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Final Report

PID controller design for Water Level Control of two Interconnected Tank

Presented to Dr. W. F. Xie

MECH 473/6021: Control System Design

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Group Members Contribution

Nima Mohammadidizgovin: Responsible for the PID controller design and tuning,mathematical modeling, linearization, and derivation of the transfer functions for both the conical and cubic tank systems. This involved applying Taylor expansion to obtain linearized models and defining the system's operating points.

Joseph Zaarour: Conducted the simulations in SIMULINK based on the derived transfer functions, analyzing the system's response. Implemented the unity feedback controller and obtained the steady-state response for the system.

Ali Mohammad Adeli: Managed the overall report, overseeing revisions and refining the text to ensure clarity and cohesiveness of the project document. Provided editorial contributions and structured the report for final submission.

Table of contents

Important note: To see the results for the controller design, which is the major part of Final report, please refer to page 14.

Table of figures

Abstract

In this study, we modeled and linearized a system consisting of a conical tank connected to a cubic tank. The mathematical models for both tanks were derived and linearized using Taylor expansion around specific operating points. Initial simulations of the system without any controller showed that both tanks naturally converged to their own steady-state water levels. A subsystem was then created in SIMULINK, where a unity feedback controller was implemented with a desired water level of 3 cm. However, the system's steady-state response reached 1.4 cm, lower than the target value. Future work will focus on designing a PID controller to achieve the desired water level, improve performance, and enhance stability for better control of this nonlinear tank system.

Keywords— Level control, Conical Tank, Unity feedback controller, PID controller.

Introduction

Effective control of variables like flow, temperature, pressure, and liquid level is crucial in industrial processes, especially in systems with interconnected tanks. Conical tanks are commonly used in industries such as food processing, chemical manufacturing, and wastewater treatment due to their better drainage and minimal residue buildup. However, their nonlinear behavior, combined with the complexity of interconnected systems, makes designing control strategies more challenging.

This project focuses on simulating the liquid levels in a system with a conical tank connected to a cubic tank. The tanks create a nonlinear system, with the conical tank feeding into the cubic tank. Models for both tanks are developed, representing their volumes and flow behavior. The system's response is analyzed using unity feedback control to evaluate how the tanks stabilize at their steady-state levels.

The study includes simulations of both tanks' steady-state responses and system behavior under feedback control. In future work, a Proportional-Integral-Derivative (PID) controller will be designed to further optimize system performance, including reducing overshoot and improving stability. This research provides insights into controlling nonlinear interconnected tank systems, with potential applications in industries requiring precise level control.

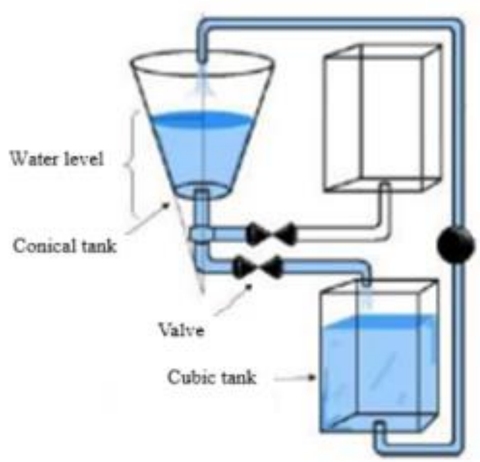

Fig 1: schematic of the system [1]

Modeling of single conic tank

The process considered here is a conical tank system shown in Figure 1 in which the level of the liquid is desired to maintain a constant value. This can be achieved by controlling the input flow rate into the tank. Here F_{in} is the inlet flow and F_{out} is the outlet flow. The objective is to control the level of tank, which can be achieved by controlling the input flow of the conical tank by using valve at inlet. At steady state both the in-flow and out-flow rates remain the same.

Variable	symbol	Unit	Value
In-flow rate (manipulated variable)	F_{in}	cm^3/s	13.3
Total height of tank	H	cm	50
Outer radius	R	cm	18
Valve Coefficient	C	-	4.3

Tab 1: Conic Tank model variables

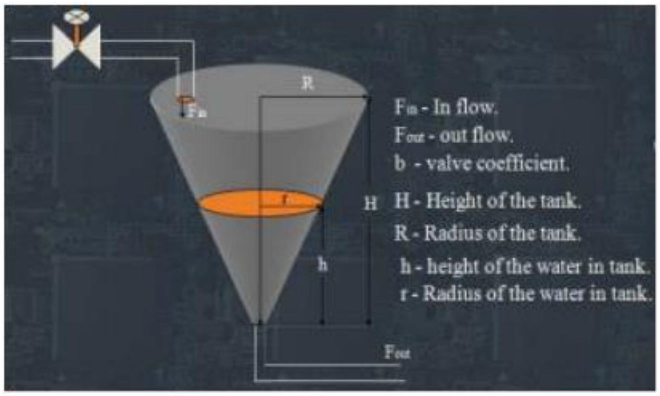

Fig 2: schematic of the conic tank [3]

