structures.

Intuitively, $P(A) = 1$ asserts that some property is true of the structure A. To prove a property P is not expressible in signature σ, we need to exhibit two σ-structures A,B which from the perspective of the logic $\mathcal{L}$ are "equivalent", but differ by property P (i.e. $P(A) = 1$ and $P(B) = 0$). Of course, by the definiton of property given, $\cong$ is too strong of an equivalence to show inexpressiblity. For a given logic, we define what it means to be equivalent from the perspective of $\mathcal{L}$.

Definition 2.1.16. Two σ structures A, B are *equivalent in logic* $\mathcal{L}$, denoted $A \equiv^{\mathcal{L}} B$ if for all sentences $\phi \in \mathcal{L}$, $A \vDash \phi \Leftrightarrow B \vDash \phi$.

Hence, for a property $\mathcal{P}$ to inexpressible in signature σ and logic $\mathcal{L}$, we must find two σ-structures A, B such that $P(A) = 1$ and $P(B) = 0$ while $A \equiv^{\mathcal{L}} B$. The purpose of "back-and-forth" style games in finite and classical model theory is to provide a methodology to prove equivalence in a logic $\mathcal{L}$.

2.2 Category Theory

Category theory is the general theory of mathematical structures and structure-preserving relationships between theses structures.

2.2.1 Definitions

Definition 2.2.1. A *category* $\mathcal{C}$ is:

- a class of objects, denoted (somewhat ambigously) as $\mathcal{C}$
- a class of $\mathcal{C}$-morphisms (or arrows) $f : A \longrightarrow B$ for every object $A, B \in \mathcal{C}$, denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

such that:

(1) For morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$, there exists a morphism $g \circ f : A \longrightarrow C$.

(2) For every object $A \in \mathcal{C}$, there exists a morphism $\mathsf{id}_A : A \longrightarrow A$ such that $\mathsf{id}_A \circ f = f$ and $g \circ \mathsf{id}_A = g$ for every $f : B \longrightarrow A$, $g : A \longrightarrow C$.

(3) For morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$, $h \circ (g \circ f) = (h \circ g) \circ f$.

We can rephrase the condition on $\mathsf{id}_A : A \longrightarrow A$ by stating that the following diagrams commute for all $A \in \mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "f"'] & A \arrow[dl, "\mathsf{id}_A"] \\ A & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\mathsf{id}_A"'] \arrow[d, "g"] \\ A \arrow[r, "g"'] & C \end{tikzcd}$$

We will make use of *commutative diagrams*, like the ones above, to express relationships between morphisms in a category. In fact, many of our proofs can be reduced to verifying a certain diagram commutes. This method is called *diagram chasing*.

Example 2.2.1. The category of sets with morphisms ordinary functions, denoted **Set**.

Example 2.2.2. The category of groups with group homomorphism.

Example 2.2.3. The category of preorders with monotone functions, denoted **Pos**.

The most important categories will be the ones that appear within finite model theory. Which are:

Example 2.2.4. Given a signature σ, the category of σ-structures with σ-morphisms will be denoted $\mathcal{R}(\sigma)$. The subcategory of finite structures in $\mathcal{R}(\sigma)$ will be denoted $\mathcal{R}_f(\sigma)$.

Modulo some set-theoretic issues, the collection of categories themselves **Cat** can itself be considered a category with morphisms called functors.

Definition 2.2.2. A *functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a map from the objects and morphisms of $\mathcal{C}$ to the objects and morphims of $\mathcal{D}$ such that:

(1) For every object $A \in \mathcal{C}$, $F(\mathsf{id}_A) = \mathsf{id}_{F(A)}$

(2) For morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$, $F(g \circ f) = F(g) \circ F(f)$.

Example 2.2.5. Forgetful functors, such as F : **Grp** $\longrightarrow$ **Set** sending a group to it's underlying set.

Example 2.2.6. Free functors in algebra, such as F : **Set** $\longrightarrow$ **Grp** sending a set X to the free group on the letters in X.

Definition 2.2.3. A functor $F : \mathcal{C} \longrightarrow \mathcal{C}$ from a category to itself is called an *endofunctor*

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can consider the collection of all functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ as a category itself with morphisms called natural transformations.

Definition 2.2.4. Suppose $\mathcal{C}$ and $\mathcal{D}$ with functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that $G : \mathcal{D} \longrightarrow \mathcal{C}$, then a *natural transformation* $\eta : F \longrightarrow G$ is a collection of $\mathcal{D}$-morphisms $\eta_A : F(A) \longrightarrow G(A)$ in $\mathcal{D}$ (called the components are η) for every $A \in \mathcal{C}$ such that the following diagram commutes for every $\mathcal{C}$-morphism $f : A \longrightarrow B$:

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & G(A) \arrow[d, "Gf"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

2.2.2 Comonads

The primary focus for this dissertation, as the name suggests, is a certain class of endofunctors with "nice" algebraic constructions called comonads. Comonads are dual to the notion of monads. Monads are analogous to the algebraic structure called a monoid; a set with an identity and associative binary operation.

Definition 2.2.5. An endofunctor $\mathbb{F} : \mathcal{C} \to \mathcal{C}$ is called *comonad* if there exists natural transformations $\epsilon : \mathbb{F} \longrightarrow id_{\mathcal{C}}$ (called the counit) and $\delta : \mathbb{F} \longrightarrow \mathbb{F} \circ \mathbb{F}$ (called the comultiplication) such that the following diagrams commute for all objects $A \in \mathcal{C}$:

$$\begin{tikzcd}
\mathbb{F}A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & \mathbb{F}\mathbb{F}A \arrow[d, "\delta_{\mathbb{F}A}"] \\
\mathbb{F}\mathbb{F}A \arrow[r, "\mathbb{F}\delta_A"'] & \mathbb{F}\mathbb{F}\mathbb{F}A
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
\mathbb{F}A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, equal] & \mathbb{F}\mathbb{F}A \arrow[d, "\mathbb{F}\epsilon_A"] \\
\mathbb{F}\mathbb{F}A \arrow[r, "\epsilon_{\mathbb{F}A}"'] & \mathbb{F}A
\end{tikzcd} \tag{2.2}$$

The two diagrams are analogous to the axioms of a monoid. Namely, the diagram (2.1) expresses the comultiplication is "coassociative'. The diagram (2.2) is analogous to the left and right identity laws of monoids.

Example 2.2.7. For a fixed set Z, there is the comonad on **Set** given by $A \mapsto Z \times A$ (on objects) and $f \mapsto < \mathsf{id}_Z, f >$. The counit $\epsilon_A : Z \times A \longrightarrow A$ is given by $(z, a) \mapsto a$ and the comultiplication $\delta_A : Z \times A \longrightarrow Z \times (Z \times A)$ is given by $(z, a) \mapsto (z, (z, a))$.

Example 2.2.8. There is the sequences, or list comonad, on **Set** given by sending $A \mapsto A^\omega$ (on objects) where A^ω is the set of finite sequences of elements in A (i.e $[a_1, \dots, a_n]$ with $n \in \omega$ and $a_i \in A$) and $f \mapsto ([a_1, \dots, a_n] \mapsto [f(a_1), \dots, f(a_n)])$. The counit is the last operation given by $[a_1, \dots, a_n] \mapsto a_n$ and the comultiplication is given by the prefixes operation $[a_1, \dots, a_n] \mapsto [[a_1], [a_1, a_2], \dots, [a_1, \dots, a_n]]$.

We will be studying infinite families of comonads with an additional "grading" relating the comonads in the family.

Definition 2.2.6. Given $(I, \leq)$ an indexing partially-ordered set, *an I-graded family of comonads* is a set $\{\mathbb{F}_i\}_{i \in I}$ such that each $\mathbb{F}_i$ is a comonad and for every $i, j \in I$ with $i \leq j$, there exists a natural transformation $i^{i,j} : \mathbb{F}_i \longrightarrow \mathbb{F}_j$.

In most cases, the indexing set is the ordinal ω with its usual total order.

Example 2.2.9. For every $k \in \omega$, there is the fixed set comonad $F_{[k]}$ where $A \mapsto [k] \times A$, as in example (2.2.7). Since $[l] \subseteq [k]$ for $l \leq k$, the components of the grading natural transformation $i^{l,k} : F_{[l]} \longrightarrow F_{[k]}$ is given by the inclusion maps $i^{l,k}_A : (i, a) \mapsto (i, a)$ where $i \in [l] \subseteq [k]$.

Example 2.2.10. For every $k \in \omega$, there is the sequences of length $\leq k$ comonad analogous to the example (2.2.8).

2.2.3 Co-Kleisli Category and Eilenberg-Moore Category

For every comonad $\mathbb{F} : \mathcal{C} \longrightarrow \mathcal{C}$, there exist two important categories associated with $\mathbb{F}$. The coKleisli category of $\mathbb{F}$ and the category of coalgebras, or Eilenberg-Moore category, of $\mathbb{F}$.

Definition 2.2.7. The *coKleisli category of* $\mathbb{F}$, denoted $\mathcal{K}(\mathbb{F})$, is a category with:

- objects are the same objects as $\mathcal{C}$
- morphisms from A to B given by $\mathcal{C}$-morphism $f : \mathbb{F}A \longrightarrow B$. That is, $\mathrm{Hom}_{\mathcal{K}(\mathbb{F})}(A, B) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{F}A, B)$

such that

(1) For every object $A \in \mathcal{K}(\mathbb{F}) = \mathcal{C}$, the identity morphism is given by the A component of the counit:

$$\epsilon_A : \mathbb{F}A \longrightarrow A \tag{2.3}$$

(2) For two $\mathcal{C}$-morphisms $f : \mathbb{F}A \longrightarrow B$ and $g : \mathbb{F}B \longrightarrow C$ (i.e. two $\mathcal{K}(F)$ morphisms) we use the comonad structure to compose them to produce a morphism $\mathbb{F}A \longrightarrow C$:

$$\mathbb{F}A \xrightarrow{\delta_A} \mathbb{F}\mathbb{F}A \xrightarrow{\mathbb{F}f} \mathbb{F}B \xrightarrow{g} C \tag{2.4}$$

The definition (2.4) makes use of the morphism $\mathbb{F}f \circ \delta_A$ to define the composition law in $\mathcal{K}(\mathbb{F})$. This operation is used often and even provides an alternative set of axioms of for the coKleisli category.

