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Homework Week 3: Gaussian Distribution

Exercise 1:

(a) Prove that the Univariate Gaussian PDF is normalized

It means: proving that:

$$\int_{-\infty}^{+\infty} p(x|\mu,\sigma^2)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1$$

Having: $A = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$

+). $z = \frac{x-\mu}{\sigma}$ (standard normalize distribution)

$\Rightarrow x = \sigma z + \mu \Rightarrow dx = \sigma dz$

Then, substitute $x = \sigma z + \mu$, we have:

$$A = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\sigma z+\mu-\mu)^2/2\sigma^2} \sigma\,dz$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\sigma^2 z^2/2\sigma^2} dz$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

+) $y^2 = \frac{z^2}{2} \Rightarrow \left[\begin{array}{l} y = z/\sqrt{2} \\ y = -z/\sqrt{2} \end{array}\right. \Rightarrow \left[\begin{array}{l} z = \sqrt{2}y \\ z = -\sqrt{2}y \end{array}\right. \Rightarrow \left[\begin{array}{l} dz = \sqrt{2}dy \\ dz = -\sqrt{2}dy \end{array}\right.$

$$\Rightarrow \left[\begin{array}{l} A = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2} \sqrt{2}\,dy \\ A = -\int_{+\infty}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2} \sqrt{2}\,dy \end{array}\right. \Rightarrow A = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2} \sqrt{2}\,dy$$

$$\Rightarrow A = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-y^2} dy$$

According to the Gaussian Integral: $\int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\pi}$

$$\Rightarrow A = \frac{1}{\sqrt{\pi}} \times \sqrt{\pi} = 1 \text{ or } p(x \mid \mu, \sigma^2) = 1$$

Hence, the Univariate Gaussian PDF is normalized

(b) A random variable X follows Gaussian distribution (notation: $X \sim N(\mu, \sigma^2)$). Prove that the value of X is μ and the standard deviation is σ

⊕ The expected value of X is μ.
That means we need to prove that

$$E(X) = \int_{-\infty}^{+\infty} x p(x \mid \mu, \sigma^2) = \mu$$

⊙ We have:

$$E(X) = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

+) $t = \frac{x-\mu}{\sqrt{2}\sigma} \Rightarrow x = \sqrt{2}\sigma t + \mu \Rightarrow dx = \sqrt{2}\sigma dt$

$$\Rightarrow E(X) = \int_{-\infty}^{+\infty} (\sqrt{2}\sigma t + \mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2} \sqrt{2}\sigma dt$$

$$= \int_{-\infty}^{+\infty} \frac{\sqrt{2}\sigma}{\sqrt{2\pi\sigma^2}} (\sqrt{2}\sigma t + \mu) e^{-t} dt$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \int_{-\infty}^{+\infty} t e^{-t} dt + \mu \int_{-\infty}^{+\infty} e^{-t^2} dt \right.$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \left[-\frac{e^{-t^2}}{2} \right]_{-\infty}^{+\infty} + \mu\sqrt{\pi} \right)$$

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$$\left(\int_{-\infty}^{+\infty} e^{-t} dt = \sqrt{\pi} \text{ \_ Gaussian Integral} \right)$$

$$= \frac{\mu\sqrt{\pi}}{\sqrt{\pi}} = \mu$$

$$\Rightarrow E(X) = \int_{-\infty}^{+\infty} x p(x|\mu, \sigma^2) = \mu$$

Hence, the expected value of X is μ

(b) The standard deviation is σ

+) $Var(X) = E(X^2) - (E(X))^2$

$$= \int_{-\infty}^{+\infty} x^2 p(x|\mu, \sigma^2) dx - \mu^2 \quad \text{(proved in a)}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x^2 e^{-(x-\mu)^2/2\sigma^2} dx - \mu^2$$

+) $t = \frac{x-\mu}{\sqrt{2}\sigma} \Rightarrow x = \sqrt{2}\sigma t + \mu \Rightarrow dx = \sqrt{2}\sigma dt$

$$\Rightarrow Var(X) = \int_{-\infty}^{+\infty} (\sqrt{2}\sigma t + \mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2} d \sqrt{2}\sigma dt - \mu^2$$

$$= \frac{\sqrt{2}\sigma}{\sqrt{2\pi\sigma^2}} \left(\int_{-\infty}^{+\infty} 2\sigma^2 t^2 e^{-t^2} dt + 2\sqrt{2}\sigma\mu \int_{-\infty}^{+\infty} t e^{-t^2} dt + \mu^2 \int_{-\infty}^{+\infty} e^{-t} dt \right) - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{+\infty} t^2 e^{-t^2} dt + 2\sqrt{2}\sigma\mu \left[\frac{-e^{t^2}}{2} \right]_{-\infty}^{+\infty} + \mu^2\sqrt{\pi} \right) - \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left(\left[-\frac{t}{2} e^{-t^2} \right]_{+\infty}^{-\infty} + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-t^2} dt \right) + \mu^2 - \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \times \frac{1}{2} \int_{-\infty}^{+\infty} e^{-t^2} dt$$

$$= \frac{2\sigma^2 \sqrt{\pi}}{2\sqrt{\pi}} = \sigma^2 \Rightarrow Var(X) = \sigma$$

$\Rightarrow$ Standard deviation $= \sqrt{Var(X)} = \sigma$

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Ex2

(a) $p(x \mid \mu, \sigma^2) = \dfrac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$

Ex2: (a) Prove ...