A. Mathematical Model:

The area of the conical tank is given by

$$A = \pi r^2 \tag{1}$$

From the figure 1,

$$\tan\theta = \frac{r}{h} = \frac{R}{H} \tag{2}$$

$$r = R\frac{h}{H} \tag{3}$$

Therefore,

$$A = \frac{\pi R^2 h^2}{H^2} \tag{4}$$

According to law of conservation of mass, Inflow rate - outflow rate = Accumulation in the tank

$$F_i - F_0 = A\frac{dh}{dt} \tag{5}$$

$$F_0 = C\sqrt{h} \tag{6}$$

Where, C is the valve coefficient on solving,

$$F_i = C\sqrt{h} = A\frac{dh}{dt} \tag{7}$$

$$\frac{dh}{dt} = \frac{F_i - C\sqrt{h}}{A} \tag{8}$$

On solving (1.4) in (1.8)

$$\frac{dh}{dt} = \frac{F_i H^2}{\pi (Rh)^2} - \frac{C\sqrt{h}H^2}{\pi (Rh)^2} \tag{9}$$

$$\frac{dh}{dt} = \alpha F_i\, h^{-2} - \beta h^{\left(-\frac{3}{2}\right)} \tag{10}$$

Where,

$$\alpha = \frac{1}{\pi}\left(\frac{H}{R}\right)^2 \tag{11}$$

$$\beta = \alpha C \tag{12}$$

For our model:

$$\alpha = \frac{1}{\pi}\left(\frac{50}{18}\right)^2 = 2.45 \tag{13}$$

$$\beta = 2.45 * 4.3 = 10.535 \tag{14}$$

B. Linearization of the model

Linearization is the process by which we approximate nonlinear systems with linear ones. It is widely used in the study of process dynamics and design of control systems for the following reasons:

The Taylor series is used, for the linearization of the nonlinearity of the conical tank [3]. In the above equation (10), a nonlinear term appears, which can be linearized using the Taylor series expansion.

In conical tank system model has two types of nonlinear $F_i h^{-2}$ a product of two functions, and $\mathbf{h}^{(-3/2)}$. We shall have to linearize each of the functions separately around the steady state (h_s, F_{is}).

The linearization of $f(h, F_1) = \mathrm{F_1}\ \mathrm{h}^{-2}$ proceed as following,

$$f(\mathrm{h},\ \mathrm{F_i}) = f(\mathrm{h_s},\ \mathrm{F_{is}}) + \left(\frac{\partial \mathrm{f}}{\partial\ \mathrm{h}}\right)_{(\mathrm{h_g},\ \mathrm{F_{in}})} (\mathrm{h} - \mathrm{h_s}) + \left(\frac{\partial \mathrm{f}}{\partial\ \mathrm{F_i}}\right)_{(\mathrm{h_g},\ \mathrm{F_{is}})} (\mathrm{F} - \mathrm{F_{is}}) + \text{ higher order system} \qquad (15)$$

Whereupon carrying out the indicated operations now signals

$$f(h, F_i) = f(h_s, F_{is}) - 2F_{is} h_s^{-3}(h - h_s) + h_s^{-2}(F_i - F_{is}) \qquad (16)$$

We ignore the higher order terms in equation (1.13). Now, the second term for linearization is $\mathrm{h}^{(-3/2)}$,

$$\mathrm{h}^{(-3/2)} = (h_s)^{\left(-\frac{3}{2}\right)} - \frac{3}{2}(h_s)^{\left(-\frac{5}{2}\right)}(h - h_s) \qquad (17)$$

Terms of equation (10),

$$\frac{d\ \mathrm{h}}{\mathrm{dt}} = \alpha\left(\mathrm{f}(\mathrm{h_s},\ \mathrm{F_{is}}) - 2\ \mathrm{F_{is}}\ \mathrm{h_s^{-3}}(\mathrm{h} - \mathrm{h_s}) + \mathrm{h_s^{-2}}(\mathrm{F_i} - \mathrm{F_{is}})\right) - \beta\left((\mathrm{h_s})^{\left(-\frac{3}{2}\right)} \frac{3}{2}(\mathrm{h_s})^{\left(-\frac{5}{2}\right)}(\mathrm{h} - \mathrm{h_s})\right) \qquad (18)$$