Definition 2.2.8. The operation $f \mapsto \mathbb{F}f \circ \delta_A$ which sends $f : \mathbb{F}A \longrightarrow B$ to $f^* : \mathbb{F}A \longrightarrow \mathbb{F}B$ is called the *Kleisli coextension.*

It is common in algebra for an algebraic structure to have an associated class of objects that the structure "acts on". For example, groups 'act on' sets via group actions and rings 'act on' modules. For comonads, the associated class of objects are *coalgebras* and this class forms a category.

Definition 2.2.9. The *category of coalgebras of* $\mathbb{F}$ or *Eilenberg-Moore category*, denoted $\mathcal{C}^{\mathbb{F}}$, is a category with:

- objects a pair (A, α) with $A \in \mathcal{C}$ and $\mathcal{C}$-morphism $\alpha : A \longrightarrow \mathbb{F}A$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \mathbb{F}A \arrow[d, "\delta_A"] \\ \mathbb{F}A \arrow[r, "\mathbb{F}\alpha"'] & \mathbb{F}\mathbb{F}A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\mathrm{id}_A"'] & \mathbb{F}A \arrow[d, "\epsilon_A"] \\ & A \end{tikzcd} \tag{2.5}$$

- morphisms from $(A, \alpha) \longrightarrow (B, \beta)$ given by $\mathcal{C}$-morphisms $h : A \longrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathbb{F}A \arrow[d, "\mathbb{F}h"] \\ B \arrow[r, "\beta"'] & \mathbb{F}B \end{tikzcd} \tag{2.6}$$

The composition law is the same as in $\mathcal{C}$ and the identity on (A, α) is just the $\mathcal{C}$-morphism $\mathsf{id}_A : A \longrightarrow A$.

The diagrams (3.10) are analogous to the presentation of group actions. Namely, the diagram on the right is analogous to the action commuting with the group operation. While the diagram on the left is analogous to the identity in the group acting trivially on a set.

Chapter 3

Ehrenfeucht-Fraïssé Comonad

3.1 Introduction

The Ehrenfeucht-Fraïssé (EF) game was the first case of a two-player game used to prove equivalence between structures in fragments of first-order logic. In particular, the EF game is used to prove equivalence between structures in the k-rank fragments of first order logic. Other games, such as the pebbling game and bisimulation game, are essentially modifications of the EF game. Given two structures A, B, the Ehrenfreuct-Fraïssé game has two players, Spoiler and Duplicator. A k-round (symmetric) EF game is played as follows: for every $i \in [k]$,

- Spoiler chooses an element in either structure $a_i \in A$ or $b_i \in B$.
- Duplicator chooses an element in the other structure $b_i \in B$ or $a_i \in A$

At the end of the k-round game, k-tuples $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ have been chosen. Duplicator wins the k-round game if the map $\chi : a_i \longmapsto b_i$ is a partial σ-isomorphism from A to B. Otherwise, Spoiler wins. The asymmetric (or existential positive) game from A to B, is the same game with the additional restriction that Spoiler must always play an element in A and the map χ obtained is a partial σ-morphism. Hence, Duplicator must always respond in B. The following result is standard in any model theory text:

Proposition 3.1.1. *The following are equivalent:*

- *Duplicator has a winning strategy in the symmetric k-round EF game*
- $A \equiv^{\mathcal{L}_{\omega,k}} B$*, i.e. for every sentence* $\phi \in \mathcal{L}_{\omega,k}$*,* $A \vDash \phi \Leftrightarrow B \vDash \phi$

The goal of the EF comonad is to construct a σ-structure $\mathbb{E}_k A$ from a σ-structure A, that "internalizes" the asymmetric and symmetric k-round EF games into the category $\mathcal{R}(\sigma)$.

3.2 Comonad laws

Let A be a σ-structure over relational signature σ, then for every $k \in \omega$ we define a σ-structure $\mathbb{E}_k A$. Intuitively, $\mathbb{E}_k A$ is the structure of Spoiler's strategies in the Ehrenfreuct-Fraïssé k-round asymmetric game from A to any σ-structure. A function $f : \mathbb{E}_k A \longrightarrow B$, then represents Duplicator's strategy (i.e. responses) to Spoiler's plays in the k-round asymmetric game from A to B. For every $i \in \omega$, let A^i be the set of i-length sequences of elements in A. Let the domain of the structure be $|\mathbb{E}_k A| = \bigcup_{i \leq k} A^i$.

Definition 3.2.1. Define, for every σ-structure A, $\epsilon_A : \mathbb{E}_k A \longrightarrow A$ by $[a_1, \dots, a_n] \mapsto a_n$ (i.e. the last move of the play).

With this definitions in place, we can define a σ-structure on $\mathbb{E}_k A$. Suppose $R \in \sigma$ is a m-ary relation, the we define the interpretation of $R^{\mathbb{E}_k A}$ such that for every $s_1, \dots, s_m \in |\mathbb{E}_k A|$,

$$(s_1, \dots, s_m) \in R^{\mathbb{E}_k A} \Leftrightarrow \text{ for every } i, j \leq m,\ s_i \sqsubseteq s_j \text{ or } s_j \sqsubseteq s_i \tag{3.1}$$

$$\text{and } R^A(\epsilon_A(s_1), \dots, \epsilon_A(s_m)) \tag{3.2}$$

Definition 3.2.2. Given a morphism $f : A \longrightarrow B$, define the morphism $\mathbb{E}_k f : \mathbb{E}_k A \longrightarrow \mathbb{E}_k B$ by $[a_1, \ldots, a_n] \mapsto [f(a_1), \ldots, f(a_n)]$

Proposition 3.2.1. *The definition (3.2.2) of $\mathbb{E}_k f : \mathbb{E}_k A \longrightarrow \mathbb{E}_k B$ given above is indeed a morphism of σ-structures.*

Proof. Suppose $R \in \sigma$, then we want to show that if $(s_1, \ldots, s_m) \in R^{\mathbb{E}_k A}$, then $(\mathbb{E}_k f(s_1), \ldots, \mathbb{E}_k f(s_m)) \in R^{\mathbb{E}_k B}$. For brevity, assume that R is a binary relation (the proof for a general m-ary relation is a straightforward generalization). Suppose $s, s' \in \mathbb{E}_k A$ such that $(s, s') \in R^{\mathbb{E}_k A}$. Let $s = [a_1, \ldots, a_n]$ and $s' = [a_1, \ldots, a'_m]$. We aim to show that $(\mathbb{E}_k f(s), \mathbb{E}_k f(s')) \in R^{\mathbb{E}_k B}$

1. Since $(s, s') \in R^{\mathbb{E}_k A}$, by condition (4.1), $s \sqsubseteq s'$ or $s' \sqsubseteq s$. Without loss of generality, assume $s \sqsubseteq s'$. Since $s \sqsubseteq s'$.
$$s' = [a_1, \ldots, a_n, a'_{n+1}, \ldots, a'_m]$$
(noting that for $i \leq n$, $a_i = a'_i$). Therefore
$$\mathbb{E}_k f(s) = [f(a_1)), \ldots, f(a_n)]$$
$$\mathbb{E}_k f(s') = [f(a_1), \ldots, f(a_n), f(a'_{n+1}), \ldots, f(a'_m)]$$
Hence, $\mathbb{E}_k f(s) \sqsubseteq \mathbb{E}_k f(s')$ and (4.1) is satisfied.

2. By condition (4.2) and $(s, s') \in R^{\mathbb{E}_k f}$, $(\epsilon_A(s), \epsilon_A(s')) = (a_n, a'_m) \in R^A$. Since $f : A \to B$ is a morphism of σ-structures, $(f(a_n), f(a'_m)) \in R^B$. That is, $(\epsilon_B \circ \mathbb{E}_k f(s), \epsilon_B \circ \mathbb{E}_k f(s')) \in R^B$. Hence, (4.2) is satisfied.

Therefore, $(\mathbb{E}_k f(s), \mathbb{E}_k f(s')) \in R^{\mathbb{E}_k B}$ and $\mathbb{E}_k f$ is indeed a morphism of σ-structures. ☐

Proposition 3.2.2. *$\epsilon : \mathbb{E}_k \longrightarrow 1_{\mathcal{R}(\sigma)}$ is a natural transformation.*

Proof. For every $A, B \in \mathcal{R}(\sigma)$ we want to show that:

$$\begin{tikzcd} \mathbb{E}_k A \arrow[r, "\epsilon_A"] \arrow[d, "\mathbb{E}_k f"'] & A \arrow[d, "f"] \\ \mathbb{E}_k B \arrow[r, "\epsilon_B"'] & B \end{tikzcd} \tag{3.3}$$

$$\begin{aligned} f \circ \epsilon_A([a_1, \ldots, a_n]) &= f(a_n) && \text{by defn (4.2.3) of } \epsilon_A \\ &= \epsilon_B([f(a_1), \ldots, f(a_n)]) && \text{by defn (4.2.3) of } \epsilon_B \\ &= \epsilon_B \circ \mathbb{E}_k f([a_1, \ldots, a_n]) && \text{by defn (3.2.2) of } \mathbb{E}_k f \end{aligned}$$

Hence, the above diagram (4.4) commutes. ☐

Definition 3.2.3. Suppose $s \in \mathbb{E}_k A$, then $s = [a_1, \ldots, a_n]$ for some $n \in \omega$ and for every $i = 1, \ldots, n$, $a_i \in A$. Let $s_i = [a_1, \ldots, a_i] \in \mathbb{E}_k A$. Define, for every σ-structure A, $\delta_A : \mathbb{E}_k A \longrightarrow \mathbb{E}_k \mathbb{E}_k A$ by $s \mapsto [s_1, \ldots, s_n]$.