- For a D-dimensional vector x, the multivariate Gaussian distribution takes the form

$$p(x \mid \mu, \sigma^2) = \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$

μ : D-dimensional mean vector
Σ : D x D ~~vector~~ covariance matrix
$|\Sigma|$: determinant of Σ

- Set. $\Delta^2 = (x-\mu)^T \Sigma^{-1}(x-\mu)$

$$= \frac{-1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + \text{const}$$

It's a quadratic form of Gaussian distribution
Consider eigenvalues and eigenvectors of Σ

$$\Sigma u_i = \lambda_i u_i \quad i = 1 \ldots D$$

Because Σ is a real, symmetric matrix, its eigenvalues will be real and its eigenvectors form an orthonormal set

$$\Sigma = \sum_{i=1}^{D} \lambda_i u_i u_i^T \Rightarrow \Sigma^{-1} = \sum_{i=1}^{D} \frac{1}{\lambda_i} u_i u_i^T$$

- So that

$$\Delta^2 = (x-\mu)^T \Sigma^{-1}(x-\mu) = \sum_{i=1}^{D} \frac{1}{\lambda_i}(x-\mu)^T u_i u_i^T (x-\mu)$$

$$= \sum_{i=1}^{D} \frac{y_i^2}{\lambda_i} \text{ with } y_i = u_i^T(x-\mu)$$

$$|\Sigma|^{1/2} = \prod_{j=1}^{D} \lambda_j^{1/2}$$

$$p(y) = \prod_{j=1}^{D} \frac{1}{(2\pi\lambda_j)}^{1/2} \exp\left(- \frac{y_j^2}{2\lambda_j}\right.$$

$$\Rightarrow \int_{-\infty}^{\infty} p(y)\, dy = \prod_{j=1}^{D} \int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_j)}^{1/2} \exp\left(\frac{-y_j^2}{2\lambda_j}\right) dy_j$$

$$= 1$$

$\Rightarrow$ Multivariate Gaussian PDF is normalized

Exercise 2

(c) Find the formula of conditional distribution in Multivariate Gaussian distribution

⊙ Suppose that x is a D-dimensional vector with Gaussian distribution $\mathcal{N}(x|\mu, \Sigma)$ and that we partition x into 2 disjoint subjet x_a and x_b

$$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

Similarly: +) $\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$ (mean vector)

+) covariance matrix Σ:

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix} \Rightarrow A = \Sigma^{-1} = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix}$$

Σ is symmetric so Σ_{aa} and Σ_{bb} are symmetric while $\Sigma_{ab} = \Sigma_{ba}^T$

We are looking for conditional distribution $p(x_a|x_b)$

⊙ We have

$$-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) = \frac{-1}{2}(x-\mu)^T A (x-\mu)$$

$$= \frac{-1}{2}(x_a-\mu_a)^T A_{aa}(x_a-\mu_a) - \frac{1}{2}(x_a-\mu_a)^T A_{ab}(x_b-\mu_b)$$

$$-\frac{1}{2}(x_b-\mu_b)^T A_{ba}(x_a-\mu_a) - \frac{1}{2}(x_b-\mu_b)^T A_{bb}(x_b-\mu_b)$$

$$= \frac{-1}{2}x_a^T A_{aa}^{-1} x_a + x_a^T(A_{aa}\mu_a - A_{ab}(x_b-\mu_b)) + \text{const}$$

It is quad-ratic form of x_a hence conditional distribution $p(x_a|x_b)$ will be Gaussian, because this distribution is characterized by its mean and its variance. Compare with Gaussian distribution

$$\Delta^2 = -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + \text{const}$$

$$\Sigma_{a|b} = A_{aa}^{-1}$$

$$\mu_{a|b} = \Sigma_{a|b} (A_{aa} \mu_a - A_{ab} (x_b - \mu_b))$$

$$= \mu_a - A_{aa}^{-1} A_{ab} (x_b - \mu_b)$$

By using Schur complement

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CMD^{-1} & D^{-1}CMBD^{-1} \end{pmatrix}, \quad M = (A - BD^{-1}C)^{-1}$$

$$\Rightarrow A_{aa} = (\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba})^{-1}$$

$$A_{ab} = -(\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba})^{-1} \Sigma_{ab} \Sigma_{bb}^{-1}$$

As a result: $\mu_{a|b} = \mu_a + \Sigma_{ab} \Sigma_{bb}^{-1} (x_b - \mu_b)$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}$$

$$\Rightarrow p(x_a | x_b) = \mathcal{N}(x_{a|b} | \mu_{a|b}, \Sigma_{a,b})$$

Exercise 2:

(a) Prove that the Multivariate Gaussian PDF is normalize.

(b) Find the formula of marginal distribution in Multivariate Gaussian Distribution

* The margional distribution given by

$$p(x_a) = \int p(x_a, x_b)\, dx_b$$

* We need to integrate out x_b by looking the quadratic form related to x_b

$$-\frac{1}{2} x_b^T A_{bb} x_b + x_b^T m$$

$$= \frac{-1}{2}(x_b - A_{bb}^{-1} m)^T A_{bb}(x_b - A_{bb}^{-1} m) + \frac{1}{2} m^T A_{bb}^{-1} m$$

with $m = A_{bb}\mu_b - A_{ba} - A_{ba}(x_a - \mu_a)$

* We can integrate over unormalized Gaussian

$$\int \exp\left(\frac{-1}{2}(x_b - A_{bb}^{-1} m)^T A_{bb}(x_b - A_{bb}^{-1} m)\right) dx_b$$

* The remaining term

$$\frac{-1}{2} x_a^T (A_{aa} - A_{ab} A_{bb}^{-1} A_{ba}) x_a + x_a^T (A_{aa} - A_{ab} \times A_{bb}^{-1} A_{ba})^{-1} \times \mu_a + \text{const}$$

Similarly, we have: $E(X) = \mu_a$

$\text{cov}[x_a] = \Sigma_{aa}$

$\Rightarrow p(x_a) = \cancel{*}\, N(x_a | \mu_a, \Sigma_{aa})$

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