At steady state,

$$\mathrm{F_i} = \mathrm{F_0} \qquad (19)$$

$$\frac{\mathrm{dh_s}}{\mathrm{dt}} = \alpha \mathrm{F_{is}}\ \mathrm{h_s^{-2}} - \beta(\mathrm{h_s})^{\left(-\frac{3}{2}\right)} = 0 \qquad (20)$$

$$\frac{\mathrm{d}(\mathrm{h} - \mathrm{h_s})}{\mathrm{dt}} = -2\alpha\ \mathrm{F_{is}}\ \mathrm{h_s^{-3}}(\mathrm{h} - \mathrm{h_s}) + \alpha \mathrm{h_s^{-2}}(\mathrm{F_i} - \mathrm{F_{is}}) + \frac{3}{2}\beta(\mathrm{h_s})^{\left(-\frac{5}{2}\right)}(\mathrm{h} - \mathrm{h_s}) \qquad (21)$$

Introduce a deviation variable $\mathrm{y} = (\mathrm{h} - \mathrm{h_s})$ and $\mathrm{u} = (\mathrm{F_i} - \mathrm{F_{is}})$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = -2\alpha\ \mathrm{F_{is}}\ \mathrm{h_s^{-3}}\mathrm{y} + \alpha \mathrm{h_s^{-2}}\mathrm{u} + \frac{3}{2}\beta(\mathrm{h_s})^{\left(-\frac{5}{2}\right)}\mathrm{y} \qquad (22)$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = -\left(\frac{1}{2}\right)\beta(\mathrm{h_s})^{\left(-\frac{5}{2}\right)}\mathrm{y} + \alpha \mathrm{h_s^{-2}}\mathrm{u} \qquad (23)$$

$$\left(\frac{2}{\beta}\right)(\mathrm{h_s})^{\left(\frac{5}{2}\right)}\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right) = -\mathrm{y} + \alpha \mathrm{h_s^{-2}}\mathrm{u} \qquad (24)$$

$$\tau \frac{\mathrm{dy}}{\mathrm{dt}} + \mathrm{y} = \left(\frac{2\alpha}{\beta}\right)(\mathrm{h_s})^{\frac{1}{2}}\mathrm{u} \qquad (25)$$

$$\tau \frac{\mathrm{dy}}{\mathrm{dt}} + \mathrm{y} = \mathrm{Ku} \qquad (26)$$

Taking the Laplace transform:

$$\frac{\mathrm{y}(\mathrm{s})}{\mathrm{u}(\mathrm{s})} = \frac{\mathrm{K}}{\tau\ \mathrm{s} + 1} = \frac{\mathrm{h}(\mathrm{s})}{\mathrm{F_i}(\mathrm{s})} \qquad (27)$$

Where,

$$K = \left(\frac{2\alpha}{\beta}\right)(h_s)^{\frac{1}{2}} \quad \text{(Steady state gain)} \tag{28}$$

$$\tau = (2/\beta)(h_s)^{(5/2)} \quad \text{(Time constant)} \tag{29}$$

At steady state $F_i = F_0$. so $F_i = c\sqrt{h}$

$$13.3 = 4.3\sqrt{h_s} \rightarrow h_s = \left(\frac{13.3}{4.3}\right)^2 = 9.56$$

C. Transfer Function of the model

So, for our model we have:

$$K = 2 \times \frac{2.45}{10.535} \times 9.56^{0.5} = 1.44$$

$$\tau = \frac{2}{10.535} \times 9.56^{2.5} = 54.2$$

So, our model transfer function will be:

$$G = \frac{y(s)}{u(s)} = \frac{K}{\tau s + 1} = \frac{1.44}{54.2s + 1}$$

Modeling of the Cubic Tank

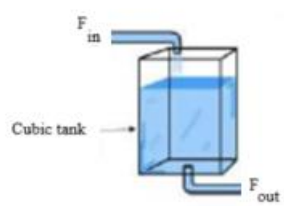

Fig 3: schematic of the cubic tank [1]

Variable	symbol	Unit	Value
Area of the tank	A	cm^2	100
Valve Coefficient	C	-	6.5
Total height of tank	H	cm	20

Tab 2: Cubic Tank Model Variables

A. mathematical model

Now we want to obtain the mathematical model of the cubic tank. F_{in} is the in-flow rate entering the rectangular tank. So:

$$\frac{dh}{dt} = \frac{F_{\text{in}} - c\sqrt{h}}{A} \tag{29}$$

B. Linearization of the model

Now, we must define operating points. First, we Let h_0 be the operating point for h and $F_{i=0}$ be the constant input flow rate at steady state. So:

$$h = h_0 + \Delta h \tag{30}$$

$$F_{\text{in}} = F_{\text{in}0} + \Delta F_{\text{in}} \tag{31}$$

Substituting h and F_{in} into the equation gives

$$\frac{d(h_0 + \Delta h)}{dt} = \frac{(F_{\text{in}0} + \Delta F_{\text{in}}) - c\sqrt{(h_0 + \Delta h)}}{A} \tag{32}$$

Linearize the Square Root Term
To linearize $\sqrt{h}$ around h_0, we use the first-order Taylor expression:

$$\sqrt{h_0 + \Delta h} \approx \sqrt{h_0} + \frac{\Delta h}{2\sqrt{h_0}} \tag{33}$$