Proposition 3.2.3. *$\delta : \mathbb{E}_k \longrightarrow \mathbb{E}_k \mathbb{E}_k$ is a natural transformation.*

Proof. For every $A, B \in \mathcal{R}(\sigma)$ we want to show that:

$$\begin{tikzcd} \mathbb{E}_k A \arrow[r, "\delta_A"] \arrow[d, "\mathbb{E}_k f"'] & \mathbb{E}_k \mathbb{E}_k A \arrow[d, "\mathbb{E}_k \mathbb{E}_k f"] \\ \mathbb{E}_k B \arrow[r, "\delta_B"'] & \mathbb{E}_k \mathbb{E}_k B \end{tikzcd} \tag{3.4}$$

$$
\begin{aligned}
\mathbb{E}_k\mathbb{E}_k f \circ \delta_A([a_1,\ldots,a_n]) &= \mathbb{E}_k\mathbb{E}_k f([s_1,\ldots,s_n]) && \text{by defn (4.2.5) of } \delta_A \\
&= [\mathbb{E}_k f(s_1),\ldots,\mathbb{E}_k f(s_n)] && \text{by defn (3.2.2) of } \mathbb{E}_k\mathbb{E}_k f \\
&= [[f(a_1)],\ldots,[f(a_1),\ldots,f(a_n)]] && \text{by defn (3.2.2) of } \mathbb{E}_k f \\
&= \delta_B([f(a_1),\ldots,f(a_n)]) && \text{by defn (4.2.5) of } \delta_B \\
&= \delta_B \circ \mathbb{E}_k f([a_1,\ldots,a_n]) && \text{by defn (3.2.2) of } \mathbb{E}_k f
\end{aligned}
$$

Hence, the above diagram (4.5) commutes. □

Theorem 3.2.4. *The triple $\langle \mathbb{E}_k, \delta, \epsilon\rangle$ is a comonad.*

Proof. By proposition (4.2.3) and (4.2.2), δ and ϵ are natural transformation. Hence, δ and ϵ are indeed the comultiplication and counit of $\mathbb{E}_k$. The associative and identity laws remain to be shown.
For associativity, for every $A \in \mathcal{R}(\sigma)$, the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{E}_k A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & \mathbb{E}_k\mathbb{E}_k A \arrow[d, "\delta_{\mathbb{E}_k A}"] \\
\mathbb{E}_k\mathbb{E}_k A \arrow[r, "\mathbb{E}_k\delta_A"'] & \mathbb{E}_k\mathbb{E}_k\mathbb{E}_k A
\end{tikzcd}
\tag{3.5}
$$

$$
\begin{aligned}
\delta_{\mathbb{E}_k A} \circ \delta_A([a_1,\ldots,a_n]) &= \delta_{\mathbb{E}_k A}([s_1,\ldots,s_n]) && \text{by defn (4.2.5) of } \delta_A \\
&= [[s_1],\ldots,[s_1,\ldots,s_n]] && \text{by defn (4.2.5) of } \delta_{\mathbb{E}_k A} \\
&= [\delta_A(s_1),\ldots,\delta_A(s_n)] && \text{by defn (4.2.5) of } \delta_A \\
&= \mathbb{E}_k\delta_A([s_1,\ldots,s_n]) && \text{by defn (3.2.2) of } \mathbb{E}_k\delta_A \\
&= \mathbb{E}_k\delta_A \circ \delta_A([a_1,\ldots,a_n]) && \text{by defn (4.2.5) of } \delta_A
\end{aligned}
$$

For identity, for every $A \in \mathcal{R}(\sigma)$, the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{E}_k A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, equal] & \mathbb{E}_k\mathbb{E}_k A \arrow[d, "\mathbb{E}_k\epsilon_A"] \\
\mathbb{E}_k\mathbb{E}_k A \arrow[r, "\epsilon_{\mathbb{E}_k A}"'] & \mathbb{E}_k A
\end{tikzcd}
\tag{3.6}
$$

$$
\begin{aligned}
\mathbb{E}_k\epsilon_A \circ \delta_A([a_1,\ldots,a_n]) &= \mathbb{E}_k\epsilon_A([s_1,\ldots,s_n]) && \text{by defn (4.2.5) of } \delta_A \\
&= [\epsilon_A(s_1),\ldots,\epsilon_A(s_n)] && \text{by defn (3.2.2) of } \mathbb{E}_k\epsilon_A \\
&= [a_1,\ldots,a_n] && \text{by defn (4.2.3) of } \epsilon_A \\
&= s_n && \text{by defn (4.2.5) of } s_n \\
&= \epsilon_{\mathbb{E}_k A}([s_1,\ldots,s_n]) && \text{by defn (4.2.3) of } \epsilon_{\mathbb{E}_k A} \\
&= \epsilon_{\mathbb{E}_k A} \circ \delta_A([a_1,\ldots,,a_n]) && \text{by defn (4.2.5) of } \delta_A
\end{aligned}
$$

By definition, $\mathbb{E}_k$ is a comonad. □

For every $l, k \in \omega$ such that $l \leq k$ and σ-structure A, there exists an inclusion $i_A^{l,k} : \mathbb{E}_l A \longrightarrow \mathbb{E}_k A$.

Proposition 3.2.5. *The inclusion maps form a natural transformation $i^{l,k} : \mathbb{E}_l \longrightarrow \mathbb{E}_k$. Further, each map preserves the counit and comultiplication (i.e. each map is a morphism of comonads).*

Proof.

$$\begin{tikzcd}
\mathbb{E}_l A \arrow[r, "i_A^{l,k}"] \arrow[d, "\mathbb{E}_l f"'] & \mathbb{E}_k A \arrow[d, "\mathbb{E}_k f"] \\
\mathbb{E}_l B \arrow[r, "i_B^{l,k}"'] & \mathbb{E}_k B
\end{tikzcd} \tag{3.7}$$

$$\begin{aligned}
\mathbb{E}_k f \circ i_A^{l,k}([a_1, \ldots, a_n]) &= \mathbb{E}_k f([a_1, \ldots, , a_n]) && \text{by defn of inclusion} \\
&= [f(a_1), \ldots, f(a_n)] && \text{by defn (3.2.2) of } \mathbb{E}_k f \\
&= i_B^{l,k}([f(a_1), \ldots, f(a_n)]) && \text{by defn of inclusion} \\
&= i_B^{l,k} \circ \mathbb{E}_l f([a_1, \ldots, a_n]) && \text{by defn (3.2.2) of } \mathbb{E}_k f
\end{aligned}$$

□

The grading given by these inclusion maps seem to suggest that there is a colimit object capturing the information of $\mathbb{E}_k A$ for every $k \in \omega$. This is indeed the case. Consider the structure $\mathbb{E}_\omega A$ with domain $|\mathbb{E}_\omega A| = \bigcup_{k \in \omega} A^k$ where A^k is the set of k-length sequences of elements in A. The structure on $\mathbb{E}_\omega A$ is similar to the structure given to $\mathbb{E}_k A$.

Proposition 3.2.6. *Let ω be considered as a poset category under the usual order. The object $\mathbb{E}_\omega A$ is the ω-colimit of the family $\{\mathbb{E}_k A\}_{k \in \omega}$ with the above inclusion maps.*

Proof. For every $k \in \omega$, define $i_A^k : \mathbb{E}_k A \to \mathbb{E}_\omega A$ as the inclusion (i.e. $[a_1, \ldots, a_n] \mapsto [a_1, \ldots, a_n]$ where $n \leq k$). Clearly, the following diagram commutes for all $l, k \in \omega$ with $l \leq k$

$$\begin{tikzcd}
\mathbb{E}_l A \arrow[rr, "i_A^{l,k}"] \arrow[dr, "i_A^l"'] & & \mathbb{E}_k A \arrow[dl, "i_A^k"] \\
& \mathbb{E}_\omega A &
\end{tikzcd} \tag{3.8}$$

Suppose that there exists a σ-structure B and for every $l, k \in \omega$ with $l \leq k$, there exist morphisms $f^l : \mathbb{E}_l A \longrightarrow B$, $f^k : \mathbb{E}_k A \longrightarrow B$ such that $f^l = f^k \circ i^{l,k}$. Consider the morphism $u : \mathbb{E}_\omega A \longrightarrow B$ given by $[a_1, \ldots, a_n] \mapsto f^n([a_1, \ldots, a_n])$. The following diagram commutes:

$$\begin{tikzcd}
\mathbb{E}_l A \arrow[rr, "i_A^{l,k}"] \arrow[dr, "i_A^l"'] \arrow[ddr, bend right, "f^l"'] & & \mathbb{E}_k A \arrow[dl, "i_A^k"] \arrow[ddl, bend left, "f^k"] \\
& \mathbb{E}_\omega A \arrow[d, dotted, "u"] & \\
& B &
\end{tikzcd} \tag{3.9}$$

Moreover, given the conditions on f^j for all $j \in \omega$, u is unique. Namely, suppose there exists a morphism $u' : \mathbb{E}_\omega A \longrightarrow B$ such that for all $j \in \omega$, $f^j = u' \circ i_A^j$. Suppose $s = [a_1, \ldots, a_k] \in \mathbb{E}_\omega A$ then for all

$j \geq k, s \in \mathbb{E}_j A$.

$$\begin{aligned} u(s) &= f^k(s) && \text{by defn of } u \\ &= f^j \circ i_A^{k,j}(s) && \text{by (6.4)} \\ &= u' \circ i_A^j \circ i_A^{k,j}(s) && \text{by hypothesis on } u' \\ &= u' \circ i_A^k(s) && \text{by (6.3)} \\ &= u'(s) && \text{by defn of inclusion} \end{aligned}$$

Since $u(s) = u'(s)$ for all $s \in \mathbb{E}_\omega A$, $u = u'$ so u is unique as desired. $\square$

3.3 Positional Form and Equivalences

In order to solidify the connection between the construction $\mathbb{E}_k$ and the Ehrenfreuct-Fraïssé game, we have to encode the strategies in the game into "positional form". This positional form is similar to the constructions of graded "back-and-forth systems" that are used in model theory texts to prove that the k-round symmetric EF game characterizes $A \equiv^{\mathcal{L}_{\omega,k}} B$.

Let $\Gamma_k(A, B) = (A \times B)^{\leq k}$ (i.e. sequences in pairs of elements in A, B with length $\leq k$). Recall that for a σ-morphism $f : \mathbb{E}_k A \longrightarrow B$, there exists the Klesli coextension $f^* : \mathbb{E}_k A \longrightarrow \mathbb{E}_k B$. Define the set function $\theta_f : |\mathbb{E}_k A| \longrightarrow \Gamma_k(A, B)$ by $s = [a_1, \ldots, a_n] \mapsto [(a_1, b_1), \ldots, (a_n, b_n)]$ where $f^*(s) = [b_1, \ldots, b_n]$. Rephrasing, $\theta_f = z \circ < \mathrm{id}_{\mathbb{E}_k A}, f^* >$ where z is the 'zipper' function sending two sequences of equal length to the natural sequence of pairs.

Definition 3.3.1. $S \subseteq \Gamma_k(A, B)$ is a *strategy in positional form* if S satisfies the following conditions:

(S0) $[] \in S$

(S1) For all $\chi \in S$ with $|\chi| < k$, $a \in A$, there exists a unique $b \in B$ such that $\chi[(a, b)] \in S$

(S2) S is reachable: For all $\chi \in S$, there is a chain

$$\chi_0 \longrightarrow \cdots \longrightarrow \chi_n$$

such that $\chi_0 = []$, $\chi_n = \chi$ and $\chi_i \in S$ with $|\chi_i| = i$ for all $i = 0, \ldots, n$.

