~~2~~, ~~4~~, ~~6~~, ~~8~~, ~~10~~, ~~12~~, ~~14~~, 16, 18

3.1

2 $\forall n \in \mathbb{Z}.\ n^3 + n + 1$ is an odd integer.

For all integers n, $n^3 + n + 1$ is an odd integer

4 $|n-2| + |n-1| = |n|$

$\exists n \in \mathbb{Z} : |n-2| + |n-1| = |n|$

6 $R(x)$: For every $x \in S$, $R(x)$ is a natural number

$\forall x \in S : R(x)$ is a natural number

$\exists x \in S : R(x)$ is a multiple of 3

8 a. There exists a nonzero rational number r such that the number $1/r$ is irrational

b. For every rational number r such that the number $r^2 \neq 2$

10 $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z} : |a-b| = 1$

b. $\exists a \in \mathbb{Z} : \forall b \in \mathbb{Z} : |a-b| = 1$

12 a. $\neg(\forall M, W) = \exists M, \neg W$

b. $\neg(\forall W, M) = \exists W, \neg M$

c. $\neg(\exists W, M) = \forall W, \neg M$

d. $\neg(\exists M, W) = \forall M, \neg W$

e. $\neg(\forall W, M) = \exists W, \neg M$

14 a. $\forall a \in S, \forall b \in T$: $a+b$ is odd (S is evens T is odds)

b. $\exists a \in S, \exists b \in T : a+b$ is odd

c. There exists an even integer a and an odd integer b such that $a+b$ is not even

16 a. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}^+ : y < x^2$

b. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}^+ : y \geq x^2$ or $\sim P(a,b)$

c. There exists a real number x for which every positive real number y such that $y > x^2$

8) $\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z} : \left| \frac{a+1}{2} - b \right| < 1$

3.2 $\cancel{2}, \cancel{4}, \cancel{6}, \cancel{8}, \cancel{10}, \cancel{12}, \cancel{14}, 16$

2 a. $P(0) = \frac{0(0+1)(2(0)+1)}{6} = 0$ is even, True

$P(3) = \frac{3(3+1)(2(3)+1)}{6} = \frac{12 \times 7}{6} = 14$, is even, true

$P(4) = \frac{4(4+1)(2(4)+1)}{6} = 30$ is even, true

b $P(0) = 0$ and 0 is an even number, True

$P(3) = 14$ and 14 is an even number, True

$P(4) = 30$ and 30 is an even number, True

For each $n \in S$: $P(n)$ $\frac{n(n+1)(2n+1)}{6}$

c $P(n)$: $\frac{n(n+1)(2n+1)}{6}$ is even is true because there exists $n \in S$ such that $P(n)$: $\frac{n(n+1)(2n+1)}{6}$ is even

- By $P(0) = 0$ is even
- By $P(3) = 14$ is even
- By $P(4) = 30$ is even

$\therefore$ There exists $n \in S$ such that $P(n)$: $\frac{n(n+1)(2n+1)}{6}$

4 $x^3 + x = 0$ is given

$\Rightarrow x^3 - x^2 + x = -x^2$

$\Rightarrow -x^2 + x = -x^2 - x^3$

$\Rightarrow x^2 - x = x^2 + x^3$

$x^2 - x \leq 0$ is given

$\therefore x^2 - x \leq 0$

6 $2n^2 + n - 1 = 0 \quad + n^3 < 0$

$2n^2 + 2n - n - 1 = 0$

$2n(n+1) - (n+1) = 0$

$(2n+1)(n+1) = 0 \quad \to \boxed{n^3 < 0}$

$n + 1 = 0$

$n = -1$

8 n can be written $n = 2k+1 \mid k \in \mathbb{Z}$

$3n+10 = 3(2k+1)+10$

$3n+10 = 6k+3+10$

$3n+10 = 6k+13$

$3n+10 = 6k+12+1$

$3n+10 = 2(3k+6)+1$, is odd.integer

Hence proved

10 $n \neq 2$ because 2 is even so $n = 3$

$3n+1 = 3(3)+1$

$3n+1 = 9+1$

$3n+1 = 10$ is even $\therefore$ if n is an odd integer, and $n \in S$ then $3n+1$ is even