Substituting this back into the equation:
And rearranging the Equation simplifies to:

$$\frac{d\Delta h}{dt} = \frac{\Delta F_{\text{in}} - c\left(\frac{\Delta h}{2\sqrt{h}}\right)}{A} \tag{34}$$

At steady state, we have $F_i = c\sqrt{h_0}$, so

$$\frac{d\Delta h}{dt} = \frac{\Delta F_{\text{in}}}{A} - \frac{c}{2A\sqrt{h_0}}\Delta h \tag{35}$$

The final linearized equation can be expressed as:

$$\frac{d\Delta h}{dt} = \frac{1}{A}\Delta F_{\text{in}} - \frac{c}{2A\sqrt{h_0}}\Delta h \tag{36}$$

To obtain the transfer function, we can take the Laplace transform of the linearized equation:

$$s\Delta H(s) = \frac{1}{A}\Delta F_{\text{in}}(s) - \frac{c}{2A\sqrt{h_0}}\Delta H(s) \tag{37}$$

Rearranging to Solve for the Transfer Function:

$$\Delta H(s)\left(s + \frac{c}{2A\sqrt{h_0}}\right) = \frac{1}{A}\Delta F_{in}(s) \tag{39}$$

Thus, the transfer function can be expressed as:

$$G(s) = \frac{\Delta H(s)}{\Delta F_{in}(s)} = \frac{1}{As + \frac{c}{2\sqrt{h_0}}} \tag{40}$$

At steady state:

$$F_{in} = F_{out} = c_1\sqrt{h_{\text{ss\_conic}}} = c_2\sqrt{h_{\text{ss\_cubic}}} \tag{41}$$

So, $4.3\sqrt{9.56} = 6.5\sqrt{h_{\text{ss\_cubic}}} \rightarrow h_{\text{ss\_cubic}} = 4.18$

C. Transfer Function

From (40), The transfer function can be expressed in the standard form:

$$G(s) = \frac{1}{\tau s + 1} \tag{42}$$

According to our model parameters for the cubic tank, we have A=100, c_2=6.5, and h_0=4.18. So:

$$G(s) = \frac{1}{As + \frac{c}{2\sqrt{h_0}}} = \frac{1}{100s + 1.58}$$

Simulation

In this section, we employed SIMULINK to model our system and asses its behavior. The objective was to obtain the steady-state response of the system when we have a step input. The simulation results indicate that both tanks achieved their steady-state heights after almost five minutes, with the conic tank reaching 9.56 cm and the cubic tank stabilizing at 4.18 cm. This outcome demonstrates the system's ability to balance water levels despite the differing geometries of the tanks.

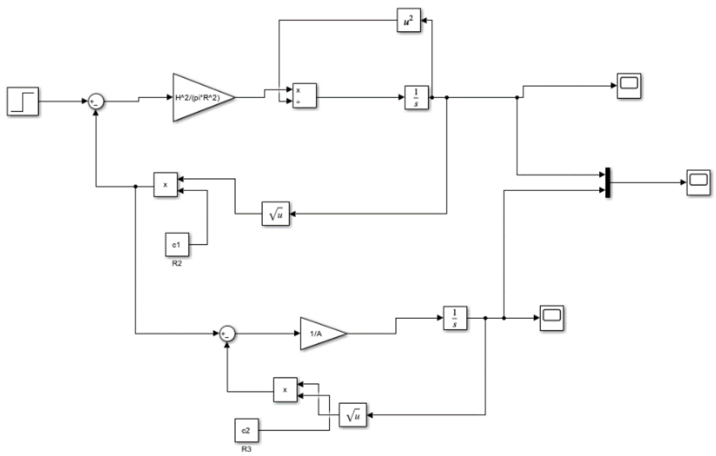

Fig 4: Simulink model of our system.