Proposition 3.3.1. *If $f : \mathbb{E}_k A \longrightarrow B$ is a σ-morphism, then there exists a strategy in positional form S_f.*

Proof. Define

$$S_f = \{\theta_f(s) \mid s \in \mathbb{E}_k A\} \cup \{[]\}$$

- $[] \in S_f$, so (S0) is satisfied.
- Suppose $\chi \in S_f$ with $|\chi| < k$ and $a \in A$. By definition of S_f either $\chi = []$ or $\chi = \theta_f(s)$ for some $s \in \mathbb{E}_k A$. In the first case, consider $\theta_f([a]) = [(a, b)] \in S$ with $b = f([a])$. In the second case, consider $\chi' = \theta_f(s[a])$. Since $|\chi| < k$, the length $|s| = j$ and the length $|s[a]| = j + 1 \leq k$. Hence, $s[a]$ is indeed in $\mathbb{E}_k A$. Therefore, $\chi' \in S_f$ and $\chi' = \chi[(a, b)]$ where $b = f(s[a])$.
- Suppose $\chi \in S_f$, then $\chi = \theta_f(s)$ (the $\chi = []$ case is trivial) for some $s \in \mathbb{E}_k A$. Since $s \in \mathbb{E}_k A$ for some $n \leq k$, $s = [a_1, \ldots, a_n]$ with $a_i \in A$ by definition of $\mathbb{E}_k A$. Let $s_i = [a_1 \ldots, a_i]$ for all $i \leq n$, then
$$[] \longrightarrow \theta_f(s_1) \longrightarrow \cdots \longrightarrow \theta_f(s_n) = \chi$$

Hence, S_f is reachable $\square$

Proposition 3.3.2. *Conversely, for every strategy in positional form S there exists a $f : \mathbb{E}_k A \to B$ such that $S = S_f$*

Proof. $S_f \subseteq S$ The strategy is to construct an appropriate f. Define $\pi_1 : \Gamma_k(A, B) \longrightarrow \mathbb{E}_k A$ by $\pi_1 : [(a_1, b_1), \ldots, (a_n, b_n)] \longrightarrow [a_1, \ldots, a_n]$ and $\pi_2 : \Gamma_k(A, B) \longrightarrow \mathbb{E}_k B$ by $\pi_2 : [(a_1, b_1), \ldots, (a_n, b_n)] \longrightarrow [b_1, \ldots, b_n]$. We construct f by recursion, up to k, on the length of a play $s \in \mathbb{E}_k A$.
Base Case: Suppose $s = [a]$ for $a \in A$. By (S0) and (S1), there exists a unique b such that $[a, b] \in S$. Let $f(s) = b$.
Recursive Step: Assume for the recursion, that $f(s)$ is defined for $|s| = n < k$ and that $\theta_f(s) \in S$. Consider $s' = s[a]$. By (S1), there exists a unique $b \in B$ such that $\chi' = \theta_f(s)[a, b]$. Let $f(s') = b$.
$S \subseteq S_f$ For every χ, we need to show that there exists an $s \in \mathbb{E}_k A$ such that $\theta_f(s) = \chi$ for the f constructed above. Consider $s = \pi_1(\chi)$. By construction, $\theta_f(s) = \theta_f(\pi_1(\chi)) = \chi$. □

3.3.1 Equivalence $\exists^+\mathcal{L}_{\omega,k}$

Definition 3.3.2. A position $\chi = [(a_1, b_1), \ldots, (a_n, b_n)] \in \Gamma_k(A, B)$ is *winning for Duplicator* if the map $\chi : a_i \longmapsto b_i$ is a partial σ-morphism from A to B. Naturally, we can extend the definition to say that a strategy in positional form $S \subseteq \Gamma_k(A, B)$ is *winning for Duplicator* if for all $\chi \in S$, χ is winning for Duplicator.

Consider the expanded signature $\sigma' = \sigma \cup \{I\}$ with I a binary relation. In order to prove the theorem, it is necessary to lift σ-structures to σ'-structures. We say a σ-structure A is a *pure σ'-structure* if I is interpreted as the identity relation on A. Note that if A is a pure σ'-structure, then by the definition of $\mathbb{E}_k A$ as a σ'-structure, $\mathbb{E}_k A$ is not a pure σ'-structure. However, two prefix comparable plays in $\mathbb{E}_k A$ are I-related if their last elements are the same. This ensures that S_f contains well-defined partial functions.

Theorem 3.3.3. *If A, B are pure σ'-structures and $f : \mathbb{E}_k A \to B$ is a function, then*

$$f : \mathbb{E}_k A \longrightarrow B \text{ is a } \sigma'\text{-morphism if and only if } S_f \text{ is winning for Duplicator}$$

Proof. $\Rightarrow$ Suppose $\chi \in S_f$, then by definition of S_f, $\chi = []$ or $\chi = \theta_f(s)$ for some $s \in \mathbb{E}_k A$. If $\chi = []$, then it is vacuously true that χ is a partial homomorphism (i.e. winning for Duplicator). If $\chi = \theta_f(s)$, then there exists some $s \in \mathbb{E}_k A$ such that $s = [a_1, \ldots, a_n]$ where $f([a_1, \ldots, a_i]) = b_i$ and $\chi = [(a_1, b_1), \ldots, (a_n, b_n)]$ for all $i = 1, \ldots, n$. For brevity, let $s_i = [a_1, \ldots, a_i] \sqsubseteq s$ (i.e. s_i is the i-th component of $\delta_A(s)$).
Suppose $R \in \sigma$ is a m-ary relation symbol, and $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $R^A(a_{i_1}, \ldots, a_{i_m})$, then:

$$\begin{aligned}
R^A(a_{i_1}, \ldots, a_{i_m}) &\Rightarrow R^A(\epsilon_A(s_{i_1}), \ldots, \epsilon_A(s_{i_m})) && \text{by defn of } \epsilon_A \\
(\text{and } s_{i_\mu} \sqsubseteq s_{i_\nu} \text{ or } s_{i_\nu} \sqsubseteq s_{i_\mu} &\text{ for all } \nu, \mu = 1, \ldots, m) && \text{by each } s_{i_*} \sqsubseteq s \\
&\Rightarrow R^{\mathbb{E}_k A}(s_{i_1}, \ldots, s_{i_m}) && \text{by interpretation of } R \text{ on } \mathbb{E}_k A \\
&\Rightarrow R^B(f(s_{i_1}), \ldots, f(s_{i_m})) && \text{by hypothesis } f \text{ is } \sigma\text{-morphism} \\
&\Rightarrow R^B(b_{i_1}, \ldots, b_{i_m})
\end{aligned}$$

Hence, χ preserves relations. Moreover, for a_i and a_j appearing in s with $a_i = a_j$, then:

$$\begin{aligned}
a_i = a_j &\Rightarrow I^A(a_i, a_j) && \text{by } A \text{ a pure } \sigma'\text{-structure} \\
&\Rightarrow I^A(\epsilon_A(s_i), \epsilon_A(s_j)) && \text{by defn of } \epsilon_A \\
(\text{and } s_i \sqsubseteq s_j &\text{ or } s_j \sqsubseteq s_i) && \text{by each } s_* \sqsubseteq s \\
&\Rightarrow I^{\mathbb{E}_k A}(s_i, s_j) && \text{by interpretation of } I \text{ on } \mathbb{E}_k A \\
&\Rightarrow I^B(f(s_i), f(s_j)) && \text{by hypothesis } f \text{ is a } \sigma'\text{-morphism} \\
&\Rightarrow I^B(b_i, b_j) \\
&\Rightarrow b_i = b_j && \text{by } B \text{ a pure } \sigma'\text{-structure}
\end{aligned}$$

Hence, χ is a well-defined partial function. Therefore, χ is indeed a partial homomorphism. Hence, for all $\chi \in S_f$, χ is winning for Duplicator. Therefore, by definition, S_f is winning for Duplicator.

$\Leftarrow$ Suppose $(s_1, \ldots, s_m) \in R^{\mathbb{E}_k A}$ for $R \in \sigma'$ (including I), then by the interpretation of R on $\mathbb{E}_k A$, there exists some prefix order greatest element $s \in \{s_1, \ldots, s_m\}$ such that for all i, $s_i \sqsubseteq s$. Consider $\chi = \theta_f(s) \in S_f$. By $s_i \sqsubseteq s$ and definition of θ_f, $(\epsilon_A(s_1), f(s_1)), \ldots, (\epsilon_A(s_m), f(s_m))$ appear in *chi*. Since $R^A(\epsilon_A(s_1), \ldots, \epsilon_A(s_m))$ and χ is a partial homomorphism (i.e. winning for Duplicator), $R^B(f(s_1), \ldots, f(s_m))$. Hence, f is a σ'-morphism. $\square$

3.3.2 Equivalence $\mathcal{L}_{\omega,k}$

3.3.3 Equivalence $\mathcal{L}_{\infty,k}(\mathsf{Cnt})$

3.4 Coalgebras and Tree-Depth

An advantage of the comonad perspective is exploring the structure of the category of coalgebras over $\mathbb{E}_k$ to give a categorical characterization of combinatorial properties of structures. In particular, we use the coalgebra category of $\mathbb{E}_k$ to give a categorical definition for tree-depth of a σ-structure A. Recall a coalgebra for $\mathbb{E}_k$ is an object A and morphism $\alpha : A \longrightarrow \mathbb{E}_k A$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \mathbb{E}_k A \arrow[d, "\delta_A"] \\ \mathbb{E}_k A \arrow[r, "\mathbb{E}_k \alpha"'] & \mathbb{E}_k \mathbb{E}_k A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\mathrm{id}_A"'] & \mathbb{E}_k A \arrow[d, "\epsilon_A"] \\ & A \end{tikzcd} \tag{3.10}$$

There are many equivalent ways to define the notion of tree-depth of a σ-structure A. We use the one given in [11] and is stated in definition (3.4.4).

Definition 3.4.1. A *forest* $\mathcal{F}$ is a disjoint union of finitely-many finite rooted trees. The *height* of the forest is the longest path between two vertices in $\mathcal{F}$

Definition 3.4.2. Given an undirected graph G, a *forest cover of* G is a forest $\mathcal{F}$ such that G is a subgraph of $\overline{\mathcal{F}}$ where $\overline{\mathcal{F}}$ is the transitive closure of $\mathcal{F}$

Definition 3.4.3. Given a σ-structure A, the *Gaifman Graph of* A, denoted $\mathcal{G}(A)$, is the undirected graph with vertices as elements of A and a, a' are connected by edge if there exists some $R \in \sigma$ with a, a' appearing in the same tuple in R^A.