12 let a and b be odd to prove $ab+a+b$

- since product of 2 odd integers is odd ab is odd
- Sum of 2 odd integers is even so $a+b$ is even
- Sum of an odd and even integer is odd so

$ab+a+b$ = odd integer

14 r and s must be of the form $r = \frac{p}{q}$ + $s = \frac{a}{b}$

$$r-s = \frac{p}{q} - \frac{a}{b} \Rightarrow r-s = \frac{pb-aq}{qb}$$

let $m = pb-aq$ + $n = qb$

$\Rightarrow r-s = \frac{m}{n}$ where $n \neq 0$

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16 $(a-b)^2 \geq 0$

$a^2+b^2-2ab \geq 0$

$a^2+b^2 \geq 2ab$

$\frac{a^2+b^2}{ab} \geq 2$

$\frac{a^2}{ab} + \frac{b^2}{ab} \geq 2 \Rightarrow \frac{a}{b} + \frac{b}{a} \geq 2$ ✓

3.3

2 $n = 2k \mid k \in \mathbb{Z}$

$\cdot\ 9n-5 = 9(2k)-5$

$= 18k-5$

(e) This shows that $9n-5$ is odd which is not true since $9n-5$ is even. so the supposition that n is even is wrong. Hence n must be odd

4 $n = 2k+1 : k \in \mathbb{Z}$

$\cdot\ 3n-11 = 3(2k+1)-11$

$= 6k+3-11$

$= 6k-8$

$= 2(3k-4)$

Since $k \in \mathbb{Z}$, $3k-4 \in \mathbb{Z}$, also $3n-11 = 2(3x-4)$ which is a multiple of 2. $\therefore$ $3n-11$ is even but this is not true since $3n-11$ is odd so the supposition that n is odd leads to a contradiction

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6 $n = 2k \mid k \in \mathbb{Z}$

- $n^4 = (2k)^4$
$= 16(k^4)$
$= 16q$

Since $16q$ is a multiple of 2. This shows that n^4 is even which leads to a contradiction.
$\therefore$ the supposition that n is even is wrong

$n = 2k+1 \mid k \in \mathbb{Z}$

- $n^4 = (2k+1)^4$
$= (8k^3 + 1 + 6k(2k+1)) \times (2k+1)$
$= (8k^3 + 1 + 12k^2 + 6k) \times (2k+1)$
$= 16k^4 + 8k^3 + 2k + 1 + 24k^3 + 12k^2 + 12k^2$
$= 16k^4 + 32k^3 + 24k^2 + 8k + 1$
$= q$

$n^4 = q$ which is not a multiple of 2 $\therefore n^4$ is odd

8 a • $n-3 = 2m$
$n+4 = 2m+3+4$
$= 2m+7$
$= 2(m+3)+1$
$= 2p+1, \ p = m+3 \in \mathbb{Z}$

- $n+4 = 2m+1$
$n-3 = (2m+1-4)-3$
$= 2m-6$
$= 2(m-3)$
$= 2p, \ p = m-3 \in \mathbb{Z}$

b. • $n-3 = 2m$
$n+4 = 2m+3+4$
$= 2m+7$
$= 2(m+3)+1$
$= 2p+1, \ p = m+3 \in \mathbb{Z}$

if $n-3$ is even then $n+4$ is odd

- $n+4$ is even
$n+4 = 2m$
$n-3 = 2m-4-3$
$= 2m-7$
$= 2(m-4)+1$
$= 2p+1, \ p = m-4 \in \mathbb{Z}$

- if $n+4$ is even then $n-3$ is odd
- Contrapositive $n+4$ is odd then $n-3$ is even