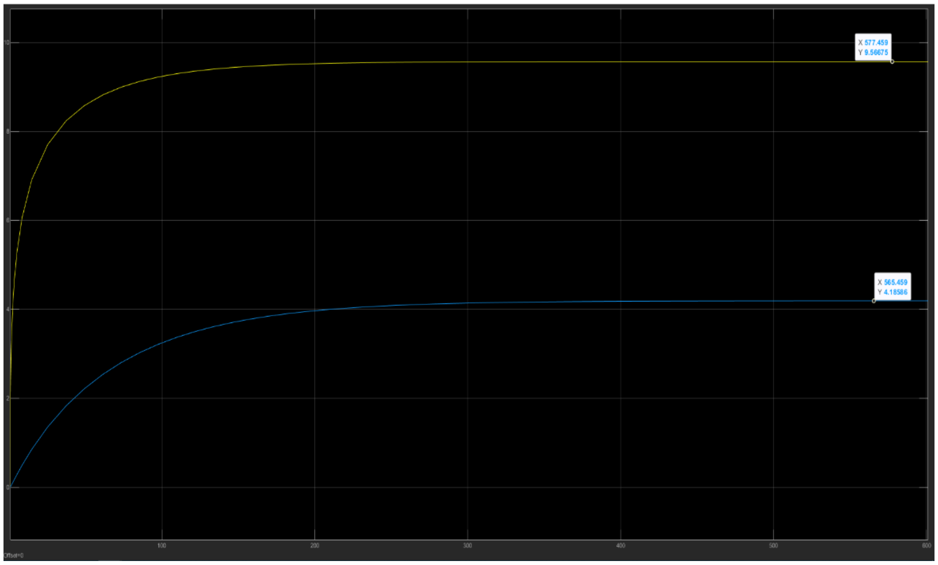

Fig 5: Steady state response of the system.

Now we create a subsystem from the two transfer functions obtained above in SIMULINK and use a unity feed back controller to reach our desired water level of 3cm.As we can see in figure 6 below, our steady state response is almost 1.4, which is much lower than our own desired value.

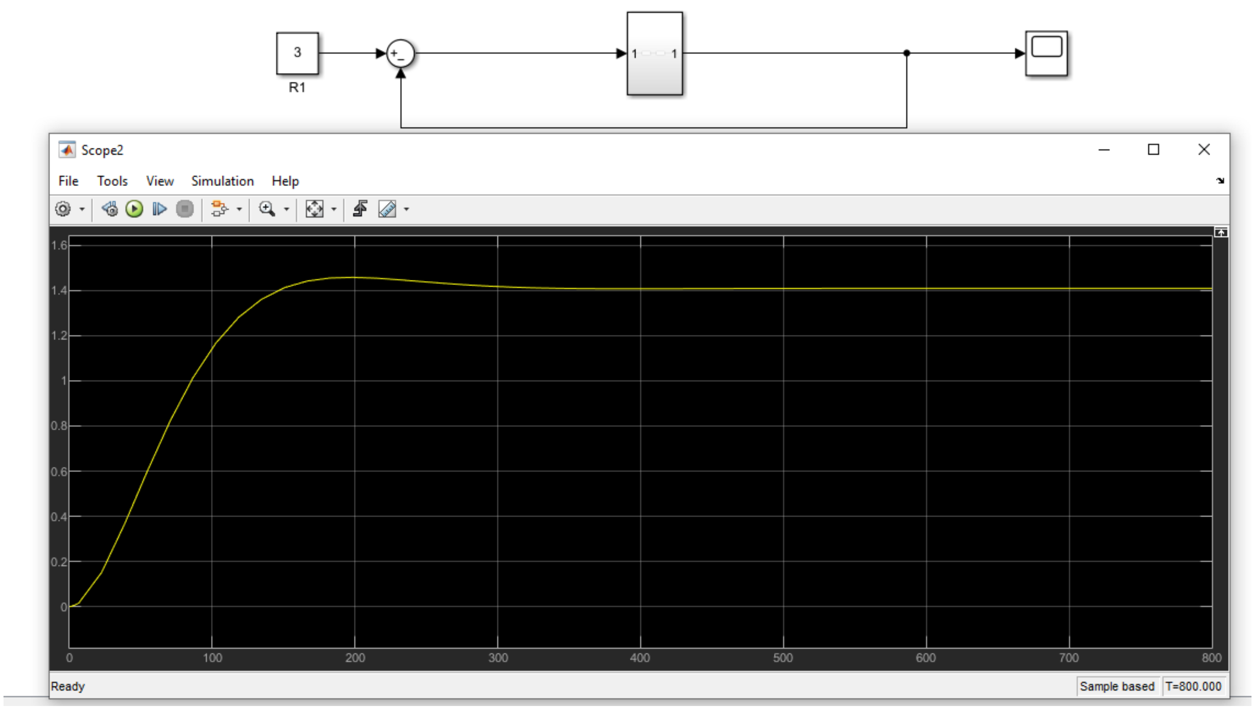

Fig 6: Steady state response of the system using unity feedback controller.

PID controller design

As observed in the initial simulation, the system with a unity feedback controller achieved a steady-state water level of 1.4 cm after approximately 300 seconds (5 minutes), which was significantly lower than the desired value of 3 cm. In order to achieve the target value of 3 cm and also improve the system's response time, a PID controller was introduced to better regulate the water levels.MATLAB Simulink was used to design and simulate the PID controller for the system. The steps followed are as follows:

Modeling the PID Controller: In Simulink, a PID controller block was added to the system. The input to the PID controller was the error between the desired water level and the actual water level, which was measured by a sensor in the simulation model.