Definition 3.4.4. Given a σ-structure A, the *tree-depth* of A, denoted $\mathrm{td}(A)$, is the least height of a forest cover $\mathcal{F}$ for $\mathcal{G}(A)$.

Proposition 3.4.1. *Let A be a finite σ-structure. There is bijective correspondence between:*

(1) coalgebras $\alpha : A \longrightarrow \mathbb{E}_k A$

(2) Forest covers $\mathcal{F}$ of $\mathcal{G}(A)$ of height k

Proof. For the (1) $\Rightarrow$ (2) direction, suppose $\alpha : A \to \mathbb{E}_k A$ is a coalgebra. Construct a forest $\mathcal{F}_n$ for every $n \leq k$ by recursion. Let $V(\mathcal{F})$ and $E(\mathcal{F})$ denote the vertices and edge relation of forest $\mathcal{F}$.
Base Case:

$$V(\mathcal{F}_1) := \{a \in A \mid \alpha(a) = [a]\}, E(\mathcal{F}_1) = \varnothing$$

Recursive Step: Assume that $\mathcal{F}_n$ is a forest that has been constructed. From the counit coalgebra law (3.10 right), it must be the case that $\alpha(a) = s$ where the last move in s is a (i.e. $\epsilon_A(s) = a$). Let a^- be the second-to-last move in s (i.e $s = t[a^-, a] = t[a^-, \epsilon_A(s)]$ for some possibly empty sequence t). Define $\mathcal{F}_{n+1}$ as follows:

$$V(\mathcal{F}_{n+1}) := V(\mathcal{F}_n) \cup \{a \in A \mid a^- \in V(\mathcal{F}_n)\}$$

$$E(\mathcal{F}_{n+1}) := E(\mathcal{F}_n) \cup \{(a^-, a) \mid a \in V(\mathcal{F}_{n+1})\}$$

Let $\mathcal{F} = \mathcal{F}_k$ be the forest constructed at the end of the recursion up to k. Suppose (a, a') is an edge in $\mathcal{G}(A)$, the by the definition of Gaifman graph there exists an $R \in \sigma$ such that a and a' appear in a tuple of R^A. By α a σ-morphism, $\alpha(a)$ and $\alpha(a')$ appear in a tuple of $R^{\mathbb{E}_k A}$. By condition (1) in the definition $R^{\mathbb{E}_k A}$, $\alpha(a) \sqsubseteq \alpha(a')$ or $\alpha(a') \sqsubseteq \alpha(a)$. Without loss of generality, assume $\alpha(a) \sqsubseteq \alpha(a')$, and let t be the suffix of $\alpha(a)$ in $\alpha(a')$. From the construction of $\mathcal{F}$, $[a]t$ describes a path between a and a' in $\mathcal{F}$. Hence, $(a, a') \in \overline{\mathcal{F}}$. Since (a, a') was arbitrary in $\mathcal{G}(A)$, $\mathcal{G}(A)$ is a subgraph of $\overline{\mathcal{F}}$. Therefore, $\mathcal{F}$ is a forest cover of $\mathcal{G}(A)$ as desired.

For the (2) $\Rightarrow$ (1) direction, suppose $\mathcal{F}$ is a forest cover of $\mathcal{G}(A)$ of height k. Consider arbitrary $a \in A$. Since $\mathcal{F}$ is a cover of $\mathcal{G}(A)$, $a \in \mathcal{F}$. By definition of forest as a disjoint union finite trees, a is in a unique tree $T \subseteq \mathcal{F}$. Let a^* be the root of tree T. By T being a tree, there exists a unique path $u = [a^*, \ldots, a]$ between a^* and a. Let $\alpha(a) = u$ with u considered as an element of $\mathbb{E}_k A$. It is straightforward to see that α as defined satisfies the coalgebra laws (3.10). $\square$

Definition 3.4.5. Given a σ-structure A, define the *coalgebra number of* A, denoted $\kappa(A)$, as the least k such that there exists a coalgebra $\alpha : A \longrightarrow \mathbb{E}_k A$.

Corollary 3.4.1.1. $\kappa(A) = \mathsf{td}(A)$

Proof. By definition of coalgebra number $\kappa(A)$ there exists a coalgebra $\alpha : A \longrightarrow \mathbb{E}_{\kappa(A)} A$. By proposition (3.4.1), there exists a corresponding forest cover $\mathcal{F}$ of $\mathcal{G}(A)$. The height of $\mathcal{F}$ is $\kappa(A)$. Hence, $\mathsf{td}(A) \leq \kappa(A)$ by definition of tree-depth as the least height of forest covers. Similarly, by definition of tree-depth $\mathsf{td}(A)$, there exists a forest cover $\mathcal{F}$ of $\mathcal{G}(A)$ of height $k = \mathsf{td}(A)$. By proposition (3.4.1), there exists a corresponding coalgebra $\alpha : A \longrightarrow \mathbb{E}_k A$. Hence, $\kappa(A) \leq k$ by definition of $\kappa(A)$. Therefore, $\mathsf{td}(A) \leq \kappa(A)$ and $\mathsf{td}(A) \geq \kappa(A)$, so $\kappa(A) = \mathsf{td}(A)$. $\square$

Hence, as the above corollary (3.4.1.1) shows, the graded family of comonads $\{\mathbb{E}_k\}_{k \in \omega}$ gives a purely categorical definition of tree-depth of a structure A. Namely, the tree-depth of A is just the least k such that a coalgebra $\alpha : A \longrightarrow \mathbb{E}_k A$ exists.

Chapter 4

Pebbling Comonad

4.1 Introduction

The Pebbling game, similar in structure to the Ehrenfreuct-Fraïssé game, is used to prove equivalence between structures in the k-variable fragments of infinitary logic. Given two structures A, B, the k-pebble game is played with Spoiler and Duplicator both having a set of k-pebbles $[k] = \{1, \ldots, k\}$.

- Spoiler places one of his pebbles $p \in [k]$ on an element in either strucure $a \in A$ or $b \in B$. If pebble p is already placed on a different element, then Spoiler removes the pebble from this element and places the pebble on the new element he chose.
- Duplicator places her corresponding pebble $p \in [k]$ on an element in the other structure $b \in B$ or $a \in A$. Just as with Spoiler, Duplicator may have to move her pebble from a previously pebbled element.

The game is played for ω many rounds. At the end of the game, $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are the pebbled elements. Duplicator wins the k-pebble game if the map $\gamma : a_p \longmapsto b_p$ is a partial σ-isomorphism from A to B. Otherwise, Spoiler wins. Just as with the case with EF games, the asymmetric (or existential positive) game from A to B, is the same game with the additional restriction that Spoiler must always pebble elements in A and that the map obtained is a partial σ-morphism. The following result holds:

Proposition 4.1.1. *The following are equivalent:*

- *Duplicator has a winning strategy in the symmetric k-pebble game*
- $A \equiv^{\mathcal{L}^k_{\infty,\omega}} B$, *i.e. for every* $\phi \in \mathcal{L}^k_{\infty,\omega}$, $A \vDash \phi \Leftrightarrow B \vDash \phi$

As was the goal with the EF game comonad, our goal is to construct a σ-structure $\mathbb{T}_k A$ from a σ-structure A, that "internalizes" the asymmetric and symmetric k-pebble games in the category $\mathcal{R}(\sigma)$. The construction of $\mathbb{T}_k$ and all of the results in this chapter were originally discovered in ([1]). The paper [1] was the inspiration to develop the other game comonads in this dissertation. We reproduce the proofs here to demonstrate the connections and differences with these other comonads.

4.2 Comonad laws

Let A be a σ-structure over relational signature σ, then for every $k \in \mathbb{N}$ we define a σ-structure $\mathbb{T}_k A$. Intuitively, $\mathbb{T}_k A$ is the structure of the finite k-pebblings on A. Let the domain of this structure be $|\mathbb{T}_k A| = (\{1, \ldots, k\} \times A)^{<\omega}$ (i.e. the set of finite sequence of elements in product $\{1, \ldots, k\} \times A$). In order, to define the σ-structure on $\mathbb{T}_k A$, a bit more notation is necessary:

Definition 4.2.1. Suppose $s, s' \in |\mathbb{T}_k A|$ such that $s = [s_1, \ldots, s_n]$ and $s' = [s'_1, \ldots, s'_m]$, then define $ss' = [s_1, \ldots, s_n, s'_1, \ldots, s'_m]$ (i.e. the concatenation of these two sequences).

Definition 4.2.2. For $s, t \in |\mathbb{T}_k A|$, say $s \sqsubseteq t$ if there exists an $s' \in |\mathbb{T}_k A|$ such that $ss' = t$; such an s' is called the suffix of s in t. $\sqsubseteq$ preorders $|\mathbb{T}_k A|$.

Definition 4.2.3. Define, for every σ-structure A, $\epsilon_A : \mathbb{T}_k A \longrightarrow A$ by $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto a_n$ (i.e. the element of the last move of the play). Further, define $\pi_A : \mathbb{T}_k A \longrightarrow \{1, \ldots, k\}$ by $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto p_n$ (i.e. the pebble index of the last move of the play).

With these definitions in place, we can define the σ-structure on $\mathbb{T}_k A$. Suppose $R \in \sigma$ is an m-ary relation, then we define the interpretation $R^{\mathbb{T}_k A}$ such that for every $s_1, \ldots, s_m \in |\mathbb{T}_k A|$,

$$(s_1, \ldots, s_m) \in R^{\mathbb{T}_k A} \Leftrightarrow \text{ for every } i, j,\ s_i \sqsubseteq s_j \text{ or } s_j \sqsubseteq s_i (\text{i.e. exists } \sqsubseteq\text{-greatest } s_*) \tag{4.1}$$
$$\text{and for every } i,\ \pi_A(s_i) \text{ does not appear in suffix of } s_i \text{ in } s_* \tag{4.2}$$
$$\text{and} R^A(\epsilon_A(s_1), \ldots, \epsilon_A(s_m)) \tag{4.3}$$

The definition of $\mathbb{T}_k A$ can be extended to morphisms of σ-structures.