$\cancel{2}, \cancel{4}, \cancel{6}, \cancel{8}, \cancel{10}, 12, 14$

3.3

8 c. $n-3 = 2m+1$

So $n+4 = (2m+1+3)-4$

$= 2m+8$

$= 2(m+4)$

$= 2p, p = m+4 \in \mathbb{Z}$

if $n-3$ is odd then $n+4$ is even

Contrapositive: if $n-3$ is even then $n+4$ is odd

$n+4 = 2m$

$n-3 = 2m-4-3$

$= 2m-7$

$= 2(m-4)+1$

$= 2p, p = m-4 \in \mathbb{Z}$

if $n+4$ is even then $n-3$ is odd

contra... $n-3$ is even then $n+4$ is odd

10 $y = 5$

$x \geq 9 - y$

$x \geq 9 - 5$

$x \geq 4$

So $x \geq 5$ is false only when $y \geq 5$

$y = 4$

$x \geq 9 - y$

$x \geq 9 - 4$

$x \geq 5$

So $x \geq 5$ is true only when $y < 5$

12 $a, b, m \in \mathbb{Z}$

$a = 3m+1, \; b = 2m+1, \; m \geq 1$

$2a+3b \geq 12m+1$

$2(3m+1) + 3(2m+1) \geq 12m+1$

$6m+2+6m+3 \geq 12m+1$

$12m+5 \geq 12m+1$

$5 \geq 1$ so here condition is true

$2, 4, 6, 8, 10$

14 Proof is correct

3.4

2 n is even, $n = 2m$

$n^3 - n = (2m)^3 - 2m$

$= 8m^3 - 2m$

$= 2m(4m^2 - 1)$ is an integer $\therefore n^3 - n$ is even in this case

n is odd, $n = 2m+1$

$n^3 - n = (2m+1)^3 - (2m+1)$

$= (2m+1)((2m+1)^2 - 1)$

$= (2m+1)(4m^2 + 1 - 4m - 1)$

$= (2m+1)(4m^2 - 4m)$

$n^3 - n = 4m(2m+1)(m-1)$ is an integer $\therefore n^3 - n$ is even

4 • $n = 2m$

$3n+1 = 3(2m) + 1$

$= 6m + 1$

$= 6m + 2 - 1$

$= 2(3m+1) - 1$

is odd

• $n = 2m+1$

$5n+1 = 5(2m+1) + 1$

$= 10m + 5 + 1$

$= 2(5m+3)$ is even

• $n = 2m+1$

$3n+1 = 3(2m+1) + 1$

$= 6m + 3 + 1$

$= 2(3m+2)$

is even

$n = 2m$

$5n+1 = 5(2m) + 1$

$= 10m + 1$

$= 10m + 2 - 1$

$= 2(5m+1) - 1$ is odd

6, 8, 10

6 $3m-n$

$m=2a$ + $n=2b$

$3m-n = 3(2a)-(2b)$

$= 6a-2b$

$= 2(3a-b) \in \mathbb{Z}$ is even

$m=2a+1$ + $n=2b+1$

$3m-n = 3(2a+1)-(2b+1)$

$= 6a+3-2b-1$

$= 6a-2b+2$

$= 2(3a-b+1) \in \mathbb{Z}$ is even

$m=2a$ + $n=2b+1$

$3m-n = 3(2a)-(2b+1)$

$= 6a-2b+1$

$= 2(3a-b)+1 \in \mathbb{Z}$ is odd

$m=2a+1$ + $n=2b$

$3m-n = 3(2a+1)-(2b)$

$= 6a+3-2b$

$= 2(3a-b-1)-1$ is odd

8 $m = 2a+1$ & $n = 2b$

$3m-4n = 3(2a+1) - 4(2b)$
$= 6a+3-8b$
$= 6a-8b+2+1$
$= 2(3a-4b+1)+1$

is odd.