Tuning the PID Parameters:

- The **Proportional (K_p)** parameter controls how aggressively the controller reacts to errors. A higher K_p reduces the error more quickly but can lead to overshoot.
- The **Integral (K_i)** parameter helps eliminate any steady-state error by integrating the error over time. While increasing K_i can reduce long-term error, it may introduce oscillations or overshoot if too high.
- The **Derivative (K_d)** parameter helps reduce overshoot and oscillations by adjusting the controller output based on the rate of change of the error. However, if K_d is too large, it can make the system too sensitive, which might cause instability.

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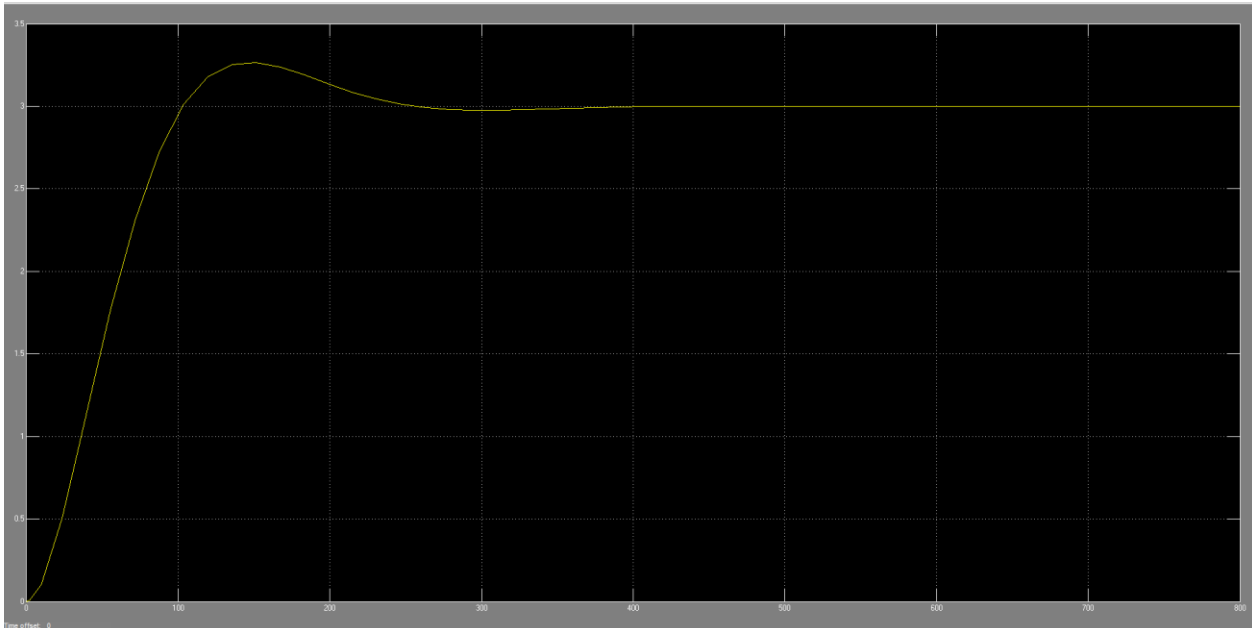


Figure7: system response after default PID parameters tuned by Matlab

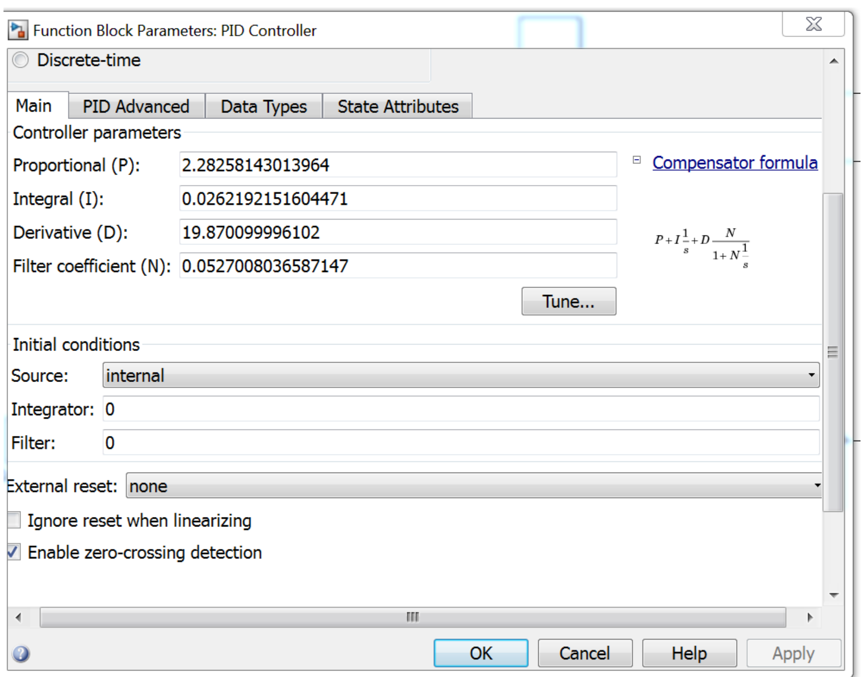


Figure 8: obtained PID parameters

Obtained Parameters are:

K_p=2.28

K_i=0.026

K_d=19.87

As we can see, while the PID controller stabilized the water level at 3 cm, the response time is low (5 minutes). To improve both the response time and reduce overshoot, the **proportional (Kp)** and **derivative (Kd)** gains were significantly increased. This adjustment sped up the system's reaction to changes in error and helped minimize overshoot by damping the system's response. As a result, the system achieved a faster response while maintaining the desired water level at 3 cm,which is shown in figure below.