Definition 4.2.4. Given a morphism $f : A \longrightarrow B$, define the morphism $\mathbb{T}_k f : \mathbb{T}_k A \longrightarrow \mathbb{T}_k B$ by $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto [(p_1, f(a_1)), \ldots, (p_n, f(a_n))]$

Proposition 4.2.1. *The definition (4.2.4) of $\mathbb{T}_k f : \mathbb{T}_k A \longrightarrow \mathbb{T}_k B$ given above is indeed a morphism of σ-structures.*

Proof. Suppose $R \in \sigma$, then we want to show that if $(s_1, \ldots, s_m) \in R^{\mathbb{T}_k A}$, then $(\mathbb{T}_k f(s_1), \ldots, \mathbb{T}_k f(s_m)) \in R^{\mathbb{T}_k B}$. For brevity, assume that R is a binary relation (the proof for a general m-ary relation is a straightforward generalization). Suppose $s, s' \in \mathbb{T}_k A$ such that $(s, s') \in R^{\mathbb{T}_k A}$. Let $s = [(p_1, a_1), \ldots, (p_n, a_n)]$ and $s' = [(q_1, a_1), \ldots, (q_m, a'_m)]$. We aim to show that $(\mathbb{T}_k f(s), \mathbb{T}_k f(s')) \in R^{\mathbb{T}_k B}$

1. Since $(s, s') \in R^{\mathbb{T}_k A}$, by condition (4.1), $s \sqsubseteq s'$ or $s' \sqsubseteq s$. Without loss of generality, assume $s \sqsubseteq s'$. Since $s \sqsubseteq s'$.
$$s' = [(p_1, a_1), \ldots, (p_n, a_n), (q_{n+1}, a'_{n+1}), \ldots, (q_m, a'_m)]$$
(noting that for $i \leq n$, $p_i = q_i$ and $a_i = a'_i$). Therefore
$$\mathbb{T}_k f(s) = [(p_1, f(a_1)), \ldots, (p_n, f(a_n))]$$
and
$$\mathbb{T}_k f(s') = [(p_1, f(a_1)), \ldots, (p_n, f(a_n)), (q_{n+1}, f(a'_{n+1})), \ldots, (q_m, f(a'_m))]$$
Hence, $\mathbb{T}_k f(s) \sqsubseteq \mathbb{T}_k f(s')$ and (4.1) is satisfied.

2. By condition (4.2) and $(s, s') \in R^{\mathbb{T}_k f}$, for $n < i \leq m$, $p_n \neq q_i$. Since $\mathbb{T}_k f$ does not affect pebble indices, (4.2) is satisfied.

3. By condition (4.3) and $(s, s') \in R^{\mathbb{T}_k f}$, $(\epsilon_A(s), \epsilon_A(s')) = (a_n, a'_m) \in R^A$. Since $f : A \to B$ is a morphism of σ-structures, $(f(a_n), f(a'_m)) \in R^B$. That is, $(\epsilon_B \circ \mathbb{T}_k f(s), \epsilon_B \circ \mathbb{T}_k f(s')) \in R^B$. Hence, (4.3) is satisfied.

Therefore, $(\mathbb{T}_k f(s), \mathbb{T}_k f(s')) \in R^{\mathbb{T}_k B}$ and $\mathbb{T}_k f$ is indeed a morphism of σ-structures. □

Proposition 4.2.2. *$\epsilon : \mathbb{T}_k \longrightarrow 1_{\mathcal{R}(\sigma)}$ is a natural transformation.*

Proof. For every $A, B \in \mathcal{R}(\sigma)$ we want to show that:

$$\begin{tikzcd} \mathbb{T}_k A \arrow[r, "\epsilon_A"] \arrow[d, "\mathbb{T}_k f"'] & A \arrow[d, "f"] \\ \mathbb{T}_k B \arrow[r, "\epsilon_B"'] & B \end{tikzcd} \tag{4.4}$$

$$
\begin{aligned}
f \circ \epsilon_A([(p_1, a_1), \ldots, (p_n, a_n)]) &= f(a_n) && \text{by defn (4.2.3) of } \epsilon_A \\
&= \epsilon_B([(p_1, f(a_1)), \ldots, (p_n, f(a_n))]) && \text{by defn (4.2.3) of } \epsilon_B \\
&= \epsilon_B \circ \mathbb{T}_k f([(p_1, a_1), \ldots, (p_n, a_n)]) && \text{by defn (4.2.4) of } \mathbb{T}_k f
\end{aligned}
$$

Hence, the above diagram (4.4) commutes. □

Definition 4.2.5. Suppose $s \in \mathbb{T}_k A$, then $s = [(p_1, a_1), \ldots, (p_n, a_n)]$ for some $n \in \omega$ and for every $i = 1, \ldots, n$, $p_i \in \{1, \ldots, k\}$, $a_i \in A$. Let $s_i = [(p_1, a_1), \ldots, (p_i, a_i)] \in \mathbb{T}_k A$. Define, for every σ-structure A, $\delta_A : \mathbb{T}_k A \longrightarrow \mathbb{T}_k \mathbb{T}_k A$ by $s \mapsto [(p_1, s_1), \ldots, (p_n, s_n)]$.

Proposition 4.2.3. *$\delta : \mathbb{T}_k \longrightarrow \mathbb{T}_k \mathbb{T}_k$ is a natural transformation.*

Proof. For every $A, B \in \mathcal{R}(\sigma)$ we want to show that:

$$
\begin{tikzcd}
\mathbb{T}_k A \arrow[r, "\delta_A"] \arrow[d, "\mathbb{T}_k f"'] & \mathbb{T}_k \mathbb{T}_k A \arrow[d, "\mathbb{T}_k \mathbb{T}_k f"] \\
\mathbb{T}_k B \arrow[r, "\delta_B"'] & \mathbb{T}_k \mathbb{T}_k B
\end{tikzcd}
\tag{4.5}
$$

$$
\begin{aligned}
\mathbb{T}_k \mathbb{T}_k f \circ \delta_A([(p_1, a_1), \ldots, (p_n, a_n)]) &= \mathbb{T}_k \mathbb{T}_k f([(p_1, s_1), \ldots, (p_n, s_n)]) && \text{by defn (4.2.5) of } \delta_A \\
&= [(p_1, \mathbb{T}_k f(s_1)), \ldots, (p_n, \mathbb{T}_k f(s_n))] && \text{by defn (4.2.4) of } \mathbb{T}_k \mathbb{T}_k f \\
&= [(p_1, [(p_1, f(a_1))]), \ldots, (p_n, [(p_1, f(a_1)), \ldots, (p_n, f(a_n))])] && \text{by defn (4.2.4) of } \mathbb{T}_k f \\
&= \delta_B([(p_1, f(a_1)), \ldots, (p_n, f(a_n))]) && \text{by defn (4.2.5) of } \delta_B \\
&= \delta_B \circ \mathbb{T}_k f([(p_1, a_1), \ldots, (p_n, a_n)]) && \text{by defn (4.2.4) of } \mathbb{T}_k f
\end{aligned}
$$

Hence, the above diagram (4.5) commutes. □

Theorem 4.2.4. *The triple $\langle \mathbb{T}_k, \delta, \epsilon \rangle$ is a comonad.*

Proof. By proposition (4.2.3) and (4.2.2), δ and ϵ are natural transformation. Hence, δ and ϵ are indeed the comultiplication and counit of $\mathbb{T}_k$. The associative and identity laws remain to be shown.
For associativity, for every $A \in \mathcal{R}(\sigma)$, the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{T}_k A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & \mathbb{T}_k \mathbb{T}_k A \arrow[d, "\delta_{\mathbb{T}_k A}"] \\
\mathbb{T}_k \mathbb{T}_k A \arrow[r, "\mathbb{T}_k \delta_A"'] & \mathbb{T}_k \mathbb{T}_k \mathbb{T}_k A
\end{tikzcd}
\tag{4.6}
$$

$$
\begin{aligned}
\delta_{\mathbb{T}_k A} \circ \delta_A([(p_1, a_1), \ldots, (p_n, a_n)]) &= \delta_{\mathbb{T}_k A}([(p_1, s_1), \ldots, (p_n, s_n)]) && \text{by defn (4.2.5) of } \delta_A \\
&= [(p_1, [(p_1, s_1)]), \ldots, (p_n, [(p_1, s_1), \ldots, (p_n, s_n)])] && \text{by defn (4.2.5) of } \delta_{\mathbb{T}_k A} \\
&= [(p_1, \delta_A(s_1)), \ldots, (p_n, \delta_A(s_n))] && \text{by defn (4.2.5) of } \delta_A \\
&= \mathbb{T}_k \delta_A([(p_1, s_1), \ldots, (p_n, s_n)]) && \text{by defn (4.2.4) of } \mathbb{T}_k \delta_A \\
&= \mathbb{T}_k \delta_A \circ \delta_A([(p_1, a_1), \ldots, (p_n, a_n)]) && \text{by defn (4.2.5) of } \delta_A
\end{aligned}
$$

For identity, for every $A \in \mathcal{R}(\sigma)$, the following diagram commutes:

$$\begin{tikzcd} \mathbb{T}_k A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, equal] & \mathbb{T}_k \mathbb{T}_k A \arrow[d, "\mathbb{T}_k \epsilon_A"] \\ \mathbb{T}_k \mathbb{T}_k A \arrow[r, "\epsilon_{\mathbb{T}_k A}"'] & \mathbb{T}_k A \end{tikzcd} \tag{4.7}$$

$$\begin{aligned} \mathbb{T}_k \epsilon_A \circ \delta_A([(p_1, a_1), \ldots, (p_n, a_n)]) &= \mathbb{T}_k \epsilon_A([(p_1, s_1), \ldots, (p_n, s_n)]) && \text{by defn (4.2.5) of } \delta_A \\ &= [(p_1, \epsilon_A(s_1)), \ldots, (p_n, \epsilon_A(s_n))] && \text{by defn (4.2.4) of } \mathbb{T}_k \epsilon_A \\ &= [(p_1, a_1), \ldots, (p_n, a_n)] && \text{by defn (4.2.3) of } \epsilon_A \\ \\ &= s_n && \text{by defn (4.2.5) of } s_n \\ &= \epsilon_{\mathbb{T}_k A}([(p_1, s_1), \ldots, (p_n, s_n)]) && \text{by defn (4.2.3) of } \epsilon_{\mathbb{T}_k A} \\ &= \epsilon_{\mathbb{T}_k A} \circ \delta_A([(p_1, a_1), \ldots, (p_n, a_n)]) && \text{by defn (4.2.5) of } \delta_A \end{aligned}$$

By definition, $\mathbb{T}_k$ is a comonad. □

For every $l, k \in \omega$ such that $l \leq k$ and σ-structure A, there exists an inclusion $i_A^{l,k} : \mathbb{T}_l A \longrightarrow \mathbb{T}_k A$.