$m = 2a+1$ & $n = 2b$

$2m+n = 2(2a+1)+2b$
$= 4a+2+2b$
$= 2(2a+b+1)$

is even

$m = 2a$ + $n = 2b+1$

$3m-4n = 3(2a) - 4(2b+1)$
$= 6a-(8b+4)$
$= 2(3a-4b-2)$

is even

$m = 2a$ & $n = 2b+1$

$2m+n = 2(2a) + 2b+1$
$= 4a+2b+1$
$= 2(2a+b+1)-1$

is odd

10 $m = 2a$ & $n = 2b$

$mn = 2a(2b)$
$= 4ab$

is even

$m+n = 2a+2b$
$= 4(a+b)$

is even

$m = 2a+1$ & $n = 2b$

$mn = (2a+1)(2b)$
$= 4ab+2b$
$= 2(ab+b)$

is even

$m+n = 2a+1+2b$
$= 2a+2b+1$
$= 2(a+b+1)-1$

is odd

~~2~~, ~~4~~, ~~6~~, ~~8~~, ~~10~~, ~~12~~, ~~14~~, 16

3.5

2 $m=2$, $n=2$

$mn = 2\times 2$

$= 4$

4 $x=0$, then $-1<x<1$

$x^2-x-2>0$ at $x=0$

$x^2-x-2 = (0)^2-(0)-2$

$= 0-0-2$

$= -2<0$ Hence not true for $x=0$

6 $A=\{2,3\}$, $B=\{1,3\}$ + $C=\{1,2\}$

$A\cup B = \{1,2,3\}$ + $A\cup C = \{1,2,3\}$

$A\cup B = A\cup C$ $B\neq C$

8 $A=\{2,3\}$ $B=\{1,2\}$ $A\cup B=\{1,2,3\}$

$P(A\cup B) = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\{\}\}$

$P(A) = \{\{2\},\{3\},\{2,3\}\{\}\}$

$P(B) = \{\{1\},\{2\},\{1,2\}\{\}\}$

$P(A\cup B) \neq P(A)\cup P(B)$

10 $m=0$ + $n=-1$

$n<m<n^2$ true when $m=0$ + $n=-1$

$-1<0<(-1)^2$

$-1<0<1$

$m=-1$ + $n=0$

$n<m<n^2$ false when $m=-1$ + $n=0$

$0<-1<(0)^2$

$0<-1<0$

a.

12 if $n^2-4=0$ then $(n+2)(n-2)=0$

either $n+2=0$ or $n-2=0$

but $n+2\neq 0$ because $n=-2$ but n must be a positive integer $\Rightarrow n-2=0$

b. if $n^2-4=0$ then $(n+2)(n-2)=0$

either $n+2=0$ or $n-2=0$ or both $=0$

But $n-2\neq 0$ because if $n=2$ it is not a negative integer

$n+2=0$

a.

14 $\frac{n^2-n}{2}=\frac{n(n-1)}{2}$. $n(n-1)$ is a multiple of 4 for $n=4, 8$ and so is $n(n-1)$ and for $n=1, 8$ $n-1$ is a multiple of 4

$\Rightarrow n(n-1)=4k$ for some $k\in\mathbb{Z}$ & $\frac{n(n-1)}{2}=2k$ for some $k\in\mathbb{Z}$ $\therefore \frac{n^2-n}{2}$ is even $\in\mathbb{Z}$