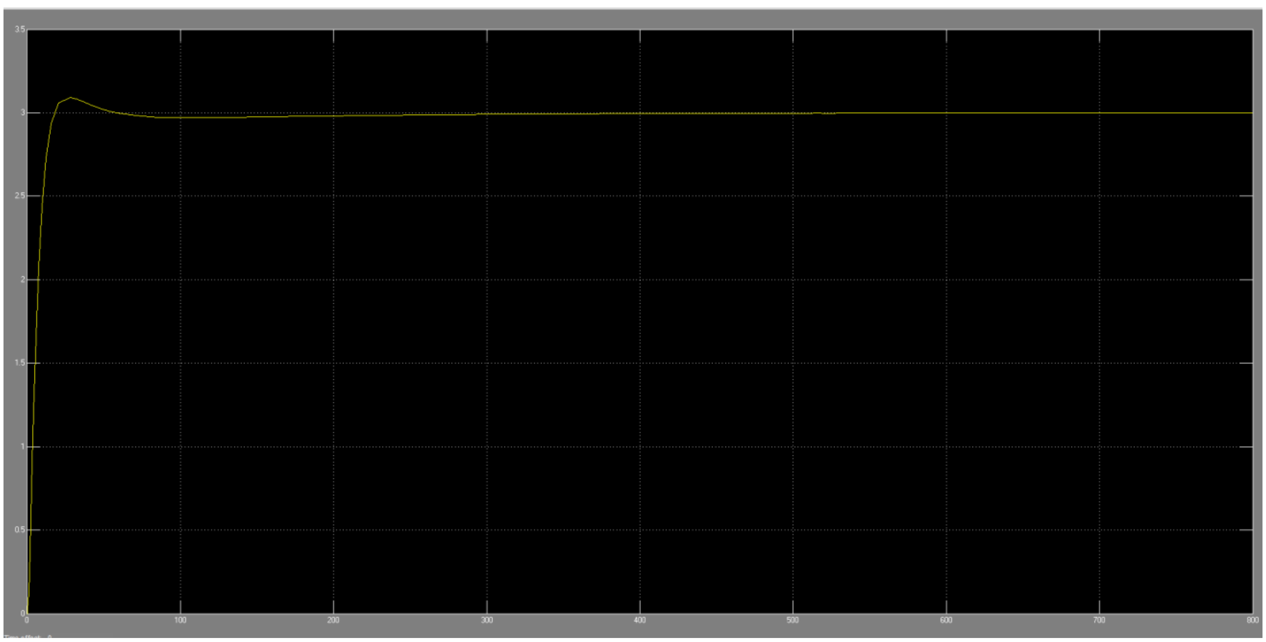


Figure 9:system response after tuning PID parameters ourselves

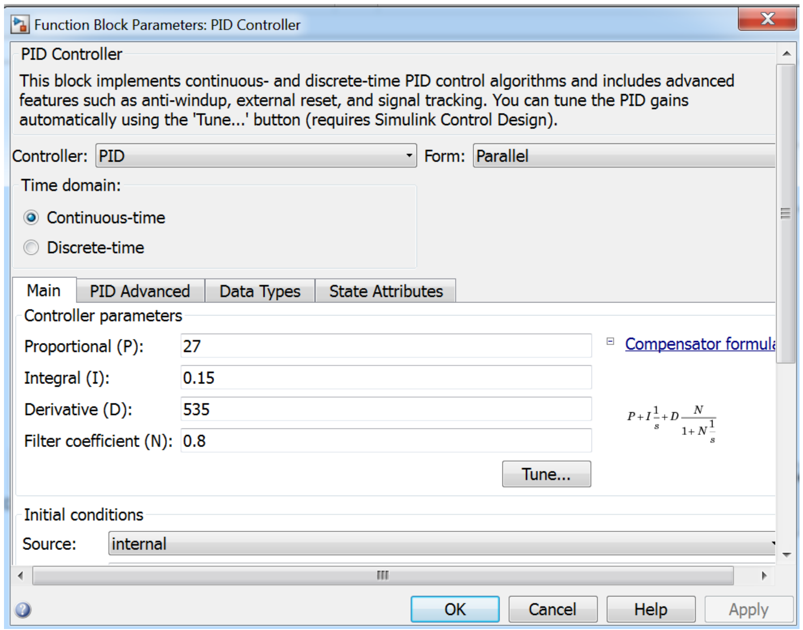


Figure 10: New obtained PID parameters

Obtained Parameters

K_p=27

K_i=0.15

K_d=534

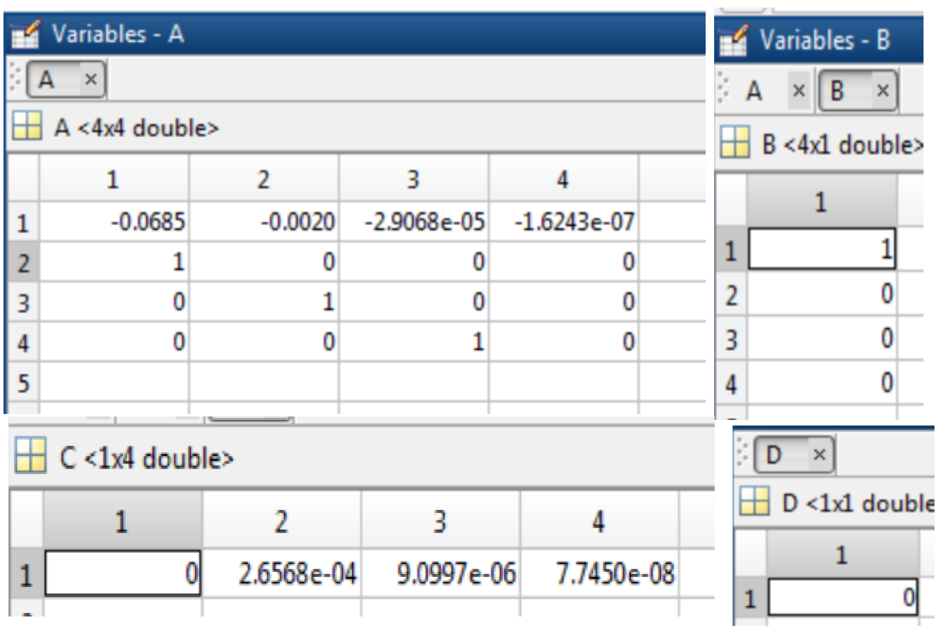
State space of the model

Dynamic systems can be represented as transfer functions or state-space models. While transfer functions describe input-output relationships in the frequency domain, state-space models represent system dynamics using state variables. So we have decided to obtain matrices A, B, C ,and D by Matlab code provided below.

```
clc;clear;close all;

num1=[1.44];

den1=[54.2 1];

num2=[1];
```

```
den2=[100 1.58];

G1=tf(num1,den1);

G2=tf(num2,den2);

G=G1*G2/(1+G1*G2);

[num, den] = tfdata(G, 'v');

[A, B, C, D] = tf2ss(num, den);
```

By running the code above, the A,B,C and D matrices are obtainded, which are shown below.



Variables - A

A <4x4 double>

	1	2	3	4
1	-0.0685	-0.0020	-2.9068e-05	-1.6243e-07
2	1	0	0	0
3	0	1	0	0
4	0	0	1	0
5				

Variables - B

B <4x1 double>

	1
1	1
2	0
3	0
4	0

C <1x4 double>

	1	2	3	4
1	0	2.6568e-04	9.0997e-06	7.7450e-08

D <1x1 double

	1
1	0

Figure 11. Obtained values for A , B , C and D matrices

Conclusion