Proposition 4.2.5. *The inclusion maps form a natural transformation $i^{l,k} : \mathbb{T}_l \longrightarrow \mathbb{T}_k$. Further, each map preserves the counit and comultiplication (i.e. each map is a morphism of comonads).*

Proof.

$$\begin{tikzcd} \mathbb{T}_l A \arrow[r, "i_A^{l,k}"] \arrow[d, "\mathbb{T}_l f"'] & \mathbb{T}_k A \arrow[d, "\mathbb{T}_k f"] \\ \mathbb{T}_l B \arrow[r, "i_B^{l,k}"'] & \mathbb{T}_k B \end{tikzcd} \tag{4.8}$$

$$\begin{aligned} \mathbb{T}_k f \circ i_A^{l,k}([(p_1, a_1), \ldots, (p_n, a_n)]) &= \mathbb{T}_k f([(p_1, a_1), \ldots, (p_n, a_n)]) && p_i \in \{1, \ldots, l\} \subseteq \{1, \ldots, k\} \\ &= [(p_1, f(a_1)), \ldots, (p_n, f(a_n))] && \text{by defn (4.2.4) of } \mathbb{T}_k f \\ &= i_B^{l,k}([(p_1, f(a_1)), \ldots, (p_n, f(a_n))]) && p_i \in \{1, \ldots, l\} \subseteq \{1, \ldots, k\} \\ &= i_B^{l,k} \circ \mathbb{T}_l f([(p_1, a_1), \ldots, (p_n, a_n)]) && \text{by defn (4.2.4) of } \mathbb{T}_k f \end{aligned}$$

□

The grading given by these inclusion maps seem to suggest that there is a colimit object capturing the information of $\mathbb{T}_k A$ for every $k \in \omega$. This is indeed the case. Consider the structure $\mathbb{T}_\omega A$ with domain $|\mathbb{T}_\omega A| = (\omega \times A)^\omega$. The structure on $\mathbb{T}_\omega A$ is similar to the structure given to $\mathbb{T}_k A$.

Proposition 4.2.6. *Let ω be considered as a poset category under the usual order. The object $\mathbb{T}_\omega A$ is the ω-colimit of the family $\{\mathbb{T}_k A\}_{k \in \omega}$ with the above inclusion maps.*

Proof. For every $k \in \omega$, define $i_A^k : \mathbb{T}_k A \to \mathbb{T}_\omega A$ as the inclusion (i.e. $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto [(p_1, a_1), \ldots, (p_n, a_n)]$). Clearly, the following diagram commutes for all $l, k \in \omega$ with $l \leq k$

$$\begin{tikzcd} \mathbb{T}_l A \arrow[rr, "i_A^{l,k}"] \arrow[dr, "i_A^l"'] & & \mathbb{T}_k A \arrow[dl, "i_A^k"] \\ & \mathbb{T}_\omega A & \end{tikzcd} \tag{4.9}$$

Suppose that there exists a σ-structure B and for every $l, k \in \omega$ with $l \leq k$, there exist morphisms $f^l : \mathbb{T}_l A \longrightarrow B$, $f^k : \mathbb{T}_k A \longrightarrow B$ such that $f^l = f^k \circ i^{l,k}$. Consider the morphism $u : \mathbb{T}_\omega A \longrightarrow B$ given by $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto f^j([(p_1, a_1), \ldots, (p_n, a_n)])$ where $j = \max(p_1, \ldots, p_n)$

$$\begin{tikzcd}
\mathbb{T}_l A \arrow[rr, "i_A^{l,k}"] \arrow[dr, "i_A^l"] \arrow[ddr, bend right, "f^l"'] & & \mathbb{T}_k A \arrow[dl, "i_A^k"'] \arrow[ddl, bend left, "f^k"] \\
& \mathbb{T}_\omega A \arrow[d, dotted, "u"] & \\
& B &
\end{tikzcd} \tag{4.10}$$

Moreover, given the conditions on f^j for all $j \in \omega$, u is unique. Namely, suppose there exists a morphism $u' : \mathbb{T}_\omega A \longrightarrow B$ such that for all $j \in \omega$, $f^j = u' \circ i_A^j$. Suppose $s = [(p_1, a_1), \ldots, (p_n, a_n)] \in \mathbb{T}_\omega A$ and $k = \max(p_1, \ldots, p_n)$, then for all $j \geq k, s \in \mathbb{T}_j A$.

$$\begin{aligned}
u(s) &= f^k(s) && \text{by defn of } u \\
&= f^j \circ i_A^{k,j}(s) && \text{by (6.4)} \\
&= u' \circ i_A^j \circ i_A^{k,j}(s) && \text{by hypothesis on } u' \\
&= u' \circ i_A^k(s) && \text{by (6.3)} \\
&= u'(s) && \text{by defn of inclusion}
\end{aligned}$$

Since $u(s) = u'(s)$ for all $s \in \mathbb{T}_\omega A$, $u = u'$ so u is unique as desired. □

4.3 Positional Form and Equivalences

4.3.1 Equivalence $\exists^+ \mathcal{L}^k_{\infty,\omega}$

4.3.2 Equivalence $\mathcal{L}^k_{\infty,\omega}$

4.3.3 Equivalence $\mathcal{L}^k_{\infty,\omega}(Cnt)$

4.4 Coalgebras and Tree-Width

Chapter 5

Modal Unfolding Comonad

5.1 Introduction

5.2 Comonad laws

5.3 Positional Form and Equivalences

5.3.1 Equivalence $\exists^{+}\mathcal{M}_{\infty,\omega}$

5.3.2 Equivalence $\mathcal{M}_{\infty,\omega}$

5.4 Guarded Unfolding

Chapter 6

Generalizations and Relationships

The aim of this chapter is to generalize some of the arguments used in previous chapters and to explain connections between the different comonads. We also discuss how to the relationships between the different comonads can used to prove results about the relationship between the corresponding logics. In particular, the first section gives an general arrow-theoretic development of (**??**), (**??**), and (**??**) which transfer equivalence in $\exists^+\mathcal{L}$, under the right conditions, to equivalence in $\mathcal{L}$.

6.1 Arrow-Theoretic from $\exists^+\mathcal{L}$ to $\mathcal{L}$

6.2 Relationship between $\mathbb{T}_k$ and $\mathbb{E}_k$

6.2.1 $\mathbb{T}_\omega$ and $\mathbb{E}_\omega$

Since full first-order logic is the set of formulas that have rank $< \omega$ and contain countably infinite number of variables, the colimit comoands $\mathbb{T}_\omega$ and $\mathbb{E}_\omega$ both capture equivalence in $\exists^+\mathcal{L}^\omega_{\omega,\omega}$. Hence, they would be "equivalent" in some sense. The following proposition and corollary captures this intuition.

Proposition 6.2.1. *$\mathbb{E}_\omega$ is a retract of $\mathbb{T}_\omega$*

Proof. Consider the natural transformation ρ with components $\rho_A : \mathbb{T}_\omega A \longrightarrow \mathbb{E}_\omega A$ given by $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto [a_1, \ldots, a_n]$. The map ρ_A is indeed a σ-morphism, since the conditions for $R^{\mathbb{E}_k A}$ to hold are nearly the same (just forget the pebbles) as the first and third condition for $R^{\mathbb{T}_k A}$ to hold. uppose A,B are σ-structures and $f : A \longrightarrow B$ is a morphism, then:the

$$\begin{tikzcd}
\mathbb{T}_\omega A \arrow[r, "\rho_A"] \arrow[d, "\mathbb{T}_\omega f"'] & \mathbb{E}_\omega A \arrow[d, "\mathbb{E}_\omega f"] \\
\mathbb{T}_\omega B \arrow[r, "\rho_B"'] & \mathbb{E}_\omega B
\end{tikzcd} \tag{6.1}$$

$$\begin{aligned}
\mathbb{E}_\omega f \circ \rho_A([(p_1, a_1), \ldots, (p_n, a_n)]) &= \mathbb{E}_\omega f([a_1, \ldots, a_n]) \\
&= [f(a_1), \ldots, f(a_n)] \\
&= \rho_B([(p_1, f(a_1)), \ldots, (p_n, f(a_n))]) \\
&= \rho_B \circ \mathbb{T}_\omega f([(p_1, a_1), \ldots, (p_n, a_n)])
\end{aligned}$$

Hence, $\rho : \mathbb{T}_\omega \longrightarrow \mathbb{E}_\omega$ is indeed a natural transformation. Consider the natural transformation i with components $i_A : \mathbb{E}_\omega A \longrightarrow \mathbb{T}_\omega A$ given by $[a_1, \ldots, a_n] \mapsto [(1, a_1), \ldots, (n, a_n)]$. Suppose $R \in \sigma$ is a m-ary relation and $s_1, \ldots, s_m \in \mathbb{E}_\omega A$: Hence, i_A is indeed a σ-homomorphism.

$$\begin{tikzcd}
\mathbb{E}_\omega A \arrow[r, "i_A"] \arrow[d, "\mathbb{E}_\omega f"'] & \mathbb{T}_\omega A \arrow[d, "\mathbb{T}_\omega f"] \\
\mathbb{E}_\omega B \arrow[r, "i_B"'] & \mathbb{T}_\omega B
\end{tikzcd} \tag{6.2}$$

$$\begin{aligned}
\mathbb{T}_\omega f \circ i_A([a_1, \ldots, a_n]) &= \mathbb{T}_\omega f([(1, a_1), \ldots, (n, a_n)]) \\
&= [(1, f(a_1)), \ldots, (n, f(a_n))] \\
&= i_B([f(a_1), \ldots, f(a_n)]) \\
&= i_B \circ \mathbb{E}_\omega f([a_1, \ldots, a_n])
\end{aligned}$$

Hence, $i : \mathbb{E}_\omega \longrightarrow \mathbb{T}_\omega$ is indeed a natural transformation. Suppose A is a σ-structure and $[a_1, \ldots, a_n] \in \mathbb{E}_\omega A$, then:

$$\rho_A \circ i_A([a_1, \ldots, a_n]) = \rho_A([(1, a_1), \ldots, (n, a_n)]) = [a_1, \ldots, a_n]$$

Since $[a_1, \ldots, a_n] \in \mathbb{E}_\omega A$ was arbitrary, $\rho_A \circ i_A = \mathrm{id}_{\mathbb{E}_\omega A}$. By A being arbitrary and ρ,i natural transformations, $\rho \circ i = \mathrm{id}_{\mathbb{E}_\omega}$. Therefore, there exists morphisms (in the endofunctor category on $\mathcal{R}(\sigma)$) $\rho : \mathbb{T}_\omega \to \mathbb{E}_\omega$ and $i : \mathbb{E}_\omega \to \mathbb{T}_\omega$ such that $\rho \circ i = \mathrm{id}_{\mathbb{E}_\omega}$. By definition, $\mathbb{E}_\omega$ is a retract of $\mathbb{T}_\omega$. □