b. $n=6$ $\frac{(6^2-6)}{2}=\frac{36-6}{2}=\frac{30}{2}=15$

when $n=6$

15 is an odd integer

$\cancel{2}, \cancel{4}, \cancel{6}, 8, 10, 12$

16 $a=2,\ b=2$

$2a+3b=2(2)+3(2)$

$=4+6$

$=10$ iseven as well as 2

3.6

2 $(m-2)^2+(n-6)^2 \le 1$ $m=3,\ n=6$

$(m-2)^2+(n-6)^2=(3-2)^2+(6-6)^2$

$=1^2+0^2$

$=1$

≤ 1

4 • $a<b$ • $\frac{a+b}{2}=r$ • $a<\frac{a+b}{2}<b$

6 $a=\sqrt{2}$ $b=1$ $a^b=\sqrt{2}^1=\sqrt{2}$ Which is irrational

8 $n=1$

$4n^2-8n+3=4(1)^2-8(1)+3$

$=4-8+3$

$=-1$

<0

10 $x^2=y$ $y^2-y+2=0$

$y=\frac{-(-1)\pm\sqrt{1-4\cdot1\cdot2}}{2\cdot1}$

$=\frac{1\pm\sqrt{-7}}{2}=\frac{1\pm i\sqrt{7}}{2}=x^2$

$x=\frac{1\pm i\sqrt{7}}{2}$ which is not a real number

✓

$2, 4, 6, 8, 10$

12 $a < b+1 \Rightarrow a < |b|+1$

$\therefore a - |b| < 1 \Rightarrow |a - |b|| < 1$

3.7

2 a, $a+2$, $a+4$ are odd integers

so $a + (a+2) + a+4 = 100$

$\Rightarrow 3a + 6 = 100$

$3a = 100 - 6 = 94$

$\frac{3a = 94}{3} = a = 94/3$ which is not an integer

4 a, $a+2$, $a+4$ are even. k = some odd integer

$a + a+2 + a+4 = k$

$3a + 6 = k$

$3a = k+6$

$a = \frac{k}{3} + 2$

Odd integer over odd integer is either odd or not integer

6 $a = 1$ + $b = \sqrt{2}$

$ab = 1 \cdot \sqrt{2}$

$= \sqrt{2}$ this is irrational

8 n is even + k is even

$7(n) + 9 = k$

$7n = k - 9$ which makes k odd

$n = k/7$

an odd integer over odd integer is either odd or Not integer

$2, 4, 6, 8$

10 $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

$(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$

$\Rightarrow a+b+2\sqrt{a}\sqrt{b} = a+b$

$\Rightarrow 2\sqrt{a}\sqrt{b} = 0$ $\quad$ & $2 \neq 0$

$\Rightarrow \sqrt{a}\sqrt{b} = 0$

this contradicts if a and b are positive real numbers

4.1 $\quad 2, 4, 6, 8$

2 for $n=1$ $\quad \frac{1}{1\cdot 3} = \frac{1}{2(1)+1}$ ✓

$\frac{1}{3} = \frac{1}{3}$

$n=k \quad \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} \cdots \frac{k}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

$n=k+1 \quad \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

$\hookrightarrow \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} =$

$\hookrightarrow \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

Checks out

$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$

$= \frac{k+1}{2k+3}$ ✓

4 first n odd positive integers is:

$$1+3+5+\cdots+(2n-1)=n^2$$

first $2n$ positive integers is

$$1+2+3+4+5+\cdots+(2n-1)+2n=\frac{2n(2n+1)}{2}$$

$\therefore$ first n even positive integers is

$$[1+2+3+4+5+\cdots+(2n-1)+2n]-[1+3+5+\cdots+(2n-1)]=\frac{2n(2n+1)}{2}-n^2$$

$$\Rightarrow 2+4+6+\cdots+2n=\frac{2n(2n+1)}{2}-n^2$$
$$=2n^2+n-n^2$$
$$=n(n+1)$$

$\therefore$ for every positive integer n

$$2+4+6+\cdots+2n=n(n+1)$$

$n=1$ $\quad 2+4+6+\cdots+2(1)=1(1+1)$
$$2=2 \checkmark$$

$n=k$ $\quad 2+4+6+\cdots+2(k)=k(k+1)$

$n=k+1$ $\quad 2+4+6+\cdots 2(k+1)=k(k+1)+2(k+1)$
$$=(k+1)(k+2)$$

$\Rightarrow \quad 2+4+6+\cdots+2k+2(k+1)=(k+1)(k+2) \checkmark$

6 $n=1$ $\quad 2+5+8+\cdots+(3n-1)=n(3n+1)/2$

$$3(1)-1 = (1(3(1)+1))/2$$
$$2 = 4/2$$
$$2 = 2$$

$n=k$ $\quad 2+5+8+\cdots+(3k-1)=k(3k+1)/2$

$n=k+1$ $\quad 2+5+8+\cdots+(3k+2)=\frac{(k+1)(3k+4)}{2}$

$$2+5+8+\cdots+(3k-1)(3k+2)=\frac{k(3k+1)}{2}+(3k+2)$$
$$=\frac{k(3k+1)+2(3k+2)}{2}$$
$$=\frac{3k^2+k+6k+4}{2}$$
$$=\frac{3k^2+7k+4}{2}$$
$$=\frac{(k+1)(3k+4)}{2}$$