This study modeled and linearized a conical tank connected to a cubic tank, deriving mathematical models and using Taylor expansion at specific operating points. Simulations verified that the system stabilized at steady-state heights. A unity feedback controller in SIMULINK produced a steady-state response of 1.4 cm, falling short of the desired 3 cm water level. By adding a PID controller, the system parameters were fine-tuned, allowing the system to reach the desired 3 cm water level. The time response improved significantly, reducing from 5 minutes to 60 seconds, while also minimizing overshoot and enhancing stability for better overall performance.

Reference

[1] C. Jauregui, M. D. Mermoud, G. Lefranc, R. Orostica, J. Carlos, and O. Beytia, “Conical Tank Level Control with Fractional PID,” *IEEE Latin America Transactions*, vol. 14, no. 6, pp. 2598–2604, Jun. 2016, doi: https://doi.org/10.1109/tla.2016.7555225.

[2] P. Krishnan, “Dynamic Modeling and Simulation of Single Conical Tank Systems for Control Optimization,” *International Journal of Advanced Research in Engineering and Technology (IJARET)*, vol. 10, no. 4, pp. 292–302, 2019.

[3] T. Pushpaveni, S. Raju, N. Archana, and M. Chandana, “Modeling and Controlling of Conical tank system using adaptive controllers and performance comparison with conventional PID,” *International Journal of Scientific & Engineering Research*, vol. 4, no. 5, 2013.