Corollary 6.2.1.1. *$Hom_{\mathcal{R}(\sigma)}(\mathbb{T}_\omega A, B) \cong Hom_{\mathcal{R}(\sigma)}(\mathbb{E}_\omega A, B)$.*

Proof. For $\mathrm{Hom}_{\mathcal{R}(\sigma)}(\mathbb{T}_\omega A, \_) \to \mathrm{Hom}_{\mathcal{R}(\sigma)}(\mathbb{E}_\omega A, \_)$ precompose with ρ. Similarly, for the opposite direction, precompose with i. □

6.2.2 Grading by both $\mathbb{O}_{k,n}$

The family of comonads $\{\mathbb{T}_k\}$ is graded by the number of variables in the formulas under consideration, so this family 'internalizes' equivalence in the fragments $\mathcal{L}^k_{\infty,\omega}$. On the other hand, $\mathbb{E}_n$ is graded by the quantifier rank, so this family 'internalizes' equivalence in the fragments $\mathcal{L}^\omega_{\omega,n}$. A natural question is to consider fragments restricted both by number of variables and rank, i.e. $\mathcal{L}^k_{\omega,n}$ for $n, k \in \omega$. This comonad will be denoted $\mathbb{O}_{k,n}$. Intuitively, $\mathbb{O}_{k,n}A$ represents the structure of Spoiler plays for k-pebble games on a σ-structure A played for at most n rounds. Let the domain of the structure be $|\mathbb{O}_{k,n}A| = \{s \in \mathbb{T}_k A : |s| = n\}$. The structure on $\mathbb{O}_{k,n}A$ is induced from $\mathbb{T}_k A$. Namely, for $R \in \sigma$ an m-ary relation, $R^{\mathbb{O}_{k,n}A} = R^{\mathbb{T}_k A} \cap |\mathbb{O}_{k,n}A|^m$.

Fixing the number of rounds n, we have a directed system of inclusions maps. Namely, for A a σ-structure and $l, k \in \omega$ with $l \leq k$, there is a canonical inclusion morphism $i^{l,k}_A : \mathbb{O}_{l,n}A \longrightarrow \mathbb{O}_{k,n}A$.

Proposition 6.2.2. *$\mathbb{E}_n A$ is the ω-colimit of the system $\{\mathbb{O}_{k,n}A\}_{k\in\omega}$.*

Fixing the number of pebbles k, we have a directed system of inclusion maps. Namely, for A a σ-structure and $m, n \in \omega$, withe $m \leq n$, there is a canonical inclusion morphism $i^A_{m,n} : \mathbb{O}_{k,m}A \longrightarrow \mathbb{O}_{k,n}A$.

Proposition 6.2.3. *$\mathbb{T}_k A$ is the ω-colimit of the system $\{\mathbb{O}_{k,n}A\}_{n\in\omega}$*

Proof. For every $k \in \omega$, define $i^A_n : \mathbb{O}_{k,n}A \to \mathbb{T}_k A$ as the inclusion (i.e. $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto [(p_1, a_1), \ldots, (p_n, a_n)]$). Clearly, the following diagram commutes for all $m, n \in \omega$ with $m \leq n$

$$\begin{tikzcd}
\mathbb{O}_{k,m}A \arrow[rr, "i^A_{m,n}"] \arrow[dr, "i^A_n"'] & & \mathbb{O}_{k,n}A \arrow[dl, "i^A_m"] \\
& \mathbb{T}_k A &
\end{tikzcd} \tag{6.3}$$

Suppose that there exists a σ-structure B and for every $m, n \in \omega$ with $m \leq n$, there exist morphisms $f_m : \mathbb{O}_{k,m}A \longrightarrow B$, $f_n : \mathbb{O}_{k,n}A \longrightarrow B$ such that $f_m = f_n \circ i^A_{m,n}$. Consider the morphism $u : \mathbb{T}_k A \longrightarrow B$ given by $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto f_n([(p_1, a_1), \ldots, (p_n, a_n)])$

$$
\begin{tikzcd}
\mathbb{O}_{k,m}A \arrow[rr, "i^A_{m,n}"] \arrow[dr, "i^A_m"] \arrow[ddr, bend right, "f_m"'] & & \mathbb{O}_{k,n}A \arrow[dl, "i^A_n"'] \arrow[ddl, bend left, "f_n"] \\
& \mathbb{T}_k A \arrow[d, dotted, "u"] & \\
& B &
\end{tikzcd}
\tag{6.4}
$$

Moreover, given the conditions on f_n for all $n \in \omega$, u is unique. Namely, suppose there exists a morphism $u' : \mathbb{T}_k A \longrightarrow B$ such that for all $n \in \omega$, $f_n = u' \circ i^A_n$. Suppose $s = [(p_1, a_1), \ldots, (p_n, a_n)] \in \mathbb{O}_{n,k}A$ then for all $j \geq n, s \in \mathbb{O}_{k,i}A$.

$$
\begin{aligned}
u(s) &= f_n(s) && \text{by defn of } u \\
&= f_j \circ i^A_{n,j}(s) && \text{by (6.4)} \\
&= u' \circ i^A_j \circ i^A_{n,j}(s) && \text{by hypothesis on } u' \\
&= u' \circ i^A_n(s) && \text{by (6.3)} \\
&= u'(s) && \text{by defn of inclusion}
\end{aligned}
$$

Since $u(s) = u'(s)$ for all $s \in \mathbb{T}_k A$, $u = u'$ so u is unique as desired. ☐

6.3 New Proofs for Common Results

A natural question when given two different logics $\mathcal{L}, \mathcal{L}'$ is whether $\mathcal{L}$ is more, less or equally expressive as $\mathcal{L}'$. The syntactic approach to show that $\mathcal{L}'$ is at least as expressive as $\mathcal{L}$ (denoted, $\mathcal{L} \subseteq \mathcal{L}'$) is to show that every sentence in $\mathcal{L}$ can be translated to a sentence $\mathcal{L}'$. The semantic approach is to show that a structure that can interpret sentences in $\mathcal{L}'$ can intrepret sentences in $\mathcal{L}$. Games, and more importantly the game comonads we developed, give a structural approach. The following proposition justifies this structural approach.

Proposition 6.3.1. *$\mathcal{L} \subseteq \mathcal{L}'$ if and only if for all σ and two σ-structures A, B, $A \equiv^{\mathcal{L}'} B \Rightarrow A \equiv^{\mathcal{L}} B$.*

Proof. $\Rightarrow$ Suppose $A \equiv^{\mathcal{L}'} B$, then for all sentences $\phi \in \mathcal{L}'$, $A \vDash \phi \Leftrightarrow B \vDash \phi$. In particular, for all sentences $\psi \in \mathcal{L}$ (since $\mathcal{L} \subseteq \mathcal{L}'$, up to translation, by hypothesis), $A \vDash \psi \Leftrightarrow B \vDash \psi$. Hence, $A \equiv^{\mathcal{L}} B$.
$\Leftarrow$ By contrapositive. Suppose that there exists σ-structures A, B such that $A \equiv^{\mathcal{L}'} B$, but $A \not\equiv^{\mathcal{L}} B$. Since $A \not\equiv^{\mathcal{L}} B$, there exists some sentence $\phi \in \mathcal{L}$ such that $A \vDash \phi$ and $B \nvDash \phi$. Clearly, $\phi \notin \mathcal{L}'$ since $A \equiv^{\mathcal{L}'} B$. Therefore, $\mathcal{L} \nsubseteq \mathcal{L}'$. ☐

Since back-and-forth games characterize $\equiv^{\mathcal{L}}$, these games give a structural approach to proving $\mathcal{L} \subseteq \mathcal{L}'$ that does not make reference to cumbersome syntactic details or coordinate-heavy intrepretations. However, even with this game approach the details could get unwieldy. Namely, to show $\mathcal{L} \subseteq \mathcal{L}'$, i.e. $A \equiv^{\mathcal{L}'} B \Rightarrow A \equiv^{\mathcal{L}} B$, we would have to construct a winning strategy for Duplicator in the $\mathcal{L}$ back-and-forth game from a winning strategy in the $\mathcal{L}'$ back-and-forth game. The advantage of having internalized these strategies into the category $\mathcal{R}(\sigma)$ is that producing a Duplicator strategy in one game from a Duplicator strategy in another game just amounts to coming up with the correct morphisms. The following propositions illustrate this technique well.

A well-known result in finite model theory is that the k-variable fragment contains the k-rank fragment. Syntactically, this result is quite easy to see on sentences in prenex normal form. Namely, a prenex normal form sentence with quantifier rank $\leq k$ has at most k many bound variables. For more general sentences, the proof can get a bit tricky. However, the proof using comonads makes this result easy.

Proposition 6.3.2. *The following results hold for all $k \in \omega$:*

(1) $\exists^+\mathcal{L}_{\omega,k} \subseteq \exists^+\mathcal{L}^k_{\infty,\omega}$

(2) $\mathcal{L}_{\omega,k} \subseteq \mathcal{L}^k_{\infty,\omega}$

(3) $\mathcal{L}_{\omega,k}(\mathsf{Cnt}) \subseteq \mathcal{L}^k_{\infty,\omega}(\mathsf{Cnt})$

Proof. By proposition (6.3.1), it suffices to show that for two σ-structures A, B, $A^+ \equiv^{\mathcal{L}^k_{\infty,\omega}} B \Rightarrow A^+ \equiv^{\mathcal{L}_{\omega,k}} B$. Suppose $A \equiv^{\mathcal{L}^k_{\infty,\omega}} B$, then by (**??**), there exists morphisms $f : \mathbb{T}_k A \longrightarrow B$ and $g : \mathbb{T}_k B \longrightarrow A$. Recall the natural transformation $i : \mathbb{E}_k \longrightarrow \mathbb{T}_k$ used in (**??**). Hence, there exists morphisms $f \circ i_A : \mathbb{E}_k A \longrightarrow B$ and $g \circ i_B : \mathbb{E}_k B \longrightarrow A$. Therefore, by (**??**), $\square$

A result by [**?**] showed that the full modal fragment of infinitary logic is contained within the two-variable fragment of infinitary logic. We prove the same result, using our comonadic formulation.

Proposition 6.3.3. $\mathcal{M}_{\infty,\omega} \subseteq \mathcal{L}^2_{\infty,\omega}$.

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