1 The Koch Snowflake

The *Koch snowflake* one of the first fractals, is based on work by the Swedish mathematician Helge von Koch [1]. It is what we get if we start with an

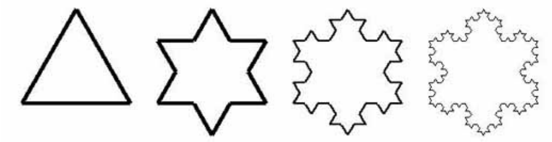

Figure 1: The initial equilateral triangle and the refinement of the Koch snowflake after one, two, and ther iterations.

equilateral triangle and repeat the following an infinite number of times:

> *Divide all line segments into three segments of equal length. Then draw, for each middle line segment an equilateral triangle that has the middle segment as its base and points outward. Finally, remove all middle segments.*

Figure 1 shows the first iterations in the construction. (Original)

1.1 Two properties

Theorem 1. *The Koch snowflake has infinite length.*
Proof. Let Δ denote a triangle, with side length s, on which we base the construction of a snowflake. Len N_i denote the number of line segments, and L_i the lenth of the segments, in iteration i of the construction. Then

$$N_n = \begin{cases} 3 & \text{if } n = 0 \text{ (i.e before any iterations), and} \\ 4N_{n-1} & \text{otherwise,} \end{cases}$$

which solves to

$$N_n = 3 \cdot 4^n, \tag{1}$$

while

$$L_n = \frac{L_{n-1}}{3} = \frac{L_{n-2}}{3^2} = \frac{L_{n-3}}{3^3} = \ldots = \frac{L_0}{3^n} = \frac{s}{3^n} \tag{2}$$

The total length

$$N_n L_n = 3 \cdot 4^n \frac{s}{3^n} = 3s\frac{4^n}{3^n} = 3s\left(\frac{4}{3}\right)^n.$$

Since $4/3 > 1$, it follows that $N_n L_n$ tends to infinity as $n \to \infty$, i.e. the Koch snowflake has infinite length. $\square$

Theorem 2. *The Koch snowflake has finite area.*
Proof. In an iteration, a triangle is added on each line segment of the previous iterations. So, in iteration n, the number of new triangle $T_n = N_{n-1}$, which, by Eq. 1, can be siplified to

$$T_n = \frac{3}{4} \cdot 4^n. \tag{3}$$

The area a_n of each such triangle, with the exception of the area $a_0 = \frac{\sqrt{3}}{4}s^2$ of Δ, is one ninith of the area of a triangle added in iteration $n-1$, or

$$a_n = \frac{a_{n-1}}{9} = \frac{a_{n-2}}{9^2} = \frac{a_{n-3}}{9^3} = \ldots = \frac{a_0}{9^n}. \tag{4}$$

This means that in iteration n be Eqs. 3 and 4, the area of all added triangles

$$b_n = T_n a_n = \left(\frac{3}{4} \cdot 4^n\right)\left(\frac{a_0}{9^n}\right) = \frac{3a_0}{4}\left(\frac{4}{9}\right)^n$$

All in all, after iteration n, the tatal are (Original)

$$\begin{aligned} A_n &= a_0 + \sum_{k=1}^{n} b_k \\ &= a_0\left(1 + \frac{3}{4}\sum_{k=1}^{n}\left(\frac{4}{9}\right)^k\right) \\ &= a_0\left(1 + \frac{1}{3}\sum_{k=0}^{n-1}\left(\frac{4}{9}\right)^k\right) \\ &= a_0\left(1 + \frac{3}{5}\left(1 - \left(\frac{4}{9}\right)^n\right)\right) \\ &= \frac{a_0}{5}\left(8 - 3\left(\frac{4}{9}\right)^n\right) \end{aligned}$$

Now, since

$$\lim_{n\to\infty} 3\left(\frac{4}{9}\right)^n = 0,$$

it follows that $lim_{n\to\infty} A_n = \frac{8a_0}{5}$, i.e. the Koch snowflake has finite area. □

Referenser

[1] H. von Koch, *Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire*, Arkiv för matematik, astronomi, och fysik, Kngliga Vetenskapsakademien. 1, 681-702, 1904