checks out

8 $n=1$ $\quad 1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

$$(1)^3=\frac{1^2(1+1)^2}{4}$$
$$1=\frac{4}{4}$$
$$1=1$$

$n=k$ $\quad 1^3+2^3+\cdots+k^3=\frac{k^2(k+1)^2}{4}$

$n=k+1$ $\quad 1^3+2^3+\cdots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3$

$$=(k+1)^2\left[\frac{k^2}{4}+(k+1)\right]$$
$$=(k+1)^2\left[\frac{k^2+4+4}{4}\right]$$
$$=\frac{(k+1)^2(k+2)^2}{4}$$

yes

4.2 2, 4, 6

2 $n=7$ $7! >? 3^7$

$5040 >? 2187$ ✓

$n=k+1$ $(k+1)! >? 3^{(k+1)}$

$(k+1)k! >? (k+1)3^k$

Since $k \in S = \{i \in \mathbb{Z} : i \geq 7\}$

$\Rightarrow k+1 \geq 8$

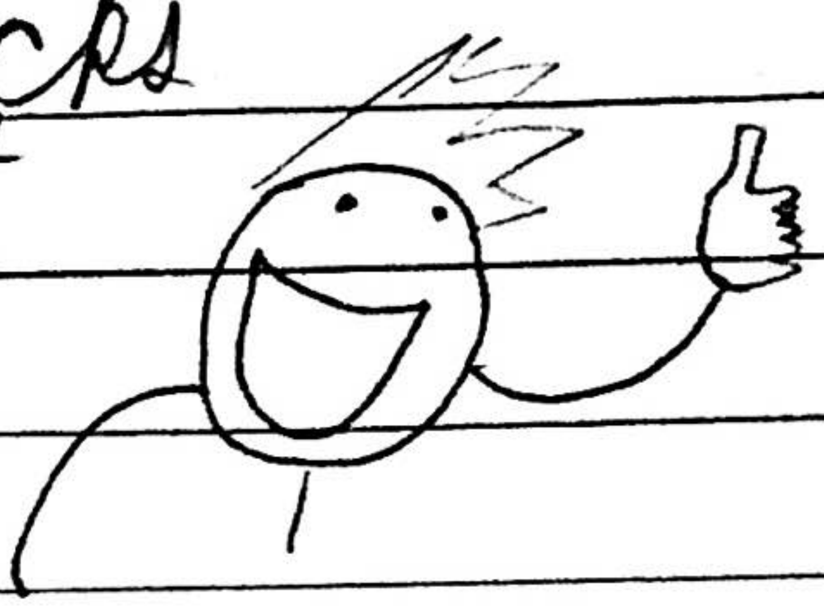
$\Rightarrow k+1 > 3$

$\Rightarrow (k+1)! > 3^k(k+1)$

$> 3^k(3)$

$= 3^{k+1}$ ✓

Checks Out



4 $n=1$ $3^1 > 1$

$3 > 1$ ✓

$n=k+1$ $3^{k+1} > k+1$

$k=0$ $3^{0+1} > 0+1$

$3^1 > 1$ ✓

$k \geq 1$ $3^{k+1} > 3k$

$3^k(3) > k+2k$

$> k+1$ ✓

Still Checks out

6 $n=2$ $2^2 >? 2+1$

$4 >? 3$ ✓ Yes

$n=k+1$ $(k+1)^2 >? (k+1)+1$

$k^2+2k+1 >? k+1+2k+1$

$k^2+2k+1 >? k+2+2k$

Since $k \in S = \{i \in \mathbb{Z} : i \geq 2\}$

$k \geq 2$

$\Rightarrow 2k \geq 4$

$2k > 0$

Checks out

$\Rightarrow (k+1)^2 > k+2$ ✓